Mathematics

Research article

Resolution of open problems via Orlicz-Zygmund spaces and new geometric properties of Morrey spaces in the Besov sense with non-standard growth

Waqar Afzal[1], Mujahid Abbas[2,3], Najla M. Aloraini[4] and Jongsuk Ro[5,6,*]

[1] Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan

[2] Department of Mechanical Engineering Sciences, Faculty of Engineering and the Built Environment, Doornfontein Campus, University of Johannesburg, South Africa

[3] Department of Medical Research, China Medical University, Taichung 406040, Taiwan, China

[4] Department of Mathematics, College of Science, Qassim University, Buraydah 52571, Saudi Arabia

[5] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea

[6] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea

* **Correspondence:** Email: jsro@cau.ac.kr.

Abstract: In this paper, we introduced a novel norm structure and corresponding modular for Orlicz-Zygmund spaces, designed to capture summability and integrability properties under non-standard growth conditions. Moreover, we established the Hermite-Hadamard inequalities and gave a positive answer to the open problem 3.11 of Almeida and Hästö in their article (Besov spaces with variable smoothness and integrability, *J. Funct. Anal.*, **258** (2010), 1628–1655), since the Hermite-Hadamard inequalities improve triangular-type inequalities. In addition, we explored some new structural properties of Besov-type Morrey spaces of summability and integrability. Furthermore, we addressed an open problem concerning the compactness of Morrey-type spaces characterized by summability and integrability properties. This problem was originally posed by Peter Hästö during the conference "Nonstandard Growth Phenomena", held in Turku, Finland, from August 29 to 31, 2017.

1. Introduction

The Hermite-Hadamard inequalities stand today as a fundamental result in mathematical analysis, originally developed by Charles Hermite and Jacques Hadamard during the late 19th and early 20th centuries, with numerous applications across a wide range of real-world problems. The Hermite-Hadamard inequalities help numerical analysis researchers determine error estimates when using the trapezoidal and midpoint rules for numerical integration. The optimization discipline uses it to analyze and create boundaries for objective functions. The inequality allows researchers in probability and statistics to conduct risk assessments through their mathematical treatment of convex utility functions and risk measures, thereby yielding beneficial findings throughout both domains.

Convex inequalities, particularly the Hermite-Hadamard inequalities, have broad applications in various scientific and practical fields. In economics, they are instrumental in estimating average utility or production costs under convexity assumptions, which are commonly used in modeling consumer preferences and temporal transformation [1]. In a multi-scale spatial-temporal interaction fusion network such as the one used for digital twin-based thermal error compensation in precision machine tools, integral inequalities (most notably Gronwall-type inequalities) are essential for establishing stability and error-bounding properties across temporal scales and interacting spatial components [2]. These inequalities offer valuable analytical frameworks to manage uncertainty, optimize resource allocation, and derive bounds for error estimates [3, 4].

The Hermite-Hadamard inequalities state that if $\varphi : [\alpha, \beta] \to \mathrm{R}$ is a convex function on the interval $[\alpha, \beta]$, then the following inequalities hold [5]:

$$\varphi\left(\frac{\alpha + \beta}{2}\right) \le \frac{1}{\beta - \alpha} \int_\alpha^\beta \varphi(\mu) \, \mathrm{d}\mu \le \frac{\varphi(\alpha) + \varphi(\beta)}{2}.$$

These inequalities have been explored through various types of generalized convex mappings, including p-convex, s-convex, harmonic convex, Godunova-Levin and h-convex functions, and several other classes exist; see, for example, [6–8]. The Hermite-Hadamard inequalities have been generalized in various mathematical settings, each offering new insights and applications. In the fractional calculus framework, integral versions involving Riemann-Liouville and Caputo operators extend classical convexity results to non-integer orders [9]. The quantum calculus version of the inequality utilizes Jackson integrals and q-analogues to formulate inequalities where traditional limits are replaced by discrete difference operators [10, 11]. A stochastic extension based on probabilistic metric spaces and random variables provides bounds under convex stochastic dominance assumptions [12]. The fuzzy version explores generalized convexity in fuzzy analysis, employing the Hukuhara differentiability to establish inequalities for fuzzy-valued functions [13]. Meanwhile, for interval-valued mappings, the inequality is reformulated using convexity for interval-valued functions in partial relation [14].

In 2015, the distinguished mathematician S. S. Dragomir [15] introduced a novel formulation of the Hermite-Hadamard inequalities. His work contributed to the advancement of the classical framework of Lebesgue spaces by establishing a refined expression for functions using norm-based terms. This formulation preserved the core structure of the inequality while maintaining convexity, leading to broader applications in analysis.

Let X be a vector space over the field $\mathbb{K}$ (either R or $\mathbb{C}$), and let $\xi, \eta \in \mathrm{X}$ with $\xi \ne \eta$, and $\rho \in [1, \infty)$.

Then, the following inequality holds:

$$\left\| \frac{\xi + \eta}{2} \right\|^\rho \leq \int_0^1 \|(1 - \tau)\xi + \tau\eta\|^\rho d\tau \leq \frac{\|\xi\|^\rho + \|\eta\|^\rho}{2}. \tag{1.1}$$

Our aim is to extend inequality (1.1) in the setting of function spaces with non-standard growth conditions on the exponent. Toward this goal, we first discuss some appropriate literature on variable exponent spaces.

Non-standard growth on exponent functions is the terminology used for applications where the behavior of the exponent's growth in the function is not constant or uniform over the domain, compared to conventional function spaces with fixed exponents. In these applications, the exponent can depend on the point or the area within the domain, and there can be more flexible and general growth patterns. A non-standard growth on the exponent $p(x)$ characterizes these spaces in which the integrability conditions change across the domain regions. The elastic character of variable exponent spaces provides great modeling ability in domains that need to deal with non-homogeneous processes in fluid dynamics [16] and image processing [17], as well as elasticity theory [18] and non-standard PDEs [19].

Non-standard growth sequence spaces ($\ell_{p(\cdot)}$) are function spaces in which the growth of sequences is regulated by a variable exponent or a varying growth function, as opposed to being constant. This comes, originally, from the groundbreaking work of Orlicz [20] who created generalized function spaces during the mid-twentieth century. The work of Orlicz (1932) expanded traditional L^p-spaces by establishing spaces that utilize growth conditions controlled by Young functions, thus enabling additional analysis on nonstandard function spaces. Variable exponent spaces emerged from this important foundational work by providing researchers with new ideas about spaces in which the exponent changes according to location. This extension broadens sequence space theory beyond classical boundaries, yet creates a comprehensive mathematical framework suitable for imaging work, signal analysis, and biological modeling.

Formally, let $p = \{p_n\}$ be a sequence of real numbers such that $p_n > 1$ for all n. The variable exponent sequence space $\ell_{p(n)}$ is defined as

$$\ell_{p(n)} = \left\{ \kappa = \{\kappa_n\} \in R^n : \sum_{n=1}^\infty |\kappa_n|^{p_n} < \infty \right\}.$$

The norm on $\ell_{p(n)}$ is given by

$$\|\kappa\|_{\ell_{p(n)}} = \inf \left\{ \lambda > 0 : \sum_{n=1}^\infty \left| \frac{\kappa_n}{\lambda} \right|^{p_n} \leq 1 \right\}.$$

These spaces have excellent flexibility for real-world usage because of their generalized summability conditions, providing finer control. Research on these spaces has revealed new challenges, as they rely on an underlying nonuniform convex norm structure and do not enjoy the duality relations characteristic of fixed p-spaces. Much attention has been drawn in recent research to investigating the topological nature of these spaces, especially for both modular sequence spaces and variable exponent Lebesgue spaces. Research on $\ell_{p(n)}$-spaces has had a number of important developments. Izuki et al. [21] conducted research on the convergence of variable exponent spaces with an additional focus on sequence transformation properties to advance our understanding of their complex topological

and analytical behavior. Rabinovich and Samko [22] analyzed the boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces.

The efforts of different authors have resulted in the extension of a number of function spaces, such as Lebesgue spaces of integrability, Morrey spaces [23], Lorentz spaces [24], Hardy spaces [25], Bergman spaces [26], Orlicz spaces [27], and Zygmund spaces [28], among many others, from their traditional definitions to the variable exponent space context. For instance, boundedness embedding and compactness theories for operators evolved substantially during recent times while gaining diverse functional-space applications across different domains. Cordes [29] introduced the study of compact commutators and bounded pseudo differential operators which formed the essential groundwork for operator theory. The research of Denk et al. [30] added R-boundedness to Fourier multiplier analysis thus revealing fresh perspectives on operator performance within variable domains. Fan [31] studied boundary trace embedding theorems for variable exponent Sobolev spaces to show how variable exponents influence this relationship between embeddings. Mihăilescu and Rădulescu [32] expanded this framework for nonhomogeneous quasilinear eigenvalue problems while continuing research on variable exponent Sobolev spaces. Samko [33] analyzed weighted Hardy and singular operators within Morrey spaces to develop essential analytic methods for bounded operator investigations. Thieme [34] examined spectral bounds in infinite-dimensional systems thereby enabling population dynamics and time-heterogeneous systems applications. Guliyev et al. [35] executed an analysis of maximal and singular integral operator boundedness within variable exponent Morrey spaces through a unified method for generalized Morrey-type spaces. Unal and Aydın [36] recently investigated compact embeddings for weighted variable exponent Sobolev spaces while examining their applications to weighted Laplacians which demonstrated applicability between theoretical findings and practical uses.

Completeness, separability, and compactness are basic concepts in function space theory. These notions are of importance in determining the structure behavior and analytical properties of the spaces, and they form the core in many areas of functional analysis, operator theory, and PDEs. Dorantes-Aldama and Shakhmatov [37] further developed these notions by studying the completeness and compactness in metric spaces, topological groups, and functions spaces, which makes it possible to approach these properties in a much more general context. The researchers built on previous work in the topology of function spaces when Holá and Zsilinszky [38] investigated the completeness standards and related topological properties of the function space graphs, further enhancing the topological exploration of function spaces. Through his research, Rodabaugh [39] developed foundational results using separation axioms with representation theorems and compactness to advance the study of functional spaces in fuzzy set theory. Almira and Luther [40] added to compactness and approximation space theory through their work which explained the application of generalized approximation spaces in numerical analysis and optimization. Carro and Soria [41] introduced weighted Lorentz spaces while investigating their relationships with the Hardy operator to extend knowledge about bounded operators in weighted environments.

Bandaliyev et al. [42] investigated the precompactness and completeness of sets within variable exponent Morrey spaces by utilizing bounded metric measure spaces.

Theorem 1.1. *[42] Let (Ω, ρ, ν) be a bounded doubling metric measure space, and let $\mathrm{p}(\cdot) \in \mathbf{P}(\Omega, \nu)$ satisfy $0 < \mathrm{p}_- \leq \mathrm{p}_+ < \infty$ and $\sup_{\sigma \in \Omega} \frac{\alpha(\sigma)}{\mathrm{p}(\sigma)} < \frac{1}{\mathrm{p}_+}$. Assume that a subset $\mathbf{Q}$ of $\mathrm{L}^{\mathrm{p}(\cdot), \alpha(\cdot)}(\Omega)$ satisfies:*

- $\mathbf{Q}$ *is bounded in* $\mathrm{L}^{\mathrm{p}(\cdot), \alpha(\cdot)}(\Omega)$*;*

- *For some* $q \in (0, p_-)$,

$$\lim_{\tau \to 0^+} \sup_{\zeta \in \mathbf{Q}} \left\| \int_{\mathbf{B}(\cdot, \tau)} |\zeta(\cdot) - \zeta(y)|^q \, d\nu(y) \right\|_{L^{\frac{p(\cdot)}{q}, \alpha(\cdot)}(\Omega)} = 0.$$

Then, $\mathbf{Q}$ *is totally bounded in* $L^{p(\cdot), \alpha(\cdot)}(\Omega)$.

Xu et al. [43] investigated the completeness, total boundedness, and separability in Bochner-Lebesgue spaces with variable exponents, extending the classical theory of compactness to the context of these generalized function spaces.

Theorem 1.2. *[43] Let* $p(\cdot) \in \mathbf{P}(\mathbb{R}^n)$. *Suppose that* Ω *is a bounded subset of* $L^{p(\cdot)}(\mathbb{R}^n, \mathbf{D})$. *If*

- *for every* $\kappa > 0$, $\sigma \in \mathbb{R}^n$,

$$\left\{ \int_{\mathbf{B}(\sigma, r)} \varphi(\alpha) d\alpha : \varphi \in \Omega \right\} \quad \text{is a totally bounded set in } \mathbf{D},$$

- *average continuity holds uniformly,*

$$\lim_{h \to 0^+} \sup_{\varphi \in \Omega} \|\mathbf{M}_h \varphi - \varphi\|_{L^{p(\cdot)}(\mathbb{R}^n, \mathbf{D})} = 0,$$

and

- *convergence at infinity holds uniformly,*

$$\lim_{\alpha \to \infty} \sup_{\varphi \in \Omega} \left\| \varphi \chi_{\mathbf{B}_\alpha^c} \right\|_{L^{p(\cdot)}(\mathbb{R}^n, \mathbf{D})} = 0,$$

where $\mathbf{B}_\alpha^c$ *is the complement of* $\mathbf{B}_\alpha = \{\sigma \in \mathbb{R}^n : |\sigma| \le \alpha\}$. *Then* Ω *is a precompact set in* $L^{p(\cdot)}(\mathbb{R}^n, \mathbf{D})$.

