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Research article

Generalized Perron complements of strictly generalized doubly diagonally dominant matrices

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Abstract: It is well known that there is an intrinsic connection between Perron complements and Schur complements. It has been demonstrated that Schur complements of strictly generalized doubly diagonally dominant matrices retain the property of strict generalized double diagonal dominance. Our primary aim of this study is to extend these findings to generalized Perron complements of nonnegative irreducible matrices. Specifically, we established that generalized Perron complements derived from strictly generalized doubly diagonally dominant and nonnegative irreducible matrices preserve strict generalized double diagonal dominance and nonnegative irreducibility. Numerical examples are provided to substantiate our theoretical results.

Keywords: nonnegative matrix; generalized Perron complement; diagonally dominant matrix; generalized doubly diagonally dominant matrix
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1. Introduction

Let $C^{n\times n}$ ($R^{n\times n}$) denote the set of all $n \times n$ complex (real) matrices and $N = \{1, 2, \dots, n\}$. For $A = \left(a_{ij}\right) \in R^{n\times n}$, we say that A is a nonnegative matrix if $a_{ij} \geq 0$ for all i, j. This fundamental concept plays a crucial role in nonnegative matrix factorization (NMF), which has become an important tool in machine learning applications. Researchers [1,2] have comprehensively documented the theoretical foundations and practical implementation of NMF.

For $n \geq 2$, an $n \times n$ matrix A is said to be reducible if there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where B and D are square submatrices of order at least one. The matrix A is called irreducible if no such permutation matrix exists. For the special case of a 1×1 complex matrix, A is irreducible if and only if its single entry is nonzero.

Recall that a square matrix A is a nonsingular M-matrix if there exists an $n \times n$ nonnegative matrix P and some real number s, such that

$$A = sI - P, \ s > \rho(P),$$

where $\rho(P)$ is the spectral radius of P and I denotes the identity matrix. The set of nonsingular M-matrix of order n is denoted by M_n[3].

Let $A = \left(a_{ij}\right) \in C^{n\times n}$. The comparison matrix $m(A) = (m_{ij})$ of A is defined by

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

As is known, A is an H-matrix if its comparison matrix $m(A) \in M_n$.

Let $\alpha, \beta \subset N$ be nonempty ordered index sets. We denote by $A(\alpha,\beta)$ the submatrix of $A \in C^{n\times n}$ with rows α and columns β, $A(\alpha) = A(\alpha,\alpha)$, and the principal submatrix when $\alpha = \beta$. For a nonsingular principal submatrix $A(\gamma)$, where $\gamma \subset N$, the Schur complement of A with respect to $A(\gamma)$ is defined as:

$$A/A(\gamma) \ = A(\gamma^c) - A(\gamma^c,\gamma)\left[A(\gamma)\right]^{-1} A(\gamma,\gamma^c),$$

where γ^c denotes the complement of γ. The classical Schur determinantal formula [4] provides the fundamental relation:

$$\det A/A(\alpha) = \frac{\det A}{\det A(\alpha)}. \tag{1.1}$$

Given a matrix $A = \left(a_{ij}\right) \in C^{n\times n}$, if

$$|a_{ii}| \geq \sum_{k\neq i}^{n} |a_{ik}|, \ i \in N, \tag{1.2}$$

then A is (row) diagonally dominant. The matrix A is strictly diagonally dominant if for every $i \in N$, the following strict inequality holds:

$$|a_{ii}| > \sum_{k\neq i}^{n} |a_{ik}|.$$

A fundamental result in matrix theory is that strictly diagonally dominant matrices are necessarily nonsingular, and this nonsingularity property is inherited by all principal submatrices.

The matrix A is called a doubly diagonally dominant matrix [5] if

$$|a_{ii}| \left|a_{jj}\right| \geq \sum_{k\neq i}^{n} |a_{ik}| \sum_{k\neq j}^{n} \left|a_{jk}\right|, \ i \neq j, \ i,j \in N. \tag{1.3}$$

A is further said to be strictly doubly diagonally dominant if all inequalities in (1.3) hold for all distinct $i, j \in N$.

Let $A = \left(a_{ij}\right) \in C^{n\times n}$ $(n \geq 2)$ and denote

$$J(A) = \left\{ i \,\middle|\, |a_{ii}| > \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}|, i \in N \right\}.$$

Suppose that α and β are nonempty index sets satisfying $\alpha \bigcup \beta = N, \alpha \bigcap \beta = \varnothing$. We define the partial absolute deleted row and column sums as follows:

$$r_i^{(\alpha)}(A) = \sum_{\substack{k\neq i \\ k\in\alpha}} |a_{ik}|, \;\; r_i^{(\beta)}(A) = \sum_{\substack{k\neq i \\ k\in\beta}} |a_{ik}|.$$

When the index set α contains only one element (i.e., $\alpha = \{i_0\}$), we adopt the convention that $r_{i_0}^{(\alpha)}(A) = 0$. Similarly, $r_{i_0}^{(\beta)}(A) = 0$ when $\beta = \{i_0\}$.

Recall that A is generalized doubly diagonally dominant if there exist proper subsets α, β of N, such that $\alpha \bigcup \beta = N, \alpha \bigcap \beta = \varnothing$ and

$$\left[|a_{ii}| - r_i^{(\alpha)}(A)\right]\left[|a_{jj}| - r_j^{(\beta)}(A)\right] \geq r_i^{(\beta)}(A) r_j^{(\alpha)}(A) \tag{1.4}$$

for all $i \in \alpha, j \in \beta$. See [6] for details. We call A a strictly generalized doubly diagonally dominant matrix if all inequalities in (1.4) hold for all $i \in \alpha, j \in \beta$.

Remark 1.1. *When $n = 2$, generalized doubly diagonally dominant matrices reduce to the classical doubly diagonally dominant case.*

Remark 1.2. *The generalized doubly diagonally dominant property is partition-dependent: a matrix A satisfying (1.4) for specific subsets α and β may fail to satisfy the condition for alternative partitions.*

Following the convention established in [6], we adopt identical notation for matrices of order $n \geq 2$.

- D_n : Diagonally dominant matrices;
- SD_n : Strictly diagonally dominant matrices;
- DD_n : Doubly diagonally dominant matrices;
- SDD_n : Strictly doubly diagonally dominant matrices;
- $GDD_n^{\alpha,\beta}$: Generalized doubly diagonally dominant matrices;
- $SGDD_n^{\alpha,\beta}$: Strictly generalized doubly diagonally dominant matrices.

