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Research article

Sharp estimates for the p-adic m-linear n-dimensional Hardy and Hilbert operators on p-adic weighted Morrey space

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Abstract: In this paper, we studied the sharp bounds for the m-linear n-dimensional p-adic integral operator with a kernel on central and noncentral p-adic Morrey spaces with power weight. As an application, the sharp bounds for p-adic Hardy and Hilbert operators on p-adic weighted Morrey spaces were obtained. Finally, we also found the sharp bound for the p-adic Hausdorff operator on p-adic weighted central and noncentral Morrey spaces, which generalizes the previous results.

Keywords: p-adic m-linear n-dimensional integral operator with kernel; Hardy operator; Hilbert operator; p-adic weighted Morrey space; sharp bound for integral operator
Mathematics Subject Classification: Primary 42B25; Secondary 42B20, 47B47, 47H60

1. Introduction

In recent years, researchers on mathematical physics have paid increasing attention to p-adic fields, because p-adic numbers are widely used in theoretical and mathematical physics [1, 2, 22]. For this reason, the harmonic analysis of p-adic fields has attracted significant attention [16–18].

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers, which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$. If any non-zero rational number x is represented as $x = p^{\gamma}\frac{m}{n}$, where m and n are integers that are not divisible by p and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \quad |x + y|_p \leqslant \max\{|x|_p, |y|_p\},$$

and

$$||x|_p y|_p = |x|_p^{-1}|y|_p, \quad d(|x|_p y) = |x|_p^{-1} dy, \quad x \in \mathbb{Q}_p^n.$$

It follows from the second property that when $|x|_p \neq |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$. From the standard p-adic analysis [22], we see that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^{\gamma} \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.1}$$

where a_j are integers, $0 \leqslant a_j \leqslant p-1$, $a_0 \neq 0$. The series (1.1) converges in the p-adic norm because $|a_j p^j|_p = p^{-\gamma}$. The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, ..., x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, ..., n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leqslant j \leqslant n} |x_j|_p.$$

Denoted by $B_\gamma = \{x \in \mathbb{Q}_p^n : |x - a|_p \leqslant p^\gamma\}$, the ball with center at $x_j \in \mathbb{Q}_p$ and radius p^γ, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a) = B_\gamma(a) \backslash B_{\gamma-1}(a)$ and

$$B_\gamma(a) = \bigcup_{k \leqslant \gamma} S_k(a), \quad \{x \in \mathbb{Q}_p^n : |x - a|_p < p^\gamma\} = \bigcup_{k < \gamma} S_k(a).$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Haar measure dx on $\mathbb{Q}_p^n$ (up to positive constant multiple) that is translation invariant. We normalize the measure dx so that

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. From this integral theory, it is easy to obtain that $|B_\gamma(a)|_H = p^{\gamma n}$ and $|S_\gamma(a)|_H = p^{\gamma n}(1 - p^{-n})$ for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to p-adic fields, see [19].

In recent years, p-adic analysis has received a lot of attention due to its application in mathematical physics (cf. [1, 2]). There are numerous papers on p-adic analysis, such as [11, 12] about Riesz potentials, and [3] about p-adic pseudo-differential equations. The harmonic analysis on p-adic fields has been drawing more and more attention (cf. [14, 15, 20] and references therein).

The p-adic m-linear n-dimensional Hardy operator [5] is defined by

$$T_1^p(f_1, ..., f_m)(x) = \frac{1}{|x|_p^{mn}} \int_{|(y_1,...,y_m)|_p \leqslant |x|_p} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

where $x \in \mathbb{Q}_p^n \backslash \{0\}$. Batbold and Sawano [5] obtained that the norm of T_1^p on p-adic Lebesgue spaces and p-adic Morrey spaces is

$$\|T_1^p\|_{L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q}) \to L^q(\mathbb{Q}_p, |x|_p^{\alpha})} = \frac{(1 - p^{-1})^m}{\prod_{j=1}^m (1 - p^{(1/q_j)+(a_j/q)-1})},$$

and

$$\|T_1^p\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1 q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_m q_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^{\beta})} = \frac{(1-p^{-1})^m}{\prod_{i=1}^m (1-p^{\delta_i})}.$$

The p-adic m-linear n-dimensional Hilbert operator [5] is defined by

$$T_2^p(f_1,...,f_m)(x) = \int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x|_p^n+|y_1|_p^n+\cdots+|y_m|_p^n)^m} dy_1\cdots dy_m,$$

where $x\in\mathbb{Q}_p^n\backslash\{0\}$. The p-adic m-linear n-dimensional Hausdorff operator [21] is defined by

$$T_\Phi^p(f_1,...,f_m)(x) = \int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n} \frac{\Phi(x/|y_1|_p,...,x/|y_m|_p)}{|y_1|_p^n\cdots|y_m|_p^n} f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m,$$

where $x\in\mathbb{Q}_p^n\backslash\{0\}$, and Φ is a nonnegative function on $\mathbb{Q}_p^n$. Chen et al. [6] obtained the norm of the multilinear Hausdorff operator on a p-adic function space.

Computation of the operator norm of integral operators is a challenging work in harmonic analysis. In 2017, Batbold and Sawano [4] studied one-dimensional m-linear Hilbert-type operators, incuding the sharp bounds for the Hardy-Littlewood-Pólya operator on weighted Morrey spaces. He et al. [13] extended the results in [4] and obtained the sharp bound for the generalized Hardy-Littlewood-Pólya operator on weighted central and noncentral homogenous Morrey spaces. In 2011, Wu and Fu [23] obtained the sharp estimate of the m-linear p-adic Hardy operator on Lebesgue spaces with power weight.

Inspired by the above, we study a more general operator, which includes the p-adic Hardy and Hilbert operator as a special case. We consider their operator norm on two power weighted p-adic Morrey space and its central version. Finally, we also find the sharp bound for the Hausdorff operator on power weighted central and noncentral Morrey spaces, which generalizes the previous results. Fu and Wu et al. [8–10] conducted many related research, which is the basis for our research.

We present the definition of the weighted p-adic Morrey space $B^{q,\lambda}(\mathbb{Q}_p^n,\omega_1,\omega_2)$ on $\mathbb{Q}_p^n$, where ω_1, ω_2: $\mathbb{Q}_p^n\to(0,\infty)$ are positive measurable functions.

Definition 1.1. *Let* $1<q<\infty$, $\frac{1}{q}<\lambda<0$*, then the weighted p-adic Morrey space* $L^{q,\lambda}(\mathbb{Q}_p^n,\omega_1,\omega_2)$ *is the set of all* $f\in L^q_{loc}(\mathbb{Q}_p^n)$ *for which the norm*

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n,\omega_1,\omega_2)} = \sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}} \left(\int_{B_\gamma(a)}\omega_1(x)dx\right)^{-\lambda-\frac{1}{q}} \left(\int_{B_\gamma(a)}|f(x)|^q\omega_2(x)dx\right)^{\frac{1}{q}} < \infty. \tag{1.2}$$

Definition 1.2. *Let* $1<q<\infty$, $\frac{1}{q}<\lambda<0$, B_γ *replace with* $B_\gamma(0)$ *in the above definition, then the weighted p-adic Morrey space* $B^{q,\lambda}(\mathbb{Q}_p^n,\omega_1,\omega_2)$ *is the set of all* $f\in L^q_{loc}(\mathbb{Q}_p^n)$ *for which the norm*

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n,\omega_1,\omega_2)} = \sup_{\gamma\in\mathbb{Z}} \left(\int_{B_\gamma}\omega_1(x)dx\right)^{-\lambda-\frac{1}{q}} \left(\int_{B_\gamma}|f(x)|^q\omega_2(x)dx\right)^{\frac{1}{q}} < \infty. \tag{1.3}$$

2. Proofs of the main results

In this section, we will study the p-adic m-linear n-dimensional integral operator with a kernel. Let $K:\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n\to(0,\infty)$ be a measurable radial kernel such that $K(y_1,...,y_m)=K(|y_1|_p^{-1},...,|y_m|_p^{-1})$, then satisfies that

$$C^p=\int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n}K(y_1,...,y_m)\prod_{i=1}^{m}|y_i|_p^{n\lambda_j-\frac{\beta_j}{q}+\alpha(1+\frac{1}{q_j})}dy_1\cdots dy_m<\infty, \tag{2.1}$$

where λ_j, β_j, q_j, q, α are pre-defined indicator and some fixed indicators, $j=1,2,...,m.$ The p-adic m-linear n-dimensional integral operator with a kernel is defined by

$$T^p(f_1,...,f_m)(x)=\int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n}K(y_1,...,y_m)f_1(|x|_p^{-1}y_1)\cdots f_m(|x|_p^{-1}y_m)dy_1\cdots dy_m, \tag{2.2}$$

where $x\in\mathbb{R}^n\setminus\{0\}$ and f_j is a measurable radial function on $\mathbb{Q}_p^n$ with $j=1,2,...,m.$ Note that T^p is in fact an integral operator with a homogeneous radial K of degree $-mn$.