Morrey spaces play a central role in relating algebraic structures to topological space. They provide for the realization of abstract algebraic structures in concrete terms and are vital to numerous applications across homology and homotopy theory. Haroske et al. [44] presented an updated treatment of Morrey spaces and their derived versions by developing a single framework that reveals their mathematical properties together with their embedding propositions. The analysis of metric measure spaces depends heavily on Morrey spaces for carrying Euclidean results into metric areas as noted in [45]. Variable exponent Morrey spaces provide a refined framework in fluid dynamics research to manage variable regularity disturbances [46]. Relevant research results demonstrated Morrey spaces prove useful for nonlinear PDE studies in [47] because they provide robust solutions for non-linear and degenerate problems. Researchers have shown through [48] that Morrey spaces act as fundamental tools for deriving a priori estimates combined with fine regularity results within elliptic equation regularity theory. Researchers have examined embedding theorems and compactness properties with rigorous detail in [49] because these foundations support many analytical applications. The field of weighted Morrey spaces constitutes an essential extension due to their natural harmonic analysis environment and weighted inequality capabilities according to [50].

Researchers [51] developed the homogeneous grand mixed Herz-Morrey spaces during their study in 2024 and investigated fundamental features of these spaces. The study proved the boundedness property of sublinear and fractional-type operators inside these spaces.

Theorem 1.3. *[51] Let $0 < \theta$, $0 \le \alpha < \infty$, and $1 < p, q_i < \infty$ for each $i = 1, 2, \ldots, n$. Assume that*

$$\sum_{i=1}^{n} \frac{1}{q_i} < \kappa < n\left(1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i}\right).$$

Define the annular region $\mathbf{A}_k$ by

$$\mathbf{A}_k := \mathbf{B}_k \setminus \mathbf{B}_{k-1},$$

where

$$\mathbf{B}_k := \left\{\sigma \in R^n : |\sigma| \le 2^k\right\}.$$

Let $\mathbf{T}$ denote a sublinear operator satisfying the following conditions:

- *$\|\mathbf{T}\varphi\|_{L^{\vec{q}}(R^n)} \le \mathbf{C}_0\|\varphi\|_{L^{\vec{q}}(R^n)'}$;*
- *For every $k \in \mathbb{Z}$, let φ be a function such that*

$$\operatorname{supp}\varphi \subset \mathbf{A}_k, \quad where \quad \operatorname{supp}\varphi := \overline{\{\sigma \in R^n : \varphi(\sigma) \ne 0\}}.$$

  Assume further that $|\sigma| \ge 2^{k+1}$. Then the sublinear operator $\mathbf{T}$ satisfies the pointwise estimate:

$$|\mathbf{T}\varphi(\sigma)| \le \mathbf{C}_1 \frac{\|\varphi\|_{L^1(R^n)}}{|\sigma|^n},$$

  where $\mathbf{C}_1$ is a positive constant independent of φ and σ.
- *For every $k \in \mathbf{Z}$, a function φ such that $\operatorname{supp}\varphi \subset \mathbf{A}_k$, and $|\sigma| \le 2^{k-2}$,*

$$|\mathbf{T}\varphi(\sigma)| \le \mathbf{C}_1 \frac{\|\varphi\|_{L^1(R^n)}}{2^{nk}}.$$

Thus, $\mathbf{T}$ is bounded on $\mathbf{M}\dot{K}_{\vec{q},\alpha}^{\kappa,p,\theta}(R^n)$.

For further recent results related to function spaces with non-standard growth, we refer the reader to [52–54] and the references therein.

This paper contributes significantly to the existing literature by generalizing inequality (1.1) within the framework of function spaces, specifically Orlicz-Zygmund spaces characterized by non-standard growth in the exponents. While this inequality has been studied through various approaches, its formulation in the setting of function spaces with non-standard growth remains novel. Moreover, since the Hermite-Hadamard inequality serves as a refinement of the triangle inequality and the Orlicz-Zygmund space extends the classical Lebesgue space, we address problem (3.11) posed by Peter Hästö [55], which concerns the triangle inequality and the norm structure in mixed Lebesgue spaces. Additionally, we explore several structural properties of Morrey spaces in the Besov sense, which were recently introduced by the authors in [56]. These spaces, owing to their intricate nature, have been scarcely studied in the literature. Our primary motivation for this investigation stems from the foundational works of [42, 43, 56], which have guided and inspired the present study.

This work is divided into four parts. The first part contains an introduction that presents an overview of the research. In Section 2, we recall basic definitions and known results, such as modular spaces, variable exponent spaces, and their mixed structures. Section 3 introduces our key contributions, which are Hermite-Hadamard-type inequalities in a new setting and several new structural properties of Besov-type Morrey spaces with non-standard growth exponents, and establish completeness, compactness, and separability of the space. Lastly, Section 4 offers the conclusions and possible directions of research based on these results.

2. Preliminary results

Here, we shall review some basic definitions and facts about mixed norm function spaces with variable exponents. Most of the properties related to function spaces can be read in the book [57], and mixed Lebesgue spaces are treated in [56, 58].

2.1. Semi-modular spaces

Variable Lebesgue spaces are classified as semi-modular spaces, which generalize normed spaces. To this end, we will first review some definitions and results related to modular spaces.

Definition 2.1. *Let* $\mathbf{M}$ *be a vector space over some field* $\mathbf{K}$ *(real or complex). The mapping* $\rho : \mathbf{M} \longrightarrow [0, \infty)$ *is said to be semi-modular on* $\mathbf{M}$ *if*

- $\rho(0) = 0$*;*
- $\rho(\alpha\mu) = \rho(\mu)$ *for all* $\mu \in \mathbf{M}$ *and* $\alpha \in \mathbf{K}, |\alpha| = 1$*;*
- *if* $\rho(\alpha\mu) = 0$ *for all* $\alpha > 0$*, then* $\mu = 0$*;*
- *function* $[0, \infty) \ni \alpha \mapsto \rho(\alpha\mu)$ *is left-continuous for every* $\mu \in \mathbf{M}$*;*
- *function* $[0, \infty) \ni \alpha \mapsto \rho(\alpha\mu)$ *is non-decreasing for every* $\mu \in \mathbf{M}$*.*

If a vector space $\mathbf{M}$ has ρ as a semi-modular on it, then the space

$$\mathbf{M}_\rho := \{\mu \in \mathbf{M} : \exists_{\alpha>0}\rho(\alpha\mu) < \infty\},$$

known as a semi-modular space, is connected to the functional $\|\cdot\|_\rho : \mathbf{K}_\rho \to [0, \infty)$ and defined as follows:

$$\|\mathsf{s}\|_\rho := \inf\{\alpha > 0 : \rho(\mathsf{s}/\alpha) \le 1\} \text{ for } \mathsf{s} \in \mathbf{K}_\rho.$$

Proposition 2.1. *Suppose that on vector space* $\mathbf{M}$*,* ρ *is a semi-modular. Consequently, for each* $\mu \in \mathbf{M}$*, we have*

$$\rho(\mu) \le 1 \iff \|\mu\|_\rho \le 1.$$

Proof. Assume that $\rho(\mu) \le 1$. Then, $\|\mu\|_\rho \le 1$ is inferred from the definition of the norm $\|\cdot\|_\rho$. Conversely, if we assume that $\|\mu\|_\rho \le 1$, then for each $\alpha > 1$ and due to the left continuity of the function, we have

$$\rho(\mu/\alpha) \le 1.$$

2.2. Variable exponent spaces

We recall the notion of the variable exponent Lebesgue space. Let Ω be the Lebesgue measurable space associated with $\mathbf{R}^n$, and let $\mathsf{p} : \Omega \to (0, \infty)$ be a measurable function, typically defined as follows:

- $\mathsf{p}^- := \mathrm{essinf}_{\sigma\in\Omega}\,\mathsf{p}(\sigma) = \sup\{\alpha : \mathsf{p}(\sigma) \ge \alpha \text{ a.e. } \sigma \in \Omega\}$.
- $\mathsf{p}^+ := \mathrm{esssup}_{\sigma\in\Omega}\,\mathsf{p}(\sigma) = \inf\{\alpha : \mathsf{p}(\sigma) \le \alpha \text{ a.e. } \sigma \in \Omega\}$.
- $\Omega_0 := \{\sigma \in \Omega : 1 < \mathsf{p}(\sigma) < \infty\} = \mathsf{p}^{-1}((1, \infty))$.
- $\Omega_1 := \{\sigma \in \Omega : \mathsf{p}(\sigma) = 1\} = \mathsf{p}^{-1}(1)$.
- $\Omega_\infty := \{\sigma \in \Omega : \mathsf{p}(\sigma) = \infty\} = \mathsf{p}^{-1}(\infty)$.

- the conjugate exponent $p'(\cdot)$:

$$p'(\sigma) := \begin{cases} \infty & (\sigma \in \Omega_1), \\ \frac{p(\sigma)}{p(\sigma)-1} & (\sigma \in \Omega_0), \\ 1 & (\sigma \in \Omega_\infty). \end{cases}$$

Namely, $\frac{1}{p(\sigma)} + \frac{1}{p'(\sigma)} = 1$ always holds for a.e. $\sigma \in \Omega$. In particular, if $p(\cdot)$ equals a constant p, then of course $p'(\cdot) = p'$ is the usual conjugate exponent. The function $p'(\cdot)$ does not represent the derivative of $p(\cdot)$; rather, it denotes the conjugate exponent of $p(\cdot)$. Many examples of exponent functions on the real number set demonstrate variable and constant exponent conditions. The expression $p(\alpha) = p$ serves as an elementary case of a constant exponent function with a fixed real number p. A variable exponent function demonstrates oscillatory behavior based on σ and has an expression of $p(\alpha) = 2 + \sin(\alpha)$. We will refer to any Lebesgue measurable function $p : \Omega \to (0, \infty)$ as a variable exponent, with the condition that $p^- > 0$. For the set Ω, $\mathbf{P}_0(\Omega)$ denotes the space of all functions with varying exponents. The notation $\mathbf{P}(\Omega)$ marks the pack of all varying exponent functions p such that $p_- \geq 1$. Consider the vector space $L^0(\Omega)$, which contains all measurable functions over Ω. For $p \in \mathbf{P}_0(\Omega)$, over the space $L^0(\Omega)$, we define the semi-modular as follows:

$$\rho_{p(\cdot)}(\varphi) := \int_\Omega \Psi_{p(\mu)}(|\varphi(\mu)|)d\mu,$$

where

$$\Psi_p(\sigma) := \begin{cases} \sigma^p, & \text{if } 0 < p < \infty, \\ 0, & \text{if } \sigma \leq 1, p = \infty, \\ \infty, & \text{if } \sigma > 1, p = \infty, \end{cases}$$

for each $\sigma \geq 0$. This modular space is described by the semi-modular $\rho_{p(\cdot)}$, which in turn defines the $L^{p(\cdot)}(\Omega)$ space as follows:

$$\varphi \in L^{p(\cdot)}(\Omega) \iff \exists_{\alpha>0} \rho_{p(\cdot)}(\alpha\varphi) = \int_\Omega \varphi_{p(\mu)}(\alpha|\varphi(\mu)|)d\mu < \infty.$$

The following Luxemburg quasi-norm makes the $L^{p(\cdot)}(\Omega)$ space as a quasi-Banach space:

$$\|\varphi\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\alpha > 0 : \rho_{p(\cdot)}(\varphi/\alpha) \leq 1\right\} \text{ for } \varphi \in L^{p(\cdot)}(\Omega).$$

Remark 2.1. *If $p(\alpha) \equiv p \in (0, \infty)$ is a constant, the space $L^{p(\cdot)}(\Omega)$ simplifies to the classical Lebesgue space $L^p(\Omega)$. The interaction between the semimodular and quasinorm space presents advanced challenges when studying variable exponent spaces instead of constant exponent spaces. However, for any variable exponent function $p \in \mathbf{P}(\mathbb{R}^n)$, the fundamental equivalence $\rho_{p(\cdot)}(\varphi) \leq 1$ iff $\left\|\varphi \mid L^{p(\cdot)}\right\| \leq 1$ follows directly from the left-continuity property of the semimodular $\rho_{p(\cdot)}$. It is essential to emphasize that the function space $L_{p(\cdot)}$ inherently satisfies the lattice property, and $\varphi \in L^{p(\cdot)}$ and $\left\|\varphi \mid L^{p(\cdot)}\right\| < \infty$ are equivalent for any $\varphi \in \mathbf{M}(\mathbb{R}^n)$ (the class of extended real-valued measurable mappings.*

It is the particular case of Musielak-Orlicz space, and whenever $p \in \mathbf{P}(\Omega)$, then $L^{p(\cdot)}(\Omega)$ becomes a Banach space. When the exponent is constant, it reduces to a classical Lebesgue space.