Upon examining condition (1.4), we observe that when the index set β (or analogously, α) is empty, all terms involving $r_i^{(\beta)}(A)$ and $r_j^{(\beta)}(A)$ vanish. In this degenerate case, we may interpret the inequality (1.4) as $|a_{ii}| - r_i^{(\alpha)}(A) \geq 0$, which corresponds to A being diagonally dominant. Following standard convention, we denote this special case as

$$GDD_n^{N,\varnothing} = GDD_n^{\varnothing,N} = D_n$$

and

$$SGDD_n^{N,\varnothing} = SGDD_n^{\varnothing,N} = SD_n.$$

The study of matrix families often focuses on whether key properties or structures are preserved under operations such as submatrix extraction or matrix transformations. Notably, several important matrix classes exhibit inheritance properties. For example, both principal submatrices and Schur complements of positive semidefinite matrices remain positive semidefinite [7]. Similar inheritance holds for M-matrices, H-matrices, inverse M-matrices, diagonally dominant matrices, and doubly diagonally dominant matrices, as demonstrated in [8–11].

In the context of developing divide and conquer algorithms for computing stationary distributions of Markov chains, Meyer [12,13] pioneered the concept of Perron complements for nonnegative irreducible matrices. Given a nonnegative irreducible matrix $A \in R^{n\times n}$ and nonempty subsets α and β, satisfying $\alpha \bigcup \beta = N, \alpha \bigcap \beta = \emptyset$, the Perron complement of $A(\alpha)$ in A is formally defined as:

$$P(A/\alpha) = A(\beta) + A(\beta,\alpha)[\rho(A)I - A(\alpha)]^{-1}A(\alpha,\beta). \tag{1.5}$$

Meyer's foundational work established two key properties. First, the Perron complement $P(A/\alpha)$ inherits the nonnegativity and irreducibility of the original matrix A. Second, the matrices $P(A/\alpha)$ and A share identical spectral radius, i.e., $\rho(P(A/\alpha)) = \rho(A)$.

In 2002, Lu [14] introduced a key modification by replacing $\rho(A)$ with λ in Eq (1.5), thereby proposing the generalized Perron complement of A with respect to $A(\alpha)$, defined as follows:

$$P_\lambda(A/\alpha) = A(\beta) + A(\beta,\alpha)[\lambda I - A(\alpha)]^{-1}A(\alpha,\beta), \tag{1.6}$$

where $\lambda > \rho(A(\alpha))$. Clearly, $P_\lambda(A/\alpha)$ is well-defined for $\lambda > \rho(A(\alpha))$.

Lu proposed a novel methodology that exploits the relationship between the spectral radius of a nonnegative irreducible matrix and its generalized Perron complement, establishing the following key result:

$$\rho(P_\lambda(A/\alpha)) \begin{cases} < \rho(A), & \text{if } \lambda > \rho(A), \\ = \rho(A), & \text{if } \lambda = \rho(A), \\ > \rho(A), & \text{if } \rho(A(\alpha)) < \lambda < \rho(A). \end{cases}$$

The estimation of eigenvalue bounds via Perron complement techniques represents a fundamental research direction with significant theoretical and practical implications. Extensive investigations have yielded substantial advancements in this domain. For example, Perron complement matrices have been utilized to refine eigenvalue bounds for Z-matrices [15]. In the case of nonnegative irreducible matrices, Huang et al. [16–18] developed innovative Perron complement approaches that achieve optimal spectral radius bounds.

Moreover, the Perron complement is a powerful analytical tool for investigating matrix properties through its intrinsic relationship with the parent matrix. For specialized treatments of this approach applied to particular matrix classes, we refer to the following comprehensive analyses in the literature. Zhong and Zhao [19] rigorously investigated the extended Perron complements of M-matrices, whereas Zhou and Huang [20] focused on Perron complements of inverse N_0-matrices, building upon Neumann's earlier work [21] on inverse M-matrices. Fallat and Neumann [22] established fundamental results regarding Perron complements of totally nonnegative matrices, which were later generalized by Adm and Garloff [23] to broader matrix classes. Collectively, these studies provide a rigorous theoretical framework for understanding Perron complements across different matrix structures, highlighting their mathematical properties and potential applications.

Building upon the theoretical framework developed in [8–11,20–23], we investigate the properties of generalized Perron complements for strictly generalized doubly diagonally dominant matrices. In Section 2, we establish that under appropriate conditions, the generalized Perron complements of strictly generalized doubly diagonally dominant and nonnegative irreducible matrices retain strict diagonal dominance as well as strict generalized doubly diagonal dominance. The theoretical results are subsequently validated through two numerical experiments presented in Section 3. We conclude with a summary of key findings in Section 4.

2. Major results

Our first theorem states that if A is a strictly generalized doubly diagonally dominant and nonnegative irreducible matrix with respect to subsets α and β of N, then the generalized Perron complement $P_\lambda(A/\alpha)$ is strictly diagonally dominant and nonnegative irreducible. Before proceeding to the main theorem, we establish some pivotal lemmas, which are crucial in our proof process.

Lemma 2.1 [3]. *Let A be an $n \times n$ nonnegative matrix, then*

$$\min_i \sum_{k=1}^{n} a_{ik} \leq \rho(A) \leq \max_i \sum_{k=1}^{n} a_{ik}.$$

This lemma demonstrates that the spectral radius of a nonnegative matrix lies between its minimum row sum and maximum row sum.

Lemma 2.2 [14]. *Let A be an $n \times n$ nonnegative irreducible matrix and nonempty subsets α and β satisfying $\alpha \cup \beta = N$, $\alpha \cap \beta = \emptyset$, then the generalized Perron complement $P_\lambda(A/\alpha)$ is nonnegative irreducible for any $\lambda > \rho(A(\alpha))$.*

Lemma 2.3 [21]. *Let A be an $n \times n$ nonnegative irreducible matrix and nonempty set $\alpha \subset N$. For any $\emptyset \neq \gamma_1, \gamma_2 \subset \alpha$ with $\gamma_1 \cup \gamma_2 = \alpha$ and $\gamma_1 \bigcap \gamma_2 = \emptyset$, the following holds:*

$$P_\lambda(A/\alpha) = P_\lambda\left(P_\lambda(A/\gamma_1)/\gamma_2\right), \ \lambda \geq \rho(A).$$

Building upon the foundation established in Lemma 2.3, the following result naturally emerges.

