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*Research article*

# Fixed point theory in elliptic-valued metric spaces: applications to Fredholm integral equations and climate change analysis

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**Abstract:** The aim of this research article was to explore the framework of elliptic-valued metric spaces and establish novel common fixed point theorems for a wide range of generalized contraction mappings. Our findings provide a significant extension and unification of existing fixed point results, offering more refined mathematical tools for analyzing nonlinear problems in abstract spaces. To illustrate the effectiveness of our theorems, we present a concrete example that underscores the originality of our approach. Additionally, we highlight the practical significance of our main result by applying it to solve a nonlinear Fredholm integral equation of the second kind, which plays a crucial role in climate modeling. This equation is essential for understanding temperature distribution dynamics and feedback mechanisms within Earth's energy balance system. By bridging theoretical developments in metric space theory with real-world climate applications, this study contributes to both mathematical research and environmental science.

**Keywords:** fixed point; elliptic-valued metric space; self mappings; nonlinear Fredholm integral equations
**Mathematics Subject Classification:** 46S40, 47H10, 54H25

## 1. Introduction

The real number system combines rational and irrational numbers into a continuous line, forming the foundation of conventional mathematics. It supports both everyday measurements and scientific modeling. To address more complex mathematical problems, the real numbers were later extended to complex numbers in the 16th century.

Pioneered by Gerolamo Cardano (1501–1575) and later refined by Rafael Bombelli (1526–1572), complex numbers introduced the imaginary unit $i$, defined by $i^2 = -1$, to solve cubic equations like $x^3 = 15x + 4$ [1, 2]. This breakthrough expanded algebraic problem-solving, enabling roots for previously "impossible" equations. Mathematicians soon reimagined the imaginary unit's properties, yielding new number systems. In the 19th century, William Kingdon Clifford [3] proposed a unit $j$ with $j^2 = 1$, creating a system where numbers take the form $a + bj$. Unlike complex numbers, hyperbolic numbers split into "light-like" components (where $a^2 - b^2 = 0$), making them useful in relativity theory and mechanical wave analysis. Their geometry models Lorentz transformations in spacetime. Introduced by Eduard Study [4] in the early 20th century, dual numbers use a nilpotent unit $\epsilon$ ($\epsilon^2 = 0$). Expressions like $a + b\epsilon$ efficiently compute derivatives in robotics and computer graphics, as $\epsilon$ "tags" infinitesimal changes–a process that is foundational to automatic differentiation in machine learning. Modern mathematicians generalize these systems by defining a unit $i$ with $i^2 = p$, where $p \in \mathbb{R}$. This parameter $p$ categorizes systems into:

- Elliptic ($p < 0$): Equivalent to complex numbers, forming a field with no zero divisors;
- Parabolic ($p = 0$): Dual numbers, featuring nilpotent elements for tangent space modeling;
- Hyperbolic ($p > 0$): Hyperbolic numbers, with zero divisors (e.g., $(1 + j)(1 - j) = 0$) applicable in split-complex algebra.

On the other hand, Fréchet [5] pioneered the idea of a metric space (MS) defined by a set equipped with a distance function that follows three core principles. First, the distance between any two distinct elements is always positive, while the distance from an element to itself is exactly zero; second, the distance between two elements remains unchanged regardless of the direction in which it is measured; and finally, the direct distance between two elements never exceeds the combined distance when measured through any intermediate third element. These principles underpin the rigorous study of continuity, convergence, and geometric relationships across mathematical theory and practical applications. In MS, one of the primary results is the Banach contraction principle [6], which asserts that any contraction mapping in a complete MS possesses a unique fixed point (FP). Later on, Fisher [7] improved upon the existing theorem by integrating rational contractions into the established principles of FP theory.

Azam et al. [8] introduced the framework of complex-valued metric spaces (C-VMSs) and established some FP results for mappings governed by rational-type contractive conditions. This concept naturally extends to defining complex-valued normed and inner product spaces, opening up significant avenues for further research. Although C-VMSs constitute a special case of cone metric spaces (CMSs) [9], they were introduced explicitly to address rational expressions that are not typically well defined in general cone metric settings. This is primarily because CMSs rely on Banach spaces, which lack a division structure. In contrast, the structure of complex numbers allows for division, making it possible to refine various analytical results that are otherwise inapplicable within the general theory of CMSs. Rouzkard et al. [10] further generalized the central theorem of Azam et al. [8] through the introduction of a supplementary term within the rational contractive condition. Sintunavarat et al. [11] later enhanced the contractive condition by introducing control functions and proved a common FP theorem, thereby extending the core result established by Azam et al. [8]. Subsequently, Ahmad et al. [12] extended the concept of control functions, originally used in the contractive inequality by Sintunavarat et al. [11], to the setting of complex-valued extended $b$-metric spaces (C-VE$b$MSs) and established new FP theorems. Mlaiki et al. [13] applied FP results in complex-valued triple controlled metric spaces (C-VTCMSs) to solve Fredholm-type integral equations. The foundational work of Öztürk and colleagues, detailed in [14], established elliptic-valued metric spaces (E-VMSs). These spaces utilize a metric defined through elliptic numbers, which are an expanded number system incorporating principles derived from elliptic function theory. These numbers, deeply connected to the complex analysis of elliptic curves, offer a more sophisticated mathematical structure than conventional real or complex numbers. In E-VMSs, the measurement of distance among points transcends simple scalar values, as it is modulated by the characteristics of elliptic functions. This innovative framework ensures a platform for examining geometric configurations in spaces exhibiting cyclic, periodic, or complex interactions, demonstrating significant applicability in fields such as number theory and algebraic geometry. Alamri et al. [15] broadened the scope of this concept and established some novel FP theorems in this context. Detailed discussions on this subject can be found in [16–19].

From a different perspective, integral equations are fundamental in real-life applications, especially in climate change modeling, as they provide a mathematical framework for understanding complex environmental processes. These equations are widely used to describe heat transfer, radiative balance, and energy exchange in the Earth's atmosphere and oceans. Nonlinear integral equations help capture the intricate feedback mechanisms that influence global temperature variations. In energy balance models (EBMs), integral equations account for non local heat transport, which is essential for predicting long-term climate patterns. They also play a key role in modeling the greenhouse effect by incorporating the absorption, emission, and scattering of radiation. Additionally, they aid in studying ice sheet dynamics, ocean circulation, and temperature distribution across different geographical regions. By integrating observational data with mathematical models, integral equations enhance the accuracy of climate predictions. Their application extends to assessing the impact of anthropogenic activities on global warming and extreme weather events. Through numerical and analytical techniques, scientists use these equations to develop more reliable climate projections. Ultimately, integral equations serve as a powerful tool for understanding and mitigating the effects of climate change. For further insights on this topic, readers may refer to [20–22].

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminary concepts and definitions related to E-VMSs. In Section 3, we establish our main FP theorems under generalized contractive conditions and provide non-trivial illustrative examples. Section 4 derives FP results in C-VMSs as direct consequences of the main theorems. Section 5 is devoted to applications: in the first part, we solve a nonlinear Fredholm integral equation of the second kind using our FP framework; in the second part, we apply this formulation to energy balance models (EBMs) to study climate dynamics. Finally, Section 6 concludes the paper with a summary of the main results and potential directions for future research.

## 2. Preliminaries

We begin by reviewing key notations and definitions that will be utilized throughout this paper. For a comprehensive treatment, the reader is referred to [2]. Let $\mathbb{E}_p$ denote the set of elliptic numbers, defined by

$$\mathbb{E}_p = \left\{ z = \nu + i\omega : \nu, \omega \in \mathbb{R},\ i^2 = p < 0 \right\}.$$

For any $z \in \mathbb{E}_p$, where $z = \nu + i\omega$, the real part is $\nu$ and the imaginary part is $\omega$. The elliptic norm of $z \in \mathbb{E}_p$ is given by

$$||z||_{\mathbb{E}} = \sqrt{z\bar{z}} = \sqrt{\nu^2 - p\omega^2},$$

where $\bar{z} = \nu - i\omega$ is the elliptic conjugate of $z$. It is straightforward to verify that $\mathbb{E}_p$, under standard addition and scalar multiplication, forms a two-dimensional real vector space. Moreover, due to its algebraic structure, $\mathbb{E}_p$ satisfies the axioms of a field. Consequently, each elliptic number $z = \nu + i\omega$ admits a unique representation in the plane $\mathbb{R}^2$, establishing a bijective correspondence between $\mathbb{E}_p$ and $\mathbb{R}^2$. This representation defines the elliptic plane. In this setting, the distance between $z_1 = (\nu_1, \omega_1)$ and $z_2 = (\nu_2, \omega_2)$ in $\mathbb{E}_p$ is given by

$$||z_1 - z_2||_{\mathbb{E}} = \sqrt{(\nu_1 - \nu_2)^2 - p(\omega_1 - \omega_2)^2},$$

provided that $(\nu_1 - \nu_2)^2 - p(\omega_1 - \omega_2)^2 > 0$ and $p < 0$.

Within this elliptic geometry, the set of points equidistant from the origin with a unit norm forms an ellipse, formulated through the equation

$$\nu^2 - p\omega^2 = 1,$$

which contrasts with the unit circle in the classical complex plane (see [2]). From this point onward, we denote the zero element of $\mathbb{E}_p$ by $\theta$.

A partial order $\preceq$ is defined on $\mathbb{E}_p$ as follows: For any $z_1 = \nu_1 + i\omega_1$, $z_2 = \nu_2 + i\omega_2 \in \mathbb{E}_p$, we write $z_1 \preceq z_2$ if and only if

$$Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus, $z_1 \preceq z_2$ holds if and only if any of the conditions listed below are true:

$$\begin{aligned}
\text{(i) } Re(z_1) &= Re(z_2) \text{ and } Im(z_1) < Im(z_2), \\
\text{(ii) } Re(z_1) &< Re(z_2) \text{ and } Im(z_1) = Im(z_2), \\
\text{(iii) } Re(z_1) &< Re(z_2) \text{ and } Im(z_1) < Im(z_2), \\
\text{(iv) } Re(z_1) &= Re(z_2) \text{ and } Im(z_1) = Im(z_2).
\end{aligned}$$

To be precise, $z_1 \precsim z_2$ (with $z_1 \neq z_2$) signifies that at least one of conditions (i), (ii), or (iii) hold, while $z_1 \prec z_2$ is used only when condition (iii) is true. The following outlines some basic properties of the partial order $\preceq$ on $\mathbb{E}_p$.

