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Research article

Szász-integral operators linking general-Appell polynomials and approximation

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Abstract: This manuscript is associated with a study of general Appell polynomials. In this research work, we introduced a new sequence of Szász-Integral type of sequence of operators via general-Appell polynomials to discuss approximation properties for Lebesgue integrable functions ($L^p[0,\infty)$). In addition, estimates are studied in view of test functions and central moments. Next, the convergence rate is discussed using the Korovkin theorem and a Voronovskaja-type theorem. Moreover, direct approximation results via modulus of continuity of first and second order, Peetre's K-functional, Lipschitz type space, and the r^{th} order Lipschitz type maximal functions are investigated. In the following section, we present weighted approximation results, and statistical approximation theorems are discussed.

Keywords: modulus of smoothness; mathematical operators; Szász operator; Appell polynomials; approximation algorithms; order of approximation; rate of convergence
Mathematics Subject Classification: 41A25, 41A30, 41A35, 41A36

1. Introduction

The systematic development of operator theory began in the late 19^{th} century. An important aspect of approximation in operator theory is to find simple, computationally tractable approximations that capture the essential properties of more complex functions. These approximations can be used for analysis, simulation, or computation in various applications such as quantum mechanics, signal processing, and control theory. It provides powerful tools for solving problems and analyzing systems involving operators. In the last decade, there has been continued research and development in approximation theory in operator theory, with a focus on more advanced techniques, with applications in data science and machine learning. In approximation theory, Weierstrass (1885) [1] formulated a sophisticated result known as the Weierstrass approximation theorem. Proving this theorem in a more straightforward and comprehensible manner has been the focus of several prominent mathematicians.

Bernstein (1912) [2] developed polynomials referred to as the Bernstein sequence of polynomials in order to give a concise demonstration of the Weierstrass approximation theorem with the aid of binomial distribution as:

$$B_\kappa(\tilde{g};u) = \sum_{\nu=0}^{\kappa} \tilde{g}\left(\frac{\nu}{\kappa}\right)\begin{pmatrix} \kappa \\ \nu \end{pmatrix} u^\nu(1-u)^{\kappa-\nu}, \quad u \in [0,1], \tag{1.1}$$

where $\tilde{g}$ is a continuous and bounded function on $[0,1]$. The sequences of operators in (1.1) restrict the approximation for continuous functions on the bounded interval $[0,1]$. In order to discuss approximation properties on the unbounded interval $[0,\infty)$, Szász [3] provided the modifications to the operators in (1.1), which has a significant role to the evolution of operator theory as follows:

$$\mathcal{S}_\kappa(\tilde{g};u) = e^{-\kappa u}\sum_{\nu=0}^{\infty}\frac{(\kappa u)^\nu}{\nu!}\tilde{g}\Big(\frac{\nu}{\kappa}\Big), \ \kappa \in \mathbb{N}, \tag{1.2}$$

where the real-valued function $\tilde{g} \in C[0,\infty)$ and $u \geq 0$. As given in (1.2), the linear positive operators are solely limited to continuous functional space. Many integral variants of these sequences of operators are obtained in order to approximate the longer space of functions, i.e., the space of Lebesgue measurable functions, e.g., Szász-Durremeyer and Szász-Kantorovich type operators ([4, 5]). Many mathematicians, e.g., *Ö*zger et al. ([6,7]), Acu et al. ([8,9]), Braha et al. ([10,11]), Ayman Mursaleen et al. [12], Khursheed et al. ([13, 14]), Mohiuddine et al. ([15, 16]), Mursaleen et al. ([17, 18]), Khan et al. [19], Nasiruzzaman [20], Rao et al. ([21, 22]), and Wafi et al. [23], provided a number of generalizations for these kinds of sequences to investigate flexibility in approximation properties across several functional spaces.