Lemma 2.4. *Let A be an $n \times n$ nonnegative irreducible matrix and nonempty set $\alpha \subset N$. For any $\emptyset \neq \gamma_1, \gamma_2 \subset \alpha$ with $\gamma_1 \cup \gamma_2 = \alpha$ and $\gamma_1 \bigcap \gamma_2 = \emptyset$, the following holds:*

$$P_\lambda(A/\alpha) = P_\lambda\left(P_\lambda(A/\gamma_1)/\gamma_2\right), \ \lambda > \rho(A(\alpha)).$$

Lemma 2.5 [6]. *Let $A \in SD_n$, SDD_n, or $A \in SGDD_n$. Then, $m(A) \in M_n$.*

This lemma proves that the comparison matrices of strictly diagonally dominant (SD) matrices, strictly doubly diagonally dominant (SDD) matrices, and strictly generalized doubly diagonally dominant ($SGDD$) matrices are all M-matrices.

Lemma 2.6 [6]. *Let $A \in SGDD_n^{\alpha,\beta}$. Then, $A(\alpha) \in SD$ and $A(\beta) \in SD$.*

Lemma 2.6 demonstrates that if $A \in SGDD_n^{\alpha,\beta}$, then both $A(\alpha)$ and $A(\beta)$ are strictly diagonally dominant matrices.

Lemma 2.7 [6]. *Let $A \in C^{n\times n}$. If $A \in SGDD_n^{\alpha,\beta}$, then $\alpha \subseteq J(A)$ or $\beta \subseteq J(A)$.*

Lemma 2.8 [3]. *Given a nonnegative matrix $A \in R^{n\times n}$, the spectral radius of any principal submatrix A_k satisfies $\rho(A_k) \leq \rho(A)$.*

Lemma 2.8 demonstrates that the spectral radius of any principal submatrix of a nonnegative matrix does not exceed that of the original matrix.

The first result of this paper is stated as follows.

Theorem 2.1. *Let $A = \left(a_{ij}\right) \in R^{n\times n}$ be a nonnegative irreducible matrix and $A \in SGDD_n^{\alpha,\beta}$. If*

$$\alpha \subseteq J(A) \text{ and } \lambda \geq 2|a_{ii}| \text{ for all } i \in \alpha,$$

then the generalized Perron complement

$$P_\lambda (A/\alpha) = A(\beta) + A(\beta,\alpha)\left[\lambda I - A(\alpha)\right]^{-1} A(\alpha,\beta),\ \lambda > \rho(A(\alpha))$$

is nonnegative irreducible and strictly diagonally dominant. Moreover,

$$\beta \subseteq J(A) \text{ and } \lambda \geq 2|a_{jj}| \text{ for all } j \in \beta,$$

then the generalized Perron complement

$$P_\lambda (A/\beta) = A(\alpha) + A(\alpha,\beta)\left[\lambda I - A(\beta)\right]^{-1} A(\beta,\alpha),\ \lambda > \rho(A(\beta))$$

is nonnegative irreducible and strictly diagonally dominant.

Proof. We suppose that $\alpha = \{i_1, i_2, \cdots, i_s\}, \beta = \{j_1, j_2, \cdots, j_t\}$ with $s + t = n$. Since $A \in SGDD_n^{\alpha,\beta}$, by Lemma 2.7, we have $\alpha \subseteq J(A)$ or $\beta \subseteq J(A)$. Here, we provide the proof for the case where $\alpha \subseteq J(A)$; the case $\beta \subseteq J(A)$ follows analogously. Since

$$\alpha \subseteq J(A) = \left\{ i \,\middle|\, |a_{ii}| > \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}|,\, i \in N \right\} \tag{2.1}$$

and $\lambda \geq 2|a_{ii}|$ for all $i \in \alpha$, we obtain

$$\lambda \geq 2|a_{ii}| = |a_{ii}| + |a_{ii}| \tag{2.2}$$

and

$$|a_{ii}| + |a_{ii}| > |a_{ii}| + \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}| = \sum_{k=1}^{n} |a_{ik}| = \sum_{k\in\alpha} |a_{ik}| + \sum_{k\in\beta} |a_{ik}| \geq \sum_{k\in\alpha} |a_{ik}|. \tag{2.3}$$

From inequalities (2.2) and (2.3), we obtain

$$\lambda > \sum_{k\in\alpha} |a_{ik}|,\ i \in \alpha.$$

This implies that

$$\lambda > \max_{i\in\alpha} \sum_{k\in\alpha} |a_{ik}|.$$

Combined with Lemma 2.1, we obtain

$$\lambda > \max_{i\in\alpha} \sum_{k\in\alpha} |a_{ik}| \geq \rho(A(\alpha)). \tag{2.4}$$

According to Lemma 2.2, the generalized Perron complement $P_\lambda(A/\alpha)$ is nonnegative and irreducible.

In the following section, we consider the strictly diagonal dominance of $P_\lambda(A/\alpha)$.

By $\lambda \geq 2|a_{ii}|$ for all $i \in \alpha$, we have $\lambda - |a_{ii}| \geq |a_{ii}|$. Furthermore, we obtain

$$\lambda - |a_{ii}| - r_i^{(\alpha)}(A) \geq |a_{ii}| - r_i^{(\alpha)}(A) > 0,\ i \in \alpha. \tag{2.5}$$

The second inequality in (2.5) holds because for any $i \in \alpha$

$$|a_{ii}| > \sum_{\substack{k=1\\k\neq i}}^{n} |a_{ik}| = \sum_{\substack{k\neq i\\k\in\alpha}} |a_{ik}| + \sum_{\substack{k\neq i\\k\in\beta}} |a_{ik}| = r_i{}^{(\alpha)}(A) + r_i{}^{(\beta)}(A)$$

and

$$|a_{ii}| - r_i^{(\alpha)}(A) > r_i^{(\beta)}(A) \geq 0.$$

Since $\lambda - |a_{ii}| - r_i{}^{(\alpha)}(A) = \lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|$ holds, applying inequality (2.5) yields

$$\lambda - \sum_{k=1}^{s} \left|a_{i,i_k}\right| \geq |a_{ii}| - r_i{}^{(\alpha)}(A) > 0,\ i \in \alpha. \tag{2.6}$$

Therefore, it follows that

$$\frac{1}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|} \leq \frac{1}{|a_{ii}| - r_i{}^{(\alpha)}(A)},\ i \in \alpha.$$

According to the definition of $r_i^{(\beta)}(A)$, we have $r_i^{(\beta)}(A) \geq 0, i \in \alpha$. Thus,

$$\frac{r_i{}^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|} \leq \frac{r_i{}^{(\beta)}(A)}{|a_{ii}| - r_i{}^{(\alpha)}(A)} \leq \max_{i\in\alpha} \frac{r_i{}^{(\beta)}(A)}{|a_{ii}| - r_i{}^{(\alpha)}(A)}.$$