(i) If $\theta \preceq z_1 \precsim z_2$, then the elliptic norm of $z_1$ is strictly less than the elliptic norm of $z_2$, i.e., $||z_1||_{\mathbb{E}} < ||z_2||_{\mathbb{E}}$;

(ii) The relation $z_1 \preceq z_2$ holds if and only if $z_1 - z_1 \preceq \theta$;

(iii) If $z_1 \preceq z_2$ and $z_2 \preceq z_3$, then $z_1 \preceq z_3$ (transitivity);

(iv) For $z_1 \preceq z_2$ and any positive real number $\lambda > 0$, we have $\lambda z_1 \preceq \lambda z_2$;

(v) The conditions $\theta \preceq z_1$ and $\theta \preceq z_2$ do not guarantee that $\theta \preceq z_1 z_2$.

The concept of a C-VMS was introduced by Azam et al. [8] in 2011. This notion is a special case of CMSs by allowing distance values in $\mathbb{C}$, where division is well defined, a property not available in cone metric settings. A C-VMS is defined as follows.

**Definition 2.1.** *[8] Let $\Xi$ be a nonempty set. A function $d : \Xi \times \Xi \to \mathbb{C}$ is called a complex-valued metric on $\Xi$ if, for all $\varkappa, \sigma, \varrho \in \Xi$, the following conditions are satisfied:*

*$(cv_1)$ $\theta \preceq d(\varkappa, \sigma)$, and $d(\varkappa, \sigma) = \theta \Leftrightarrow \varkappa = \sigma$;*

*$(cv_2)$ $d(\varkappa, \sigma) = d(\sigma, \varkappa)$;*

*$(cv_3)$ $d(\varkappa, \sigma) \preceq d(\varkappa, \varrho) + d(\varrho, \sigma)$.*

*If d satisfies these conditions, then the pair $(\Xi, d)$ is called a C-VMS.*

Azam et al. [8] established the following theorem in the scope of C-VMS.

**Theorem 2.1.** *[8] Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$ be two self-mappings. If there exist the constants $\varpi_1, \varpi_2 \in [0, 1)$ such that $\varpi_1 + \varpi_2 < 1$ and*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_2 \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

We acknowledge the insightful discussion by Al-Mezel et al. [23] regarding the limitations of C-VMS and its relation to CMSs. However, in our work, we extend this framework by considering E-VMS, where the distances lie in the set $\mathbb{E}_p$ of elliptic numbers. These numbers generalize complex numbers by allowing the imaginary unit to satisfy $i^2 = p < 0$, resulting in a distinct geometry. Our setting permits rational-type contractive conditions and is structurally different from CMSs, enabling us to obtain results that are not directly derivable from the existing theories in CMSs.

Rouzkard et al. [10] generalized the theorem above in this way.

**Theorem 2.2.** *[10] Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$ be two self-mappings. If there exist the constants $\varpi_1, \varpi_2, \varpi_3 \in [0, 1)$ such that $\varpi_1 + \varpi_2 + \varpi_3 < 1$ and*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_2 \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3 \frac{d(\sigma, \mathcal{H}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

Sintunavarat et al. [11] proved the following theorem in the framework of C-VMS.

**Theorem 2.3.** *[11] Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$ be two self-mappings. If there exist the functions $\varpi_1, \varpi_2 : \Xi \to [0, 1)$ that satisfy the following conditions:*

*(i)*

$$\varpi_1(\mathcal{H}\varkappa) \le \varpi_1(\varkappa) \text{ and } \varpi_1(\mathfrak{F}\varkappa) \le \varpi_1(\varkappa),$$

$$\varpi_2(\mathcal{H}\varkappa) \le \varpi_2(\varkappa) \text{ and } \varpi_2(\mathfrak{F}\varkappa) \le \varpi_2(\varkappa);$$

*(ii)* $\varpi_1(\varkappa) + \varpi_2(\varkappa) < 1$;

*(iii)*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1(\varkappa)\, d(\varkappa, \sigma) + \varpi_2(\varkappa)\, \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

Öztürk et al. [14], inspired by the methodology employed by Azam et al. [8], introduced a novel category of MSs, specifically E-VMSs.

**Definition 2.2.** *[14] Let $\Xi$ be a nonempty set. A function $d : \Xi \times \Xi \to \mathbb{E}_p$ is called an elliptic-valued metric on $\Xi$ if, for all $\varkappa, \sigma, \varrho \in \Xi$, the following conditions are satisfied:*

*$(E_1)$ $\theta \preceq d(\varkappa, \sigma)$, and $d(\varkappa, \sigma) = \theta \Leftrightarrow \varkappa = \sigma$;*

*$(E_2)$ $d(\varkappa, \sigma) = d(\sigma, \varkappa)$;*

*$(E_3)$ $d(\varkappa, \sigma) \preceq d(\varkappa, \varrho) + d(\varrho, \sigma)$.*

*Then the couple $(\Xi, d)$ is said to be an E-VMS.*

**Example 2.1.** [14] Let $\Xi = \mathbb{E}_p$. We introduce $d : \mathbb{E}_p \times \mathbb{E}_p \to \mathbb{E}_p$ which is defined as follows:

$$d(z_1, z_2) = \|\nu_1 - \nu_2\|_{\mathbb{E}} + i\,\|\omega_1 - \omega_2\|_{\mathbb{E}},$$

where $z_1 = \nu_1 + i\omega_1, z_2 = \nu_2 + i\omega_2$ in $\mathbb{E}_p$ and $i^2 = p < 0$, then $\left(\mathbb{E}_p, d\right)$ satisfies the characteristics of an E-VMS.

**Example 2.2.** Let $\Xi = C[0, 1]$ the set of all real-valued continuous functions on the interval $[0, 1]$, and define

$$d(f, g) = \int_0^1 \left(|f(t) - g(t)| + i\left|f'(t) - g'(t)\right|\right) dt,$$

where $i^2 = p < 0$, so $d(f, g) \in \mathbb{E}_p$, and we assume that $f$ and $g$ are continuously differentiable on $[0, 1]$, and thus $(C[0, 1], d)$ is an E-VMS.

Öztürk et al. [14] proved the following FP theorem.

**Theorem 2.4.** *[14] Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. If there exists a constant $\varpi \in [0, 1)$ such that*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \preceq \varpi d(\varkappa, \sigma),$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathfrak{F}$ admits a unique FP.*

**Definition 2.3.** *[14] A point $\varkappa$ in the E-VMS $(\Xi, d)$ is defined as an e-interior element of a subset A within $\Xi$ if a non-zero elliptic number r, satisfying $\theta \prec r$ for some r in $\mathbb{E}_p$ can be found, allowing for*

$$B(\varkappa, r) = \{\sigma \in \Xi : d(\varkappa, \sigma) \prec r\} \subseteq A.$$

*The notation $B(\varkappa, r)$ represents an open ball in the E-VMS $(\Xi, d)$, centered at $\varkappa$ and having a radius of r. Consequently, a closed ball, denoted as $\overline{B(\varkappa, r)}$, is defined by the set*

$$\overline{B(\varkappa, r)} = \{\sigma \in \Xi : d(\varkappa, \sigma) \preceq r\}.$$

*A point $\varkappa$ in the set $\Xi$ is said to be a limit point of a subset $A \subset \Xi$ if, for every nonzero elliptic number r, there exists at least one point in A that lies within the punctured elliptic ball with a center $\varkappa$ and radius r. Mathematically, this is expressed as*

$$(B(\varkappa, r) - \{\varkappa\}) \cap A \neq \emptyset.$$

*The characteristic of openness for a subset A in $\Xi$ is determined by whether every point within A is an interior point. A subset A is designated as closed if it encompasses all of its limit points. The set $\Omega$, consisting of all open balls $B(\varkappa, r)$, where $\varkappa$ belongs to $\Xi$ and $\theta \prec r$ for some r in $\mathbb{E}_p$, establishes a sub-basis that generates a Hausdorff topology on $\Xi$.*

**Definition 2.4.** *[14] Let* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *be a sequence in an E-VMS* $(\Xi, d)$.

*(i) We say that the sequence* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *converges to* $\varkappa \in \Xi$ *if, given every non-zero elliptic number* $\delta \in \mathbb{E}_p$ *with* $\theta \prec \delta$*, there is a* $\jmath_0 \in \mathbb{N}$ *such that* $d(\varkappa_\jmath, \varkappa) \prec \delta$, $\forall \jmath > \jmath_0$*. This is written as* $\varkappa_\jmath \to \varkappa$ *as* $\jmath \to \infty$ *or* $\lim_{\jmath\to\infty} \varkappa_\jmath = \varkappa$.

*(ii) A sequence* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *in* $(\Xi, d)$*is a Cauchy sequence if, for every non zero elliptic number* $\delta \in \mathbb{E}_p$ *with* $\theta \prec \delta$*, there exists* $\jmath_0 \in \mathbb{N}$ *such that* $d(\varkappa_\jmath, \varkappa_{\jmath+m}) \prec \delta$ *for all* $\jmath > \jmath_0$ *and* $m \in \mathbb{N}$.

*(iii) The E-VMS* $(\Xi, d)$ *is defined as complete if and only if every Cauchy sequence within* $\Xi$ *converges to an element of* $\Xi$.

The upcoming lemmas will be employed in the proof of our leading results.

**Lemma 2.1.** *[14] Let* $(\Xi, d)$ *be an E-VMS and let* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *be a sequence in* $\Xi$*. The sequence* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *converges to* $\varkappa \in \Xi$ *if and only if the elliptic norm of* $d(\varkappa_\jmath, \varkappa)$ *tends to zero as* $\jmath \to \infty$*, i.e.,*

$$\left\| d(\varkappa_\jmath, \varkappa) \right\|_{\mathbb{E}} \to 0,$$

*as* $\jmath \to \infty$.

**Lemma 2.2.** *[14] A sequence* $\{\varkappa_\jmath\}_{\jmath\in\mathbb{N}}$ *in an E-VMS* $(\Xi, d)$ *is said to be a Cauchy sequence if and only if the elliptic norm of* $d(\varkappa_\jmath, \varkappa_{\jmath+m})$ *approaches to zero as* $\jmath$ *tends to infinity, for any natural number m. This condition is mathematically stated as*

$$\left\| d(\varkappa_\jmath, \varkappa_{\jmath+m}) \right\|_{\mathbb{E}} \to 0,$$

*as* $\jmath \to \infty$*, with* $m \in \mathbb{N}$.

**Remark 2.1.** *[14] It is a fundamental property of convergent sequences within an E-VMS that they converge to one, and only one, limit.*

## 3. Main results

In this section, we develop a comprehensive FP framework within the setting of E-VMSs. We begin by investigating FP theorems for two self-mappings under generalized contractive conditions, which are then supported by a non trivial illustrative example. Subsequently, we extend our results by establishing an FP theorem involving the control functions of one variable. Finally, we derive further FP results under contractive conditions that incorporate constant parameters.