In this paper, we will obatin the sharp bound of the p-adic m-linear n-dimensional integral operator with a kernel on p-adic weighted Morrey space. Finally, by taking a particular kernel K in operator C^p defined by (2.1), then we obtain the sharp bounds of the p-adic Hardy and Hilbert operators.

Lemma 2.1. *Let $\alpha\in\mathbb{R}$, $1<q<q_j<\infty$, $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$, $\lambda=\lambda_1+\cdots+\lambda_m$, $\beta=\beta_1+\cdots+\beta_m$, $-\frac{1}{q_j}<\lambda_j<0$, $-\frac{1}{q}<\lambda<0$, $y_j\in\mathbb{Q}_p^n$, if $f_j\in L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})$ or $B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})$, where $j=1,2,...,m$, then we have*

$$\left\|f_j(|y_j|_p^{-1}\cdot x)\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})}=|y_j|_p^{n\lambda_j-\frac{\beta_j}{q}+\alpha(\lambda_j+\frac{1}{q_j})}\left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})}, \tag{2.3}$$

and

$$\left\|f_j(|y_j|_p^{-1}\cdot x)\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})}=|y_j|_p^{n\lambda_j-\frac{\beta_j}{q}+\alpha(\lambda_j+\frac{1}{q_j})}\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{q_j\beta_j/q})}. \tag{2.4}$$

For convenience of this paper, we define the dilation index in (2.3) and (2.4) by

$$d(\lambda,q,\alpha,\beta)=n\lambda-\frac{\beta}{q}+\alpha(\lambda+\frac{1}{q}),\quad d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})=n\lambda_j-\frac{\beta_j}{q}+\alpha(\lambda_j+\frac{1}{q_j}).$$

Lemma 2.2. *Assume that real parameters q, q_j, λ, λ_j, β, β_j with $j=1,2,...,m$ are the same as in Lemma 2.1, $\alpha+n>0$. Then we have $|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\in B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$. Moreover,*

$$\left\||x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}=\frac{(1-p^{-n})^{-\lambda_j}}{(1-p^{-(\alpha+n)})^{-\lambda_j-\frac{1}{q_j}}(1-p^{-(q_j\lambda_j+1)(\alpha+n)})^{\frac{1}{q_j}}}. \tag{2.5}$$

Lemma 2.3. *Assume that real parameters q, q_j, λ, λ_j, β, β_j with $j=1,2,...,m$ are the same as in Lemma 2.1, $\alpha+n>0$. Then we have $|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\in L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$. Moreover,*

$$\left\||x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}=\max\left\{1,\frac{(1-p^{-n})^{-\lambda_j}}{(1-p^{-(\alpha+n)})^{-\lambda_j-\frac{1}{q_j}}(1-p^{-(q_j\lambda_j+1)(\alpha+n)})^{\frac{1}{q_j}}}\right\}. \tag{2.6}$$

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $1 < q < q_j < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, $\lambda = \lambda_1 + \cdots + \lambda_m$, $\beta = \beta_1 + \cdots + \beta_m$, $-\frac{1}{q_j} < \lambda_j < 0$, $-\frac{1}{q} < \lambda < 0$ with $j = 1, 2, ..., m$, f_j be a radial function in $B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$. We obtain*

$$\|T^p(f_1, ..., f_m)(x)\|_{B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} \leqslant C^p \prod_{j=1}^{m} \left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}, \tag{2.7}$$

where C^p is the constant defined by (2.1). Moreover, if $\alpha + n > 0$ and $q\lambda = q_1\lambda_1 = \cdots = q_m\lambda_m$, then C^p is the sharp constant in (2.7), then

$$\|T^p(f_1, ..., f_m)(x)\|_{\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} = C^p.$$

Theorem 2.2. *Assume that the real parameters q, q_j, λ, λ_j, β, β_j with $j = 1, 2, ..., m$ are the same as in Theorem 2.1, f_j be a radial function in $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$. We get*

$$\|T^p(f_1, ..., f_m)(x)\|_{L^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} \leqslant C^p \prod_{j=1}^{m} \left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}, \tag{2.8}$$

where C^p is the constant defined by (2.1). Moreover, if $\alpha + n > 0$ and $q\lambda = q_1\lambda_1 = \cdots = q_m\lambda_m$, then C^p is the sharp constant in (2.8), and we have

$$\|T^p(f_1, ..., f_m)(x)\|_{\prod_{j=1}^m L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})\to L^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} = C^p.$$

Corollary 2.1. *Assume that the real parameters q, q_j, λ, λ_j, β, β_j with $j = 1, 2, ..., m$ are the same as in Theorem 2.1, f_j be a radial function in $B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$. Assume also that $d(\lambda_j, q_j, \alpha, \frac{q_j\beta_j}{q}) + n > 0$. Then T_1^p is bounded from $\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$ to $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. Furthermore, if $\alpha + n > 0$, then*

$$\left\|T_1^p(f_1, ..., f_m)(x)\right\|_{\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} = \frac{(1 - p^{-n})^m}{\prod_{k=1}^m (1 - p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}. \tag{2.9}$$

The weighted Morrey space $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$ is similar.

Corollary 2.2. *Assume that the real parameters q, q_j, λ, λ_j, β, β_j with $j = 1, 2, ..., m$ are the same as in Theorem 2.1, f_j be a radial function in $B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$. Assume also that $d(\lambda_j, q_j, \alpha, \frac{q_j\beta_j}{q}) + n > 0$ and $d(\lambda, q, \alpha, \beta) < 0$. Then T_2^p is bounded from $\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$ to $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. Furthermore, if $\alpha + n > 0$, then*

$$\begin{aligned} &\left\|T_2^p(f_1, ..., f_m)(x)\right\|_{\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} \\ =&(1 - p^{-n})^m \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} \cdots \sum_{k_m=-\infty}^{+\infty} \frac{1}{(1 + p^{k_1 n} + \cdots + p^{k_m n})^m} \prod_{j=1}^{m} p^{k_j(d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n)} \\ \leqslant&\frac{(1 - p^{-n})^m (1 - p^{-mn})}{(1 - p^{d(\lambda,q,\alpha,\beta)}) \prod_{k=1}^m (1 - p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})} < \infty. \end{aligned} \tag{2.10}$$

The weighted Morrey space $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j\beta_j}{q}})$ *is similar.*

2.1. Sharp bound for the p-adic integral operator with kernel

Proof of Lemma 2.1. We only prove the scaling in $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j\beta_j}{q}})$, as the other case is similar. By computing, we obtain

$$
\begin{aligned}
&\| f_j(|y|_p^{-1}\cdot x) \|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma(a)}|f_j(|y_j|_p^{-1}x)|^{q_j}\,|x|^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
=&|y_j|^{-\frac{n}{q_j}-\frac{\beta_j}{q}}\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{|x-a|_p\leqslant p^\gamma}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{\left||y_j|_pz-a\right|_p\leqslant p^\gamma}\left|f_j(z)\right|^{q_j}|z|^{\frac{q_j\beta_j}{q}}dz\right)^{\frac{1}{q_j}}\\
=&|y_j|^{n\lambda_j-\frac{\beta_j}{q}+\alpha(\lambda_j+\frac{1}{q_j})}\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{\left||y_j|_pz-a\right|_p\leqslant p^\gamma}|z|_p^{\alpha}dz\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{\left||y_j|_pz-a\right|_p\leqslant p^\gamma}\left|f_j(z)\right|^{q_j}|z|^{\frac{q_j\beta_j}{q}}dz\right)^{\frac{1}{q_j}}\\
=&|y_j|^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_{\gamma+\log_p|y_j|_p}(a/|y_j|_p)}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_{\gamma+\log_p|y_j|_p}(a/|y_j|_p)}\left|f_j(x)\right|^{q_j}|x|^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
=&|y_j|^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}.
\end{aligned}
$$

This finishes the proof of Lemma 2.1. □

Next, we will give the proofs of Lemmas 2.2 and 2.3.