Therefore, it holds that

$$\max_{i\in\alpha} \frac{r_i{}^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|} \leq \max_{i\in\alpha} \frac{r_i{}^{(\beta)}(A)}{|a_{ii}| - r_i{}^{(\alpha)}(A)}. \tag{2.7}$$

Denote the column vector

$$X = (x_1, x_2, \cdots, x_s)^{\mathrm{T}} = [\lambda I - A(\alpha)]^{-1} \left(\sum_{k=1}^{t} a_{i_1 j_k}, \sum_{k=1}^{t} a_{i_2 j_k}, \cdots, \sum_{k=1}^{t} a_{i_s j_k} \right)^{\mathrm{T}}, \tag{2.8}$$

or equivalently written as

$$\left(\sum_{k=1}^{t} a_{i_1 j_k}, \sum_{k=1}^{t} a_{i_2 j_k}, \cdots, \sum_{k=1}^{t} a_{i_s j_k} \right)^{\mathrm{T}} = [\lambda I - A(\alpha)]\, X. \tag{2.9}$$

Let $x_q = \max\{x_1, x_2, \cdots, x_s\}$, where x_i denotes the i-th component of vector X. From Eq. (2.9), we obtain the following result:

$$
\begin{aligned}
\sum_{k=1}^{t} a_{i_q j_k} &= \left(\lambda - a_{i_q i_q}\right) x_q + \sum_{\substack{k=1 \\ k\neq q}}^{s} \left(-a_{i_q i_k}\right) x_k \\
&\geq \left(\lambda - a_{i_q i_q}\right) x_q + \sum_{\substack{k=1 \\ k\neq q}}^{s} \left(-a_{i_q i_k}\right) x_q \\
&= \lambda x_q - \left(a_{i_q i_q} + \sum_{\substack{k=1 \\ k\neq q}}^{s} a_{i_q i_k} \right) x_q \\
&= \left(\lambda - \sum_{k=1}^{s} a_{i_q i_k} \right) x_q.
\end{aligned}
$$

According to inequality (2.6), we have

$$
\lambda - \sum_{k=1}^{s} |a_{i_q i_k}| > 0, \ i_q \in \alpha.
$$

Therefore, we obtain

$$
x_q \leq \frac{\sum\limits_{k=1}^{t} \left|a_{i_q j_k}\right|}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i_q i_k}\right|} = \frac{r_{i_q}^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i_q i_k}\right|}.
$$

This implies that

$$
x_q \leq \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|}. \tag{2.10}
$$

Furthermore, we derive

$$
\begin{aligned}
X &\leq \left(\max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|}, \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|}, \cdots, \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|} \right)^{\mathrm{T}} \\
&= \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum\limits_{k=1}^{s} \left|a_{i,i_k}\right|} (1, 1, \cdots, 1)^{\mathrm{T}}.
\end{aligned} \tag{2.11}
$$

Note that $A(\alpha)$ is nonnegative and that $\lambda > \rho(A(\alpha))$ (by inequality (2.4)), so that $\lambda I - A(\alpha)$ is an M-matrix. This shows that $[\lambda I - A(\alpha)]^{-1} \geq 0$.

Since $A \in SGDD_n^{\alpha,\beta}$, for all $i \in \alpha$, $j \in \beta$, we have

$$\left(|a_{ii}| - r_i^{(\alpha)}(A)\right)\left(|a_{jj}| - r_j^{(\beta)}(A)\right) > r_i^{(\beta)}(A) r_j^{(\alpha)}(A).$$

By inequality (2.6), for arbitrary $i \in \alpha$, $|a_{ii}| - r_i^{(\alpha)}(A) > 0$, it consequently follows that

$$|a_{jj}| - r_j^{(\beta)}(A) > \frac{r_j^{(\beta)}(A)}{|a_{ii}| - r_i^{(\alpha)}(A)} r_j^{(\alpha)}(A).$$

Thus, for arbitrary $j \in \beta$, we obtain

$$|a_{jj}| - r_j^{(\beta)}(A) > \max_{i\in\alpha} \frac{r_j^{(\beta)}(A)}{|a_{ii}| - r_i^{(\alpha)}(A)} r_j^{(\alpha)}(A). \tag{2.12}$$

Denote the element of $P_\lambda(A/\alpha)$ located in the j_m-th row and j_k-th column as $(b_{j_m j_k})$. Thus, for any $j_m \in \beta$, we have

$$\begin{aligned}
&\left|b_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|b_{j_m j_k}\right| \\
=&\left|a_{j_m j_m} + \left(a_{j_m i_1}, a_{j_m i_2}, \cdots, a_{j_m i_s}\right)[\lambda I - A(\alpha)]^{-1}\left(a_{i_1 j_m}, a_{i_2 j_m}, \cdots, a_{i_s j_m}\right)^{\mathrm{T}}\right| \\
&- \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k} + \left(a_{j_m i_1}, a_{j_m i_2}, \cdots, a_{j_m i_s}\right)[\lambda I - A(\alpha)]^{-1}\left(a_{i_1 j_k}, a_{i_2 j_k}, \cdots, a_{i_s j_k}\right)^{\mathrm{T}}\right| \\
\geq&\left|a_{j_m j_m}\right| - \left(\left|a_{j_m i_1}\right|, \left|a_{j_m i_2}\right|, \cdots, \left|a_{j_m i_s}\right|\right)[\lambda I - A(\alpha)]^{-1}\left(\left|a_{i_1 j_m}\right|, \left|a_{i_2 j_m}\right|, \cdots, \left|a_{i_s j_m}\right|\right)^{\mathrm{T}} \\
&- \sum_{\substack{k=1\\k\neq m}}^{t} \left[\left|a_{j_m j_k}\right| + \left(\left|a_{j_m i_1}\right|, \left|a_{j_m i_2}\right|, \cdots, \left|a_{j_m i_s}\right|\right)[\lambda I - A(\alpha)]^{-1}\left(\left|a_{i_1 j_k}\right|, \left|a_{i_2 j_k}\right|, \cdots, \left|a_{i_s j_k}\right|\right)^{\mathrm{T}}\right] \\
=&\left|a_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k}\right| - \left(\left|a_{j_m i_1}\right|, \left|a_{j_m i_2}\right|, \cdots, \left|a_{j_m i_s}\right|\right)[\lambda I - A(\alpha)]^{-1}\left(\sum_{k=1}^{t} a_{i_1 j_k}, \sum_{k=1}^{t} a_{i_2 j_k}, \cdots, \sum_{k=1}^{t} a_{i_s j_k}\right)^{\mathrm{T}} \\
=&\left|a_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k}\right| - \left(\left|a_{j_m i_1}\right|, \left|a_{j_m i_2}\right|, \cdots, \left|a_{j_m i_s}\right|\right) X \qquad \text{(by Eq. (2.8))} \\
\geq&\left|a_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k}\right| - \left(\left|a_{j_m i_1}\right|, \left|a_{j_m i_2}\right|, \cdots, \left|a_{j_m i_s}\right|\right) \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum_{k=1}^{s}\left|a_{i,i_k}\right|}(1, 1, \cdots, 1)^{\mathrm{T}} \qquad \text{(by (2.11))} \\
=&\left|a_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k}\right| - \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{\lambda - \sum_{k=1}^{s}\left|a_{i,i_k}\right|} \sum_{k=1}^{s}\left|a_{j_m i_k}\right| \\
\geq&\left|a_{j_m j_m}\right| - \sum_{\substack{k=1\\k\neq m}}^{t} \left|a_{j_m j_k}\right| - \max_{i\in\alpha} \frac{r_i^{(\beta)}(A)}{|a_{ii}| - r_i^{(\alpha)}(A)} \sum_{k=1}^{s}\left|a_{j_m i_k}\right| \qquad \text{(by (2.7))}
\end{aligned}$$