**Theorem 3.1.** *Let* $(\Xi, d)$ *be a complete E-VMS and let* $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$*. If there exists a constant* $\varpi \in [0, 1)$ *such that*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi \mathcal{M}(\varkappa, \sigma), \tag{3.1}$$

*for all* $\varkappa, \sigma \in \Xi$*, where*

$$\mathcal{M}(\varkappa, \sigma) \in \left\{ d(\varkappa, \sigma), d(\varkappa, \mathcal{H}\varkappa), d(\sigma, \mathfrak{F}\sigma), \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} \right\}, \tag{3.2}$$

*then* $\mathcal{H}$ *and* $\mathfrak{F}$ *have a unique common FP in* $\Xi$.

*Proof.* We will first demonstrate that an FP of one mapping is also an FP of the other. We assume that $\mathfrak{F}\varkappa = \varkappa$ and we suppose on the contrary that $\mathcal{H}\varkappa \neq \varkappa$ . Then from (3.1) and (3.2), we have

$$d(\mathcal{H}\varkappa, \varkappa) = d(\mathcal{H}\varkappa, \mathfrak{F}\varkappa) \preceq \varpi \mathcal{M}(\varkappa, \varkappa),$$

where

$$\mathcal{M}(\varkappa, \varkappa) \in \left\{ d(\varkappa, \varkappa), d(\varkappa, \mathcal{H}\varkappa), d(\varkappa, \mathfrak{F}\varkappa), \frac{d(\varkappa, \mathcal{H}\varkappa) d(\varkappa, \mathfrak{F}\varkappa)}{1 + d(\varkappa, \varkappa)} \right\}.$$

Now we investigate four different cases.

Case i. If $\mathcal{M}(\varkappa, \varkappa) = d(\varkappa, \varkappa)$, then

$$d(\mathcal{H}\varkappa, \varkappa) \preceq \varpi d(\varkappa, \varkappa) = \theta$$

implies that $\mathcal{H}\varkappa = \varkappa$.

Case ii. If $\mathcal{M}(\varkappa, \varkappa) = d(\varkappa, \mathcal{H}\varkappa)$, then

$$d(\mathcal{H}\varkappa, \varkappa) \preceq \varpi d(\varkappa, \mathcal{H}\varkappa),$$

which is possible only where $\mathcal{H}\varkappa = \varkappa$ because $\varpi < 1$.

Case iii. If $\mathcal{M}(\varkappa, \varkappa) = d(\varkappa, \mathfrak{F}\varkappa)$, then

$$d(\mathcal{H}\varkappa, \varkappa) \preceq \varpi d(\varkappa, \mathfrak{F}\varkappa) = \theta$$

implies that $\mathcal{H}\varkappa = \varkappa$.

Case iv. If $\mathcal{M}(\varkappa, \varkappa) = \frac{d(\varkappa, \mathcal{H}\varkappa) \cdot d(\varkappa, \mathfrak{F}\varkappa)}{1 + d(\varkappa, \varkappa)}$, then

$$d(\mathcal{H}\varkappa, \varkappa) \preceq \varpi \frac{d(\varkappa, \mathcal{H}\varkappa) d(\varkappa, \mathfrak{F}\varkappa)}{1 + d(\varkappa, \varkappa)} = \theta$$

implies that $\mathcal{H}\varkappa = \varkappa$. Similarly, it can be demonstrated that any FP of $\mathcal{H}$ is also an FP of $\mathfrak{F}$. Now, let $\varkappa_0 \in \Xi$ be an arbitrary point. Define

$$\varkappa_{2\jmath+1} = \mathfrak{F}\varkappa_{2\jmath} \text{ and } \varkappa_{2\jmath+2} = \mathcal{H}\varkappa_{2\jmath+1},$$

for all $\jmath \geq 0$. We hypothesize that $\varkappa_\jmath \neq \varkappa_{\jmath+1}$ for all values of $\jmath$. Assuming the existence of an integer $\jmath$ such that $\varkappa_{2\jmath} = \varkappa_{2\jmath+1}$. Then

$$\varkappa_{2\jmath} = \mathfrak{F}\varkappa_{2\jmath},$$

and $\varkappa_{2\jmath}$ is an FP of $\mathfrak{F}$ and hence an FP of $\mathcal{H}$. Furthermore, if $\varkappa_{2\jmath+1} = \varkappa_{2\jmath+2}$ for a particular value of $\jmath$, it follows that $\varkappa_{2\jmath+1}$ represents a common FP of the mappings $\mathfrak{F}$ and $\mathcal{H}$. Now by (3.1) and (3.2), we have

$$d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) = d(\mathcal{H}\varkappa_{2\jmath-1}, \mathfrak{F}\varkappa_{2\jmath}) \preceq \varpi \mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}), \tag{3.3}$$

where

$$\mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \in \left\{ \begin{array}{c} d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}), d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1}), d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath}), \\ \frac{d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1}) d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath})}{1 + d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath})} \end{array} \right\}.$$

Next, we examine the following four cases.

Case i. If $\mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) = d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath})$, then, from (3.3), we have

$$d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \leq \varpi d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}),$$

which implies that

$$\left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \right\|_{\mathbb{E}}.$$

Case ii. If $\mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) = d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1})$, then, from (3.3), we have

$$d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \leq \varpi d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1}) = \varpi d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}),$$

which implies that

$$\left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \right\|_{\mathbb{E}}.$$

Case iii. If $\mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) = d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath})$, then, from (3.3), we have

$$d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \leq \varpi d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath}) = \varpi d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}),$$

which yields that

$$\left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} < \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}},$$

which is a contradiction.

Case iv. If $\mathcal{M}(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) = \frac{d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1}) d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath})}{1 + d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath})}$, then, from (3.3), we have

$$\begin{aligned} d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) &\leq \varpi \frac{d(\varkappa_{2\jmath-1}, \mathcal{H}\varkappa_{2\jmath-1}) d(\varkappa_{2\jmath}, \mathfrak{F}\varkappa_{2\jmath})}{1 + d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath})} \\ &= \varpi \frac{d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1})}{1 + d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath})}, \end{aligned}$$

which implies that

$$\begin{aligned} \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} &\leq \varpi \frac{\left\| d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \right\|_{\mathbb{E}}}{\left\| 1 + d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \right\|_{\mathbb{E}}} \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \\ &\leq \varpi \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}}. \end{aligned}$$

Therefore, it can be concluded that in all instances

$$\left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} < \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}},$$

for all $\jmath$. Hence, we have

$$\left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath-1}, \varkappa_{2\jmath}) \right\|_{\mathbb{E}}, \tag{3.4}$$

for all $\jmath$. Similarly, we can prove that

$$\left\| d(\varkappa_{2\jmath+1}, \varkappa_{2\jmath+2}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{2\jmath}, \varkappa_{2\jmath+1}) \right\|_{\mathbb{E}}, \tag{3.5}$$

for all $\jmath$. Thus from (3.4) and (3.5), we can conclude that

$$\left\| d(\varkappa_{\jmath}, \varkappa_{\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{\jmath-1}, \varkappa_{\jmath}) \right\|_{\mathbb{E}},$$

for every $\jmath$. This further yields that

$$\left\| d(\varkappa_{\jmath}, \varkappa_{\jmath+1}) \right\|_{\mathbb{E}} \leq \varpi \left\| d(\varkappa_{\jmath-1}, \varkappa_{\jmath}) \right\|_{\mathbb{E}} \leq \varpi^2 \left\| d(\varkappa_{\jmath-2}, \varkappa_{\jmath-1}) \right\|_{\mathbb{E}} \leq \ldots \leq \varpi^{\jmath} \left\| d(\varkappa_0, \varkappa_1) \right\|_{\mathbb{E}}, \tag{3.6}$$

for every $\jmath$. Given that $m$ and $\jmath$ are natural numbers satisfying $m > \jmath$, the implication is that

$$\begin{aligned}
\left\|d\left(\varkappa_J,\varkappa_m\right)\right\|_{\mathbb{E}} &\leq \left\|d\left(\varkappa_J,\varkappa_{J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\varkappa_{J+1},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa_J,\varkappa_{J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\varkappa_{J+1},\varkappa_{J+2}\right)\right\|_{\mathbb{E}}+\left\|d\left(\varkappa_{J+2},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa_J,\varkappa_{J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\varkappa_{J+1},\varkappa_{J+2}\right)\right\|_{\mathbb{E}}+...+\left\|d\left(\varkappa_{m-1},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \varpi^{J}\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}+\varpi^{J+1}\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}+...+\varpi^{m-1}\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}\\
&\leq \left(\varpi^{J}+\varpi^{J+1}+...+\varpi^{m-1}\right)\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}\\
&\leq \left[\frac{\varpi^{J}}{1-\varpi}\right]\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}},
\end{aligned}$$

and so

$$\left\|d\left(\varkappa_J,\varkappa_m\right)\right\|_{\mathbb{E}} \leq \left[\frac{\varpi^{J}}{1-\varpi}\right]\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}} \to 0 \text{ as } m, J \to \infty.$$

Applying Lemma 2.2, we ascertain that the sequence $\left\{\varkappa_J\right\}$ is a Cauchy sequence. Due to the fact that $\Xi$ is complete, there exists a point $\varkappa^*$ such that $\varkappa_J \to \varkappa^* \in \Xi$ as $J \to \infty$, that is,

$$\lim_{J\to\infty} d\left(\varkappa_J,\varkappa^*\right)=\theta. \tag{3.7}$$

We now proceed to prove that $\varkappa^*$ is an FP of $\mathcal{H}$. Assume, for the sake of contradiction, that $\varkappa^*$ is not an FP of the mapping $\mathcal{H}$. Then $\varkappa^* \neq \mathcal{H}\varkappa^*$, implying that $0 < \left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}$. Now by (3.1), we have

$$\begin{aligned}
\left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}} &\leq \left\|d\left(\varkappa^*,\varkappa_{2J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\varkappa_{2J+1},\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa^*,\varkappa_{2J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\mathfrak{F}\varkappa_{2J},\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}\\
&= \left\|d\left(\varkappa^*,\varkappa_{2J+1}\right)\right\|_{\mathbb{E}}+\left\|d\left(\mathcal{H}\varkappa^*,\mathfrak{F}\varkappa_{2J}\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa^*,\varkappa_{2J+1}\right)\right\|_{\mathbb{E}}+\varpi\left\|\mathcal{M}\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}},
\end{aligned} \tag{3.8}$$

where

$$\left\|\mathcal{M}\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}} \in \left\{\begin{array}{c} \left\|d\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}}, \left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}, \left\|d\left(\varkappa_{2J},\mathfrak{F}\varkappa_{2J}\right)\right\|_{\mathbb{E}},\\ \frac{\left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}\cdot\left\|d\left(\varkappa_{2J},\mathfrak{F}\varkappa_{2J}\right)\right\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}}} \end{array}\right\},$$

that is,

$$\left\|\mathcal{M}\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}} \in \left\{\begin{array}{c} \left\|d\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}}, \left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}, \left\|d\left(\varkappa_{2J},\varkappa_{2J+1}\right)\right\|_{\mathbb{E}},\\ \frac{\left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}\left\|d\left(\varkappa_{2J},\varkappa_{2J+1}\right)\right\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa^*,\varkappa_{2J}\right)\right\|_{\mathbb{E}}} \end{array}\right\}. \tag{3.9}$$