Proof of Lemma 2.2. For the case when $\alpha+n>0$, let $f_j(x)=|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}$ for all $x\in\mathbb{Q}_p^n\backslash\{0\}$ and $f_j(0):=0\,(j=1,2,...,m)$. Then for any $B_\gamma=B(0,p^\gamma)$, we need to show that $f_j\in B_\gamma^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$. By computing, we have

$$
\begin{aligned}
&\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\,|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\sum_{k=-\infty}^{\gamma}\int_{S_k}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\sum_{k'=-\infty}^{\gamma}\int_{S_{k'}}|x|_p^{n\lambda_jq_j+\alpha(\lambda_jq_j+1)}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\sum_{k=-\infty}^{\gamma}p^{k\alpha}\int_{S_k}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\sum_{k'=-\infty}^{\gamma}p^{(\lambda_jq_j(\alpha+n)+\alpha)k'}\int_{S_{k'}}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\sum_{k=-\infty}^{\gamma}p^{k\alpha}\times p^{kn}(1-p^{-n})\right)^{-\lambda_j-\frac{1}{q_j}}\left(\sum_{k'=-\infty}^{\gamma}p^{(\lambda_jq_j(\alpha+n)+\alpha)k'}\times p^{k'n}(1-p^{-n})\right)^{\frac{1}{q_j}}
\end{aligned}
$$

$$=(1-p^{-n})^{-\lambda_j}\left(\sum_{k=-\infty}^{\gamma}p^{k(\alpha+n)}\right)^{-\lambda_j-\frac{1}{q_j}}\left(\sum_{k'=-\infty}^{\gamma}p^{(q_j\lambda_j+1)(\alpha+n)k'}\right)^{\frac{1}{q_j}}.$$

Since $\alpha+n>0$ and $1+q_j\lambda_j>0$, we have

$$\sum_{k=-\infty}^{\gamma}p^{(\alpha+n)k}=\frac{p^{(\alpha+n)\gamma}(1-(p^{-(\alpha+n)})^{\infty})}{1-p^{-(\alpha+n)}}=\frac{p^{(\alpha+n)\gamma}}{1-p^{-(\alpha+n)}},$$

$$\sum_{k'=-\infty}^{\gamma}p^{(q_j\lambda_j+1)(\alpha+n)k'}=\frac{p^{(q_j\lambda_j+1)(\alpha+n)\gamma}(1-(p^{-(q_j\lambda_j+1)(\alpha+n)})^{\infty})}{1-p^{-(q_j\lambda_j+1)(\alpha+n)}}=\frac{p^{(q_j\lambda_j+1)(\alpha+n)\gamma}}{1-p^{-(q_j\lambda_j+1)(\alpha+n)}}.$$

Notice that $(\alpha+n)\,\gamma(-\lambda_j-\frac{1}{q_j})+\frac{1}{q_j}\gamma(q_j\lambda_j+1)\,(\alpha+n)=0$, so we obtain

$$\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}=\frac{(1-p^{-n})^{-\lambda_j}}{(1-p^{-(\alpha+n)})^{-\lambda_j-\frac{1}{q_j}}(1-p^{-(q_j\lambda_j+1)(\alpha+n)})^{\frac{1}{q_j}}}<\infty. \tag{2.11}$$

Using (2.11), we have that $f_j\in B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$.
This finishes the proof of Lemma 2.2. □

Proof of Lemma 2.3. For the case when $\alpha+n>0$, let $f_j(x)=|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}$ for all $x\in\mathbb{Q}_p^n\backslash\{0\}$ and $f_j(0):=0\,(j=1,2,...,m)$. Then for any $B_\gamma(a)=B(a,p^\gamma)$, we need to show that $f_j\in L_\gamma^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$. We will consider the following two cases:
(i) If $|a|_p>p^\gamma$ and $x\in B_\gamma(a)$, since $|x-a|_p\leqslant p^\gamma$, it follows that $|x|_p\leqslant\max\{|x-a|_p,|a|_p\}=|a|_p>p^\gamma$. Thus, we have

$$\begin{aligned}
&\left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma(a)}|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|a|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma(a)}|a|_p^{n\lambda_jq_j+\alpha(\lambda_jq_j+1)}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}|a|_p^{\alpha(-\lambda_j-\frac{1}{q_j})+\frac{1}{q_j}(n\lambda_jq_j+\alpha(\lambda_jq_j+1))}\left(\int_{B_\gamma(a)}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma(a)}dx\right)^{\frac{1}{q_j}}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}|a|_p^{n\lambda_j}|B_\gamma(a)|_H^{-\lambda_j}=\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}|a|_p^{n\lambda_j}\times p^{-n\lambda_j\gamma}\leqslant(p^\gamma)^{n\lambda_j}\times p^{-n\lambda_j\gamma}=1.
\end{aligned}$$

(ii) If $|a|_p\leqslant p^\gamma$ and $x\in B_\gamma(a)$, since $|x-a|_p\leqslant p^\gamma$ and $|a|_p\leqslant p^\gamma$, then $|x|_p=\max\{|x-a|_p,|a|_p\}\leqslant p^\gamma$, so we have $x\in B_\gamma$. For $x\in B_\gamma$, then $|x|_p\leqslant p^\gamma$, we have $|x-a|_p\leqslant p^\gamma$ and $x\in B_\gamma(a)$. So we have $B_\gamma(a)=B_\gamma$, thus

$$\left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}=\sup_{a\in Q_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma(a)}|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}$$

$$= \sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}|x|_p^{n\lambda_j q_j+\alpha(\lambda_j q_j+1)}dx\right)^{\frac{1}{q_j}}.$$

By the argument in the proof of Lemma 2.2, we have that for $|a|_p \leqslant p^{\gamma}$, $x \in B_\gamma(a)$,

$$\left\|f_j\right\|_{L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})} < \infty. \tag{2.12}$$

In conclusion, we can see that $f_j \in L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j\beta_j}{q}})$.
This finishes the proof of Lemma 2.3. □

Proof of Theorem 2.1. First, we claim that the operator $T^p(f_1,..,f_m)(x)$ and its restriction to the functions $g(x) = g(|x|_p^{-1})$ have the same operator norm. In fact, set

$$g_j(x) = \frac{1}{1-p^{-n}}\int_{|\xi_j|_{p=1}} f_j(|x|_p^{-1}\xi_j)d\xi_j, \quad x \in \mathbb{Q}_p^n, \quad j = 1,...,m.$$

Obviously, g_j satisfies $g_j(x) = g_j(|x|_p^{-1})$, $T^p(g_1,...,g_m)(x)$ is equal to