$= \left|a_{j_m j_m}\right| - r_{j_m}{}^{(\beta)}(A) - \max_{i\in\alpha} \dfrac{r_i^{(\beta)}(A)}{|a_{ii}| - r_i^{(\alpha)}(A)} r_{j_m}{}^{(\alpha)}(A)$

$>0.$ (by (2.12))

The above proof demonstrates that $P_\lambda(A/\alpha)$ is strictly diagonally dominant. This completes the proof.

Our first main result (Theorem 2.1) demonstrates that the generalized Perron complement of a strictly generalized doubly diagonally dominant ($SGDD_n^{\alpha,\beta}$) matrix inherits strict diagonal dominance when certain structural conditions are satisfied. We now extend this analysis to prove a more comprehensive property.

Theorem 2.2. *Let $A = \left(a_{ij}\right) \in R^{n\times n}$ be a nonnegative irreducible matrix and $A \in SGDD_n^{\alpha,\beta}$. If*

$$\alpha \subseteq J(A) \text{ and } \lambda \geq 2|a_{ii}| \text{ for all } i \in \alpha,$$

then the generalized Perron complement $P_\lambda(A/\gamma)$ is nonnegative irreducible and strictly generalized doubly diagonally dominant for any proper subset γ of α. Moreover,

$$\beta \subseteq J(A) \text{ and } \lambda \geq 2|a_{jj}| \text{ for all } j \in \beta,$$

then the generalized Perron complement $P_\lambda(A/\gamma)$ is nonnegative irreducible and strictly generalized doubly diagonally dominant for any proper subset γ of β.

Proof. We suppose that $\alpha = \{i_1, i_2, \cdots, i_s\}, \beta = \{j_1, j_2, \cdots, j_t\}$ with $s + t = n$. Since $A \in SGDD_n^{\alpha,\beta}$, by Lemma 2.7, we have $\alpha \subseteq J(A)$ or $\beta \subseteq J(A)$. Here, we provide the proof for the case where $\alpha \subseteq J(A)$; the case $\beta \subseteq J(A)$ follows analogously. Let

$$\alpha \subseteq J(A) = \left\{ i \,\middle|\, |a_{ii}| > \sum_{\substack{k=1\\k\neq i}}^{n} |a_{ik}|, i \in N \right\}. \tag{2.13}$$

From inequality (2.4), we obtain

$$\lambda > \rho(A(\alpha)).$$

Additionally, since $\gamma \subseteq \alpha$ and $A(\alpha)$ is nonnegative, according to Lemma 2.8, we have $\rho(A(\alpha)) \geq \rho(A(\gamma))$. Therefore, we derive $\lambda > \rho(A(\gamma))$. It follows from Lemma 2.2 that the generalized Perron complement $P_\lambda(A/\gamma)$ is nonnegative irreducible.

Next, we prove the strictly generalized diagonal dominance of $P_\lambda(A/\gamma)$. The proof proceeds by considering two distinct cases.

Case 1. γ contains only one element.

If $\gamma = \{i_1\} = \alpha$, by Theorem 2.1, we obtain

$$P_\lambda(A/\gamma) = P_\lambda(A/\alpha) \in SD_{n-1} = SGDD_{n-1}^{\varnothing,\beta}.$$

If $\gamma = \{i_1\} \subset \alpha$, for any fixed $j_h \in \beta$ and $i_m \in \alpha - \{i_1\}$, let

$$A' = \begin{pmatrix} \lambda - |a_{i_1i_1}| & -r_{i_1}{}^{(\alpha)}(A) & -r_{i_1}{}^{(\beta)}(A) \\ -|a_{i_mi_1}| & |a_{i_mi_m}| - \sum\limits_{\substack{k\neq 1,m \\ i_k\in\alpha}} |a_{i_mi_k}| & -r_{i_m}{}^{(\beta)}(A) \\ -|a_{j_hi_1}| & -\sum\limits_{\substack{k\neq 1 \\ i_k\in\alpha}} |a_{j_hi_k}| & |a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A) \end{pmatrix}. \tag{2.14}$$

Through direct computation, we obtain the Schur complement of A' with respect to $A'(1)$ as follows:

$$A'/A'(1) = \begin{pmatrix} |a_{i_mi_m}| - \sum\limits_{\substack{k\neq 1,m \\ i_k\in\alpha}} |a_{i_mi_k}| - \left|\dfrac{a_{i_mi_1}}{\lambda - a_{i_1i_1}}\right| r_{i_1}{}^{(\alpha)}(A) & -r_{i_m}{}^{(\beta)}(A) - \left|\dfrac{a_{i_mi_1}}{\lambda - a_{i_1i_1}}\right| r_{i_1}{}^{(\beta)}(A) \\ -\sum\limits_{\substack{k\neq 1 \\ i_k\in\alpha}} |a_{j_hi_k}| - \left|\dfrac{a_{j_hi_1}}{\lambda - a_{i_1i_1}}\right| r_{i_1}{}^{(\alpha)}(A) & |a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A) - \left|\dfrac{a_{j_hi_1}}{\lambda - a_{i_1i_1}}\right| r_{i_1}{}^{(\beta)}(A) \end{pmatrix}.$$