Now, taking the limit as $J \to \infty$ in the inequalities (3.8) and (3.9) and applying the result from (3.7), we deduce the following conclusion from (3.8) that

$$\left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}} \leq \varpi\left\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}},$$

which is a contradiction because $\varpi < 1$. Hence, $\varkappa^* = \mathcal{H}\varkappa^*$. Similarly, we can prove that $\varkappa^*$ is an FP of $\mathfrak{F}$; that is, $\varkappa^* = \mathfrak{F}\varkappa^*$. Thus, $\varkappa^*$ is a common FP of $\mathcal{H}$ and $\mathfrak{F}$. For the purpose of proving the uniqueness of $\varkappa^*$, let us presume that $\mathfrak{F}$ and $\mathcal{H}$ possess a distinct common FP, denoted as $\varkappa^/$. Then

$$\varkappa^/ = \mathcal{H}\varkappa^/ = \mathfrak{F}\varkappa^/,$$

but $\varkappa^* \neq \varkappa^/$. Now from (3.1), we have

$$d\left(\varkappa^*, \varkappa^/\right) = d\left(\mathcal{H}\varkappa^*, \mathfrak{F}\varkappa^/\right) \leq \varpi\mathcal{M}\left(\varkappa^*, \varkappa^/\right), \tag{3.10}$$

where

$$\mathcal{M}\left(\varkappa^*, \varkappa^/\right) \in \left\{ \begin{array}{c} d\left(\varkappa^*, \varkappa^/\right), d\left(\varkappa^*, \mathcal{H}\varkappa^*\right), d\left(\varkappa^/, \mathfrak{F}\varkappa^/\right), \\ \frac{d(\varkappa^*, \mathcal{H}\varkappa^*)d\left(\varkappa^/, \mathfrak{F}\varkappa^/\right)}{1+d(\varkappa,\sigma)} \end{array} \right\} = \left\{d\left(\varkappa^*, \varkappa^/\right), \theta\right\}.$$

Let us now assess the following two possibilities.

Case i. If $\mathcal{M}\left(\varkappa^*, \varkappa^/\right) = d\left(\varkappa^*, \varkappa^/\right)$, then, from (3.10), we have

$$d\left(\varkappa^*, \varkappa^/\right) \leq \varpi d\left(\varkappa^*, \varkappa^/\right),$$

which implies that

$$\left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} \leq \varpi \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}};$$

that is,

$$(1 - \varpi) \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} \leq 0.$$

Since $(1 - \varpi) \neq 0$, so $\left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} = 0$ implies that $\varkappa^* = \varkappa^/$.

Case ii. If $\mathcal{M}\left(\varkappa^*, \varkappa^/\right) = \theta$, then, from (3.10), we have

$$d\left(\varkappa^*, \varkappa^/\right) \leq \theta,$$

which implies that

$$\left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} \leq 0.$$

Thus, $\varkappa^* = \varkappa^/$. Hence the common FP of $\mathfrak{F}$ and $\mathcal{H}$ is unique. □

**Corollary 3.1.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. If there exists a constant $\varpi \in [0, 1)$ such that*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \leq \varpi\mathcal{M}(\varkappa, \sigma), \tag{3.11}$$

*for all $\varkappa, \sigma \in \Xi$, where*

$$\mathcal{M}(\varkappa, \sigma) \in \left\{d(\varkappa, \sigma), d(\varkappa, \mathfrak{F}\varkappa), d(\sigma, \mathfrak{F}\sigma), \frac{d(\varkappa, \mathfrak{F}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)}\right\},$$

*then $\mathfrak{F}$ has a unique FP.*

*Proof.* Take $\mathcal{H} = \mathfrak{F}$ in Theorem 3.1. □

**Example 3.1.** Let $\Xi=E_{-2}=\{z=\varkappa+i\sigma:\varkappa,\sigma\in\mathbb{R},i^2=p=-2<0\}$ and $d:E_{-2}\times E_{-2}\to E_{-2}$ is defined as

$$d(z_1,z_2)=\|\varkappa_1-\varkappa_2\|_{\mathbb{E}}+i\,\|\sigma_1-\sigma_2\|_{\mathbb{E}},$$

where $z_1=\varkappa_1+i\sigma_1$ and $z_2=\varkappa_2+i\sigma_2$ in $\mathbb{E}_p$ and $i^2=p=-2<0$, in which case $(E_{-2},d)$ is a complete E-VMS. Define a mapping $\mathfrak{F}:E_{-2}\to E_{-2}$ by $\mathfrak{F}(z)=\frac{z}{3}$. Now $\mathfrak{F}(z_1)=\frac{1}{3}(\varkappa_1+i\sigma_1)$ and $\mathfrak{F}(z_2)=\frac{1}{3}(\varkappa_2+i\sigma_2)$. Then

$$\begin{aligned}d(\mathfrak{F}z_1,\mathfrak{F}z_2)&=\|\mathfrak{F}z_1-\mathfrak{F}z_2\|_{\mathbb{E}}\\&=\left\|\frac{\varkappa_1}{3}-\frac{\varkappa_2}{3}\right\|_{\mathbb{E}}+i\left\|\frac{\sigma_1}{3}-\frac{\sigma_2}{3}\right\|_{\mathbb{E}}\\&=\frac{1}{3}\left(\|\varkappa_1-\varkappa_2\|_{\mathbb{E}}+i\,\|\sigma_1-\sigma_2\|_{\mathbb{E}}\right)\\&=\frac{1}{3}d(z_1,z_2).\end{aligned}$$

Therefore, the mapping $\mathfrak{F}(z)=\frac{z}{3}$ satisfies the contractive condition (3.11) with $\varpi=\frac{1}{3}<1$ and $0+0i$ is an FP of the mapping $\mathfrak{F}$.

**Theorem 3.2.** *Let $(\Xi,d)$ be a complete E-VMS and let $\mathcal{H},\mathfrak{F}:\Xi\to\Xi$. Assume that there exist the functions $\varpi_1,\varpi_2,\varpi_3:\Xi\to[0,1)$ satisfying the following conditions:*

*(i)*

$$\begin{aligned}&\varpi_1(\mathcal{H}\varkappa)\le\varpi_1(\varkappa)\ \text{and}\ \varpi_1(\mathfrak{F}\varkappa)\le\varpi_1(\varkappa),\\&\varpi_2(\mathcal{H}\varkappa)\le\varpi_2(\varkappa)\ \text{and}\ \varpi_2(\mathfrak{F}\varkappa)\le\varpi_2(\varkappa),\\&\varpi_3(\mathcal{H}\varkappa)\le\varpi_3(\varkappa)\ \text{and}\ \varpi_3(\mathfrak{F}\varkappa)\le\varpi_3(\varkappa);\end{aligned}$$

*(ii)* $\varpi_1(\varkappa)+\varpi_2(\varkappa)+\varpi_3(\varkappa)<1$;

*(iii)*

$$d(\mathcal{H}\varkappa,\mathfrak{F}\sigma)\le\varpi_1(\varkappa)d(\varkappa,\sigma)+\varpi_2(\varkappa)\frac{d(\varkappa,\mathcal{H}\varkappa)d(\sigma,\mathfrak{F}\sigma)}{1+d(\varkappa,\sigma)}+\varpi_3(\varkappa)\frac{d(\sigma,\mathcal{H}\varkappa)d(\varkappa,\mathfrak{F}\sigma)}{1+d(\varkappa,\sigma)},\qquad(3.12)$$

*for all $\varkappa,\sigma\in\Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Now, let $\varkappa_0\in\Xi$ be an arbitrary point. Define

$$\varkappa_{2\jmath+1}=\mathfrak{F}\varkappa_{2\jmath}\ \text{and}\ \varkappa_{2\jmath+2}=\mathcal{H}\varkappa_{2\jmath+1},$$

for all $\jmath\ge0$. Assuming the existence of an integer $\jmath$ such that $\varkappa_{2\jmath}=\varkappa_{2\jmath+1}$. Then

$$\varkappa_{2\jmath}=\mathfrak{F}\varkappa_{2\jmath},$$

and $\varkappa_{2\jmath}$ is an FP of $\mathfrak{F}$ and hence an FP of $\mathcal{H}$. Furthermore, if $\varkappa_{2\jmath+1}=\varkappa_{2\jmath+2}$ for a particular value of $\jmath$, it follows that $\varkappa_{2\jmath+1}$ represents a common FP of the mappings $\mathfrak{F}$ and $\mathcal{H}$. By (3.12), we have

$$\begin{aligned}d(\varkappa_{2\jmath},\varkappa_{2\jmath+1})&=d(\mathcal{H}\varkappa_{2\jmath-1},\mathfrak{F}\varkappa_{2\jmath})\\&\le\varpi_1(\varkappa_{2\jmath-1})d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_{2\jmath-1}\right)\frac{d\left(\varkappa_{2\jmath-1},\mathcal{H}\varkappa_{2\jmath-1}\right)d\left(\varkappa_{2\jmath},\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\end{aligned}$$

$$
\begin{aligned}
&+\varpi_3\left(\varkappa_{2\jmath-1}\right)\frac{d\left(\varkappa_{2\jmath},\mathcal{H}\varkappa_{2\jmath-1}\right)d\left(\varkappa_{2\jmath-1},\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\\
=\ & \varpi_1(\varkappa_{2\jmath-1})d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_{2\jmath-1}\right)\frac{d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\\
&+\varpi_3\left(\varkappa_{2\jmath-1}\right)\frac{d\left(\varkappa_{2\jmath},\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\\
=\ & \varpi_1(\mathfrak{F}\varkappa_{2\jmath-2})d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})+\varpi_2\left(\mathfrak{F}\varkappa_{2\jmath-2}\right)\frac{d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\\
\leq\ & \varpi_1(\varkappa_{2\jmath-2})d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_{2\jmath-2}\right)\frac{d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})}\\
\leq\ & \cdots\leq\varpi_1(\varkappa_0)d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_0\right)\frac{d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left\|d(\varkappa_{2\jmath},\varkappa_{2\jmath+1})\right\|_{\mathbb{E}} &\leq \varpi_1(\varkappa_0)\left\|d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}+\varpi_2\left(\varkappa_0\right)\frac{\left\|d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)\right\|_{\mathbb{E}}}{\left\|1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}}\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}\\
&\leq \varpi_1(\varkappa_0)\left\|d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}+\varpi_2\left(\varkappa_0\right)\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}.
\end{aligned}
$$