Notably, the definition of $L^{p(\cdot)}(\Omega)$ space becomes simpler if $p_+ < \infty$, so that $\varphi \in L^{p(\cdot)}(\Omega)$ if, and only if,

$$\int_\Omega |\varphi(\mu)|^{p(\mu)} d\mu < \infty.$$

Certain presumptions on the exponent function p are required for a few claims in this work. Specifically, function $p : \Omega \longrightarrow R$ is locally log-Hölder continuous on Ω, if there exists $c_{\log}(p) > 0$ and for all $\mu, \nu \in \Omega$, we have

$$|p(\mu) - p(\nu)| \leq \frac{c_{\log}(p)}{\log(e + 1/|\mu - \nu|)}.$$

We say that p satisfies the log-Hölder continuous at infinity (or has a log decay at infinity), if there exists $p_\infty \in \Omega$ and a constant $c_{\log} > 0$ and for all $\nu \in \Omega$, we have

$$|p(\mu) - p_\infty| \leq \frac{c_{\log}(p)}{\log(e + |\mu|)}.$$

If both of the aforementioned criteria are met by the function p, we declare that p is globally log-Hölder continuous $\mathbf{C}_{\log}(p)$ on Ω. The corresponding class of functions is defined as follows:

$$\mathbf{P}_0^{\log}(\Omega) := \left\{ p \in \mathbf{P}_0(\Omega) : \frac{1}{p} \in \mathbf{C}_{\log}(p) \right\}.$$

For the semi-modular $\rho_{p(\cdot)}$ and the quasi-norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$, we now formulate several well-known and practical statements.

Proposition 2.2. *Let* $p \in \mathbf{P}_0(\Omega)$ *with* $p_+ < \infty$ *and let* $\varphi \in L^{p(\cdot)}(\Omega)$. *Then,*

$$\min \left\{ (\rho_{p(\cdot)}(\varphi))^{\frac{1}{p_-}}, (\rho_{p(\cdot)}(\varphi))^{\frac{1}{p_+}} \right\} \leq \|\varphi\|_{L^{p(\cdot)}(\Omega)} \leq \max \left\{ (\rho_{p(\cdot)}(\varphi))^{\frac{1}{p_-}}, (\rho_{p(\cdot)}(\varphi))^{\frac{1}{p_+}} \right\}.$$

We now recall the definition of Morrey spaces with variable exponents, which serve as a generalization of the classical Morrey spaces.

Definition 2.2. *For* $p, u \in \mathbf{P}(R^n)$ *with* $0 < p^- \leq p(\alpha) \leq u(\alpha) \leq \infty$, *the variable exponent Morrey space* $M_{p(\cdot),u(\cdot)} := M_{p(\cdot),u(\cdot)}(R^n)$ *consists of all functions* $\varphi \in \mathbf{M}(R^n)$ *with a finite quasinorm as follows:*

$$\left\| \varphi \left| M_{p(\cdot),u(\cdot)} \right\| := \sup_{\alpha \in R^n, \kappa > 0} \kappa^{\frac{n}{u(\alpha)} - \frac{n}{p(\alpha)}} \left\| \varphi \chi_{\mathbf{B}(\alpha,\kappa)} \right| L_{p(\cdot)} \right\|. \tag{2.1}$$

By the definition of $L^{p(\cdot)}$, *it becomes evident that the expression given in (2.1) can be equivalently reformulated in the following manner:*

$$\left\| \varphi \mid M_{p(\cdot),u(\cdot)} \right\| = \sup_{\alpha \in R^n, \kappa > 0} \inf \left\{ \eta > 0 : \rho_{p(\cdot)} \left(\kappa^{\frac{n}{u(\alpha)} - \frac{n}{p(\alpha)}} \frac{\varphi}{\eta} \chi_{\mathbf{B}(\alpha,\kappa)} \right) \leq 1 \right\}.$$

Remark 2.2. *A study of variable exponent Morrey spaces first appeared in [59] for Euclidean domains while the same concept emerged in [60] as part of the research on quasimetric spaces. Almeida et al. [59] presented embedding results and equivalent norms for bounded domains and*

proved them under the condition $p(\alpha) \geq 1$ *while assuming log-Hölder continuity of exponents. A study of Morrey spaces with variable exponents was featured in [61] while in [62], the authors established generalized versions of these spaces. In [63], researchers introduced partition norms that enable researchers to transform local findings into global results for variable exponent Morrey spaces on* R^n. *Different publications in the field provide multiple definitions although some of those definitions do not fully match each other. The definition employs the quasinorm (2.1) to present the framework which resembles [64] while investigating classical operators on unbounded domains.*

2.3. Mixed norm function spaces

Now, we define function spaces that exhibit a mixed convex modular structure, incorporating both sequence-based and integrability properties simultaneously.

Definition 2.3. *[55] Let* $p, q \in \mathbf{P}(\mathsf{R}^n)$. *The functional of mixed Lebesgue sequence space* $\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)$ *is defined on sequences of* $L^{p(\cdot)}$, *as follows:*

$$\rho_{\ell_{q(\cdot)}(L^{p(\cdot)})}\left(\left(\varphi_j\right)\right) := \sum_{j \geq 0}^{\infty} \inf\left\{\alpha > 0 : \rho_{p(\cdot)}\left(\frac{\varphi_j}{\alpha^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}. \tag{2.2}$$

A left-continuous semimodular ingredient exists according to the convention $\alpha^{\frac{1}{\infty}} = 1$ together with its corresponding Luxemburg norm.

$$\left\|\left(\varphi_j\right) \mid \ell_{q(\cdot)}\left(L_{p(\cdot)}\right)\right\| = \left\|\left(\varphi_j\right)\right\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$$
$$:= \inf\left\{\nu > 0 : \rho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\left(\varphi_j\right) / \nu\right) \leq 1\right\}, \quad \left(\varphi_j\right) \in \ell_{q(\cdot)}\left(L_{p(\cdot)}\right). \tag{2.3}$$

Under certain assumptions [65], the space $\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)$ becomes a normed space. When q^+ is finite, then Eq (2.2) simplifies into a reduced form as follows:

$$\rho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\left(\varphi_\alpha\right)_\alpha\right) = \sum_{\alpha \geq 0} \left\|\left|\varphi_\alpha\right|^{q(\cdot)} \mid L_{\frac{p(\cdot)}{q(\cdot)}}\right\|. \tag{2.4}$$

Remark 2.3. *This formulation establishes a quasinorm in the space* $\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)$ *for all* $p, q \in \mathbf{P}(\mathsf{R}^n)$. *Furthermore, it guarantees that the functional* $\left\|\cdot \mid \ell_{q(\cdot)}\left(L^{p(\cdot)}\right)\right\|$ *defines a norm under specific conditions: when* $\frac{1}{p(\alpha)} + \frac{1}{q(\alpha)} \leq 1$, *almost everywhere. More recently, Afzal et al. [65] observed that this also holds for Hadamard inequalities if* $1 \leq q(\alpha) \leq p(\alpha) \leq \infty$. *However, unlike the case where* q *is constant, the expression (2.4) does not necessarily define a norm in terms of the Jensen inequality when* $\min\{p^-, q^-\} \geq 1$, *as shown in [65], where an example in Theorem 10 demonstrates the failure of the triangle-type Jensen inequality in this scenario.*

Now, we re-examine Morrey spaces endowed with a mixed modular structure.

Definition 2.4. *[56] Consider exponent functions* $p, q, u \in \mathbf{P}(\mathsf{R}^n)$ *such that the condition* $p(\alpha) \leq u(\alpha)$. *The functional of mixed norm Morrey-sequence space* $\ell_{q(\cdot)}\left(M_{p(\cdot),u(\cdot)}\right)$ *encompass all measurable sequences* $\left(\varphi_\eta\right)_\eta \subset \mathbf{M}(\mathsf{R}^n)$, *and then one has*

$$\rho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}\left(\left(\varphi_\eta\right)_\eta\right) := \sum_{\eta \geq 0} \sup_{\alpha \in \mathsf{R}^n, \kappa > 0} \inf\left\{\sigma > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{u(\alpha)} - \frac{n}{p(\alpha)}} \varphi_\eta \chi_{\mathbf{B}(\alpha,\kappa)}}{\sigma^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\},$$

and the associated Luxemburg norm, typically defined as

$$\left\|(\varphi_\eta)_\eta \mid \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})\right\| := \inf\left\{\alpha > 0 : \rho_{\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})}\left(\frac{1}{\alpha}(\varphi_\eta)_\eta\right) \le 1\right\}.$$

Remark 2.4. *It can be shown that the space $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ constitutes a vector space, as established in [56]. The fact that $\ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})$ is closed under addition follows directly from the proof of the (quasi) triangle inequality for the norm $\left\|\cdot \mid \ell_{q(\cdot)}(M_{p(\cdot),u(\cdot)})\right\|$, as established in Proposition 3.11 and Theorem 3.12.*

2.4. Zygmund spaces, definitions, and preparatory tools

In the following part, we will present helpful characteristics of the logarithm function as described in [66]. For values of μ, ν that are greater than or equal to 0 and constants A, p that are greater than 1, we will define

$$\log(e + \nu\mu) \le \log(e + \nu) + \log(e + \mu),$$
$$\log(e + A\nu) \le A\log(e + \nu),$$
$$(\nu + \mu)^p \log(e + \nu + \mu) \le (2\nu)^p \log(e + 2\nu) + (2\mu)^p \log(e + 2\mu)$$
$$\le 2^{p+1}\nu^p \log(e + \nu) + 2^{p+1}\mu^p \log(e + \mu).$$

The Zygmund space $L^{p(\cdot)} \log^\beta L(\Omega; R^n)$, for $1 \le p < \infty$, $\beta \in R$ ($\beta \ge 0$ for $p = 1$), is defined as the Orlicz-Zygmund space $L^\Phi(\Omega; R^n)$ generated by the Young function

$$\Phi(\mu) \simeq \mu^p \log^\beta(e + \mu) \quad \text{for every } \mu \ge \mu_0 \ge 0.$$

Therefore, the Orlicz-Zygmund space $L^{p(\cdot)} \log^\beta L(\Omega; R^n)$ is defined as follows:

$$L^{p(\cdot)} \log^\beta L(\Omega; R^n) := \left\{\zeta : \Omega \to R^n \text{ measurable} : \int_\Omega |\zeta|^p \log^\beta(e + |\zeta|)d\nu < \infty\right\}$$

and it becomes a Banach space with the Luxemburg norm

$$\|\zeta\|_{L^{p(\cdot)} \log^\beta L(\Omega; R^n)} := \inf\left\{\alpha > 0 : \zeta_\Omega \left|\frac{\zeta}{\alpha}\right|^p \log^\beta\left(e + \left|\frac{\zeta}{\alpha}\right|\right)d\nu \le 1\right\}.$$

Iwaniec and Verde in [67, Theorem 8.1] showed the following estimates:

$$\|\zeta\|_{L^{p(\cdot)} \log^\beta L} \le \left[\int_\Omega |\zeta| \log\left(e + \frac{|\zeta|}{\|\zeta\|_{L^p(\Omega)}}\right)d\nu\right]^{1/p} \le 2\|\zeta\|_{L^p \log L}.$$

In [67, Theorem 8.1], they focused solely on $\beta = 1$ as the exponent of the logarithm. It is evident that their proof holds true for any $\beta \ge 0$. Therefore, we have

$$\|\zeta\|_{L^{p(\cdot)} \log^\beta L} \le \left[\int_\Omega |\zeta|^p \log^\beta\left(e + \frac{|\zeta|}{\|\zeta\|_{L^p(\Omega)}}\right)d\nu\right]^{1/p} \le 2\|\zeta\|_{L^p \log^\beta L}.$$

The following Hölder-type inequality is applicable (see, for example, [68]):

$$\|\zeta\eta\|_{L^c \log^\gamma L} \le C(\alpha, \beta, \gamma)\|\zeta\|_{L^\alpha \log^\alpha L}\|\eta\|_{L^\beta \log^\beta L},$$

whenever $\alpha, \beta, \gamma > 1$ and $\alpha, \beta, \gamma \in R$.

3. The major results

Building on the previously defined Orlicz-Zygmund space with variable integrability and the concept of Lebesgue sequence spaces, we now introduce a new modular along with its associated norm. These tools will allow for a more refined development of our results. To validate this space, we proceed by examining some of its structural properties.