Because $A \in SGDD_n^{\alpha,\beta}$, for $i_1, i_m \in \alpha$, $j_h \in \beta$, we have

$$\left[|a_{i_1i_1}| - r_{i_1}{}^{(\alpha)}(A)\right]\left[|a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A)\right] \geq r_{i_1}{}^{(\beta)}(A)\, r_{j_h}{}^{(\alpha)}(A)$$

and

$$\left[|a_{i_mi_m}| - r_{i_m}{}^{(\alpha)}(A)\right]\left[|a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A)\right] \geq r_{i_m}{}^{(\beta)}(A)\, r_{j_h}{}^{(\alpha)}(A),$$

that is

$$\left[|a_{i_1i_1}| - r_{i_1}{}^{(\alpha)}(A)\right]\left[|a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A)\right] \geq r_{i_1}{}^{(\beta)}(A)\left(|a_{j_hi_1}| + \sum_{\substack{k\neq 1 \\ i_k\in\alpha}} |a_{j_hi_k}|\right) \tag{2.15}$$

and

$$\left(|a_{i_mi_m}| - \sum_{\substack{k\neq 1,m \\ i_k\in\alpha}} |a_{i_mi_k}| - |a_{i_mi_1}|\right)\left[|a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A)\right] \geq r_{i_m}{}^{(\beta)}(A)\left(|a_{j_hi_1}| + \sum_{\substack{k\neq 1 \\ i_k\in\alpha}} |a_{j_hi_k}|\right). \tag{2.16}$$

According to $\lambda \geq 2|a_{ii}|$ for all $i \in \alpha$ and Eq. (2.13), we derive

$$\lambda - |a_{i_1i_1}| \geq |a_{i_1i_1}| > 0. \tag{2.17}$$

Since $A \in SGDD_n^{\alpha,\beta}$, it follows from Lemma 2.6 that $A(\alpha) \in SD, A(\beta) \in SD$. This shows that for $i_1 \in \alpha$, $j_h \in \beta$, we have

$$|a_{i_1i_1}| - r_{i_1}{}^{(\alpha)}(A) > 0 \tag{2.18}$$

and

$$|a_{j_hj_h}| - r_{j_h}{}^{(\beta)}(A) > 0. \tag{2.19}$$

From inequalities (2.17) and (2.18), we obtain

$$\lambda - |a_{i_1i_1}| - r_{i_1}{}^{(\alpha)}(A) \geq |a_{i_1i_1}| - r_{i_1}{}^{(\alpha)}(A) > 0. \tag{2.20}$$

Therefore, combining inequalities (2.19), (2.20), and (2.15), we obtain

$$\begin{aligned}\left[\lambda - |a_{i_1i_1}| - r_{i_1}^{(\alpha)}(A)\right]\left[|a_{j_hj_h}| - r_{j_h}^{(\beta)}(A)\right] &\geq \left[|a_{i_1i_1}| - r_{i_1}^{(\alpha)}(A)\right]\left[|a_{j_hj_h}| - r_{j_h}^{(\beta)}(A)\right] \\ &\geq r_{i_1}^{(\beta)}(A)\left(|a_{j_hi_1}| + \sum_{\substack{k\neq 1\\ i_k\in\alpha}} |a_{j_hi_k}|\right). \qquad (2.21)\end{aligned}$$

According to inequalities (2.16) and (2.21), we observe that

$$A' \in SGDD_3^{\{1,2\},\{3\}}.$$

Moreover, it is straightforward to observe that $A' = m(A')$. Since $A' \in SGDD_3^{\{1,2\},\{3\}}$, it follows from Lemma 2.5 that $A' = m(A') \in M_3$. Consequently, we derive

$$\det A' > 0. \qquad (2.22)$$

Identify the (m,h)-entry of $P_\lambda(A/\gamma)$ by $b_{i_mj_h}$. Upon computation, for $i_m, j_h \in N \setminus \gamma$, we obtain

$$\begin{aligned}&\left(|b_{i_mi_m}| - \sum_{\substack{k\neq 1,m\\ i_k\in\alpha}} |b_{i_mi_k}|\right)\left(|b_{j_hj_h}| - \sum_{\substack{u\neq h\\ j_u\in\beta}} |b_{j_hj_u}|\right) - \sum_{j_u\in\beta} |b_{i_mj_u}| \sum_{\substack{k\neq 1\\ i_k\in\alpha}} |b_{j_hi_k}| \\
=&\left(\left|a_{i_mi_m} + \frac{a_{i_mi_1}a_{i_1i_m}}{\lambda - a_{i_1i_1}}\right| - \sum_{\substack{k\neq 1,m\\ i_k\in\alpha}} \left|a_{i_mi_k} + \frac{a_{i_mi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right|\right) \times \left(\left|a_{j_hj_h} + \frac{a_{j_hi_1}a_{i_1j_h}}{\lambda - a_{i_1i_1}}\right| - \sum_{\substack{u\neq h\\ j_u\in\beta}} \left|a_{j_hj_u} + \frac{a_{j_hi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right|\right) \\
&- \sum_{j_u\in\beta} \left|a_{i_mj_u} + \frac{a_{i_mi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right| \sum_{\substack{k\neq 1\\ i_k\in\alpha}} \left|a_{j_hi_k} + \frac{a_{j_hi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right| \\
\geq&\left(|a_{i_mi_m}| - \left|\frac{a_{i_mi_1}a_{i_1i_m}}{\lambda - a_{i_1i_1}}\right| - \sum_{\substack{k\neq 1,m\\ i_k\in\alpha}} |a_{i_mi_k}| - \sum_{\substack{k\neq 1,m\\ i_k\in\alpha}} \left|\frac{a_{i_mi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right|\right) \times \left(|a_{j_hj_h}| - \left|\frac{a_{j_hi_1}a_{i_1j_h}}{\lambda - a_{i_1i_1}}\right| - \sum_{\substack{u\neq h\\ j_u\in\beta}} |a_{j_hj_u}| - \sum_{\substack{u\neq h\\ j_u\in\beta}} \left|\frac{a_{j_hi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right|\right) \\
&- \left(\sum_{j_u\in\beta} |a_{i_mj_u}| + \sum_{j_u\in\beta} \left|\frac{a_{i_mi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right|\right) \times \left(\sum_{\substack{k\neq 1\\ i_k\in\alpha}} |a_{j_hi_k}| + \sum_{\substack{k\neq 1\\ i_k\in\alpha}} \left|\frac{a_{j_hi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right|\right) \\
=&\left(|a_{i_mi_m}| - \sum_{\substack{k\neq 1,m\\ i_k\in\alpha}} |a_{i_mi_k}| - \sum_{\substack{k\neq 1\\ i_k\in\alpha}} \left|\frac{a_{i_mi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right|\right) \times \left(|a_{j_hj_h}| - \sum_{\substack{u\neq h\\ j_u\in\beta}} |a_{j_hj_u}| - \sum_{j_u\in\beta} \left|\frac{a_{j_hi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right|\right) \\
&- \left(\sum_{j_u\in\beta} |a_{i_mj_u}| + \sum_{j_u\in\beta} \left|\frac{a_{i_mi_1}a_{i_1j_u}}{\lambda - a_{i_1i_1}}\right|\right) \times \left(\sum_{\substack{k\neq 1\\ i_k\in\alpha}} |a_{j_hi_k}| + \sum_{\substack{k\neq 1\\ i_k\in\alpha}} \left|\frac{a_{j_hi_1}a_{i_1i_k}}{\lambda - a_{i_1i_1}}\right|\right)\end{aligned}$$