Since $\frac{\left\|d\left(\varkappa_{2\jmath-1},\varkappa_{2\jmath}\right)\right\|_{\mathbb{E}}}{\left\|1+d(\varkappa_{2\jmath-1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}}<1$. Thus, from the above inequality, we have

$$
\left\|d(\varkappa_{2\jmath},\varkappa_{2\jmath+1})\right\|_{\mathbb{E}}\leq\frac{\varpi_1(\varkappa_0)}{1-\varpi_2\left(\varkappa_0\right)}\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}. \tag{3.13}
$$

Similarly, by (3.12), we have

$$
\begin{aligned}
d(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}) &= d(\mathfrak{F}\varkappa_{2\jmath},\mathcal{H}\varkappa_{2\jmath+1})=d(\mathcal{H}\varkappa_{2\jmath+1},\mathfrak{F}\varkappa_{2\jmath})\\
&\leq \varpi_1(\varkappa_{2\jmath+1})d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_{2\jmath+1}\right)\frac{d\left(\varkappa_{2\jmath+1},\mathcal{H}\varkappa_{2\jmath+1}\right)d\left(\varkappa_{2\jmath},\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}\\
&\quad+\varpi_3\left(\varkappa_{2\jmath+1}\right)\frac{d\left(\varkappa_{2\jmath},\mathcal{H}\varkappa_{2\jmath+1}\right)d\left(\varkappa_{2\jmath+1},\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}\\
&= \varpi_1(\varkappa_{2\jmath+1})d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})+\varpi_2\left(\varkappa_{2\jmath+1}\right)\frac{d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}\\
&\quad+\varpi_3\left(\varkappa_{2\jmath+1}\right)\frac{d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}\\
&= \varpi_1(\mathfrak{F}\varkappa_{2\jmath})d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})+\varpi_2\left(\mathfrak{F}\varkappa_{2\jmath}\right)\frac{d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varpi_1(\varkappa_{2\jmath})d(\varkappa_{2\jmath+1},\varkappa_{2\jmath}) + \varpi_2\left(\varkappa_{2\jmath}\right)\frac{d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})}\\
&\leq \cdots \leq \varpi_1(\varkappa_0)d(\varkappa_{2\jmath+1},\varkappa_{2\jmath}) + \varpi_2\left(\varkappa_0\right)\frac{d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left\|d(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2})\right\|_{\mathbb{E}} &\leq \varpi_1(\varkappa_0)\left\|d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})\right\|_{\mathbb{E}} + \varpi_2\left(\varkappa_0\right)\left\|d(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2})\right\|_{\mathbb{E}}\frac{\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}}{\left\|1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}}\\
&\leq \varpi_1(\varkappa_0)\left\|d(\varkappa_{2\jmath},\varkappa_{2\jmath+1})\right\|_{\mathbb{E}} + \varpi_2\left(\varkappa_0\right)\left\|d(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2})\right\|_{\mathbb{E}},
\end{aligned}
$$

since $\frac{\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}}{\left\|1+d(\varkappa_{2\jmath+1},\varkappa_{2\jmath})\right\|_{\mathbb{E}}} < 1$. Thus, from the above inequality, we have

$$
\left\|d(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2})\right\|_{\mathbb{E}} \leq \frac{\varpi_1(\varkappa_0)}{1-\varpi_2\left(\varkappa_0\right)}\left\|d(\varkappa_{2\jmath},\varkappa_{2\jmath+1})\right\|_{\mathbb{E}}. \tag{3.14}
$$

Let $\lambda = \frac{\varpi_1(\varkappa_0)}{1-\varpi_2(\varkappa_0)} < 1$. Then, from (3.13) and (3.14), we have

$$
\left\|d\left(\varkappa_\jmath,\varkappa_{\jmath+1}\right)\right\|_{\mathbb{E}} \leq \lambda\left\|d\left(\varkappa_{\jmath-1},\varkappa_\jmath\right)\right\|_{\mathbb{E}}
$$

for all $\jmath \in \mathbb{N}$. Inductively, we can construct a sequence $\{\varkappa_\jmath\}$ in $\Xi$ such that

$$
\begin{aligned}
\left\|d\left(\varkappa_\jmath,\varkappa_{\jmath+1}\right)\right\|_{\mathbb{E}} &\leq \lambda\left\|d\left(\varkappa_{\jmath-1},\varkappa_\jmath\right)\right\|_{\mathbb{E}}\\
&\leq \lambda^2\left\|d\left(\varkappa_{\jmath-2},\varkappa_{\jmath-1}\right)\right\|_{\mathbb{E}}\\
&\quad.\\
&\quad.\\
&\quad.\\
&\leq \lambda^\jmath\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}},
\end{aligned}
$$

for all $\jmath \in \mathbb{N}$. Therefore, for arbitrary natural numbers $m$ and $\jmath$ satisfying the condition $m > \jmath$, it is demonstrable that

$$
\begin{aligned}
\left\|d\left(\varkappa_\jmath,\varkappa_m\right)\right\|_{\mathbb{E}} &\leq \left\|d\left(\varkappa_\jmath,\varkappa_{\jmath+1}\right)\right\|_{\mathbb{E}} + \left\|d\left(\varkappa_{\jmath+1},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa_\jmath,\varkappa_{\jmath+1}\right)\right\|_{\mathbb{E}} + \left\|d\left(\varkappa_{\jmath+1},\varkappa_{\jmath+2}\right)\right\|_{\mathbb{E}} + \left\|d\left(\varkappa_{\jmath+2},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \left\|d\left(\varkappa_\jmath,\varkappa_{\jmath+1}\right)\right\|_{\mathbb{E}} + \left\|d\left(\varkappa_{\jmath+1},\varkappa_{\jmath+2}\right)\right\|_{\mathbb{E}} + ... + \left\|d\left(\varkappa_{m-1},\varkappa_m\right)\right\|_{\mathbb{E}}\\
&\leq \varpi^\jmath\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}} + \varpi^{\jmath+1}\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}} + ... + \varpi^{m-1}\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}\\
&\leq \left(\varpi^\jmath + \varpi^{\jmath+1} + ... + \varpi^{m-1}\right)\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}}\\
&\leq \left[\frac{\varpi^\jmath}{1-\varpi}\right]\left\|d\left(\varkappa_0,\varkappa_1\right)\right\|_{\mathbb{E}},
\end{aligned}
$$

and so

$$\left\|d\left(\varkappa_{\jmath},\varkappa_m\right)\right\|_{\mathbb{E}} \leq \left[\frac{\varpi^{\jmath}}{1-\varpi}\right]\|d\left(\varkappa_0,\varkappa_1\right)\|_{\mathbb{E}} \to 0 \text{ as } m,\jmath \to \infty.$$

Utilizing Lemma 2.2, we conclude that the sequence $\left\{\varkappa_{\jmath}\right\}$ satisfies the definition of a Cauchy sequence. Given the completeness of the space $\Xi$, there exists a point $\varkappa^*$ such that $\varkappa_{\jmath} \to \varkappa^* \in \Xi$ as $\jmath \to \infty$. Now, we show that $\varkappa^*$ is FP of $\mathcal{H}$. From (3.12), we have

$$\begin{aligned}
d\left(\varkappa^*,\mathcal{H}\varkappa^*\right) &\leq d\left(\varkappa^*,\mathfrak{F}\varkappa_{2\jmath}\right)+d\left(\mathfrak{F}\varkappa_{2\jmath},\mathcal{H}\varkappa^*\right)\\
&= d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)+d\left(\mathcal{H}\varkappa^*,\mathfrak{F}\varkappa_{2\jmath}\right)\\
&\leq \left(\begin{array}{c} d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)+\varpi_1\left(\varkappa^*\right)d\left(\varkappa^*,\varkappa_{2\jmath}\right)\\ +\varpi_2\left(\varkappa^*\right)\frac{d(\varkappa^*,\mathcal{H}\varkappa^*)d\left(\varkappa_{2\jmath},\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)}\\ +\varpi_3\left(\varkappa^*\right)\frac{d\left(\varkappa_{2\jmath},\mathcal{H}\varkappa^*\right)d\left(\varkappa^*,\mathfrak{F}\varkappa_{2\jmath}\right)}{1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)}\end{array}\right)\\
&= \left(\begin{array}{c} d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)+\varpi_1\left(\varkappa^*\right)d\left(\varkappa^*,\varkappa_{2\jmath}\right)\\ +\varpi_2\left(\varkappa^*\right)\frac{d(\varkappa^*,\mathcal{H}\varkappa^*)d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)}{1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)}\\ +\varpi_3\left(\varkappa^*\right)\frac{d\left(\varkappa_{2\jmath},\mathcal{H}\varkappa^*\right)d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)}{1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)}\end{array}\right).
\end{aligned}$$

This implies that

$$\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\|_{\mathbb{E}} \leq \left(\begin{array}{c} \left\|d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}+\varpi_1\left(\varkappa^*\right)\left\|d\left(\varkappa^*,\varkappa_{2\jmath}\right)\right\|_{\mathbb{E}}\\ +\varpi_2\left(\varkappa^*\right)\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\|_{\mathbb{E}}\frac{\left\|d\left(\varkappa_{2\jmath},\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)\right\|_{\mathbb{E}}}\\ +\varpi_3\left(\varkappa^*\right)\left\|d\left(\varkappa_{2\jmath},\mathcal{H}\varkappa^*\right)\right\|_{\mathbb{E}}\frac{\left\|d\left(\varkappa^*,\varkappa_{2\jmath+1}\right)\right\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa^*,\varkappa_{2\jmath}\right)\right\|_{\mathbb{E}}}\end{array}\right).$$