In 2013, Khan and Raza [24] also introduced the family of 2-variable general-Appell polynomials $A_{p,\nu}(\kappa u, h)$ defined by the generating function as

$$A(t)e^{ut}\phi(y,t) = \sum_{\nu=0}^{\infty} A_{p,\nu}(u,y)\frac{t^\nu}{\nu!},$$

where $A(t)$ can be expressed as $A(t) = \sum_{\nu=0}^{\infty} A_\nu t^\nu$, $A_0 \neq 0$, and $\phi(y,t)$ can be expressed as

$$\phi(y,t) = \sum_{\nu=0}^{\infty}\phi_\nu(y)\frac{t^\nu}{\nu!}, \quad \phi_0(y) \neq 0.$$

Recently, Raza et al. [25] provided a class of sequence of operators $G_{\kappa,A}(.;.)$, $\kappa \in \mathbb{N}$, given by the formula

$$G_{\kappa,A}(\tilde{g};u) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}\tilde{g}\Big(\frac{\nu}{\kappa}\Big), \quad u \in \mathbb{R}_0^+, \tag{1.3}$$

where $A_{p,\nu}$ are the two variable degenerate Appell polynomials (see [25]).

Appell polynomials constitute a class of special functions defined by a characteristic generating function and notable for their closure under composition, forming an abelian group. They find applications in diverse fields like approximation theory, theoretical physics, and related areas. The need to study operators with more complex forms, including Appell polynomials, stems from their ability to model and analyze complex phenomena, their connections to other important mathematical structures, and their utility in solving problems in various scientific disciplines. Also, the beauty of these types of polynomials is that they bridge two fields of research, i.e., special functions and operators theory.

The sequence presented by (1.3) are positive and linear. The basic information about positive linear operators, including their modifications and applications, can be found in [26].

As the operators described in (1.3) are limited for continuous functions only, we present a sequence of positive linear operators to provide approximations in the larger class of functions, i.e., $(L^p[0,\infty))$, $1 \leq p < \infty$ (space of Lebesgue measurable function) endowed with the pth norm $\|f\|_p = (\int |f|^p)^{\frac{1}{p}}$, which are termed as Szász-Integral operators in the context of general Appell Polynomials as

$$\mathcal{H}_\kappa^A(\tilde{g};u) \;=\; \sum_{\nu=0}^{\infty} A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\tilde{g}(y)dy, \qquad for\; u\in\mathbb{R}_0^+,\ \kappa\in\mathbb{N}, \tag{1.4}$$

where

$$A_\nu^p(\kappa u,h) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\frac{A_{p,\nu}(\kappa u,h)}{\nu!} \;\; and \;\; Q_\kappa^\nu(y) = \frac{\kappa^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}y^{\nu+\lambda}e^{-\kappa y},$$

with the Γ (Gamma) function, which is given as

$$\Gamma(m) = \int_0^\infty t^{m-1}e^{-t}dt = (m-1)\Gamma(m-1) = (m-1)!.$$

Lemma 1.1. *The sequence of operators introduced in* (1.4) *are linear.*

Proof. In view of (1.4) and for $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathcal{H}_\kappa^{\mathcal{A}}(\lambda_1\tilde{g}_1+\lambda_2\tilde{g}_2;u) &= \sum_{\nu=0}^{\infty} A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)(\lambda_1\tilde{g}_1+\lambda_2\tilde{g}_2)(y)dy\\
&= \lambda_1\sum_{\nu=0}^{\infty} A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\tilde{g}_1(y)dy\\
&+ \lambda_2\sum_{\nu=0}^{\infty} A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\tilde{g}_2(y)dy\\
&= \lambda_1\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1;u)+\lambda_2\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2;u).
\end{aligned}$$