Definition 3.1. *Let* $p, q \in P(R^n)$. *The mixed generalized Orlicz-Zygmund sequence space* $\ell_{q(\cdot)} \log^\beta \left(L^{p(\cdot)} \right)$ *consists of sequences* (ζ_j), *where each* $\zeta_j \in L^{p(\cdot)}$, *and is defined via the modular functional:*

$$\rho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} ((\zeta_j)) := \sum_{j=1}^\infty \inf \left\{ \alpha_j > 0 : \rho_{p(\cdot)} \left(\frac{\zeta_j}{\alpha_j^{1/q(\cdot)}} \right) \log^\beta \left(e + \frac{|\zeta_j|}{\alpha_j^{1/q(\cdot)}} \right) \le 1 \right\}.$$

If we consider the following convention $\alpha^{\frac{1}{\infty}} = 1$, the norm is defined as follows:

$$\left\| (\zeta_j)_j \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} := \inf \left\{ \nu > 0 \mid \varrho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \left(\frac{1}{\nu} (\zeta_j)_j \right) \log^\beta \left(e + \frac{|\zeta^j|}{\nu} \right) \le 1 \right\}.$$

If $q^+ < \infty$, then

$$\inf \left\{ \alpha > 0 \; \middle| \; \rho_{p(\cdot)} \left(\frac{\zeta}{\alpha_j^{\frac{1}{q(\cdot)}}} \right) \log^\beta \left(e + \frac{|\zeta|}{\alpha^{\frac{1}{q(\cdot)}}} \right) \le 1 \right\} = \left\| |\zeta|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

If we consider the following convention $\alpha^{\frac{1}{\infty}} = 1$, the norm is defined as follows:

$$\left\| (\zeta_j)_j \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} := \inf \left\{ \nu > 0 \mid \rho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \left(\frac{1}{\nu} (\zeta_j)_j \right) \log^\beta \left(e + \frac{|\zeta^j|}{\nu} \right) \le 1 \right\}.$$

If $q^+ < \infty$, then

$$\inf \left\{ \alpha > 0 \; \middle| \; \rho_{p(\cdot)} \left(\frac{\zeta}{\alpha_j^{\frac{1}{q(\cdot)}}} \right) \log^\beta \left(e + \frac{|\zeta|}{\alpha^{\frac{1}{q(\cdot)}}} \right) \le 1 \right\} = \left\| |\zeta|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Remark 3.1. *Zygmund-type spaces are typically defined using expressions of the form:*

$$\Phi(t) = t^p (\log(e + t))^\beta,$$

where $\beta \in R$ *and* $p > 1$. *When* $\beta = 0$, *the logarithmic term vanishes, simplifying the expression to*

$$\Phi(t) = t^p \cdot (\log(e + t))^0 = t^p.$$

This is exactly the Young function used to define the classical Lebesgue space L^p. *Hence, it represents a natural generalization of the Lebesgue space.*

In the variable exponent setting, where we consider spaces with variable summability and integrability, this generalization extends in a similar manner. Specifically, when $\beta = 0$, *the norm and associated modular in the variable exponent case coincide with Definition 3.1 as presented in [55].*

This represents our first key result for the Hermite-Hadamard-type inequality.

3.1. A positive resolution to an open problem: Hermite-Hadamard inequalities in Orlicz-Zygmund spaces of summability and integrability

Theorem 3.1. *Let* $p, q \in P(\mathbb{R}^n)$ *satisfy one of the following conditions:*

- $p(\kappa) \geq 1$ *and* q *is a constant function with* $q \geq 1$ *almost everywhere,*
- $1 \leq q(\kappa) \leq p(\kappa) \leq \infty$ *for almost all* $\kappa \in \mathbb{R}^n$, *or*
- $\frac{1}{p(\kappa)} + \frac{1}{q(\kappa)} \leq 1$ *for almost all* $\kappa \in \mathbb{R}^n$.
- $\beta \geq 0$ *and* $\Omega = [0, 1]$.

Then, the functional $\|\cdot\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}$ *satisfies the Hermite-Hadamard inequalities*

$$\left\| \frac{\eta_\nu + \Psi_\nu}{2} \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \leq \int_\Omega \|(1 - \sigma)\eta_\nu + \sigma\Psi_\nu\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \, d\sigma \leq \frac{\|\eta_\nu\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} + \|\Psi_\nu\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}}{2}.$$

Proof. If $1 \leq p(\kappa)$ and $q \in [1, \infty)$ are constants, the proof becomes straightforward. In the remaining part, we begin by analyzing two double inequalities under the assumption of variable exponents. Subsequently, we establish that the middle term of the inequality is either equal to or less than its latter component:

$$\left\| \frac{\eta_\nu + \Psi_\nu}{2} \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \leq \frac{\|\eta_\nu\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} + \|\Psi_\nu\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}}{2}. \tag{3.1}$$

For every sequence of measurable functions $(\eta_\nu)_{\nu=1}^\infty$ and $(\Psi_\nu)_{\nu=1}^\infty$, let $2\alpha_1 > 0$ and $2\alpha_2 > 0$ be given such that

$$\rho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \left(\frac{\eta_\nu}{2\alpha_1} \right) \log^\beta \left(e + \frac{|\eta_\nu|}{2\alpha_1} \right) \leq 1 \quad \text{and} \quad \rho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \left(\frac{\Psi_\nu}{2\alpha_2} \right) \log^\beta \left(e + \frac{|\Psi_\nu|}{2\alpha_2} \right) \leq 1.$$

Our objective now is to demonstrate that

$$\rho_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \left(\frac{\eta_\nu + \Psi_\nu}{2\alpha_1 + 2\alpha_2} \right) \log^\beta \left(e + \frac{|\eta_\nu + \Psi_\nu|}{2\alpha_1 + 2\alpha_2} \right) \leq 1.$$

For every $\epsilon > 0$, there exist sequences of positive numbers $\{\varsigma_\nu\}_{\nu=1}^\infty$ and $\{\kappa_\nu\}_{\nu=1}^\infty$ such that

$$\rho_{p(\cdot)} \left(\frac{\eta_\nu(\kappa)}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) \log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) \leq 1$$

$$\text{and} \quad \rho_{p(\cdot)} \left(\frac{\Psi_\nu(\kappa)}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right) \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right) \leq 1, \tag{3.2}$$

along with

$$\sum_{\nu=0}^\infty \varsigma_\nu \leq 1 + \frac{\epsilon}{2} \quad \text{and also} \quad \sum_{\nu=0}^\infty \kappa_\nu \leq 1 + \frac{\epsilon}{2}.$$

We define

$$A_\nu := \frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2}, \quad \text{that is,} \quad \sum_{\nu=0}^\infty A_\nu \leq 1 + \frac{\epsilon}{2}.$$

We will demonstrate that

$$\rho_{p(\cdot)}\left(\frac{\eta_\nu(\kappa) + \Psi_\nu(\kappa)}{\left(\frac{\alpha_1\varsigma_\nu + \alpha_2\kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}}(2\alpha_1 + 2\alpha_2)}\right)\log^\beta\left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1\varsigma_\nu + \alpha_2\kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}}(2\alpha_1 + 2\alpha_2)}\right) \leq 1, \quad \forall \nu \in \mathbb{N}. \tag{3.3}$$

Define the sets

$$\Omega_0 := \{\kappa \in \mathbb{R}^n \mid p(\kappa) < \infty\}$$

and

$$\Omega_\infty := \{\kappa \in \mathbb{R}^n \mid p(\kappa) = \infty\}.$$

Now, for each $\kappa \in \Omega_0$, consider

$$H_\nu(\kappa) := \left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)$$

and

$$J_\nu(\kappa) := \left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\right).$$

Then, (3.2) can be rewritten as

$$\int_{\Omega_0} H_\nu(\kappa)d\kappa = \int_{\Psi_0}\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)d\kappa \leq 1$$

and

$$\mathrm{ess}-\sup_{\kappa\in\Omega_\infty}\frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1\varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) \leq 1. \tag{3.4}$$

Also,

$$\int_{\Omega_0}\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\right)d\kappa \leq 1$$

$$\text{and} \quad \mathrm{ess}-\sup_{\kappa\in\Omega_\infty}\left\{\frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2\kappa_\nu^{\frac{1}{q(\kappa)}}}\right)\right\} \leq 1. \tag{3.5}$$

Our aim is to establish (3.3), that is,

$$\int_{\Omega_0}\left(\frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1\varsigma_\nu + \alpha_2\kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}}(2\alpha_1 + 2\alpha_2)}\right)^{p(\kappa)}\log^\beta\left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1\varsigma_\nu + \alpha_2\kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}}(2\alpha_1 + 2\alpha_2)}\right)d\kappa \leq 1$$

and $\text{ess} - \sup\limits_{\kappa \in \Omega_\infty} \left\{ \dfrac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\dfrac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}} (2\alpha_1 + 2\alpha_2)} \log^\beta \left(e + \dfrac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\dfrac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}} (2\alpha_1 + 2\alpha_2)} \right) \right\} \leq 1. \quad (3.6)$

We begin by proving the second part of (3.6). Initially, we note that (3.2) together with (3.4) and (3.5) leads to

$$\frac{|\eta_\nu(\kappa)|}{2} \leq \frac{\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}{\log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right)} \quad \text{and} \quad \frac{|\Psi_\nu(\kappa)|}{2} \leq \frac{\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}{\log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)},$$

which are valid for almost all $\kappa \in \Omega_\infty$. Applying $q(\kappa) \geq 1$ and utilizing Hölder's inequality in the form

$$\left(\frac{\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}} + \alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}{2\alpha_1 + 2\alpha_2} \right) \left[\log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) + \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right) \right]$$

$$\leq \left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2} \right)^{\frac{1}{q(\kappa)}} \log^\beta \left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{(2\alpha_1 + 2\alpha_2)^{\frac{1}{q(\kappa)}}} \right),$$

we get

$$\frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2} \right)^{\frac{1}{q(\kappa)}} (2\alpha_1 + 2\alpha_2)} \log^\beta \left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{(2\alpha_1 + 2\alpha_2)^{\frac{1}{q(\kappa)}}} \right) \leq 1.$$

Now, we proceed to confirm the first relation in (3.6). Let $1 \leq q(\kappa) \leq p(\kappa) < \infty$ for every $\kappa \in \Omega_0$. We then apply Hölder's inequality as follows:

$$\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right)^{\frac{1}{p(\kappa)}} \varsigma_\nu^{\frac{1}{q(\kappa)}} 2\alpha_1 \log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right)^{\frac{1}{p(\kappa)}} \kappa_\nu^{\frac{1}{q(\kappa)}} 2\alpha_2 \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)$$

$$\leq (2\alpha_1 + 2\alpha_2)^{1 - \frac{1}{q(\kappa)}} (2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu)^{\frac{1}{q(\kappa)} - \frac{1}{p(\kappa)}} \times \left(\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right) \varsigma_\nu 2\alpha_1 \log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) \right.$$

$$\left. + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right) \kappa_\nu 2\alpha_2 \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) \right)^{\frac{1}{p(\kappa)}}. \quad (3.7)$$

If $\frac{1}{p(\kappa)} \leq 1 - \frac{1}{q(\kappa)}$ holds almost everywhere for every $\kappa \in \Omega_0$, then we substitute (3.7) with

$$\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right)^{\frac{1}{p(\kappa)}} \varsigma_\nu^{\frac{1}{q(\kappa)}} 2\alpha_1 \log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right)^{\frac{1}{p(\kappa)}} \kappa_\nu^{\frac{1}{q(\kappa)}} 2\alpha_2 \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)$$

$$\leq (2\alpha_1 + 2\alpha_2)^{1 - \frac{1}{p(\kappa)} - \frac{1}{q(\kappa)}} (2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu)^{\frac{1}{q(\kappa)}}$$

$$\times \left(\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right) \alpha_1 \log^\beta \left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}} \right) + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right)^{p(\kappa)} \right) \alpha_2 \log^\beta \left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}} \right) \right)^{\frac{1}{p(\kappa)}}.$$

Utilizing (3.7), we can proceed as follows:

$$\int_{\Omega_0} \left(\frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}} (2\alpha_1 + 2\alpha_2)} \right)^{p(\kappa)} \log^\beta\left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{(2\alpha_1 + 2\alpha_2)^{\frac{1}{q(\kappa)}}}\right) d\kappa$$

$$\leq \int_{\Omega_0} \left(\frac{\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right)^{\frac{1}{p(\kappa)}} \varsigma_\nu^{\frac{1}{q(\kappa)}} 2\alpha_1 \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right)^{\frac{1}{p(\kappa)}} \kappa_\nu^{\frac{1}{q(\kappa)}} 2\alpha_2 \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 + 2\alpha_2} \right)^{p(\kappa)}$$

$$\cdot \left(\frac{2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2} \right)^{-\frac{p(\kappa)}{q(\kappa)}} d\kappa$$

$$\leq \int_{\Omega_0} \frac{\left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right) \varsigma_\nu 2\alpha_1 \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) + \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right) \kappa_\nu 2\alpha_2 \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu} d\kappa$$

$$\leq \frac{2\alpha_1 \varsigma_\nu}{2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu} \int_{\Omega_0} \left(\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right) \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) d\kappa$$

$$+ \frac{2\alpha_2 \kappa_\nu}{2\alpha_1 \varsigma_\nu + 2\alpha_2 \kappa_\nu} \int_{\Omega_0} \left(\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)}\right) \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) d\kappa \leq 1, \tag{3.8}$$

where we also applied (3.4) and (3.5). Alternatively, if we begin with (3.8), we proceed as follows:

$$\int_{\Omega_0} \left(\frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{\left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{2\alpha_1 + 2\alpha_2}\right)^{\frac{1}{q(\kappa)}} (2\alpha_1 + 2\alpha_2)} \right)^{p(\kappa)} \times \log^\beta\left(e + \frac{|\eta_\nu(\kappa) + \Psi_\nu(\kappa)|}{(2\alpha_1 + 2\alpha_2)^{\frac{1}{q(\kappa)}}}\right) d\kappa$$

$$= \int_{\Omega_0} \left(\frac{\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) \varsigma_\nu^{\frac{1}{q(\kappa)}} 2\alpha_1 \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 + 2\alpha_2} \left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{\alpha_1 + \alpha_2}\right)^{-\frac{p(\kappa)}{q(\kappa)}} \right.$$

$$\left. + \frac{\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right) \kappa_\nu^{\frac{1}{q(\kappa)}} 2\alpha_2 \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 + 2\alpha_2} \left(\frac{\alpha_1 \varsigma_\nu + \alpha_2 \kappa_\nu}{\alpha_1 + \alpha_2}\right)^{-\frac{p(\kappa)}{q(\kappa)}} \right)^{p(\kappa)} d\kappa$$

$$\leq \int_{\Omega_0} \frac{\left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)} 2\alpha_1 \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 + 2\alpha_2} d\kappa + \int_{\Omega_0} \frac{\left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)} 2\alpha_2 \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)}{2\alpha_1 + 2\alpha_2} d\kappa$$

$$= \frac{2\alpha_1}{2\alpha_1 + 2\alpha_2} \int_{\Omega_0} \left(\frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)} \log^\beta\left(e + \frac{|\eta_\nu(\kappa)|}{2\alpha_1 \varsigma_\nu^{\frac{1}{q(\kappa)}}}\right) d\kappa$$

$$+ \frac{2\alpha_2}{2\alpha_1 + 2\alpha_2} \int_{\Omega_0} \left(\frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right)^{p(\kappa)} \log^\beta\left(e + \frac{|\Psi_\nu(\kappa)|}{2\alpha_2 \kappa_\nu^{\frac{1}{q(\kappa)}}}\right) d\kappa \leq 1. \tag{3.9}$$