$$\begin{aligned}
&T^p(g_1,...,g_m)(x)\\
=&\int_{\mathbb{Q}_p^{nm}} K(y_1,...,y_m)\, g_1(|x|_p^{-1}y_1)\cdots g_m(|x|_p^{-1}y_m)dy_1\cdots dy_m\\
=&\int_{\mathbb{Q}_p^{nm}} K(y_1,...,y_m)\prod_{j=1}^{m}\left(\frac{1}{1-p^{-n}}\int_{|\xi_j|_p=1} f_j(|\,|x|_p^{-1}y_j|_p^{-1}\xi_j)d\xi_j\right)dy_1\cdots dy_m\\
=&\frac{1}{(1-p^{-n})^m}\int_{\mathbb{Q}_p^{nm}} K(y_1,...,y_m)\prod_{j=1}^{m}\left(\int_{|\xi_j|_p=1} f_j(|x|_p^{-1}|y_j|_p^{-1}\xi_j)d\xi_j\right)dy_1\cdots dy_m\\
=&\frac{1}{(1-p^{-n})^m}\int_{\mathbb{Q}_p^{nm}} K(y_1,...,y_m)\prod_{j=1}^{m}\left(\int_{|z_j|_p=|y_j|_p} f_j(|x|_p^{-1}z_j)|y_j|_p^{-n}dz_j\right)dy_1\cdots dy_m\\
=&\frac{1}{(1-p^{-n})^m}\int_{\mathbb{Q}_p^{nm}}\int_{|y_1|_p=|z_1|_p}\cdots\int_{|y_m|_p=|z_m|_p} K(|y_1|_p^{-1},...,|y_m|_p^{-1})\prod_{j=1}^{m} f_j(|x|_p^{-1}z_j)|y_j|_p^{-n}dy_1\cdots dy_m dz_m\cdots dz_1\\
=&\frac{1}{(1-p^{-n})^m}\int_{\mathbb{Q}_p^{nm}}\int_{|t_1|_p=1}\cdots\int_{|t_m|_p=1} K(|z_1|_p^{-1},...,|z_m|_p^{-1})\prod_{j=1}^{m} f_j(|x|_p^{-1}z_j)dt_1\cdots dt_m dz_m\cdots dz_1\\
=&\int_{\mathbb{Q}_p^{nm}} K(z_1,...,z_m)f_1(|x|_p^{-1}z_1)\cdots f_m(|x|_p^{-1}z_m)dz_1\cdots dz_m\\
=&T^p(f_1,...,f_m)(x).
\end{aligned}$$

In the fourth to fifth lines, we let $z_j = |y_j|_p^{-1}\xi_j$. From the fifth to sixth lines, we perform an integral permutation. In the sixth to seventh lines, we set $y_j = |z_j|_p^{-1}t_j$. On the other hand, using the Holder's inequality, we conclude

$$\left\|g_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}$$

$$
\begin{aligned}
&=\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\left|\frac{1}{1-p^{-n}}\int_{|\xi_j|=1}f_j(|x|_p^{-1}\xi_j)d\xi_j\right|^{q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&=\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\frac{1}{1-p^{-n}}\left(\int_{B_\gamma}\left|\int_{|\xi_j|=1}f_j(|x|_p^{-1}\xi_j)d\xi_j\right|^{q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&\leqslant\frac{1}{1-p^{-n}}\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\int_{|\xi_j|=1}|f_j(|x|_p^{-1}\xi_j)|^{q_j}d\xi_j\left(\int_{|\xi_j|=1}dx\right)^{q_j-1}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&=(1-p^{-n})^{-\frac{1}{q_j}}\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\int_{|\xi_j|=1}|f_j(|x|_p^{-1}\xi_j)|^{q_j}d\xi_j\,|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&=(1-p^{-n})^{-\frac{1}{q_j}}\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\int_{|z_j|_p=|x|_p}|f_j(z_j)|^{q_j}\,|x|_p^{-n}\,dz_j\,|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&=(1-p^{-n})^{-\frac{1}{q_j}}\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\int_{|x|_p=|z_j|_p}|x|_p^{-n}\,|x|_p^{\frac{q_j\beta_j}{q}}dx|f_j(z_j)|^{q_j}dz_j\right)^{\frac{1}{q_j}}\\
&=(1-p^{-n})^{-\frac{1}{q_j}}\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}\int_{|t_j|=1}|z_j|_p^{-n}|t_j|^{-n}|z_j|^{\frac{q_j\beta_j}{q}}|t_j|^{\frac{q_j\beta_j}{q}}|f_j(z_j)|^{q_j}|z_j|_p^n dt_jdz_j\right)^{\frac{1}{q_j}}\\
&=\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_\gamma}|f_j(z_j)|^{q_j}|z_j|^{\frac{q_j\beta_j}{q}}dz_j\right)^{\frac{1}{q_j}}\\
&=\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}.
\end{aligned}
$$

From the third to fourth lines, we apply Hölder's inequality. In the fifth to sixth lines, we let $z_j=|x|_p^{-1}\xi_j$. From the sixth to seventh lines, we perform an integral permutation. In the seventh to eighth lines, we set $x=|z_j|_p^{-1}t_j$. Therefore, we have

$$
\frac{\|T^p(f_1,...,f_m)\|_{B^{q_j,\lambda_j}\left(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta\right)}}{\prod_{j=1}^m\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}}\leqslant\frac{\|T^p(g_1,...,g_m)\|_{B^{q_j,\lambda_j}\left(\mathbb{Q}_p^{nm},|x|_p^\alpha,|x|_p^\beta\right)}}{\prod_{j=1}^m\left\|g_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}},
$$

which implies that the operators T^p and their restriction to the function g satisfying $g_j(x)=g_j(|x|_p^{-1})$ have the same operator norm in $B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)$. So without loss of generality, we assume that $f_j\in B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})$ with $j=1,2,...,m$ satisfies that $f_j(x)=f_j(|x|_p^{-1})$ in the rest of the proof.

The following sequence is obtained by Minkowski's inequality and Holder's inequality; notice that $\frac{1}{q}=\frac{1}{q_j}+\cdots+\frac{1}{q_m}$, $\lambda=\lambda_1+\cdots+\lambda_m$, $\beta=\beta_1+\cdots+\beta_m$, then $\frac{q}{q_j}+\cdots+\frac{q}{q_m}=1$ and $f(|x|_p^{-1}y_j)=f(x|y_j|_p^{-1})$, thus we have

$$
\begin{aligned}
&\|T^p(f_1,...,f_m)(x)\|_{B^{q,\lambda}\left(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta\right)}\\
&=\sup_{\gamma>0}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma}\left|\int_{\mathbb{Q}_p^{mn}}|x|_p^{\frac{\beta}{q}}K(y_1,...y_m)f_1(|x|_p^{-1}y_1)\cdots f_m(|x|_p^{-1}y_m)dy_1\cdots dy_m\right|^q dx\right)^{\frac{1}{q}}\\
&\leqslant\sup_{\gamma>0}\left(\int_{B_\gamma}|x|_p^\alpha dx\right)^{-\lambda-\frac{1}{q}}\int_{\mathbb{Q}_p^{mn}}\left(\int_{B_\gamma}|x|_p^\beta|K(y_1,...y_m)f_1(|x|_p^{-1}y_1)\cdots f_m(|x|_p^{-1}y_m)|^q dx\right)^{\frac{1}{q}}dy_1\cdots dy_m
\end{aligned}
$$

$$=\sup_{\gamma>0}\int_{\mathbb{Q}_p^{mn}} K(y_1,...,y_m)\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma}|x|_p^{\beta}|f_1(x|y_1|_p^{-1})\cdots f_m(x|y_m|_p^{-1})|^q dx\right)^{\frac{1}{q}} dy_1\cdots dy_m$$

$$\leqslant\sup_{\gamma>0}\int_{\mathbb{Q}_p^{mn}} K(y_1,...,y_m)\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda-\frac{1}{q}}\prod_{j=1}^{m}\left(\int_{B_\gamma}|f_j(x|y_j|_p^{-1})|^{q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}} dy_1\cdots dy_m$$

$$=\sup_{\gamma>0}\int_{\mathbb{Q}_p^{mn}} K(y_1,...,y_m)\prod_{j=1}^{m}\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda_j-\frac{1}{q_j}}\prod_{j=1}^{m}\left(\int_{B_\gamma}|f_j(x|y_j|_p^{-1})|^{q_j}|x|_p^{\frac{q_j\beta_j}{q}}dx\right)^{\frac{1}{q_j}} dy_1\cdots dy_m$$

$$\leqslant\int_{\mathbb{Q}_p^{mn}} K(y_1,...,y_m)\prod_{j=1}^{m}\left\|f_j(x|y_j|_p^{-1})\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})} dy_1\cdots dy_m.$$

Using Lemma 2.1, we can deduce that

$$\|T^p(f_1,...,f_m)(x)\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})}\leqslant\int_{\mathbb{Q}_p^{mn}} K(y_1,...y_m)\prod_{j=1}^{m}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_1\cdots dy_m\prod_{j=1}^{m}\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}$$

$$= C^p\prod_{j=1}^{m}\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})}.$$