□

Lemma 1.2. *As discussed by Raza et al. in [25], we can have the following equalities:*

$$\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!}=\tilde{\Lambda}(1)e^{\kappa u}\tilde{\xi}(h,1);$$

$$\sum_{\nu=0}^{\infty}\nu\frac{A_{p,\nu}(\kappa u,h)}{\nu!}=\Big[\kappa u\tilde{\Lambda}(1)\tilde{\xi}(h,1)+\tilde{\Lambda}(1)\tilde{\xi}'(h,1)+\tilde{\Lambda}'(1)\tilde{\xi}(h,1)\Big]e^{\kappa u};$$

$$\begin{aligned}\sum_{\nu=0}^{\infty}\nu^2\frac{A_{p,\nu}(\kappa u,h)}{\nu!}&=\Big[(\kappa^2u^2+\kappa u)\tilde{\Lambda}(1)\tilde{\xi}(h,1)+(2\kappa u+1)[\tilde{\Lambda}(1)\tilde{\xi}'(h,1)\\&+\tilde{\Lambda}'(1)\tilde{\xi}(h,1)]+2\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)+\tilde{\xi}''(h,1)\tilde{\Lambda}(1)\\&+\tilde{\xi}(h,1)\tilde{\Lambda}''(1)\Big]e^{\kappa u}.\end{aligned}$$

Lemma 1.3. *Let* $\tilde{g}_\theta(y)=y^\theta$, $\theta\in\{0,1,2\}$ *be the test functions by* (1.3). *Then, we have*

$$\begin{aligned}\mathcal{H}_\kappa^{\mathcal{A}}(1;u)&=1;\\ \mathcal{H}_\kappa^{\mathcal{A}}(y;u)&=u+\frac{1}{\kappa}\Big[\lambda+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big];\\ \mathcal{H}_\kappa^{\mathcal{A}}(y^2;u)&=u^2+\frac{1}{\kappa^2}\Big[\kappa(2\lambda+4)u+(2\kappa u+2\lambda+4)\Big\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}\\ &+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big\}+2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\lambda^2+3\lambda+2\Big];\end{aligned}$$

for each $u\in\mathbb{R}_0^+$.

Proof. In the direction of (1.4), we have

$$\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_\theta;u)=\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)\tilde{g}_\theta(y)dy.$$

Now, for $\theta=0$,

$$\begin{aligned}\mathcal{H}_\kappa^{\mathcal{A}}(1;u)&=\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)dy\\&=\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\frac{\kappa^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_0^\infty y^{\nu+\lambda}e^{-\kappa y}dy\\&=\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)=1.\end{aligned}$$

For $\theta=1$,

$$\mathcal{H}_\kappa^{\mathcal{A}}(y;u)=\sum_{\nu=0}^{\infty}A_\nu^p(\kappa u,h)\int_0^\infty Q_\kappa^\nu(y)ydy$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{\infty} A_{\nu}^{p}(\kappa u,h)\frac{\kappa^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_{0}^{\infty} y^{\nu+\lambda+1}e^{-\kappa y}dy \\
&= \frac{1}{\kappa}\sum_{\nu=0}^{\infty}\nu A_{\nu}^{p}(\kappa u,h)+\Big(\frac{\lambda+1}{\kappa}\Big)\sum_{\nu=0}^{\infty}A_{\nu}^{p}(\kappa u,h) \\
&= \frac{1}{\kappa}\frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\nu\frac{A_{p,\nu}(\kappa u,h)}{\nu!}+\Big(\frac{\lambda+1}{\kappa}\Big)\frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}(\kappa u,h)}{\nu!} \\
&= u+\frac{1}{\kappa}\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\Big].
\end{aligned}
$$