Thus, the proof of (3.1) is complete in both cases. Next, we examine the intermediate term and demonstrate that it is at most equal to the final component of the inequality. Now, let us consider

$$
\int_\Omega \left\| (1 - \sigma)\eta_\nu + \sigma\Psi_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \, d\sigma
$$

$$
\leq \int_\Omega (1 - \sigma) \left\| \eta_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \, d\sigma + \int_\Omega \sigma \left\| \Psi_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} \, d\sigma
$$

$$
\leq \frac{\left\| \eta_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}}{2} + \frac{\left\| \Psi_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}}{2} = \frac{\left\| \eta_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})} + \left\| \Psi_\nu \right\|_{\ell_{q(\cdot)} \log^\beta (L^{p(\cdot)})}}{2}. \tag{3.10}
$$

Considering Eqs (3.9) and (3.8), we derive the desired result. □

Remark 3.2.
- *If we set* $q(\cdot) = \infty$, $\beta = 0$, *and* $p^- = p^+$, *then we obtain inequality (1.3) as presented by Dragomir in [15].*
- *If we consider* $q(\cdot) = \infty$, $\beta = 0$, *and the equality of the essential supremum and essential infimum, then it refines the double inequality given in [69, p. 51].*
- *If we consider* $\beta = 0$, *then this resolves the open problem 3.11 presented in [55, p. 11].*

Remark 3.3. *As is well known, function spaces characterized by summability and integrability with variable exponents provide a more general framework than both the variable Lebesgue spaces of integrability and variable sequence spaces. These spaces naturally unify the structural features of both, with the variable Lebesgue space emerging as a special case. In [70], the authors show on page 15 that when the variable exponent is taken to be constant, the space reduces to the classical Lebesgue space. Additionally, in [15], related results were developed within the setting of classical Lebesgue spaces. Building upon these foundations, our work offers a natural and substantial generalization of these results within a more comprehensive modular framework.*

Note: As explicitly stated, we adopt the same approach as the authors in [55] when dealing with the iterated function space in the proof of Theorem 3.6, where one of the exponents in the sequence space is assumed to be constant. In that context, we apply the Hölder conjugacy condition

$$
\frac{1}{p(\kappa)} + \frac{1}{q(\kappa)} \leq 1 \quad \text{for almost all } \kappa \in R^n,
$$

which is precisely the same strategy employed when proving the Hermite-Hadamard-type result presented in this paper.

The primary objective of this section is to explore various new structural properties of Besov-type Morrey spaces, as established by the authors in [56].

3.2. Completeness of Morrey spaces of summability and integrability

First, we demonstrate that this modular space is a quasi-Banach space.

Theorem 3.2. *If* $p, q, v \in P_0$ *with* $p(\alpha) \leq v(\alpha)$, *then* $\ell_{q(\cdot)} (M_{p}(\cdot))$ *is a quasi-Banach space.*

Proof. To establish that $\ell_{q(\cdot)} (M_{p}(\cdot))$ is a quasi-Banach space, we first prove that it is a quasi-normed space. By Definition 2.4, we just need to consider quasi-convexity. Let $\kappa \in \left(0, \min\left\{ \frac{p^-}{2}, \frac{q^-}{2}, 1 \right\} \right]$ and

define $\tilde{p} = \frac{p}{\kappa}$ and $\tilde{q} = \frac{q}{\kappa}$. Then clearly $\frac{1}{\tilde{p}(\cdot)} + \frac{1}{\tilde{q}(\cdot)} \leqslant 1$. Thus we have

$$
\begin{aligned}
\left\| (\eta_\nu)_\nu + (\varphi_\nu)_\nu \right\|_{\ell_{q(\cdot)}(M_p(\cdot))} &= \left\| \left| (\eta_\nu)_\nu + (\varphi_\nu)_\nu \right|^\kappa \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))}^{\frac{1}{\kappa}} \\
&\leqslant \left\| (|\eta_\nu|^\kappa)_\nu + (|\varphi_\nu|^\kappa)_\nu \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))}^{\frac{1}{\kappa}} \\
&\leqslant \left(\left\| (|\eta_\nu|^\kappa)_\nu \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))} + \left\| (|\varphi_\nu|^\kappa)_\nu \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))} \right)^{\frac{1}{\kappa}} \\
&= \left(\left\| (\eta_\nu)_\nu \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))}^\kappa + \left\| (\varphi_\nu)_\nu \right\|_{\ell_{\tilde{q}(\cdot)}(M_{\tilde{p}}(\cdot))}^\kappa \right)^{\frac{1}{\kappa}} \\
&\leqslant 2^{\frac{1-\kappa}{\kappa}} \left(\left\| (\eta_\nu)_\nu \right\|_{\ell_{q(\cdot)}(M_p(\cdot))} + \left\| (\varphi_\nu)_\nu \right\|_{\ell_{q(\cdot)}(M_p(\cdot))} \right),
\end{aligned}
$$

which concludes the proof. The next step is to establish the completeness of the space. For this purpose, we fix a Cauchy sequence φ^i in $\ell_{q(\cdot)}(M_p(\cdot))$. Hence, for any $\varepsilon > 0$, there exists $\wp$ such that for all $i, j \geq \wp$, we have

$$
\left\| \varphi^i - \varphi^j \right\|_{\ell_{q(\cdot)}(M_p(\cdot))} < \varepsilon.
$$

By the quasi-norm's definition,

$$
\sup_{\alpha \in R^n, \kappa > 0} \rho_{\ell_{q(\cdot)}(M_p(\cdot))} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi^i - \varphi^j) \chi_{B(\alpha,\kappa)}}{\varepsilon} \right) \leq 1, \quad \text{for} \quad i, j \geq \wp,
$$

and hence

$$
\sum_{j \geq 0} \sup_{\alpha \in R^n, \kappa > 0} \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi^i_\mu - \varphi^j_\mu) \chi_{B(\alpha,\kappa)}}{\varepsilon \lambda^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq 1.
$$

In particular, for each $\delta > 1$, we have

$$
\sup_{\alpha \in R^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi^i_\mu - \varphi^j_\mu) \chi_{B(\alpha,\kappa)}}{\varepsilon \delta^{\frac{1}{q(\cdot)}}} \right) \leq 1.
$$

Hence, by the definition of the Morrey space of integrability $M_p(\cdot)$, we get

$$
\frac{1}{\delta^{\frac{1}{q^-}}} \left\| \varphi^i_\mu - \varphi^j_\mu \right\|_{M_p(\cdot)} \leq \left\| \frac{\varphi^i_\mu - \varphi^j_\mu}{\delta^{\frac{1}{q(\cdot)}}} \right\|_{M_p(\cdot)} \leq \varepsilon.
$$

Thus, passing with $\delta \to 1$, we get $\left\| \varphi^i_\mu - \varphi^j_\mu \right\|_{M_p(\cdot)} \leq \varepsilon$. Thus, we demonstrated that for each μ, φ^i_μ is a Cauchy sequence in $M_p(\cdot)$. Thus, for every μ, there exists $\varphi_\mu \in M_p(\cdot)$ such that

$$
\varphi^i_\mu \to \varphi_\mu \quad \text{in} \quad M_p(\cdot).
$$

Now, we shall prove that

$$
\varphi^i \to \varphi \quad \text{in} \quad \ell_{q(\cdot)}(M_p(\cdot)), \tag{3.11}
$$

where $\varphi = (\varphi_\mu)_\mu$. For this purpose, we require the following:

$$\inf\left\{\lambda_\mu : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}$$

$$\leq \liminf_{j\to\infty}\inf\left\{\lambda_\mu(j) : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu(j)^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}. \tag{3.12}$$

In order to prove (3.12), we introduce the following notation:

$$\mathbf{A} = \liminf_{j\to\infty}\inf\left\{\lambda_\mu(j) : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu(j)^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}.$$

Let j_k be a subsequence such that

$$\mathbf{A} = \liminf_{k\to\infty}\inf\left\{\lambda_\mu(j_k) : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^{j_k})\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}.$$

Hence, for each $v > 0$, there exists $\mathbb{N}$ such that for all $k \geq \mathbb{N}$, one has

$$\left\{\lambda_\mu(j_k) : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^{j_k})\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu(j_k)^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} < \mathbf{A} + v.$$

Thus,

$$\sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^{j_k})\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon(\mathbf{A} + v)^{\frac{1}{q(\cdot)}}}\right) \leq 1.$$

Next, we apply the Fatou lemma and the fundamental characteristics of the L^∞ norm, and we obtain

$$\sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon(\mathbf{A} + v)^{\frac{1}{q(\cdot)}}}\right) \leq 1.$$

Hence,

$$\inf\left\{\lambda_\mu : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} \leq \mathbf{A} + v.$$

Finally, applying $v \to 0$, we obtain (3.12). Now, we can prove (3.11). Since φ^i is a Cauchy sequence in $\ell_{q(\cdot)}(\mathbb{M}_{p(\cdot)})$, for $i, j \geq \wp$, we have

$$\sum_{j\geq 0}\inf\left\{\lambda_\mu : \sup_{\alpha\in\mathbb{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu^j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} \leq 1.$$

Therefore, by (3.12), for each $\mathbb{N}$ and $i \geq \wp$, we have

$$\sum_{j\geq 0}\sup_{\alpha\in\mathbb{R}^n,\kappa>0}\inf\left\{\lambda_\mu : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_\mu^i - \varphi_\mu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\lambda_\mu^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} \leq 1.$$

Then, passing $\mathbb{N} \to \infty$, we obtain

$$\left\|\varphi^i - \varphi\right\|_{\ell_{q(\cdot)}(\mathbb{M}_{p(\cdot)})} \leq \varepsilon \quad \text{for} \quad i \geq \wp.$$

Remark 3.4. *If we take* $v(\alpha)) = p(\alpha)$, *the results reduce to the completeness of mixed Lebesgue spaces* $\ell_{q(\cdot)}\left(L^{p(\cdot)}\right)$. *Additionally, choosing* $v(\alpha)) = p(\alpha)$ *and* $q(\cdot) = \infty$ *refines Theorem 2.71 concerning the completeness of variable Lebesgue spaces of integrability in* [70]. *On the other hand, if we only set* $q(\cdot) = \infty$, *it provides a refinement of the integrability Morrey space in the variable exponent setting.*

3.3. Separability of Morrey spaces of summability and integrability

The following result establishes that the mixed-norm space $\ell_{q(\cdot)}(M_{p}(\cdot))$ is separable when the exponents satisfy $p^+, q^+, v^+ < \infty$.

Theorem 3.3. *If* $p, q, v \in \mathbf{P}_0$ *and* $p^+, q^+, v^+ < \infty$, *then* $\ell_{q(\cdot)}(M_{p}(\cdot))$ *is separable.*

Proof. For each $k \in \mathbb{N}$, we define

$$\mathbf{V}_k = \left\{ \varphi \in \ell_{q(\cdot)}(M_{p}(\cdot)) ; \quad \varphi_i = 0, i > k \right\}.$$

It is clear that $\mathbf{V}_k$ represents a closed subset of $\ell_{q(\cdot)}(M_{p}(\cdot))$ and

$$\underbrace{M_{p}(\cdot) \times M_{p}(\cdot) \times \ldots \times M_{p}(\cdot)}_{k \text{ times}} \times \{0\} \times \cdots \times \{0\} \times \cdots \subset \mathbf{V}_k.$$

We divide the proof into two parts. First, we show that for each $k \in \mathbb{N}$, the space $\mathbf{V}_k$ is separable. Since $p^+ < \infty$, we know that $M_{p}(\cdot)$ is separable. Therefore, there exists a countable subset of $M_{p}(\cdot)$, say $\mathbf{K}$, such that $\overline{\mathbf{K}}^{\|\cdot\|_{M_{p}(\cdot)}} = M_{p}(\cdot)$. Set

$$\mathbf{K}_k := \underbrace{\mathbf{K} \times \mathbf{K} \times \ldots \times \mathbf{K}}_{k \text{ times}} \times \{0\} \times \cdots \times \{0\} \times \cdots \subset \mathbf{V}_k.$$

Obviously, $\mathbf{K}_k$ is countable, so it is enough to show that $\overline{\mathbf{K}}^{\|\cdot\|_{\ell_{q(\cdot)}(M_{p}(\cdot))}} = \mathbf{V}_k$. Let us fix $\varepsilon > 0$ and $\varphi \in \mathbf{V}_k$. Then, there exists $\eta \in \mathbf{K}_k$ such that for $\nu \in \{1, 2, \cdots, k\}$, we have

$$\|\varphi_\nu - \eta_\nu\|_{M_{p}(\cdot)} \leq \frac{\varepsilon}{k^{\frac{1}{q^-}}}.$$