Now, we will show that the operator norm of $T^p(f_1,...,f_m)(x)$ is equal to C^p. Taking

$$f_j=|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})},\ j=1,2,...,m.$$

Since $\alpha+n>0$, using Lemma 2.2, then $f_j\in B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})$, we calculate that

$$\begin{aligned}T^p(f_1,...,f_m)(x)&=\int_{\mathbb{Q}_p^{mn}} K(y_1,...y_m)f_1(|x|_p^{-1}y_1)\cdots f_m(|x|_p^{-1}y_m)dy_1\cdots dy_m\\&=\int_{\mathbb{Q}_p^{mn}} K(y_1,...y_m)\prod_{j=1}^{m}(|x|_p|y_j|_p)^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_1\cdots dy_m\\&=C^p\prod_{j=1}^{m}|x|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}=C^p|x|_p^{d(\lambda,q,\alpha,\beta)}.\end{aligned}$$

Since $\lambda=\lambda_1+\cdots+\lambda_m$, $\frac{1}{q}=\frac{1}{q_j}+\cdots+\frac{1}{q_m}$, $q\lambda=q_1\lambda_1=\cdots=q_m\lambda_m$, we have

$$\begin{aligned}&\|T^p(f_1,...,f_m)(x)\|_{B^{q_j,\lambda_j}\left(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta}\right)}\\&=\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma}|C^p|x|_p^{d(\lambda,q,\alpha,\beta)}|^q|x|_p^{\beta}dx\right)^{\frac{1}{q}}\\&=C^p\sup_{\gamma\in\mathbb{Z}}\left(\int_{B_\gamma}|x|_p^{\alpha}dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma}|x|_p^{nq\lambda+\alpha(q\lambda+1)}dx\right)^{\frac{1}{q}}\\&=C^p\sup_{\gamma\in\mathbb{Z}}\prod_{j=1}^{m}\left(\int_{B_\gamma}|x_j|_p^{\alpha}dx_j\right)^{-\lambda_j-\frac{1}{q_j}}\prod_{j=1}^{m}\left(\int_{B_\gamma}|x_j|_p^{nq_j\lambda_j+\alpha(q_j\lambda_j+1)}dx_j\right)^{\frac{1}{q_j}}.\end{aligned}$$

Notice that $(nq_j\lambda_j + \alpha(q_j\lambda_j + 1) + n)\frac{1}{q_j} + (n+\alpha)(-\lambda_j - \frac{1}{q_j}) = 0$, we obtain

$$\begin{aligned}
&\|T^p(f_1,...,f_m)(x)\|_{B^{q_j,\lambda_j}\left(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta\right)}\\
=&C^p\prod_{j=1}^m \sup_{\gamma_j\in\mathbb{Z}}\left(\int_{B_0}|y_j|_p^\alpha dy_j\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_0}|y_j|_p^{nq_j\lambda_j+\alpha(q_j\lambda_j+1)}dy_j\right)^{\frac{1}{q_j}}\\
=&C^p\prod_{j=1}^m \sup_{\gamma_j\in\mathbb{Z}}\left(\int_{B_{\gamma_j}}|z_j|_p^\alpha dz_j\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_{\gamma_j}}\left||z_j|_p^{d(\lambda_j,q_j,\alpha_j,\frac{q_j\beta_j}{q})}\right|^{q_j}|z_j|_p^{\frac{q_j\beta_j}{q}}dz_j\right)^{\frac{1}{q_j}}\\
=&C^p\prod_{j=1}^m\left\|f_j\right\|_{B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})}.
\end{aligned}$$

In the first to second lines, we let $x_j = p^{-\gamma}y_j$. From the second to third lines, we let $y_j = z_jp^{\gamma_j}$. Through the above steps, we have completed the proof of Theorem 2.1. It is

$$\|T^p(f_1,...,f_m)(x)\|_{\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta)} = C^p.$$

□

Proof of Theorem 2.2. The previous step is similar to the proof of Theorem 2.1, using Lemma 2.3, then $f_j \in L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^{\frac{q_j\beta_j}{q}})$, finaly we obtain that

$$\begin{aligned}
&\|T^p(f_1,...,f_m)(x)\|_{L^{q_j,\lambda_j}\left(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta\right)}\\
=&\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|x|_p^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma(a)}|C^p\,|x|_p^{d(\lambda,q,\alpha,\beta)}\,|^q\,|x|_p^\beta\,dx\right)^{\frac{1}{q}}\\
=&C^p\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\int_{B_\gamma(a)}|x|_p^\alpha dx\right)^{-\lambda-\frac{1}{q}}\left(\int_{B_\gamma(a)}|x|_p^{nq\lambda+\alpha(q\lambda+1)}\,dx\right)^{\frac{1}{q}}\\
=&C^p\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\prod_{j=1}^m\left(\int_{B_\gamma(a)}|x_j|_p^\alpha dx_j\right)^{-\lambda_j-\frac{1}{q_j}}\prod_{j=1}^m\left(\int_{B_\gamma(a)}|x_j|_p^{nq_j\lambda_j+\alpha(q_j\lambda_j+1)}dx_j\right)^{\frac{1}{q_j}}.
\end{aligned}$$

Notice that $(nq_j\lambda_j + \alpha(q_j\lambda_j + 1) + n)\frac{1}{q_j} + (n+\alpha)(-\lambda_j - \frac{1}{q_j}) = 0$, we obtain

$$\begin{aligned}
&\|T^p(f_1,...,f_m)(x)\|_{L^{q_j,\lambda_j}\left(\mathbb{Q}_p^n,|x|_p^\alpha,|x|_p^\beta\right)}\\
=&C^p\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\prod_{j=1}^m\left(\int_{B_0(p^\gamma a)}|y_j|_p^\alpha dy_j\right)^{-\lambda_j-\frac{1}{q_j}}\prod_{j=1}^m\left(\int_{B_0(p^\gamma a)}|y_j|_p^{nq_j\lambda_j+\alpha(q_j\lambda_j+1)}dy_j\right)^{\frac{1}{q_j}}\\
=&C^p\prod_{j=1}^m\sup_{a_j\in\mathbb{Q}_p^n}\left(\int_{B_0(a_j)}|y_j|_p^\alpha dy_j\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_0(a_j)}|y_j|_p^{nq_j\lambda_j+\alpha(q_j\lambda_j+1)}dy_j\right)^{\frac{1}{q_j}}\\
=&C^p\prod_{j=1}^m\sup_{a_j\in\mathbb{Q}_p^n,\gamma_j\in\mathbb{Z}}\left(\int_{B_{\gamma_j}(p^{-\gamma_j}a_j)}|z_j|_p^\alpha dz_j\right)^{-\lambda_j-\frac{1}{q_j}}\left(\int_{B_{\gamma_j}(p^{-\gamma_j}a_j)}\left||z_j|_p^{d(\lambda_j,q_j,\alpha_j,\frac{q_j\beta_j}{q})}\right|^{q_j}|z_j|_p^{\frac{q_j\beta_j}{q}}dz_j\right)^{\frac{1}{q_j}}
\end{aligned}$$

$$
\begin{aligned}
&= C^p \prod_{j=1}^{m} \sup_{a_j \in \mathbb{Q}_p^n, \gamma_j \in \mathbb{Z}} \left(\int_{B_{\gamma_j}(a_j)} |z_j|_p^{\alpha} dz_j \right)^{-\lambda_j - \frac{1}{q_j}} \left(\int_{B_{\gamma_j}(a_j)} \left| |z_j|_p^{d(\lambda_j, q_j, \alpha_j, \frac{q_j \beta_j}{q})} \right|^{q_j} |z_j|_p^{\frac{q_j \beta_j}{q}} dz_j \right)^{\frac{1}{q_j}} \\
&= C^p \prod_{j=1}^{m} \left\| f_j \right\|_{L^{q_j, \lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j \beta_j}{q}})}.
\end{aligned}
$$

In the first to second lines, we let $x_j = p^{-\gamma} y_j$. From the third to fourth lines, we let $y_j = z_j p^{\gamma_j}$. Through the above steps, we have completed the proof of Theorem 2.2. It is

$$
\| T^p (f_1, ..., f_m)(x) \|_{\prod_{j=1}^{m} L^{q_j, \lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j \beta_j}{q}}) \to L^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\beta})} = C^p.
$$