$$=\left(\left|a_{i_m i_m}\right|-\sum_{\substack{k\neq 1,m\\ i_k\in\alpha}}\left|a_{i_m i_k}\right|-\left|\frac{a_{i_m i_1}}{\lambda-a_{i_1 i_1}}\right|\sum_{\substack{k\neq 1\\ i_k\in\alpha}}\left|a_{i_1 i_k}\right|\right)\times\left(\left|a_{j_h j_h}\right|-\sum_{\substack{u\neq h\\ j_u\in\beta}}\left|a_{j_h j_u}\right|-\left|\frac{a_{j_h i_1}}{\lambda-a_{i_1 i_1}}\right|\sum_{j_u\in\beta}\left|a_{i_1 j_u}\right|\right)$$

$$-\left(\sum_{j_u\in\beta}\left|a_{i_m j_u}\right|+\left|\frac{a_{i_m i_1}}{\lambda-a_{i_1 i_1}}\right|\sum_{j_u\in\beta}\left|a_{i_1 j_u}\right|\right)\times\left(\sum_{\substack{k\neq 1\\ i_k\in\alpha}}\left|a_{j_h i_k}\right|+\left|\frac{a_{j_h i_1}}{\lambda-a_{i_1 i_1}}\right|\sum_{\substack{k\neq 1\\ i_k\in\alpha}}\left|a_{i_1 i_k}\right|\right)$$

$$=\left(\left|a_{i_m i_m}\right|-\sum_{\substack{k\neq 1,m\\ i_k\in\alpha}}\left|a_{i_m i_k}\right|-\left|\frac{a_{i_m i_1}}{\lambda-a_{i_1 i_1}}\right|r_{i_1}{}^{(\alpha)}(A)\right)\times\left(\left|a_{j_h j_h}\right|-r_{j_h}{}^{(\beta)}(A)-\left|\frac{a_{j_h i_1}}{\lambda-a_{i_1 i_1}}\right|r_{i_1}{}^{(\beta)}(A)\right)$$

$$-\left(r_{i_m}{}^{(\beta)}(A)+\left|\frac{a_{i_m i_1}}{\lambda-a_{i_1 i_1}}\right|r_{i_1}{}^{(\beta)}(A)\right)\times\left(\sum_{\substack{k\neq 1\\ i_k\in\alpha}}\left|a_{j_h i_k}\right|+\left|\frac{a_{j_h i_1}}{\lambda-a_{i_1 i_1}}\right|r_{i_1}{}^{(\alpha)}(A)\right)$$

$$=\det\left(A'/A'(1)\right)$$

$$=\frac{\det A'}{\det A'(1)}\qquad\text{(by Eq. (1.1))}$$

$$=\frac{\det A'}{\lambda-\left|a_{i_1 i_1}\right|}$$

$$>0.\qquad\text{(by (2.17) and (2.22))}$$

Therefore, for $\gamma=\{i_1\}\subset\alpha$, we obtain

$$P_\lambda(A/\gamma)=P_\lambda(A/i_1)\in SGDD_{n-1}^{\alpha-\{i_1\},\beta}.$$

Case 2. γ contains two elements. Without loss of generality, let $\gamma=\{i_1,i_2\}\subseteq\alpha$. Applying Lemma 2.4, we have

$$P_\lambda(A/\gamma)=P_\lambda(P_\lambda(A/i_1)/i_2)\in SGDD_{n-2}^{\alpha-\{i_1,i_2\},\beta}.$$

If γ contains more than two elements and $\gamma\subseteq\alpha$, we can deduce, by induction, that $P_\lambda(A/\gamma)\in SGDD$.

3. Numerical examples

The following numerical examples serve to validate the theoretical results established in Theorems 2.1 and 2.2.

Example 3.1. *Consider nonnegative irreducible*

$$A=\begin{pmatrix}4&2&0\\4&6&2\\4&0&4\end{pmatrix},$$

and it has been verified that matrix $A\in SGDD_3^{\alpha,\beta}$, *where* $\alpha=\{1\}\subseteq J(A)=\{1\}$, $\beta=\{2,3\}$. *Since* $\lambda\geq 2|a_{ii}|, i\in\alpha$, *we may set* $\lambda=9$. *Through direct computation, we obtain the generalized Perron* *complement of A with respect to $A(\alpha)$ as follows:*

$$P_\lambda(A/\alpha) = \begin{pmatrix} 6 & 2 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} (9-4)^{-1} \begin{pmatrix} 2 & 0 \end{pmatrix} = \begin{pmatrix} 7.6 & 2 \\ 1.6 & 4 \end{pmatrix}.$$

Clearly, $P_\lambda(A/\alpha)$ is nonnegative and irreducible. Moreover, it is straightforward to verify that $P_\lambda(A/\alpha)$ is strictly diagonally dominant, which agrees with the conclusion of Theorem 2.1.

Example 3.2. *Consider nonnegative irreducible*

$$B = \begin{pmatrix} 9 & 3 & 3 & 0 \\ 3 & 9 & 0 & 3 \\ 6 & 3 & 12 & 3 \\ 3 & 6 & 3 & 12 \end{pmatrix},$$

and it has been verified that matrix $B \in SGDD_4^{\alpha,\beta}$, where $\alpha = \{1,2\} \subseteq J(A) = \{1,2\}$, $\beta = \{3,4\}$. Since $\lambda \geq 2|a_{ii}|, i \in \alpha$, we may set $\lambda = 19$. For a proper subset $\gamma = \{1\} \subset \alpha$, through direct computation, we obtain the generalized Perron complement of B with respect to $B(\gamma)$ as follows:

$$\begin{aligned} P_\lambda(B/\gamma) &= \begin{pmatrix} 9 & 0 & 3 \\ 3 & 12 & 3 \\ 6 & 3 & 12 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} (19-9)^{-1} \begin{pmatrix} 3 & 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 9.9 & 0.9 & 3 \\ 4.8 & 13.8 & 3 \\ 6.9 & 3.9 & 12 \end{pmatrix}. \end{aligned}$$

Clearly, $P_\lambda(B/\gamma)$ is nonnegative and irreducible. Moreover, it is straightforward to verify that $P_\lambda(B/\gamma) \in SGDD_3^{\{1\},\{2,3\}}$, which agrees with the conclusion of Theorem 2.2.