Then, we have

$$\sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi_\nu - \eta_\nu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon \cdot \left(\frac{1}{k}\right)^{\frac{1}{q^-}}} \right) \leq 1,$$

and we obtain

$$\inf\left\{ \alpha_\nu; \quad \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi_\nu - \eta_\nu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon \cdot (\alpha_\nu)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq \frac{1}{k}.$$

Therefore,

$$\rho_{\ell_{q(\cdot)}(M_{p}(\cdot))}\left(\frac{\varphi - \eta}{\varepsilon} \right) = \sum_{\nu \geq 0}^{\infty} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf\left\{ \alpha_\nu > 0; \quad \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi_\nu - \eta_\nu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon \cdot (\alpha_\nu)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\}$$

$$= \sum_{\nu \geq 0}^{k} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf\left\{ \alpha_\nu > 0; \quad \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi_\nu - \eta_\nu)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon \cdot (\alpha_\nu)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\}$$

$$\leq 1.$$

Now, we have to show that

$$\overline{\bigcup_{k=1}^{\infty} \mathbf{V}_k}^{\|\cdot\|_{\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))}} = \ell_{q(\cdot)}\left(\mathbf{M}_p(\cdot)\right).$$

To prove it, we let $\varphi \in \ell_{q(\cdot)}\left(\mathbf{M}_p(\cdot)\right)$, and then

$$\sum_{i \geq 0} \inf \left\{ \alpha_i > 0; \quad \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\alpha_i^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq 1,$$

where $\mathbf{M} = \|\varphi\|_{\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))}$. Therefore, for every $0 < \varepsilon < 1$, $\exists\, k \in \mathbb{N}$ such that

$$\sum_{i=k+1}^{\infty} \inf \left\{ \alpha_i > 0; \quad \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\alpha_i^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq \varepsilon.$$

Thus, we obtain

$$\sum_{i=k+1}^{\infty} \inf \left\{ \tau_i > 0; \quad \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\left(\varepsilon\tau_i\right)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq \varepsilon. \tag{3.13}$$

On the other hand, we have

$$\sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}} \right) \leq \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\left(\varepsilon\tau_i\right)^{\frac{1}{q(\cdot)}}} \right) \leq 1. \tag{3.14}$$

Thanks to (3.14), we have

$$\left\{ \tau_i; \quad \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\left(\varepsilon\tau_i\right)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \subset \left\{ \tau_i; \quad \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\}. \tag{3.15}$$

Then in view of (3.13), we have

$$\sum_{i=k+1}^{\infty} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf \left\{ \tau_i; \quad \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\varepsilon^{\frac{1}{q^+}}\tau_i^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\}$$

$$\leq \sum_{i=k+1}^{\infty} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf \left\{ \tau_i; \quad \rho_{p(\cdot)} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}}(\varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{2\mathbf{M}\left(\varepsilon\tau_i\right)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \leq 1. \tag{3.16}$$

Finally, let us define

$$\eta = \begin{cases} \varphi_i, & \text{if } i \leq k, \\ 0, & \text{if } i > k. \end{cases}$$

It is clear that $\varphi \in \mathbf{V}_k$, and by (3.16), we obtain $\|\varphi - \eta\|_{\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))} \leq 2\mathbf{M}\varepsilon^{\frac{1}{q^+}}$, which is the desired result. $\qquad\square$

Remark 3.5. *If we set* $v(\alpha)) = p(\alpha)$*, the results simplify to the separability of mixed Lebesgue spaces* $\ell_{q(\cdot)}\left(L^{p,\cdot}\right)$*. Furthermore, choosing* $v(\alpha)) = p(\alpha)$ *and* $q(\cdot) = \infty$ *enhances the separability of variable Lebesgue spaces of integrability, as discussed in [70]. Conversely, setting only* $q(\cdot) = \infty$ *leads to a refinement of the separability of Morrey space within the variable exponent framework. Moreover, if* $v(\alpha)) = p(\alpha) = c$ *are constant, all these results reduce to the classical sense of Lebesgue and Morrey spaces.*

Proposition 3.1. *Let* $p, q, v \in \mathbf{P}_0(\Omega)$ *and let* $(\varphi_j), (\eta_j) \subset M_{p(\cdot)}(\mathbb{R}^n)$*. If* $0 \leq \varphi_j \leq \eta_j$ *for each* $j \in \mathbb{N}$*, then*

$$\left\|(\varphi_j)\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})} \leq \left\|(\eta_j)\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})}.$$

Proof. It suffices to show that

$$\left\{\alpha > 0 : \rho_{\ell_{q(\cdot)}(M_{p(\cdot)})}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right) \leq 1\right\} \subset \left\{\alpha > 0 : \rho_{\ell_{q(\cdot)}(M_{p(\cdot)})}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right) \leq 1\right\}.$$

$$(3.17)$$

Suppose that $\alpha > 0$ is such that

$$\rho_{\ell_{q(\cdot)}(M_{p(\cdot)})}\left(\frac{\eta_j}{\alpha}\right) = \sum_{j \geq 0} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} \leq 1.$$

Let us fix $j \in \mathbb{N}$ and let $\tau_j > 0$ be such that

$$\rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1.$$

Due to the assumption that $0 \leq \varphi_j \leq \eta_j$, we get

$$\rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1 \leq \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1.$$

Therefore,

$$\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\} \subset \left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}. \quad (3.18)$$

Finally, it easily follows that

$$\rho_{\ell_{q(\cdot)}(M_{p(\cdot)})}\left(\frac{\varphi_j}{\alpha}\right) = \sum_{j \geq 0} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}$$

$$\leq \sum_{j \geq 0} \sup_{\alpha \in \mathbb{R}^n, \kappa > 0} \inf\left\{\tau_j > 0 : \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\eta_j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\alpha\tau_j^{\frac{1}{q(\cdot)}}}\right) \leq 1\right\}$$

$$= \rho_{\ell_{q(\cdot)}(M_{p(\cdot)})}\left(\frac{\eta_j}{\alpha}\right) \leq 1,$$

and this concludes the proof. □

Lemma 3.1. *Assume that* $p, q, v \in \mathbf{P}_0(\Omega)$ *with* $(\varphi_i)_{i=1}^\infty \in \ell_{q(\cdot)}(M_{p(\cdot)})$. *Then, for every* $j \in \mathbb{N}$, *we have*

$$\left\|\varphi_j\right\|_{M_{p(\cdot)}(\Omega)} \leq \left\|(\varphi_j)\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})}.$$

Proof. Suppose $j \in \mathbb{N}$, and we define the sequence $(\eta_i(j))_{i=1}^\infty$ in such a way that $\eta_i(j) := \varphi_j \delta_{ij}$. Then, for each $i \in \mathbb{N}$, we have $|\eta_i(j)| \leq |\varphi_j|$, and as sequences preserve their order, it follows that

$$\left\|(\eta_j(j))\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})} \leq \left\|(\varphi_j)\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})}.$$

Finally, by the Proposition 3.1, we get

$$\left\|(\eta_j(j))\right\|_{\ell_{q(\cdot)}(M_{p(\cdot)})} = \left\|\varphi_j\right\|_{M_{p(\cdot)}(\Omega)},$$

and the proof follows. □

3.4. Compactness of Morrey spaces of summability and integrability

Theorem 3.4. *Let* $p, v \in \mathbf{P}_0^{\log}(\Omega)$ *and* $q \in \mathbf{P}_0(\Omega)$, *and suppose that* $p_+, q_+, v_+ < \infty$. *Then the set* $\mathbf{Q} \subset \ell_{q(\cdot)}(M_{p(\cdot)})$ *is precompact (i.e., totally bounded) in* $\ell_{q(\cdot)}(M_{p(\cdot)})$ *if and only if the following conditions are satisfied:*

(a) $\mathbf{Q}$ *is bounded in* $\ell_{q(\cdot)}(M_p(\cdot))$.
(b) *For every* $\varepsilon > 0$, *there exists* $\mathbf{K} \in \mathbb{N}$ *such that for all* $\varphi \in \mathbf{Q}$,

$$\sum_{r=\mathbf{K}+1}^\infty \left\| |\varphi_r|^{q(\cdot)} \right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \leq \varepsilon.$$

(c) *For all* $r \in \mathbb{N}$ *and* $0 < u < p_-$, *we have*

$$\lim_{\alpha \to 0^+} \sup_{\varphi \in \mathbf{Q}} \int_\Omega \left(\frac{1}{|\mathbf{B}(\mu, \alpha)|} \int_{\mathbf{B}(\mu,\alpha)} |\varphi_r(\mu) - \varphi_r(v)|^u \, dv \right)^{\frac{p(\mu)}{u}} d\mu = 0.$$

(d) *For all* $r \in \mathbb{N}$, *we have*

$$\lim_{R \to \infty} \sup_{\varphi \in \mathbf{Q}} \int_{\Omega \setminus \mathbf{B}(0,R)} |\varphi_r(\mu)|^{p(\mu)} d\mu = 0.$$

Proof. First of all, we shall show the sufficiency of conditions (a)–(d). The space $\ell_{q(\cdot)}(M_p(\cdot))$ is a quasi-Banach space, so it suffices to establish that the sequence (φ^k) has a convergent subsequence.

For every $r \in \mathbb{N}$, the sequence $(\varphi_r^k)_{k=1}^\infty$ is bounded in $M_{p(\cdot)}(\Omega)$. By condition (a), there exists $\alpha \in \mathbb{R}$ such that

$$\sup_{k \in \mathbb{N}} \left\|\varphi^k\right\|_{\ell_{q(\cdot)}(M_p(\cdot))} \leq \alpha.$$

Fixing $r \in \mathbb{N}$, we apply Lemma 3.1 to deduce that

$$\sup_{k \in \mathbb{N}} \left\|\varphi_r^k\right\|_{M_{p(\cdot)}(\Omega)} \leq \sup_{k \in \mathbb{N}} \left\|\varphi^k\right\|_{\ell_{q(\cdot)}(M_p(\cdot))} \leq \alpha.$$

This confirms that the sequence $\left(\varphi_r^k\right)_{k=1}^\infty$ is bounded in $\mathtt{M}_p(\cdot)$. Finally, for a fixed $r \in \mathtt{N}$, we define

$$\mathbf{K}_r := \overline{\left\{\varphi_r^k \in \mathtt{M}_p(\cdot)(\Omega) : k \in \mathtt{N}\right\}}.$$

Since the sets $(\mathbf{K}_r)_{j=1}^\infty$ are closed and bounded in $\mathtt{M}_p(\cdot)(\Omega)$, we apply the Cantor diagonal argument. Consequently, there exist a strictly increasing sequence $(k_1) \subset \mathtt{N}$ and a sequence of measurable functions $(\varphi_r) \subset \mathtt{M}_p(\cdot)(\Omega)$ such that for every $r \in \mathtt{N}$, we have

$$\varphi_r^{k_1} \xrightarrow{l \to \infty} \varphi_r \text{ in } \mathtt{M}_p(\cdot)(\Omega).$$

Let $\varphi = (\varphi_r)$, and we shall prove that

$$\varphi^{k_1} \xrightarrow{l \to \infty} \varphi \text{ in } \ell_{q(\cdot)}(\mathtt{M}_p(\cdot)).$$

We denote $\left(\varphi^{k_1}\right)$ by $\left(\varphi^k\right)$. Fixing $\varepsilon \in (0,1)$, it suffices to show that

$$\rho_{\ell_{q(\cdot)}(\mathtt{M}_p(\cdot))}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi^k - \varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right) \le 1,$$

for sufficiently large $k \in \mathtt{N}$. From (b), there exists $\mathbf{K} \in \mathtt{N}$ such that

$$\sum_{r=\mathbf{K}+1}^\infty \left\|\left|\varphi_r^k\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \le \frac{\varepsilon^{q_+}}{2^{q_+}4\mathbf{M}} \le \varepsilon. \tag{3.19}$$

Subsequently, we can split the modular as follows:

$$\rho_{\ell_{q(\cdot)}(\mathtt{M}_p(\cdot))}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi^k - \varphi)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right) = \sum_{r=1}^{\mathbf{K}} \left\|\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k - \varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$$
$$+ \sum_{r=\mathbf{K}+1}^\infty \left\|\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k - \varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}. \tag{3.20}$$

Next, applying the triangle inequality for $\|\cdot\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$, along with Lemma 3.1, Fatou's lemma for the counting measure, and Eq (3.19), we obtain

$$\sum_{r=\mathbf{K}+1}^\infty \left\|\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k - \varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$$
$$\le \sum_{r=\mathbf{K}+1}^\infty \left\|2^{q_+}\left(\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)} + \left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right)\right\|_{\mathtt{L}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$$
$$\le 2^{q_+}\mathbf{M}\left(\sum_{r=\mathbf{K}+1}^\infty \left\|\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \sum_{r=\mathbf{K}+1}^\infty \left\|\left|\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{\mathtt{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)$$

$$\leq \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \left(\sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right)$$

$$= \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \left(\sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right.$$

$$\left. + \sum_{r=\mathbf{K}+1}^{\infty} \inf\left\{ \lambda_r : \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\lambda_r^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \right)$$

$$\leq \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \left(\sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right.$$

$$\left. + \sum_{r=\mathbf{K}+1}^{\infty} \liminf_{j\to\infty} \inf\left\{ \lambda_r(j) : \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\lambda_r(j)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \right)$$

$$\leq \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \left(\sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right.$$

$$\left. + \liminf_{j\to\infty} \sum_{r=\mathbf{K}+1}^{\infty} \inf\left\{ \lambda_r(j) : \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^j)\chi_{\mathbf{B}(\alpha,\kappa)}}{\lambda_r(j)^{\frac{1}{q(\cdot)}}} \right) \leq 1 \right\} \right)$$

$$= \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \left(\sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \liminf_{j\to\infty} \sum_{r=\mathbf{K}+1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k)\chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right)$$

$$\leq \frac{2^{q_+}\mathbf{M}}{\varepsilon^{q_+}} \cdot \frac{\varepsilon^{q_+}}{2^{q_++1}\mathbf{M}} = \frac{1}{2} \leq \varepsilon.$$