Letting $\jmath \to \infty$, we have $\|d\left(\varkappa^*,\mathcal{H}\varkappa^*\right)\|_{\mathbb{E}} = 0$. Thus, $\varkappa^* = \mathcal{H}\varkappa^*$. Now we prove that $\varkappa^*$ is FP of $\mathfrak{F}$. By (3.12), we have

$$\begin{aligned}
d\left(\varkappa^*,\mathfrak{F}\varkappa^*\right) &\leq d\left(\varkappa^*,\mathcal{H}\varkappa_{2\jmath+1}\right)+d\left(\mathcal{H}\varkappa_{2\jmath+1},\mathfrak{F}\varkappa^*\right)\\
&= d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)+d\left(\mathcal{H}\varkappa_{2\jmath+1},\mathfrak{F}\varkappa^*\right)\\
&\leq \left(\begin{array}{c} d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)+\varpi_1\left(\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath+1},\varkappa^*\right)\\ +\varpi_2\left(\varkappa_{2\jmath}\right)\frac{d\left(\varkappa_{2\jmath+1},\mathcal{H}\varkappa_{2\jmath+1}\right)d(\varkappa^*,\mathfrak{F}\varkappa^*)}{1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)}\\ +\varpi_3\left(\varkappa_{2\jmath}\right)\frac{d\left(\varkappa^*,\mathcal{H}\varkappa_{2\jmath+1}\right)d\left(\varkappa_{2\jmath+1},\mathfrak{F}\varkappa^*\right)}{1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)}\end{array}\right)\\
&= \left(\begin{array}{c} d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)+\varpi_1\left(\varkappa_{2\jmath}\right)d\left(\varkappa_{2\jmath+1},\varkappa^*\right)\\ +\varpi_2\left(\varkappa_{2\jmath}\right)\frac{d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)d(\varkappa^*,\mathfrak{F}\varkappa^*)}{1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)}\\ +\varpi_3\left(\varkappa_{2\jmath}\right)\frac{d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)d\left(\varkappa_{2\jmath+1},\mathfrak{F}\varkappa^*\right)}{1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)}\end{array}\right).
\end{aligned}$$

This implies that

$$\|d\left(\varkappa^*,\mathfrak{F}\varkappa^*\right)\|_{\mathbb{E}} \leq \left(\begin{array}{c} \left\|d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)\right\|_{\mathbb{E}}+\varpi_1\left(\varkappa_{2\jmath}\right)\left\|d\left(\varkappa_{2\jmath+1},\varkappa^*\right)\right\|_{\mathbb{E}}\\ +\varpi_2\left(\varkappa_{2\jmath}\right)\frac{\left\|d\left(\varkappa_{2\jmath+1},\varkappa_{2\jmath+2}\right)\right\|_{\mathbb{E}}\|d(\varkappa^*,\mathfrak{F}\varkappa^*)\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)\right\|_{\mathbb{E}}}\\ +\varpi_3\left(\varkappa_{2\jmath}\right)\frac{\left\|d\left(\varkappa^*,\varkappa_{2\jmath+2}\right)\right\|_{\mathbb{E}}\left\|d\left(\varkappa_{2\jmath+1},\mathfrak{F}\varkappa^*\right)\right\|_{\mathbb{E}}}{\left\|1+d\left(\varkappa_{2\jmath+1},\varkappa^*\right)\right\|_{\mathbb{E}}}\end{array}\right).$$

Letting $\jmath \to \infty$, we have $\|d(\varkappa^*, \mathfrak{F}\varkappa^*)\|_{\mathbb{E}} = 0$. Thus $\varkappa^* = \mathfrak{F}\varkappa^*$. Thus $\varkappa^*$ is a common FP of $\mathcal{H}$ and $\mathfrak{F}$. Now we prove that $\varkappa^*$ is unique. We suppose that there exists another common FP $\varkappa^/$ of $\mathcal{H}$ and $\mathfrak{F}$, that is,

$$\varkappa^/ = \mathcal{H}\varkappa^/ = \mathfrak{F}\varkappa^/,$$

but $\varkappa^* \neq \varkappa^/$. Now, from (3.12), we have

$$\begin{aligned} d\left(\varkappa^*, \varkappa^/\right) &= d\left(\mathcal{H}\varkappa^*, \mathfrak{F}\varkappa^/\right) \\ &\leq \varpi_1(\varkappa^*) d\left(\varkappa^*, \varkappa^/\right) + \varpi_2(\varkappa^*) \frac{d(\varkappa^*, \mathcal{H}\varkappa^*) d\left(\varkappa^/, \mathfrak{F}\varkappa^/\right)}{1 + d(\varkappa^*, \varkappa^/)} + \varpi_3(\varkappa^*) \frac{d\left(\varkappa^/, \mathcal{H}\varkappa^*\right) d\left(\varkappa^*, \mathfrak{F}\varkappa^/\right)}{1 + d(\varkappa^*, \varkappa^/)} \\ &= \varpi_1(\varkappa^*) d\left(\varkappa^*, \varkappa^/\right) + \varpi_2(\varkappa^*) \frac{d(\varkappa^*, \varkappa^*) d\left(\varkappa^/, \varkappa^/\right)}{1 + d(\varkappa^*, \varkappa^/)} + \varpi_3(\varkappa^*) \frac{d\left(\varkappa^/, \varkappa^*\right) d\left(\varkappa^*, \varkappa^/\right)}{1 + d(\varkappa^*, \varkappa^/)}. \end{aligned}$$

This implies that we have

$$\begin{aligned} \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} &\leq \varpi_1(\varkappa^*) \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} + \varpi_3(\varkappa^*) \frac{\left\|d\left(\varkappa^/, \varkappa^*\right)\right\|_{\mathbb{E}} \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}}}{\left\|1 + d(\varkappa^*, \varkappa^/)\right\|_{\mathbb{E}}} \\ &= (\varpi_1(\varkappa^*) + \varpi_3(\varkappa^*)) \left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}}. \end{aligned}$$

As $\varpi_1(\varkappa^*) + \varpi_3(\varkappa^*) < 1$, we have

$$\left\|d\left(\varkappa^*, \varkappa^/\right)\right\|_{\mathbb{E}} = 0.$$

Thus $\varkappa^* = \varkappa^/$. □

To illustrate the generality of our framework, we consider an example in an E-VMS that extends beyond the traditional C-VMSs by employing a generalized imaginary unit where $i^2 = p < 0$.

**Example 3.2.** Let $\Xi = E_p = \{z = \varkappa + i\sigma : \varkappa, \sigma \in \mathbb{R}, i^2 = p < 0\}$ and let $d : E_p \times E_p \to E_p$ be defined as

$$d(z_1, z_2) = \|z_1 - z_2\|_{\mathbb{E}} e^{i\theta_p},\ \theta_p \in \left[0, \frac{\pi(p-1)}{8p}\right],$$

where $z_1 = \varkappa_1 + i\sigma_1$ and $z_2 = \varkappa_2 + i\sigma_2$ in $\mathbb{E}_p$, and $i^2 = p < 0$ and $\theta_p$ is an argument of $z_1$ and $z_2$, and thus $(E_p, d)$ is a complete E-VMS (see [14]). Define the mappings $\mathcal{H}, \mathfrak{F} : E_p \to E_p$ by

$$\mathcal{H}(z) = \frac{z}{3} = \frac{1}{3}(\varkappa + i\sigma)$$

and

$$\mathfrak{F}(z) = \frac{z}{4} = \frac{1}{4}(\varkappa + i\sigma),$$

for any $z = \varkappa + i\sigma \in \Xi$. Define the control functions $\varpi_1, \varpi_2, \varpi_3 : \Xi \to [0, 1)$ as follows:

$$\varpi_1(z) = \varpi_1(\varkappa + i\sigma) = \frac{1}{2},$$

$$\varpi_2(z) = \varpi_2(\varkappa + i\sigma) = \frac{1}{3},$$

$$\varpi_3(z) = \varpi_3(\varkappa + i\sigma) = \frac{1}{9}.$$

It is easy to verify that

$$\varpi_1(\mathcal{H}z) \le \varpi_1(z) \text{ and } \varpi_1(\mathfrak{F}z) \le \varpi_1(z),$$
$$\varpi_2(\mathcal{H}z) \le \varpi_2(z) \text{ and } \varpi_2(\mathfrak{F}z) \le \varpi_2(z),$$
$$\varpi_3(\mathcal{H}z) \le \varpi_3(z) \text{ and } \varpi_3(\mathfrak{F}z) \le \varpi_3(z).$$

Now,

$$\varpi_1(z) + \varpi_2(z) + \varpi_3(z) = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{9+6+2}{18} = \frac{17}{18} < 1.$$

Moreover, for $z_1, z_2 \in \Xi$, we have

$$d(\mathcal{H}z_1, \mathfrak{F}z_2) = \sqrt{\left(\frac{\varkappa_1}{3} - \frac{\varkappa_2}{4}\right)^2 - p\left(\frac{\sigma_1}{3} - \frac{\sigma_2}{4}\right)^2} e^{i\theta_p},$$

$$d(z_1, z_2) = \sqrt{(\varkappa_1 - \varkappa_2)^2 - p(\sigma_1 - \sigma_2)^2} e^{i\theta_p},$$

$$d(z_1, \mathcal{H}z_1) = \sqrt{\left(\frac{2}{3}\varkappa_1\right)^2 - p\left(\frac{2}{3}\sigma_1\right)^2} e^{i\theta_p},$$

$$d(z_2, \mathfrak{F}z_2) = \sqrt{\left(\frac{3}{4}\varkappa_2\right)^2 - p\left(\frac{3}{4}\sigma_2\right)^2} e^{i\theta_p},$$

$$d(z_2, \mathcal{H}z_1) = \sqrt{\left(\varkappa_2 - \frac{\varkappa_1}{3}\right)^2 - p\left(\sigma_2 - \frac{\sigma_1}{3}\right)^2} e^{i\theta_p},$$

$$d(z_1, \mathfrak{F}z_2) = \sqrt{\left(\varkappa_1 - \frac{\varkappa_2}{4}\right)^2 - p\left(\sigma_1 - \frac{\sigma_2}{4}\right)^2} e^{i\theta_p}.$$

It is evident that the contractive condition (3.12), given by

$$d(\mathcal{H}z_1, \mathfrak{F}z_2) \le \varpi_1(z_1) d(z_1, z_2) + \varpi_2(z_1) \frac{d(z_1, \mathcal{H}z_1) d(z_2, \mathfrak{F}z_2)}{1 + d(z_1, z_2)} + \varpi_3(z_1) \frac{d(z_2, \mathcal{H}z_1) d(z_1, \mathfrak{F}z_2)}{1 + d(z_1, z_2)},$$

is satisfied by the mappings $\mathcal{H}$ and $\mathfrak{F}$. Therefore, it follows that $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP, namely $0 + 0i$.