4. Conclusions

In this study, we elucidate the fundamental properties of generalized Perron complements for strictly generalized doubly diagonally dominant matrices. Our principal findings demonstrate that for nonnegative irreducible matrices, the generalized Perron complement preserves strict diagonal dominance. Additionally, it preserves the strict generalized double diagonal dominance and the nonnegative irreducibility of the parent matrix. These results extend the classical Perron complement theory to generalized Perron complement while maintaining essential structural properties. Looking ahead, these results naturally lead to important open questions regarding Perron complements and generalized Perron complements of strictly γ-diagonally and product γ-diagonally dominant matrices.

Author contributions

Qin Zhong: Writing-original draft, Methodology, Conceptualization, Data curation; Ling Li: Formal analysis, Writing-review & editing; Gufang Mou: Methodology, Investigation, Data curation. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interests.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

References

1. F. Saberi-Movahed, K. Berahman, R. Sheikhpour, Y. F. Li, S. R. Pan, Nonnegative matrix factorization in dimensionality reduction: A Survey, 2024, arXiv:2405.03615. https://doi.org/10.48550/arXiv.2405.03615
2. S. Ahmadian, K. Berahmand, M. Rostami, S. Forouzandeh, P. Moradi, M. Jalili, Recommender Systems based on Non-negative Matrix Factorization: A Survey, *IEEE Trans. Artif. Intell.*, (2025). https://doi.org/10.1109/TAI.2025.3559053
3. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Philadelphia: Society for Industrial and Applied Mathematics, 1994. https://doi.org/10.1137/1.9781611971262
4. F. Z. Zhang, *Matrix Theory: Basic Results and Techniques*, New York: Springer, 1999. https://link.springer.com/book/10.1007/978-1-4614-1099-7
5. B. S. Li, M. J. Tsatsomeros, Doubly diagonally dominant matrices, *Linear Algebra Appl.*, **261** (1997), 221–235. https://doi.org/10.1016/S0024-3795(96)00406-5
6. J. Z. Liu, Y. Q. Huang, F. Z. Zhang, The Schur complements of generalized doubly diagonally dominant matrices, *Linear Algebra Appl.*, **378** (2004), 231–244. https://doi.org/10.1016/j.laa.2003.09.012
7. R. L. Smith, Some interlacing properties of the Schur complements of a Hermitian matrix, *Linear Algebra Appl.*, **177** (1992), 137–144. https://doi.org/10.1016/0024-3795(92)90321-Z
8. C. R. Johnson, Inverse M-matrices, *Linear Algebra Appl.*, **47** (1982), 195–216. https://doi.org/10.1016/0024-3795(82)90238-5
9. J. Z. Liu, Y. Q. Huang, Some properties on Schur complements of H-matrices and diagonally dominant matrices, *Linear Algebra Appl.*, **389** (2004), 365–380. https://doi.org/10.1016/j.laa.2004.04.012
10. D. Carlson, T. L. Markham, Schur complements of diagonally dominant matrices, *Czech. Math. J.*, **29** (1979), 246–251. https://dml.cz/bitstream/handle/10338.dmlcz/101601/CzechMathJ_29-1979-2_9.pdf

11. K. D. Ikramov, Invariance of the Brauer diagonal dominance in gaussian elimination, *Moscow Univ. Comput. Math. Cybernet.*, **N** (1989), 91–94.

12. C. D. Meyer, Uncoupling the Perron eigenvector problem, *Linear Algebra Appl.*, **114–115** (1989), 69–94. https://doi.org/10.1016/0024-3795(89)90452-7

13. C. D. Meyer, Stochastic complementation, uncoupling Markov chains, and the theory of nearly reducible systems, *SIAM Rev.*, **31** (1989), 240–272. https://doi.org/10.1137/1031050

14. L. Z. Lu, Perron complement and Perron root, *Linear Algebra Appl.*, **341** (2002), 239–248. https://doi.org/10.1016/S0024-3795(01)00378-0

15. Z. G. Ren, T. Z. Huang, X. Y. Cheng, A note on generalized Perron complements of Z-matrices, *Electron. J. Linear Al.*, **15** (2006), 8–13. https://doi.org/10.13001/1081-3810.1217

16. Z. M. Yang, Some closer bounds of Perron root basing on generalized Perron complement, *J. Comput. Appl. Math.*, **235** (2010), 315–324. https://doi.org/10.1016/j.cam.2010.06.012

17. S. M. Yang, T. Z. Huang, A note on estimates for the spectral radius of a nonnegative matrix, *Electron. J. Linear Al.*, **13** (2005), 352–358. https://doi.org/10.13001/1081-3810.1168

18. G. X. Huang, F. Yin, K. Guo, The lower and upper bounds on Perron root of nonnegative irreducible matrices, *J. Comput. Appl. Math.*, **217** (2008), 259–267. https://doi.org/10.1016/j.cam.2007.06.034

19. Q. Zhong, C. Y. Zhao, Extended Perron complements of M-matrices, *AIMS Math.*, **8** (2023), 26372–26383. https://doi.org/10.3934/math.20231346

20. S. W. Zhou, T. Z. Huang, On Perron complements of inverse N_0-matrices, *Linear Algebra Appl.*, **434** (2011), 2081–2088. https://doi.org/10.1016/j.laa.2010.12.004

21. M. Neumann, Inverses of Perron complements of inverse M-matrices, *Linear Algebra Appl.*, **313** (2000), 163–171. https://doi.org/10.1016/S0024-3795(00)00128-2

22. S. M. Fallat, M. Neumann, On Perron complements of totally nonnegative matrices, *Linear Algebra Appl.*, **327** (2001), 85–94. https://doi.org/10.1016/S0024-3795(00)00312-8

23. M. Adm, J. Garloff, Total nonnegativity of the extended Perron complement, *Linear Algebra Appl.*, **508** (2016), 214–224. https://doi.org/10.1016/j.laa.2016.07.002