Therefore, for each $k \in \mathsf{N}$, we have

$$\rho_{\ell_{q(\cdot)}(\mathtt{M}_{p(\cdot)})}\left(\frac{\varphi^k-\varphi}{\varepsilon} \right) \leq \left\| \left| \frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \frac{1}{2}. \tag{3.21}$$

Additionally, we assert that for any $r \in \{1,\ldots,\mathbf{K}\}$, we also have

$$\left\| \left| \frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \xrightarrow{k\to\infty} 0.$$

And for some specific $r \in \{1,\ldots,\mathbf{K}\}$, and $\nu \in (0,1)$, we arrive at

$$\rho_{\frac{p(\cdot)}{q(\cdot)}}\left(\left| \frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon} \right|^{q(\cdot)} \cdot \frac{1}{\nu} \right) \leq \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \int_\Omega \left| \frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\nu^{\frac{1}{q_-}}} \right|^{p(x)} \mathrm{d}x$$

$$= \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\nu^{\frac{1}{q_-}}} \right). \tag{3.22}$$

Since $\varphi_r^k \xrightarrow{k\to\infty} \varphi_r$ in $\mathtt{M}_{p(\cdot)}(\Omega)$, for sufficiently big $k \in \mathsf{N}$, we have

$$\rho_{\frac{p(\cdot)}{q(\cdot)}}\left(\left| \frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon} \right|^{q(\cdot)} \cdot \frac{1}{\nu} \right) \leq \sup_{\alpha\in\mathsf{R}^n,\kappa>0} \rho_{p(\cdot)}\left(\frac{\kappa^{\frac{n}{v(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon\nu^{\frac{1}{q_-}}} \right) \leq 1, \tag{3.23}$$

which proves that

$$\left\|\left|\frac{\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\xrightarrow{k\to\infty}0.$$

According to Eq (3.21), for sufficiently large $k\in\mathbb{N}$, we have

$$\rho_{\ell_{q(\cdot)}(M_p(\cdot))}\left(\frac{\varphi^k-\varphi}{\varepsilon}\right)\leq\left\|\left|\frac{\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^k-\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}}{\varepsilon}\right|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\leq 1,$$

which indicate that $\varphi^k\xrightarrow{k\to\infty}\varphi$ in $\ell_{q(\cdot)}(M_p(\cdot))$ and it proves that the family $\mathbf{Q}$ of measurable functions is bounded. Then we will prove that conditions (a)–(d) are necessary, assuming $\mathbf{Q}$ is totally bounded. Since (a) is evident, we shift our focus to (b). Let us set $\varepsilon\in(0,1)$ and define

$$\kappa:=\frac{\varepsilon}{2^{q_+}2\mathbf{M}},$$

where the constant $\mathbf{M}$ is once more used for the triangle inequality $\left\||\varphi_r^k|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$. According to our assumptions, there exists $\varphi^1,\ldots,\varphi^k\in\mathbf{Q}$, such that

$$\mathbf{Q}\subset\bigcup_{i=1}^{k}\mathbf{B}\left(\varphi^i,\left(\frac{\varepsilon}{2^{q_++1}2\mathbf{M}}\right)^{\frac{1}{q_-}}\right),$$

where $\mathbf{B}\left(\varphi^i,\left(\frac{\varepsilon}{2^{q_++1}2\mathbf{M}}\right)^{\frac{1}{q_-}}\right)$ are open balls in $\ell_{q(\cdot)}(M_p(\cdot))$.

Since $\varphi^i\in\ell_{q(\cdot)}(M_p(\cdot))$, there exists $\mathbf{K}\in\mathbb{N}$ such that

$$\sum_{r=\mathbf{K}+1}^{\infty}\left\||\varphi_r^i|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}<\frac{\varepsilon}{2^{q_+}2\mathbf{M}},$$

for $i=1,\ldots,k$. Let $\varphi\in\mathbf{Q}$, and then $\exists i\in\{1,\ldots,k\}$, such that

$$\left\|\varphi-\varphi^i\right\|_{\ell_{q(\cdot)}(M_p(\cdot))}<\left(\frac{\varepsilon}{2^{q_++1}2\mathbf{M}}\right)^{\frac{1}{q_-}}.$$

If we consider the unit ball property, then it follows that

$$\rho_{\ell_{q(\cdot)}(M_p(\cdot))}\left(\frac{\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi-\varphi^i)\chi_{\mathbf{B}(\alpha,\kappa)}}{\left(\frac{\varepsilon}{2^{q_++1}2\mathbf{M}}\right)^{\frac{1}{q_-}}}\right)\leq 1.$$

Finally, the triangle inequality yields

$$\sum_{r=\mathbf{K}+1}^{\infty}\left\|\left|\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r)\chi_{\mathbf{B}(\alpha,\kappa)}\right|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}$$

$$\leq 2^{q_+}\mathbf{M}\left(\sum_{r=\mathbf{K}+1}^{\infty}\left\|\left|\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r-\varphi_r^i)\chi_{\mathbf{B}(\alpha,\kappa)}\right|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}+\sum_{r=\mathbf{K}+1}^{\infty}\left\|\left|\kappa^{\frac{n}{\nu(\alpha)}-\frac{n}{p(\alpha)}}(\varphi_r^i)\chi_{\mathbf{B}(\alpha,\kappa)}\right|^{q(\cdot)}\right\|_{M^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)}\right)$$

$$\leq 2^{q_+} \mathbf{M} \left(\sum_{r=1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi_r - \varphi_r^i) \chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \sum_{r=K_++1}^{\infty} \left\| \left| \kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi_r^i) \chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} \right)$$

$$\leq 2^{q_+} \mathbf{M} \left(\frac{\varepsilon}{2^{q_+} 2\mathbf{M}} \sum_{r=1}^{\infty} \frac{2^{q_+} 2\mathbf{M}}{\varepsilon} \left\| \left| \kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi_r - \varphi_r^i) \chi_{\mathbf{B}(\alpha,\kappa)} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \frac{\varepsilon}{2^{q_++1} \mathbf{M}} \right)$$

$$\leq 2^{q_+} \mathbf{M} \left(\frac{\varepsilon}{2^{q_+} 2\mathbf{M}} \sum_{r=1}^{\infty} \frac{2^{q_+} 2\mathbf{M}}{\varepsilon} \left\| \left| \frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi_r - \varphi_r^i) \chi_{\mathbf{B}(\alpha,\kappa)}}{\kappa^{\frac{1}{q_-}}} \right|^{q(\cdot)} \right\|_{\mathbf{M}^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)} + \frac{\varepsilon}{2^{q_++1} \mathbf{M}} \right)$$

$$= 2^{q_+} \left(\frac{\varepsilon}{2^{q_+} 2\mathbf{M}} \rho_{\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))} \left(\frac{\kappa^{\frac{n}{v(\alpha)} - \frac{n}{p(\alpha)}} (\varphi - \varphi^i) \chi_{\mathbf{B}(\alpha,\kappa)}}{\left(\frac{\varepsilon}{2^{q_++1} \mathbf{M}} \right)^{\frac{1}{q_-}}} \right) + \frac{\varepsilon}{2^{q_++1} \mathbf{M}} \right) \leq 2^{q_++1} \mathbf{M} \kappa = \varepsilon,$$

and this establishes (b). To verify the validity of conditions (c) and (d), let us consider $r \in \mathbb{N}$ and analyze Lemma 3.1. From this, it follows that

$$\| \varphi_r \|_{\mathbf{M}_p(\cdot)}(\Omega) \leq \| \varphi \|_{\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))},$$

which is valid for each $\varphi \in \ell_{q(\cdot)}(\mathbf{M}_p(\cdot))$. As a result, the projective map

$$\pi_r : \ell_{q(\cdot)}(\mathbf{M}_p(\cdot)) \to \mathbf{M}_p(\cdot)(\Omega)$$

is defined as follows:

$$\pi_r(\varphi) := \varphi_r \quad \text{for } \varphi \in \ell_{q(\cdot)}(\mathbf{M}_p(\cdot)).$$

As $\ell_{q(\cdot)}(\mathbf{M}_p(\cdot))$ is a quasi-Banach space, it is closed and bounded subset of $\mathbf{Q}$, which is equivalent to its precompactness. Following that, the continuous image of the projective map is defined as

$$\pi_r(\mathbf{Q}) = \{ \varphi_r : \varphi \in \mathbf{Q} \},$$

which is precompact in $\mathbf{M}_p(\cdot)(\Omega)$, and the proof is complete.

The results of Theorem 3.4 show that if $p \in \mathbf{P}_0(\Omega)$ instead of $p \in \mathbf{P}_0^{\log}(\Omega)$, the assumptions (a)–(d) imply the closedness and boundedness. As a result, the theorem holds true. $\square$

Remark 3.6. *Theorem 3.4 provides a refinement of Theorem 3.1 presented in [71], which pertains to the integrability of variable Lebesgue spaces on metric measure spaces.*

4. Conclusions and future remarks

In this paper, we introduced a novel norm structure and a corresponding modular framework for Orlicz-Zygmund spaces characterized by non-standard growth exponents. This foundation opens the door for deeper exploration of such spaces within the landscape of modern functional analysis. A key contribution of our work is the formulation of the Hermite-Hadamard inequalities in this setting, offering a refined extension of classical triangular-type inequalities. Additionally, we investigated the intricate structure of Besov-type Morrey spaces, where summability and integrability are interwoven through a modular approach. Given the limited existing literature on these spaces, our results significantly advance the understanding of their fundamental properties.

The innovation embedded in these newly defined spaces offers substantial opportunities for future research. Potential directions include the analysis of bounded linear operators, embedding of theorems into classical and modular function spaces, and the reinterpretation of classical inequalities under this new mathematical lens. Properties such as uniform convexity, the topological framework, and the Kadec-Klee property deserve particular attention, as they are pivotal to the functional characteristics of these spaces. A more detailed examination of the norm's complex structure may also yield powerful tools for advanced analytical techniques. Furthermore, the application of these spaces in the study of partial differential equations (PDEs) and variational problems, particularly those modeling non-homogeneous or anisotropic media, is of significant interest, with implications for fields such as fluid dynamics and image processing. Continued research into embeddings, duality principles, and compactness criteria could provide new insights in harmonic analysis and potential theory. Finally, the extension of these ideas to fractional and nonlocal operators promises to enrich the theoretical foundation of inequalities and broaden their applicability across mathematical physics and applied analysis.

Author contributions

Waqar Afzal and Mujahid Abbas: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing–original draft preparation, Writing–review and editing, Visualization; Najla M. Aloraini and Jongsuk Ro: Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

Conflict of interest

The authors declare no conflicts of interest.

References

1. Y. Xiao, Y. L. Yang, D. Ye, J. Q. Zhang, Quantitative precision second-order temporal transformation-based pose control for spacecraft proximity operations, *IEEE Trans. Aerosp. Electron. Syst.*, **61** (2025), 1931–1941. https://doi.org/10.1109/TAES.2024.3469167

2. C. Ma, R. F. Mu, M. M. Li, J. L. He, C. L. Hua, L. Wang, et al., A multi-scale spatial-temporal interaction fusion network for digital twin-based thermal error compensation in precision machine tools, *Expert Syst. Appl.*, **286** (2025), 127812. https://doi.org/10.1016/j.eswa.2025.127812

3. S. S. Dragomir, *Advances in inequalities of the Schwarz, Grüss and Bessel type in inner product spaces*, New York: Nova Science Publishers Inc., 2005.