**Corollary 3.2.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. Suppose there exist the functions $\varpi_1, \varpi_2, \varpi_3 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i) $\varpi_1(\mathfrak{F}\varkappa) \le \varpi_1(\varkappa)$, $\varpi_2(\mathfrak{F}\varkappa) \le \varpi_2(\varkappa)$, $\varpi_3(\mathfrak{F}\varkappa) \le \varpi_3(\varkappa)$;*

*(ii) $\varpi_1(\varkappa) + \varpi_2(\varkappa) + \varpi_3(\varkappa) < 1$;*

*(iii)*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \le \varpi_1(\varkappa) d(\varkappa, \sigma) + \varpi_2(\varkappa) \frac{d(\varkappa, \mathfrak{F}\varkappa) d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3(\varkappa) \frac{d(\sigma, \mathfrak{F}\varkappa) d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, and then $\mathfrak{F}$ has a unique FP.*

*Proof.* Take $\mathcal{H} = \mathfrak{F}$ in Theorem 3.2. □

**Corollary 3.3.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the functions $\varpi_1, \varpi_2 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i)*

$$\varpi_1 (\mathcal{H}\varkappa) \leq \varpi_1 (\varkappa) \text{ and } \varpi_1 (\mathfrak{F}\varkappa) \leq \varpi_1 (\varkappa),$$
$$\varpi_2 (\mathcal{H}\varkappa) \leq \varpi_2 (\varkappa) \text{ and } \varpi_2 (\mathfrak{F}\varkappa) \leq \varpi_2 (\varkappa);$$

*(ii)* $\varpi_1 (\varkappa) + \varpi_2 (\varkappa) < 1$;

*(iii)*

$$d (\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 (\varkappa) d (\varkappa, \sigma) + \varpi_2 (\varkappa) \frac{d (\varkappa, \mathcal{H}\varkappa) d (\sigma, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, and then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Define $\varpi_3 : \Xi \to [0, 1)$ as $\varpi_3 (\varkappa) = 0$ in Theorem 3.2. □

**Corollary 3.4.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the functions $\varpi_1, \varpi_2 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i)* $\varpi_1 (\mathfrak{F}\varkappa) \leq \varpi_1 (\varkappa)$, $\varpi_2 (\mathfrak{F}\varkappa) \leq \varpi_2 (\varkappa)$;

*(ii)* $\varpi_1 (\varkappa) + \varpi_2 (\varkappa) < 1$;

*(iii)*

$$d (\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 (\varkappa) d (\varkappa, \sigma) + \varpi_2 (\varkappa) \frac{d (\varkappa, \mathfrak{F}\varkappa) d (\sigma, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, and then $\mathfrak{F}$ has a unique FP.*

*Proof.* Take $\mathcal{H} = \mathfrak{F}$ in the corollary above. □

**Corollary 3.5.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the functions $\varpi_1, \varpi_3 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i)*

$$\varpi_1 (\mathcal{H}\varkappa) \leq \varpi_1 (\varkappa) \text{ and } \varpi_1 (\mathfrak{F}\varkappa) \leq \varpi_1 (\varkappa),$$
$$\varpi_3 (\mathcal{H}\varkappa) \leq \varpi_3 (\varkappa) \text{ and } \varpi_3 (\mathfrak{F}\varkappa) \leq \varpi_3 (\varkappa);$$

*(ii)* $\varpi_1 (\varkappa) + \varpi_3 (\varkappa) < 1$;

*(iii)*

$$d (\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 (\varkappa) d (\varkappa, \sigma) + \varpi_3 (\varkappa) \frac{d (\sigma, \mathcal{H}\varkappa) d (\varkappa, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, in which case $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Define $\varpi_2 : \Xi \to [0, 1)$ by $\varpi_2 (\varkappa) = 0$ in Theorem 3.2. □

**Corollary 3.6.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the functions $\varpi_1, \varpi_3 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i)* $\varpi_1 (\mathfrak{F}\varkappa) \leq \varpi_1 (\varkappa)$, $\varpi_3 (\mathfrak{F}\varkappa) \leq \varpi_3 (\varkappa)$;

*(ii)* $\varpi_1 (\varkappa) + \varpi_3 (\varkappa) < 1$;

*(iii)*

$$d (\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 (\varkappa) d (\varkappa, \sigma) + \varpi_3 (\varkappa) \frac{d (\sigma, \mathfrak{F}\varkappa) d (\varkappa, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, in which case $\mathfrak{F}$ has a unique FP.*

*Proof.* Take $\mathcal{H} = \mathfrak{F}$ in the corollary above. □

**Corollary 3.7.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the constants $\varpi_1, \varpi_2, \varpi_3 \in [0, 1)$ such that $\varpi_1 + \varpi_2 + \varpi_3 < 1$ and*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_2 \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3 \frac{d(\sigma, \mathcal{H}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Define $\varpi_1, \varpi_2, \varpi_3 : \Xi \to [0, 1)$ by $\varpi_1(\varkappa) = \varkappa, \varpi_2(\varkappa) = \varpi_2$ and $\varpi_3(\varkappa) = \varkappa$ in Theorem 3.2. □

**Corollary 3.8.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the constants $\varpi_1, \varpi_2, \varpi_3 \in [0, 1)$ such that $\varpi_1 + \varpi_2 + \varpi_3 < 1$ and*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_2 \frac{d(\varkappa, \mathfrak{F}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3 \frac{d(\sigma, \mathfrak{F}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathfrak{F}$ has a unique FP.*

*Proof.* Take $\mathcal{H} = \mathfrak{F}$ in Corollary 3.7. □

**Corollary 3.9.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the constants $\varpi_1, \varpi_2 \in [0, 1)$ such that $\varpi_1 + \varpi_2 < 1$ and*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_2 \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Take $\varpi_3 = 0$ in Corollary 3.7. □

**Corollary 3.10.** *Let $(\Xi, d)$ be a complete E-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. Suppose that there exist the constants $\varpi_1, \varpi_3 \in [0, 1)$ such that $\varpi_1 + \varpi_3 < 1$ and*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi_1 d(\varkappa, \sigma) + \varpi_3 \frac{d(\sigma, \mathcal{H}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

*Proof.* Take $\varpi_2 = 0$ in Corollary 3.7. □

## 4. Fixed-point theory in complex-valued metric spaces

When $p$ is assigned the value $-1$ in Definition 2.2, the E-VMS becomes a C-VMS, consequently, we arrive at the results presented hereafter.

**Corollary 4.1.** *Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. If there exists a constant $\varpi \in [0, 1)$ such that*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \preceq \varpi \mathcal{M}(\varkappa, \sigma),$$

*for all* $\varkappa, \sigma \in \Xi$, *where*

$$\mathcal{M}(\varkappa, \sigma) \in \left\{ d(\varkappa, \sigma), d(\varkappa, \mathcal{H}\varkappa), d(\sigma, \mathfrak{F}\sigma), \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} \right\},$$

*then* $\mathcal{H}$ *and* $\mathfrak{F}$ *have a unique common FP.*

**Corollary 4.2.** *Let* $(\Xi, d)$ *be a complete C-VMS and* $\mathfrak{F} : \Xi \to \Xi$. *If there exists a constant* $\varpi \in [0, 1)$ *such that*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \leq \varpi \mathcal{M}(\varkappa, \sigma),$$

*for all* $\varkappa, \sigma \in \Xi$, *where*

$$\mathcal{M}(\varkappa, \sigma) \in \left\{ d(\varkappa, \sigma), d(\varkappa, \mathfrak{F}\varkappa), d(\sigma, \mathfrak{F}\sigma), \frac{d(\varkappa, \mathfrak{F}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} \right\},$$

*then* $\mathfrak{F}$ *has a unique FP.*

**Corollary 4.3.** *Let* $(\Xi, d)$ *be a complete C-VMS and let* $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. *If there exist the functions* $\varpi_1, \varpi_2, \varpi_3 : \Xi \to [0, 1)$ *satisfying the following conditions:*

*(i)*

$$\varpi_1(\mathcal{H}\varkappa) \leq \varpi_1(\varkappa) \text{ and } \varpi_1(\mathfrak{F}\varkappa) \leq \varpi_1(\varkappa),$$

$$\varpi_2(\mathcal{H}\varkappa) \leq \varpi_2(\varkappa) \text{ and } \varpi_2(\mathfrak{F}\varkappa) \leq \varpi_2(\varkappa),$$

$$\varpi_3(\mathcal{H}\varkappa) \leq \varpi_3(\varkappa) \text{ and } \varpi_3(\mathfrak{F}\varkappa) \leq \varpi_3(\varkappa);$$

*(ii)* $\varpi_1(\varkappa) + \varpi_2(\varkappa) + \varpi_3(\varkappa) < 1$;

*(iii)*

$$d(\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1(\varkappa)\, d(\varkappa, \sigma) + \varpi_2(\varkappa) \frac{d(\varkappa, \mathcal{H}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3(\varkappa) \frac{d(\sigma, \mathcal{H}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all* $\varkappa, \sigma \in \Xi$, *then* $\mathcal{H}$ *and* $\mathfrak{F}$ *have a unique common FP.*

**Corollary 4.4.** *Let* $(\Xi, d)$ *be a complete C-VMS and let* $\mathfrak{F} : \Xi \to \Xi$. *If there exist the functions* $\varpi_1, \varpi_2, \varpi_3 : \Xi \to [0, 1)$ *satisfying the following conditions:*

*(i)* $\varpi_1(\mathfrak{F}\varkappa) \leq \varpi_1(\varkappa)$, $\varpi_2(\mathfrak{F}\varkappa) \leq \varpi_2(\varkappa)$, $\varpi_3(\mathfrak{F}\varkappa) \leq \varpi_3(\varkappa)$;

*(ii)* $\varpi_1(\varkappa) + \varpi_2(\varkappa) + \varpi_3(\varkappa) < 1$;

*(iii)*

$$d(\mathfrak{F}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1(\varkappa)\, d(\varkappa, \sigma) + \varpi_2(\varkappa) \frac{d(\varkappa, \mathfrak{F}\varkappa)\, d(\sigma, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)} + \varpi_3(\varkappa) \frac{d(\sigma, \mathfrak{F}\varkappa)\, d(\varkappa, \mathfrak{F}\sigma)}{1 + d(\varkappa, \sigma)},$$

*for all* $\varkappa, \sigma \in \Xi$, *then* $\mathfrak{F}$ *has a unique FP.*

As a direct consequence of Theorem 3.2, we obtain the primary result of Sintunavarat et al. [11] in this manner.