□

2.2. *Sharp bound for p-adic Hardy operator*

Proof of Corollary 2.1. Next, we will use the methods in [23]. If we take the kernel

$$
K(y_1, ..., y_m) = \chi_{\{|(y_1, ..., y_m)|_p \leqslant 1\}}(y_1, ..., y_m) \tag{2.13}
$$

in Theorems 2.1 and 2.2, by a change of variables, it is easy to verify that $T^p(f_1, ..., f_m)(x) = T_1^p(f_1, ..., f_m)(x)$, then $T_1^p(f_1, ..., f_m)(x)$ can be denoted by

$$
T_1^p(f_1, ..., f_m)(x) = \int_{|(y_1, ..., y_m)|_p \leqslant 1} f_1(|x|_p^{-1} y_1) \cdots f_m(|x|_p^{-1} y_m) dy_1 \cdots dy_m,
$$

respectively, then it is all reduced to calculating

$$
C_1^p = \int_{|(y_1, ..., y_m)|_p \leqslant 1} \prod_{j=1}^{m} |y_j|_p^{d(\lambda_j, q_j, \alpha, \frac{q_j \beta_j}{q})} dy_1 \cdots dy_m.
$$

To calculate this integral, we divide the integral into m parts. Let

$$
\begin{aligned}
D_1 &= \{(y_1, ..., y_m) \in \mathbb{Q}_p^n \cdots \mathbb{Q}_p^n : |y_1|_p \leqslant 1, |y_k|_p \leqslant |y_1|_p, 1 < k \leqslant m\}, \\
D_i &= \{(y_1, ..., y_m) \in \mathbb{Q}_p^n \cdots \mathbb{Q}_p^n : |y_i|_p \leqslant 1, |y_j|_p < |y_i|_p, |y_k|_p \leqslant |y_i|_p, 1 \leqslant j < i < k \leqslant m\}, \\
D_m &= \{(y_1, ..., y_m) \in \mathbb{Q}_p^n \cdots \mathbb{Q}_p^n : |y_m|_p \leqslant 1, |y_j|_p < |y_m|_p, 1 \leqslant j < m\}.
\end{aligned}
$$

It is clear that

$$
\bigcup_{j=1}^{m} D_j = \{(y_1, ..., y_m) \in \mathbb{Q}_p^n \cdots \mathbb{Q}_p^n : |(y_1, ..., y_m)|_p \leqslant 1\},
$$

and $D_i \cap D_j = \varnothing \, (i \neq j)$. Let

$$
I_j := \int_{D_j} \prod_{k=1}^{m} |y_k|_p^{d(\lambda_k, q_k, \alpha, \frac{q_k \beta_k}{q})} dy_1 \cdots dy_m.
$$

Then

$$
C_1^p = \sum_{j=1}^{m} I_j : = \sum_{j=1}^{m} \int_{D_j} \prod_{k=1}^{m} |y_k|_p^{d(\lambda_k, q_k, \alpha, \frac{q_k \beta_k}{q})} dy_1 \cdots dy_m.
$$

Now, let us calculate $I_j,\ j=1,2,...,m$. Since $d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n>0$, then $d(\lambda,q,\alpha,\beta)+mn>0$, so we have

$$\begin{aligned}
I_1 &= \int_{D_1}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_1\cdots dy_m\\
&= \int_{|y_1|_p\leqslant 1}\int_{|y_2|_p\leqslant|y_1|_p}\cdots\int_{|y_m|_p\leqslant|y_1|_p}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_m\cdots dy_1\\
&= \int_{|y_1|_p\leqslant 1}|y_1|_p^{d(\lambda_1,q_1,\alpha,\frac{q_1\beta_1}{q})}\left(\prod_{k=2}^{m}\int_{|y_k|_p\leqslant|y_1|_p}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_1\\
&= \int_{|y_1|_p\leqslant 1}|y_1|_p^{d(\lambda_1,q_1,\alpha,\frac{q_1\beta_1}{q})}\prod_{k=2}^{m}\left(\sum_{i=-\infty}^{\log_p|y_1|_p}\int_{S_i}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_1\\
&= \int_{|y_1|_p\leqslant 1}|y_1|_p^{d(\lambda_1,q_1,\alpha,\frac{q_1\beta_1}{q})}\prod_{k=2}^{m}\left(\sum_{i=-\infty}^{\log_p|y_1|_p}p^{id(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}\times\int_{S_i}dy_k\right)dy_1\\
&= \frac{(1-p^{-n})^{m-1}}{\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,q_k\beta_k/q)-n})}\int_{|y_1|_p\leqslant 1}|y_1|_p^{d(\lambda,q,\alpha,\beta)+(m-1)n}dy_1\\
&= \frac{(1-p^{-n})^{m-1}}{\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,q_k\beta_k/q)-n})}\sum_{i=-\infty}^{0}\left(p^{i(d(\lambda,q,\alpha,\beta)+(m-1)n)}\int_{S_i}dy_1\right)\\
&= \frac{(1-p^{-n})^{m}}{(1-p^{-d(\lambda,q,\alpha,\beta)-mn})\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,q_k\beta_k/q)-n})}.
\end{aligned}$$

Similarly, for $i=2,...,m-1$, we have

$$\begin{aligned}
I_i &= \int_{D_i}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_1\cdots dy_m\\
&= \int_{|y_i|_p\leqslant 1}|y_i|_p^{d(\lambda_i,q_i,\alpha,\frac{q_i\beta_i}{q})}\left(\prod_{j=1}^{i-1}\int_{|y_j|_p<|y_i|_p}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_j\right)\left(\prod_{k=i+1}^{m}\int_{|y_k|_p\leqslant|y_i|_p}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_i\\
&= \int_{|y_i|_p\leqslant 1}|y_i|_p^{d(\lambda_i,q_i,\alpha,\frac{q_i\beta_i}{q})}\left(\prod_{j=1}^{i-1}\sum_{u=-\infty}^{\log_p|y_i|_p-1}\int_{S_u}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_j\right)\left(\prod_{k=i+1}^{m}\sum_{v=-\infty}^{\log_p|y_i|_p}\int_{S_v}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_i\\
&= (1-p^{-n})^{m-1}\int_{|y_i|_p\leqslant 1}|y_i|_p^{d(\lambda_i,q_i,\alpha,\frac{q_i\beta_i}{q})}\left(\prod_{j=1}^{i-1}\sum_{u=-\infty}^{\log_p|y_i|_p-1}p^{u(d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n)}\right)\left(\prod_{k=i+1}^{m}\sum_{v=-\infty}^{\log_p|y_i|_p}p^{v(d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})+n)}\right)dy_i\\
&= \frac{(1-p^{-n})^{m-1}\prod_{j=1}^{i-1}p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{\prod_{1\leqslant k\leqslant m,k\neq i}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}\int_{|y_i|\leqslant 1}|y_i|_p^{d(\lambda,q,\alpha,\beta)+(m-1)n}dy_i\\
&= \frac{(1-p^{-n})^{m}\prod_{j=1}^{i-1}p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{(1-p^{-d(\lambda,q,\alpha,\beta)-mn})\prod_{1\leqslant k\leqslant m,k\neq i}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}.
\end{aligned}$$

The case of $i = m$ is similar to the previous step, and we have

$$I_m = \int_{|y_m|_p\leqslant 1} |y_m|^{d(\lambda_m,q_m,\alpha,\frac{q_m\beta_m}{q})}\left(\prod_{j=1}^{m-1}\int_{|y_j|<|y_m|_p}|y_j|^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_i\right)dy_m$$
$$= \frac{(1-p^{-n})^m\prod_{j=1}^{m-1}p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{\left(1-p^{-d(\lambda,q,\alpha,\beta)-mn}\right)\prod_{j=1}^{m-1}\left(1-p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}\right)}.$$

Now, we will calculate their sum, let

$$A_m = \frac{(1-p^{-n})^m}{\left(1-p^{-d(\lambda,q,\alpha,\beta)-mn}\right)\prod_{k=1}^{m}\left(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n}\right)},\quad -d_k = \sum_{i=1}^{k}-d(\lambda_i,q_i,\alpha,\frac{q_i\beta_i}{q}).$$