4. A. A. H. Ahmadini, W. Afzal, M. Abbas, E. S. Aly, Weighted Fejér, Hermite-Hadamard, and trapezium-type inequalities for (h_1, h_2)-Godunova-Levin preinvex function with applications and two open problems, *Mathematics*, **12** (2024), 1–28. https://doi.org/10.3390/math12030382

5. Z. A. Khan, W. Afzal, W. Nazeer, J. K. K. Asamoah, Some new variants of Hermite-Hadamard and Fejér-type inequalities for Godunova-Levin preinvex class of interval-valued functions, *J. Math.*, **2024** (2024), 8814585. http://dx.doi.org/10.1155/2024/8814585

6. A. Fahad, Z. Ali, S. Furuichi, S. I. Butt, Ayesha, Y. H. Wang, New inequalities for GA-h convex functions via generalized fractional integral operators with applications to entropy and mean inequalities, *Fractal Fract.*, **8** (2024), 1–21. https://doi.org/10.3390/fractalfract8120728

7. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

8. E. Nwaeze, Set inclusions of the Hermite-Hadamard type for m polynomial harmonically convex interval valued functions, *Constr. Math. Anal.*, **4** (2021), 260–273. http://dx.doi.org/10.33205/cma.793456

9. T. S. Du, Y. Peng, Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Comput. Appl. Math.*, **440** (2024), 115582. https://doi.org/10.1016/j.cam.2023.115582

10. J. L. Cardoso, E. M. Shehata, Hermite-Hadamard inequalities for quantum integrals: a unified approach, *Appl. Math. Comput.*, **463** (2024), 128345. https://doi.org/10.1016/j.amc.2023.128345

11. W. Afzal, M. Abbas, W. Hamali, A. M. Mahnashi, M. D. L. Sen, Hermite-Hadamard-type inequalities via Caputo-Fabrizio fractional integral for h-Godunova-Levin and (h_1, h_2)-convex functions, *Fractal Fract.*, **7** (2023), 1–19. http://dx.doi.org/10.3390/fractalfract7090687

12. W. Afzal, E. Y. Prosviryakov, S. M. El-Deeb, Y. Almalki, Some new estimates of Hermite-Hadamard, Ostrowski and Jensen-type inclusions for h-convex stochastic process via interval-valued functions, *Symmetry*, **15** (2023), 1–17. https://doi.org/10.3390/sym15040831

13. L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Anal.*, **71** (2009), 1311–1328. https://doi.org/10.1016/j.na.2008.12.005

14. D. F. Zhao, M. A. Ali, A. Kashuri, H. Budak, M. Z. Sarikaya, Hermite-Hadamard-type inequalities for the interval-valued approximately h-convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2020** (2020), 1–38. https://doi.org/10.1186/s13660-020-02488-5

15. S. S. Dragomir, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, *Proyecciones*, **34** (2015), 323–341. https://doi.org/10.4067/S0716-09172015000400002

16. R. Niu, H. T. Zheng, B. L. Zhang, Navier-Stokes equations with variable viscosity in variable exponent spaces of Clifford-valued functions, *Bound. Value Probl.*, **2015** (2015), 1–17. https://doi.org/10.1186/s13661-015-0291-y

17. F. Li, Z. B. Li, L. Pi, Variable exponent functionals in image restoration, *Appl. Math. Comput.*, **216** (2010), 870–882. https://doi.org/10.1016/j.amc.2010.01.094

18. D. Y. Zheng, X. J. Cao, Provably efficient service function chain embedding and protection in edge networks, *IEEE Trans. Netw.*, **33** (2025), 178–193. http://dx.doi.org/10.1109/TNET.2024.3475248

19. Z. W. Chen, W. H. Zhou, H. L. Kuang, Z. G. Chen, J. Z. Yang, Z. H. Chen, et al., Dynamic model and vibration of rack vehicle on curve line, *Veh. Syst. Dyn.*, 2025, 1–19. https://doi.org/10.1080/00423114.2025.2494835

20. W. Orlicz, Über konjugierte exponentenfolgen, *Stud. Math.*, **3** (1931), 200–211.

21. M. Izuki, E. Nakai, Y. Sawano, Function spaces with variable exponents–an introduction, *Sci. Math. Japon.*, **77** (2014), 187–315. https://doi.org/10.32219/isms.77.2_187

22. V. Rabinovich, S. Samko, Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces, *Integral Equ. Operator Theory*, **60** (2008), 507–537. https://doi.org/10.1007/s00020-008-1566-9

23. D. Anick, B. Gray, Small *H* spaces related to Morre spaces, *Topology*, **34** (1995), 859–881. http://dx.doi.org/10.1016/0040-9383(95)00001-1

24. Ö. Kulak, The inclusion theorems for variable exponent Lorentz spaces, *Turkish J. Math.*, **40** (2016), 605–619. http://dx.doi.org/10.3906/mat-1502-23

25. Z. W. Hao, Y. Jiao, Fractional integral on martingale Hardy spaces with variable exponents, *Fract. Calc. Appl. Anal.*, **18** (2015), 1128–1145. https://doi.org/10.1515/fca-2015-0065

26. C. J. Wang, On a maximum principle for Bergman spaces with small exponents, *Integral Equ. Operator Theory*, **59** (2007), 597–601. https://doi.org/10.1007/s00020-007-1539-4

27. S. Baasandorj, S. S. Byun, H. S. Lee, Gradient estimates for Orlicz double phase problems with variable exponents, *Nonlinear Anal.*, **221** (2022), 112891. https://doi.org/10.1016/j.na.2022.112891

28. V. Kokilashvili, A. Meskhi, Maximal and Calderón-Zygmund operators in grand variable exponent Lebesgue spaces, *Georgian Math. J.*, **21** (2014), 447–461. http://dx.doi.org/10.1515/gmj-2014-0047

29. H. O. Cordes, On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators, *J. Funct. Anal.*, **18** (1975), 115–131. https://doi.org/10.1016/0022-1236(75)90020-8

30. R. Denk, M. Hieber, J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, American Mathematical Society, 2003. http://dx.doi.org/10.1090/memo/0788

31. X. L. Fan, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.*, **339** (2008), 1395–1412. https://doi.org/10.1016/j.jmaa.2007.08.003

32. M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.*, **135** (2007), 2929–2937. http://dx.doi.org/10.1090/S0002-9939-07-08815-6

33. N. Samko, Weighted Hardy and singular operators in Morrey spaces, *J. Math. Anal. Appl.*, **350** (2009), 56–72. http://dx.doi.org/10.1016/j.jmaa.2008.09.021

34. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211. https://doi.org/10.1137/080732870

35. V. S. Guliyev, J. J. Hasanov, S. G. Samko, Boundedness of maximal, potential type, and singular integral operators in the generalized variable exponent Morrey type spaces, *J. Math. Sci.*, **170** (2010), 423–443. https://doi.org/10.1007/s10958-010-0095-7

36. C. Unal, I. Aydın, Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted $p(\cdot)$-Laplacian, *Complex Var. Elliptic Equ.*, **66** (2021), 1755–1773. http://dx.doi.org/10.1080/17476933.2020.1781831

37. A. Dorantes-Aldama, D. Shakhmatov, Completeness and compactness properties in metric spaces, topological groups and function spaces, *Topol. Appl.*, **226** (2017), 134–164. https://doi.org/10.1016/j.topol.2017.04.012

38. L. Holá, L. Zsilinszky, Completeness and related properties of the graph topology on function spaces, 2013, arXiv:1304.6628.

39. S. E. Rodabaugh, Separation axioms: representation theorems, compactness, and compactifications, In: *Mathematics of fuzzy sets*, Boston: Springer, 1999, 481–552. https://doi.org/10.1007/978-1-4615-5079-2_9

40. J. M. Almira, U. Luther, Compactness and generalized approximation spaces, *Numer. Funct. Anal. Optim.*, **23** (2002), 1–38. https://doi.org/10.1081/NFA-120003668

41. M. J. Carro, J. Soria, Weighted Lorentz spaces and the Hardy operator, *J. Funct. Anal.*, **112** (1993), 480–494. https://doi.org/10.1006/jfan.1993.1042

42. R. A. Bandaliyev, P. Górka, V. S. Guliyev, Y. Sawano, Relatively compact sets in variable exponent Morrey spaces on metric spaces, *Mediterr. J. Math.*, **18** (2021), 1–23. https://doi.org/10.1007/s00009-021-01828-z

43. J. Xu, Precompact sets in Bochner-Lebesgue spaces with variable exponent, *Math. Notes*, **110** (2021), 932–941. http://dx.doi.org/10.1134/S0001434621110298

44. D. D. Haroske, S. D. Moura, L. Skrzypczak, Some embeddings of Morrey spaces with critical smoothness, *J. Fourier Anal. Appl.*, **26** (2020), 1–31. https://doi.org/10.1007/s00041-020-09758-2

45. K. Stempak, X. X. Tao, Local Morrey and Campanato spaces on quasimetric measure spaces, *J. Funct. Spaces*, **2014** (2014), 172486. https://doi.org/10.1155/2014/172486

46. R. S. Puri, D. Morrey, A Krylov-Arnoldi reduced order modelling framework for efficient, fully coupled, structural-acoustic optimization, *Struct. Multidiscip. Optim.*, **43** (2011), 495–517. https://doi.org/10.1007/s00158-010-0588-5

47. A. L. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE, *Trans. Amer. Math. Soc.*, **355** (2003), 1297–1364. https://doi.org/10.1090/S0002-9947-02-03214-2

48. G. D. Fazio, D. K. Palagachev, M. A. Ragusa, Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, *J. Funct. Anal.*, **166** (1999), 179–196. https://doi.org/10.1006/jfan.1999.3425

49. J. Lang, V. Musil, M. Olšák, L. Pick, Maximal non-compactness of Sobolev embeddings, *J. Geom. Anal.*, **31** (2021), 9406–9431. http://dx.doi.org/10.1007/s12220-020-00522-y

50. Y. Komori, S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282** (2009), 219–231. https://doi.org/10.1002/mana.200610733

51. X. X. Xia, J. Zhou, Homogeneous grand mixed Herz-Morrey spaces and their applications, *Axioms*, **13** (2024), 1–14. https://doi.org/10.3390/axioms13100713

52. M. J. Beltrán-Meneu, J. Bonet, E. Jordá, Generalized Hilbert operators acting on weighted spaces of holomorphic functions with sup-norms, *Banach J. Math. Anal.*, **19** (2025), 10. https://doi.org/10.1007/s43037-024-00395-1

53. C. Bellavita, V. Daskalogiannis, S. Miihkinen, D. Norrbo, G. Stylogiannis, J. Virtanen, Generalized Hilbert matrix operators acting on Bergman spaces, *J. Funct. Anal.*, **288** (2025), 110856. https://doi.org/10.1016/j.jfa.2025.110856

54. N. Apollonio, Cantelli's bounds for generalized tail inequalities, *Axioms*, **14** (2025), 1–18. https://doi.org/10.3390/axioms14010043

55. A. Almeida, P. Hästö, Besov spaces with variable smoothness and integrability, *J. Funct. Anal.*, **258** (2010), 1628–1655. https://doi.org/10.1016/j.jfa.2009.09.012

56. A. Almeida, A. Caetano, Variable exponent Besov-Morrey spaces, *J. Fourier Anal. Appl.*, **26** (2020), 5. https://doi.org/10.1007/s00041-019-09711-y

57. G. A. Chacón, G. R. Chacón, Variable exponent spaces of analytic functions, In: *Advances in complex analysis and applications*, IntechOpen, 2020. https://doi.org/10.5772/intechopen.92617

58. K. Saibi, Variable Besov-Morrey spaces associated with operators, *Mathematics*, **11** (2023), 1–22. https://doi.org/10.3390/math11092038

59. A. Almeida, J. Hasanov, S. Samko, Maximal and potential operators in variable exponent Morrey spaces, *Georgian Math. J.*, **15** (2008), 195–208. http://dx.doi.org/10.1515/GMJ.2008.195

60. V. Kokilashvili, A. Meskhi, Maximal functions and potentials in variable exponent Morrey spaces with non-doubling measure, *Complex Var. Elliptic Equ.*, **55** (2010), 923–936. https://doi.org/10.1080/17476930903276068

61. T. Ohno, Continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent, *Hiroshima Math. J.*, **38** (2008), 363–383. https://doi.org/10.32917/hmj/1233152775

62. V. S. Guliyev, J. J. Hasanov, S. G. Samko, Boundedness of maximal, potential type, and singular integral operators in the generalized variable exponent Morrey type spaces, *J. Math. Sci.*, **170** (2010), 423–443. http://dx.doi.org/10.1007/s10958-010-0095-7

63. P. Hästö, Local-to-global results in variable exponent spaces, *Math. Res. Lett.*, **16** (2009), 263–278.

64. V. S. Guliyev, S. G. Samko, Maximal, potential and singular operators in the generalized variable exponent Morrey spaces on unbounded sets, *J. Math. Sci.*, **193** (2013), 228–248. https://doi.org/10.1007/s10958-013-1449-8

65. W. Afzal, M. Abbas, O. M. Alsalami, Bounds of different integral operators in tensorial Hilbert and variable exponent function spaces, *Mathematics*, **12** (2024), 1–33. https://doi.org/10.3390/math12162464

66. P. Harjulehto, P. Hästö, R. Klén, Generalized Orlicz spaces and related PDE, *Nonlinear Anal.*, **143** (2016), 155–173. http://dx.doi.org/10.1016/j.na.2016.05.002

67. T. Iwaniec, A. Verde, On the operator $L(f) = f \log |f|$, *J. Funct. Anal.*, **169** (1999), 391–420. https://doi.org/10.1006/jfan.1999.3443

68. T. Iwaniec, G. Martin, *Geometric function theory and non-linear analysis*, Clarendon Press, 2001. https://doi.org/10.1093/oso/9780198509295.001.0001

69. D. R. Sahu, D. O'Regan, R. P. Agarwal, *Fixed point theory for Lipschitzian-type mappings with applications*, New York: Springer, 2009. https://doi.org/10.1007/978-0-387-75818-3

70. D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces*, Basel: Birkhäuser, 2013. https://doi.org/10.1007/978-3-0348-0548-3

71. R. Bandaliyev, P. Górka, Relatively compact sets in variable-exponent Lebesgue spaces, *Banach J. Math. Anal.*, **12** (2018), 331–346. http://dx.doi.org/10.1215/17358787-2017-0039

AIMS Press