**Corollary 4.5.** *[11] Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. If there exist the functions $\varpi_1, \varpi_2 : \Xi \to [0, 1)$ satisfying the following conditions:*

*(i)*

$$\varpi_1 (\mathcal{H}\varkappa) \leq \varpi_1 (\varkappa) \text{ and } \varpi_1 (\mathfrak{F}\varkappa) \leq \varpi_1 (\varkappa),$$
$$\varpi_2 (\mathcal{H}\varkappa) \leq \varpi_2 (\varkappa) \text{ and } \varpi_2 (\mathfrak{F}\varkappa) \leq \varpi_2 (\varkappa);$$

*(ii)* $\varpi_1 (\varkappa) + \varpi_2 (\varkappa) < 1$;

*(iii)*

$$d (\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 (\varkappa) d (\varkappa, \sigma) + \varpi_2 (\varkappa) \frac{d (\varkappa, \mathcal{H}\varkappa) d (\sigma, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

We present a derivation of the central theorem established by Rouzkard et al. [10].

**Corollary 4.6.** *[10] Let $(\Xi, d)$ be a complete C-VMS and $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. If there exist the constants $\varpi_1, \varpi_2, \varpi_3 \in [0, 1)$ such that $\varpi_1 + \varpi_2 + \varpi_3 < 1$ and*

$$d (\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 d (\varkappa, \sigma) + \varpi_2 \frac{d (\varkappa, \mathcal{H}\varkappa) d (\sigma, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)} + \varpi_3 \frac{d (\sigma, \mathcal{H}\varkappa) d (\varkappa, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

We now obtain the main theorem of Azam et al. [8] as a direct consequence of our leading theorem.

**Corollary 4.7.** *[8] Let $(\Xi, d)$ be a complete C-VMS and let $\mathcal{H}, \mathfrak{F} : \Xi \to \Xi$. If there exist the constants $\varpi_1, \varpi_2 \in [0, 1)$ such that $\varpi_1 + \varpi_2 < 1$ and*

$$d (\mathcal{H}\varkappa, \mathfrak{F}\sigma) \leq \varpi_1 d (\varkappa, \sigma) + \varpi_2 \frac{d (\varkappa, \mathcal{H}\varkappa) d (\sigma, \mathfrak{F}\sigma)}{1 + d (\varkappa, \sigma)},$$

*for all $\varkappa, \sigma \in \Xi$, then $\mathcal{H}$ and $\mathfrak{F}$ have a unique common FP.*

This finding was also established in the study by Azam et al. [8].

## 5. Applications

To illustrate the applicability of our main results, we now focus on a concrete problem where FP theory provides an effective analytical tool.

### *5.1. Fixed-point approach for solving the nonlinear Fredholm integral equations*

The study of nonlinear Fredholm integral equations of the second kind plays a crucial role in various scientific and engineering applications, including physics, biology, and climate modeling. One of the most effective techniques for solving these equations is the FP approach, which leverages contraction mappings to ensure the existence and uniqueness of the solutions. The Banach contraction principle and other FP theorems provide a solid mathematical foundation for iteratively approximating solutions to these nonlinear integral equations. This approach is particularly useful when dealing with complex integral kernels and nonlinear dependencies, making it a powerful tool for both theoretical analysis and numerical computations. In this section, we discuss the solution of nonlinear Fredholm integral equations of the second kind.

**Theorem 5.1.** *Let $\Xi = C([a,b],\mathbb{R})$ and let $d : \Xi \times \Xi \to \mathbb{E}_p$ be an E-VM given in this way*

$$d(\varkappa,\sigma) = (\|\varkappa(t) - \sigma(t)\|_\infty)\, e^{i\frac{\pi}{4}} = \left(\max_{t\in[a,b]} \|\varkappa(t) - \sigma(t)\|\right) e^{i\frac{\pi}{4}},$$

*where $i^2 = p < 0$. Let us consider the nonlinear Fredholm integral equation of the second kind*

$$\varkappa(t) = f(t) + \mu \int_a^b K(t,s,\varkappa(s))ds. \tag{5.1}$$

*Assume that the following conditions are satisfied:*

*($a_1$) $f : [a,b] \to \Xi$ is a continuous function;*
*($a_2$) $K(\cdot,\cdot,\cdot) : [a,b] \times [a,b] \times \Xi \to \Xi$ is a continuous kernel;*
*($a_3$) there exists a constant $L \geq 0$ such that*

$$\|K(t,s,\varkappa) - K(t,s,\sigma)\| \leq L\,\|\varkappa - \sigma\|,$$

*for all $t,s \in [a,b]$, $\varkappa,\sigma \in \Xi$.*
*($a_4$) The parameter $\mu$ satisfies the condition*

$$\varpi = |\mu|\,L\,(b-a) < 1.$$

*Then the Fredholm integral equation (5.1) has a unique solution in $C([a,b],\mathbb{R})$.*

*Proof.* Define the operator $\mathfrak{F} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$(\mathfrak{F}\varkappa)(t) = f(t) + \mu \int_a^b K(t,s,\varkappa(s))ds.$$

For any $\varkappa,\sigma \in C([a,b],\mathbb{R})$, we compute

$$\begin{aligned} \|(\mathfrak{F}\varkappa)(t) - (\mathfrak{F}\sigma)(t)\| &= \left\|\mu \int_a^b (K(t,s,\varkappa(s)) - K(t,s,\sigma(s)))\,ds\right\| \\ &\leq |\mu| \int_a^b L\,\|\varkappa(s) - \sigma(s)\|\,ds. \end{aligned}$$

Taking the supremum over $t \in [a,b]$, we get

$$\|\mathfrak{F}\varkappa - \mathfrak{F}\sigma\|_\infty \leq |\mu|\,L\,(b-a)\,\|\varkappa - \sigma\|_\infty,$$

which implies

$$d(\mathfrak{F}\varkappa,\mathfrak{F}\sigma) = (\|\mathfrak{F}\varkappa - \mathfrak{F}\sigma\|_\infty)\,e^{i\frac{\pi}{4}} \leq |\mu|\,L\,(b-a)\,(\|\varkappa - \sigma\|_\infty)\,e^{i\frac{\pi}{4}} = \varpi d(\varkappa,\sigma),$$

for every $\varkappa,\sigma \in \Xi$. Consequently, all the hypotheses of Corollary 3.1 are fulfilled, and $\mathfrak{F}$ possesses a unique FP, i.e., $\exists\, \varkappa \in \Xi$ such that

$$\varkappa = \mathfrak{F}\varkappa.$$

Hence, the Fredholm integral equation of the second kind (5.1) has a unique solution. □

**Example 5.1.** Consider the Fredholm integral equation of the second kind

$$\varkappa(t) = f(t) + \mu \int_a^b K(t, s, \varkappa(s))ds, \tag{5.2}$$

where $f(t) = e^t$, $K(t, s, \varkappa(s)) = e^{t+s} \sin(\varkappa(s))$, $\mu = \frac{1}{10}$, $L = e^2 > 0$ and $[a, b] = [0, 1]$. Clearly, $f(t)$ and $K(t, s, \varkappa(s))$ are continuous, and hence the conditions $(a_1)$ and $(a_2)$ of Theorem 5.1 are satisfied. Now

$$|K(t, s, \varkappa) - K(t, s, \sigma)| = e^{t+s} |\sin(\varkappa(s)) - \sin(\sigma(s))|. \tag{5.3}$$

By the mean value theorem, we have

$$|\sin(\varkappa(s)) - \sin(\sigma(s))| \leq |\cos(\varsigma)| |\varkappa(s) - \sigma(s)|$$

for some $\varsigma$ between $\varkappa(s)$ and $\sigma(s)$. Since $|\cos(\varsigma)| \leq 1$.

$$\begin{aligned} |\sin(\varkappa(s)) - \sin(\sigma(s))| &\leq |\cos(\varsigma)| |\varkappa(s) - \sigma(s)| \\ &\leq |\varkappa(s) - \sigma(s)|. \end{aligned}$$

Thus by Eq (5.3), we have

$$|K(t, s, \varkappa) - K(t, s, \sigma)| \leq e^{t+s} |\varkappa(s) - \sigma(s)|.$$

Then

$$\begin{aligned} \|K(t, s, \varkappa) - K(t, s, \sigma)\| &= \sup_{t,s\in[0,1]} |K(t, s, \varkappa) - K(t, s, \sigma)| \\ &\leq \sup_{t,s\in[0,1]} e^{t+s} |\varkappa(s) - \sigma(s)| = e^2 \|\varkappa - \sigma\|. \end{aligned}$$

Hence, the condition $(a_3)$ is fulfilled. Moreover, we have

$$\varpi = |\mu| L(b - a) = \frac{1}{10} e^2(1 - 0) < 1.$$

Therefore, the aforementioned theorem's condition $(a_4)$ holds. Thus, the nonlinear Fredholm integral equation of the second kind (5.1) has a unique solution.

### *5.2. Applications of nonlinear Fredholm integral equations of the second kind in energy balance models and climate systems*

Nonlinear Fredholm integral equations of the second kind find several applications in the modeling of climate change and temperature distribution, particularly in capturing the non local interactions and nonlinear feedback mechanisms inherent in climate systems. One prominent application is in energy balance models (EBMs), which are simplified mathematical frameworks used to study Earth's climate by balancing incoming solar radiation with outgoing thermal radiation and heat redistribution. Traditional EBMs often rely on partial differential equations (PDEs) to model heat diffusion. However, nonlinear Fredholm integral equations of the second kind are particularly suited for capturing non-local heat transport, where the temperature at a location depends on heat exchange with distant regions, such as via atmospheric or oceanic circulation [24, 25].

The general form of the nonlinear Fredholm integral equation for EBMs is

$$T(t) = f(t) + \mu \int_a^b K(t, s, T(s))ds,$$

where

- $T(t)$ is the temperature at the spatial coordinate $t$ (e.g., latitude or longitude);
- $f(t)$ is the external forcing (e.g., solar radiation absorbed at location $t$);
- $\mu$ is the coupling parameter controlling the strength of heat transport, and
- $K(t, s, T(s))$ is the kernel function encoding non-local heat transport and nonlinear feedbacks.

These equations show how Earth's temperature is affected by both local factors and heat transfer across regions, like through ocean currents or winds. They help capture important climate effects, such as how melting ice can speed up warming, and predict sudden changes like rapid ice loss. The models are detailed enough to reflect real climate behavior but simple enough for long-term forecasts, making them useful for research and policy.

## 6. Conclusions

In this work, we investigated the framework of E-VMSs and developed new common FP theorems for different types of generalized contractions. Our results extend and generalize several existing findings in the field, including the main results of Azam et al. [8], Rouzkard et al. [10], and Sintunavarat et al. [11]. To emphasize the uniqueness of our main theorem, we presented a concrete example illustrating its significance. Additionally, we applied our primary result to solve a nonlinear Fredholm integral equation of the second kind, demonstrating its practical utility. These integral equations are particularly relevant in modeling climate change dynamics.

## Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The author declares no conflict of interest.

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