Notice that $-d_m = -d\,(\lambda,q,\alpha,\beta)$, then

$$\begin{aligned}C_1^p &= I_1 + \sum_{i=2}^{m-1} I_i + I_m\\ &= A_m\left((1-p^{-d_1-n}) + (p^{-d_1-n}-p^{-d_2-2n}) + \cdots + (p^{-d_{m-1}-(m-1)n}-p^{-d_m-mn})\right) = A_m\left(1-p^{-d_m-mn}\right)\\ &= \frac{(1-p^{-n})^m}{\prod_{k=1}^{m}\left(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n}\right)}.\end{aligned}$$

This finishes the proof of Corollary 2.1. □

2.3. *Sharp bound for p-adic Hilbert operator*

Proof of Corollary 2.2. If we take the kernel

$$K\,(y_1,...,y_m) = \frac{1}{(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m} \tag{2.14}$$

in Theorems 2.1 and 2.2, by a change of variables, it is easy to verify that $T^p(f_1,...,f_m)(x) = T_2^p(f_1,...,f_m)(x)$, then $T_2^p(f_1,...,f_m)(x)$ can be denoted by

$$T_2^p(f_1,...,f_m)(x) = \int_{\mathbb{Q}_p^{nm}}\frac{1}{(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m}f_1(|x|_p^{-1}y_1)\cdots f_m(|x|_p^{-1}y_m)dy_1\cdots dy_m,$$

respectively, then it is all reduced to calculating

$$C_2^p = \int_{\mathbb{Q}_p^{nm}}\frac{1}{(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m}\prod_{j=1}^{m}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_1\cdots dy_m.$$

After a series of simple operations, we have

$$C_2^p = \int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n}\sum_{k_m=-\infty}^{+\infty}\int_{S_{k_m}}\frac{1}{(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m}\prod_{j=1}^{m}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_m\cdots dy_1$$

$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\cdots\sum_{k_m=-\infty}^{+\infty}\int_{S_{k_1}}\cdots\int_{S_{k_m}}\frac{1}{(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m}\prod_{j=1}^{m}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_m\cdots dy_1$$
$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\cdots\sum_{k_m=-\infty}^{+\infty}\frac{1}{(1+p^{k_1n}+\cdots+p^{k_mn})^m}\prod_{j=1}^{m}p^{k_jd(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\int_{S_{k_1}}\cdots\int_{S_{k_m}}dy_m\cdots dy_1$$
$$= \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\cdots\sum_{k_m=-\infty}^{+\infty}\frac{1}{(1+p^{k_1n}+\cdots+p^{k_mn})^m}\prod_{j=1}^{m}p^{k_jd(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}\prod_{j=1}^{m}p^{k_jn}(1-p^{-n})$$
$$= (1-p^{-n})^m\sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\cdots\sum_{k_m=-\infty}^{+\infty}\frac{1}{(1+p^{k_1n}+\cdots+p^{k_mn})^m}\prod_{j=1}^{m}p^{k_j(d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n)}.$$

What we want to prove is that the sum of this series is bounded. Because it is a challenging problem to calculate the sum of this series, we can indirectly prove that this series sum is bounded by an inequality. Clearly,

$$[\max(1,|y_1|_p^n,\cdots,|y_m|_p^n)]^m=\max_{1\leqslant j\leqslant m}\{1,|y_j|_p^{mn}\}\leqslant(1+|y_1|_p^n+\cdots+|y_m|_p^n)^m.$$

Then we have

$$C_2^p\leqslant D^p=:\int_{\mathbb{Q}_p^{nm}}\frac{1}{[\max(1,|y_1|_p^n,...,|y_m|_p^n)]^m}\prod_{j=1}^{m}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_1\cdots dy_m.$$

So if we prove that D^p is bounded, it means that C_2^p is bounded. Next, we refer to the methods in [7] to calculate D^p.

To calculate this integral, we divide the integral into m parts. Let

$$E_0=\{(y_1,...,y_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n:|y_k|_p\leqslant 1,1\leqslant k\leqslant m\};$$
$$E_1=\{(y_1,...,y_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n:|y_1|_p>1,|y_k|_p\leqslant|y_1|_p,1<k\leqslant m\};$$
$$E_i=\{(y_1,...,y_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n:|y_i|_p>1,|y_j|_p<|y_i|_p,|y_k|_p\leqslant|y_i|_p,1\leqslant j<i<k\leqslant m\};$$
$$E_m=\{(y_1,...,y_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n:|y_m|_p>1,|y_j|_p<|y_m|_p,1\leqslant j\leqslant m\}.$$

Its clear that

$$\bigcup_{j=0}^{m}E_j=\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n,$$

and $E_i\cap E_j=\varnothing\,(i\neq j)$. Let

$$J_j:=\int_{E_j}\frac{1}{[\max(1,|y_1|_p^n,...,|y_m|_p^n)]^m}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_1\cdots dy_m,$$

then we have

$$D^p=\sum_{j=1}^{m}J_j:=\sum_{j=1}^{m}\int_{E_j}\frac{1}{[\max(1,|y_1|_p^n,...,|y_m|_p^n)]^m}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_1\cdots dy_m.$$

Now, let us calculate $J_j,\ j=1,2,...,m$. Since $d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n>0$, we have

$$\begin{aligned}
J_0 &= \prod_{k=1}^{m}\int_{|y_k|_p\leqslant 1}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k=\prod_{k=1}^{m}\left(\sum_{i=-\infty}^{0}\int_{S_i}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)\\
&=\prod_{k=1}^{m}\left(\sum_{i=-\infty}^{0}p^{id(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}p^{in}(1-p^{-n})\right)\\
&=\frac{(1-p^{-n})^m}{\prod_{k=1}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}.
\end{aligned}$$

For $j=1$, since $d\,(\lambda,q,\alpha,\beta)<0$, $d(\lambda_1,q_1,\alpha,\frac{q_1\beta_1}{q})+n>0$, then we have

$$\begin{aligned}
J_1 &= \int_{E_1}\frac{1}{[\max(1,|y_1|_p^n,...,|y_m|_p^n)]^m}\prod_{k=1}^{m}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_1\cdots dy_m\\
&=\int_{|y_1|_p>1}|y_1|_p^{d(\lambda_1,q_1,\alpha,\frac{q_1\beta_1}{q})-mn}\prod_{k=2}^{m}\left(\int_{|y_k|_p\leqslant|y_1|_p}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_1\\
&=\frac{(1-p^{-n})^{m-1}}{\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}\int_{|y_1|_p>1}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}dy_1\\
&=B_m\left(\int_{|y_1|_p<\infty}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}\,dy_1-\int_{|y_1|_p\leqslant 1}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}\,dy_1\right)\\
&=B_m\left(\sum_{i=-\infty}^{+\infty}\int_{S_i}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}dy_1-\sum_{j=-\infty}^{0}\int_{S_j}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}dy_1\right)\\
&=B_m\sum_{j=1}^{+\infty}\int_{S_j}|y_1|_p^{d(\lambda,q,\alpha,\beta)-n}dy_1=\frac{(1-p^{-n})^m\,p^{d(\lambda,q,\alpha,\beta)}}{(1-p^{d(\lambda,q,\alpha,\beta)})\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})},
\end{aligned}$$

where

$$B_m=\frac{(1-p^{-n})^{m-1}}{\prod_{k=2}^{m}(1-p^{-d(\lambda_k,q_k,\alpha,q_k\beta_k/q)-n})}.$$

Similarly for $i=2,...,m-1$, we have

$$\begin{aligned}
J_i &= \int_{|y_i|_p>1}|y_i|_p^{d(\lambda_i,q_i,\alpha,\frac{q_i\beta_i}{q})-mn}\left(\prod_{j=1}^{i-1}\int_{|y_j|_p<|y_i|_p}|y_j|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_j\right)\left(\prod_{k=i+1}^{m}\int_{|y_k|_p\leqslant|y_i|_p}|y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})}dy_k\right)dy_i\\
&=\frac{(1-p^{-n})^{m-1}\prod_{j=1}^{i-1}p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{\prod_{1\leqslant k\leqslant m,k\neq i}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}\int_{|y_i|_p>1}|y_i|_p^{d(\lambda,q,\alpha,\beta)-n}dy_i\\
&=\frac{p^{d(\lambda,q,\alpha,\beta)}\,(1-p^{-n})^m\prod_{j=1}^{i-1}p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{(1-p^{d(\lambda,q,\alpha,\beta)})\prod_{1\leqslant k\leqslant m,k\neq i}(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}.
\end{aligned}$$

When $i = m$, similarly to the previous step, we show that

$$J_m = \int_{|y_m|_p} |y_m|^{d(\lambda_m,q_m,\alpha,\frac{q_m\beta_m}{q})-mn} \left(\prod_{j=1}^{i-1} \int_{|y_j|_p<|y_i|_p} |y_i|_p^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})} dy_j\right)\left(\prod_{k=i+1}^{m} \int_{|y_k|_p\leqslant|y_i|_p} |y_k|_p^{d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})} dy_k\right) dy_m$$
$$= \frac{p^{d(\lambda,q,\alpha,\beta)} (1-p^{-n})^m \prod_{j=1}^{m-1} p^{-d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})-n}}{(1-p^{d(\lambda,q,\alpha,\beta)}) \prod_{k=1}^{m-1} (1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}.$$

Now, we will calculate their sum, let

$$D_m = \frac{(1-p^{-n})^m p^{d(\lambda,q,\alpha,\beta)}}{(1-p^{d(\lambda,q,\alpha,\beta)}) \prod_{k=1}^{m} (1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})}, \quad \textit{and} \quad -d_k = \sum_{i=1}^{k} -d(\lambda_i, q_i, \alpha, \frac{q_i\beta_i}{q}).$$

Notice that $-d_m = -d(\lambda, q, \alpha, \beta)$, then

$$\begin{aligned} D^p &= J_0 + J_1 + \sum_{i=2}^{m-1} J_i + J_m \\ &= D_m\left(\frac{1-p^{d(\lambda,q,\alpha,\beta)}}{p^{d(\lambda,q,\alpha,\beta)}} + (1-p^{-d_1-n}) + (p^{-d_1-n} - p^{-d_2-2n}) + \cdots + (p^{-d_{m-1}-(m-1)n} - p^{-d_m-mn})\right) \\ &= D_m\left(\frac{1-p^{d_m}}{p^{d_m}} + 1 - p^{-d_m-mn}\right) = D_m p^{-d_m}(1-p^{-mn}) \\ &= \frac{(1-p^{-n})^m (1-p^{-mn})}{(1-p^{d(\lambda,q,\alpha,\beta)}) \prod_{k=1}^{m} (1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n})} < \infty. \end{aligned}$$

In conclusion, we prove that D^p is bounded, which also means that C_2^p is bounded, that is

$$(1-p^{-n})^m \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} \cdots \sum_{k_m=-\infty}^{+\infty} \frac{1}{(1+p^{k_1 n} + \cdots + p^{k_m n})^m} \prod_{j=1}^{m} p^{k_j(d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n)} < \infty.$$

Our results also show that D^p is Hilbert's bound. This finishes the proof of Corollary 2.2. □

3. Further results: sharp estimate for the Hausdorff operator

In this section, we will use the previous results to give the sharp bound for the p-adic m-linear n-dimensional Hausdorff operator on p-adic weighted Morrey spaces.

Corollary 3.1. *Assume that the real parameters q, q_j, λ, λ_j, β, and β_j with $j = 1, 2, ..., m$ are the same as in Theorem 2.1, and a nonnegative function Φ on $\mathbb{R}^n$ satisfies*

$$C_\Phi^p = \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, ..., y_m)}{|y_1|_p^n \cdots |y_m|_p^n} \prod_{j=1}^{m} |y_j|^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})} dy_1 \cdots dy_m < \infty, \tag{3.1}$$

then T_Φ^p is bounded from $\prod_{j=1}^m B^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_j\beta_j}{q}})$ to $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$.

Furthermore, if $\alpha + n > 0$, then

$$\left\|T_{\Phi}^{p}\left(f_{1},...,f_{m}\right)(x)\right\|_{\prod_{j=1}^{m} B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})} = C_{\Phi}^{p}.$$

The weighted Morrey space $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j\beta_j}{q}})$ is similar.

Proof of Corollary 3.1. By a change of variables, the p-adic m-linear n-dimensional Hausdorff operator becomes

$$T_{\Phi}^{p} = \int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n}\frac{\Phi\left(y_1,...,y_m\right)}{|y_1|_p^n\cdots|y_m|_p^n} f_1(x\,|y_1|_p^{-1})\cdots f_m(x\,|y_m|_p^{-1})dy_1\cdots dy_m.$$

According to the proof of Theorem 2.1 and Lemma 2.2, notice that there is no need to let f_j be a radial function, and we have

$$\left\|T_{\Phi}^{p}\left(f_{1},...,f_{m}\right)(x)\right\|_{\prod_{j=1}^{m} B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})} = C_{\Phi}^{p} < \infty.$$

The weighted Morrey space $L^{q_j,\lambda_j}(\mathbb{Q}_p^n, |x|_p^{\alpha}, |x|_p^{\frac{q_j\beta_j}{q}})$ is similar, so we omit the details. This finishes the proof of Corollary 3.1. □

4. Conclusions

First, the m-linear n-dimensional integral operator with a kernel has a sharp estimate. The sharp estimate on central and noncentral p-adic Morrey spaces with power weighted is given by

$$\begin{aligned}
&\|T^{p}\left(f_{1},...,f_{m}\right)(x)\|_{\prod_{j=1}^{m} B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})}\\
=&\|T^{p}\left(f_{1},...,f_{m}\right)(x)\|_{\prod_{j=1}^{m} L^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to L^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})}\\
=&\int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n} K\left(y_1,...,y_m\right)\prod_{i=1}^{m}|y_i|_p^{n\lambda_j-\frac{\beta_j}{q}+\alpha(1+\frac{1}{q_j})}dy_1\cdots dy_m := C^p,
\end{aligned}$$

where the kernel $K(y_1,..,y_m)$ satisfies $C^p < \infty$.

Second, as an application, we derive the sharp bounds for the m-linear n-dimensional Hardy operator and Hilbert operators on weighted Morrey spaces, that is

$$\left\|T_{1}^{p}\left(f_{1},...,f_{m}\right)(x)\right\|_{\prod_{j=1}^{m} B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})} = \frac{\left(1-p^{-n}\right)^m}{\prod_{k=1}^{m}\left(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n}\right)},$$

and

$$\begin{aligned}
&\left\|T_{2}^{p}\left(f_{1},...,f_{m}\right)(x)\right\|_{\prod_{j=1}^{m} B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})}\\
=&\left(1-p^{-n}\right)^m\sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}\cdots\sum_{k_m=-\infty}^{+\infty}\frac{1}{\left(1+p^{k_1 n}+\cdots+p^{k_m n}\right)^m}\prod_{j=1}^{m}p^{k_j(d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})+n)}\\
\leqslant&\frac{\left(1-p^{-n}\right)^m\left(1-p^{-mn}\right)}{\left(1-p^{d(\lambda,q,\alpha,\beta)}\right)\prod_{k=1}^{m}\left(1-p^{-d(\lambda_k,q_k,\alpha,\frac{q_k\beta_k}{q})-n}\right)} < \infty.
\end{aligned}$$

Finally, based on the previous result, we also find the estimate for the Hausdorff operator on weighted Morrey spaces:

$$\left\|T_{\Phi}^{p}(f_1,\ldots,f_m)(x)\right\|_{\prod_{j=1}^{m}B^{q_j,\lambda_j}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\frac{q_j\beta_j}{q}})\to B^{q,\lambda}(\mathbb{Q}_p^n,|x|_p^{\alpha},|x|_p^{\beta})}\int_{\mathbb{Q}_p^n}\cdots\int_{\mathbb{Q}_p^n}\frac{\Phi(y_1,\ldots,y_m)}{|y_1|_p^n\cdots|y_m|_p^n}\prod_{j=1}^{m}|y_j|^{d(\lambda_j,q_j,\alpha,\frac{q_j\beta_j}{q})}dy_1\cdots dy_m:=C_{\Phi}^{p},$$

where the nonnegative function Φ on $\mathbb{R}^n$ satisfies $C_{\Phi}^{p}<\infty$.

Author contributions

Tingting Xu: Conceptualization, methodology; Zaiyong Feng: Writing-original draft; Tianyang He: Writing-review and editing, validation, methodology; Xiaona Fan: Theoretical derivation, proof verification. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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