For $\theta = 2$,

$$
\begin{aligned}
\mathcal{H}_{\kappa}^{\mathcal{A}}(y^2;u) &= \sum_{\nu=0}^{\infty} A_{\nu}^{p}(\kappa u,h)\int_{0}^{\infty} Q_{\kappa}^{\nu}(y)y^2dy \\
&= \sum_{\nu=0}^{\infty} A_{\nu}^{p}(\kappa u,h)\frac{\kappa^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_{0}^{\infty} y^{\nu+\lambda+2}e^{-\kappa y}dy \\
&= \frac{1}{\kappa^2}\sum_{\nu=0}^{\infty}\nu^2 A_{\nu}^{p}(\kappa u,h)+\Big(\frac{2\lambda+3}{\kappa^2}\Big)\sum_{\nu=0}^{\infty}\nu A_{\nu}^{p}(\kappa u,h) \\
&\quad +\Big(\frac{\lambda^2+3\lambda+2}{\kappa^2}\Big)\sum_{\nu=0}^{\infty}A_{\nu}^{p}(\kappa u,h) \\
&= u^2+\frac{1}{\kappa^2}\Big[\kappa(2\lambda+4)u+(2\kappa u+2\lambda+4)\Big\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big\} \\
&\quad +2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\lambda^2+3\lambda+2\Big].
\end{aligned}
$$

Hence, the proof of Lemma 1.3 is completed. □

Lemma 1.4. *Let $\gamma_u^{\theta}(y) = (y-u)^{\theta}$, $\theta = 0, 1, 2$. Then, for the operators* (1.4)*, we have the central moments $\mathcal{H}_{\kappa}^{\mathcal{A}}(\gamma_u^{\theta}(y)(y), u)$ as:*

$$
\begin{aligned}
\mathcal{H}_{\kappa}^{\mathcal{A}}(\gamma_u^{0}(y),u) &= 1; \\
\mathcal{H}_{\kappa}^{\mathcal{A}}(\gamma_u^{1}(y),u) &= \frac{1}{\kappa}\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\Big]; \\
\mathcal{H}_{\kappa}^{\mathcal{A}}(\gamma_u^{2}(y),u) &= \frac{1}{\kappa^2}\Big[2u\kappa+(2\lambda+4)\Big\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big\}+ \\
&\quad +2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\lambda^2+3\lambda+2\Big]; \\
\mathcal{H}_{\kappa}^{\mathcal{A}}(\gamma_u^{4}(y),u) &= o(\frac{1}{\kappa^2});
\end{aligned}
$$

for each $u \in \mathbb{R}_0^+$.

Proof. With the aid of the Lemma 1.3 and the linearity property, we can easily complete the proof of Lemma 1.4. □

In the following sections, we examine the convergence rate of operators and their approximation order. Specifically, we discuss direct results both globally and locally in several spaces. In the final section, we explore some results of the A-statistical approximation in various functional spaces.

2. Uniform rate of convergence and order of approximation

Definition 2.1. *[26] The modulus of smoothness for $\tilde{g} \in C_B[0,\infty)$ is given by*

$$\omega(\tilde{g};\delta) = \sup_{|u_1-u_2|\leq\delta} |\tilde{g}(u_1) - \tilde{g}(u_2)|, \qquad u_1, u_2 \in [0,\infty).$$

Theorem 2.1. *Let $\mathcal{H}_\kappa^{\mathcal{A}}(.;.)$ be operators described in Eq (1.4). Then, on the compact sub interval of $[0,\infty)$, $\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};.) \rightrightarrows \tilde{g}$, for all $\tilde{g} \in C_B[0,\infty)$, where $\rightrightarrows$ denotes uniform convergence.*

Proof. In the Korovkin-type theorem given in [27], which characterizes the uniform convergence for the sequence of positive linear operators, it is enough to note that

$$\lim_{\kappa\to\infty} \mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_\theta; u) = u^\theta, \ \theta = 0, 1, 2,$$

uniformly on subsets of $[0,\infty)$. We can easily establish this result with the help of Lemma 1.3. □

Now, we show that the Voronovskaja-type asymptotic approximation theorem for the $\mathcal{H}_\kappa^{\mathcal{A}}(.;.)$ given in (1.4).

Theorem 2.2. *Let $\tilde{g} \in C_B[0,\infty)$ and $\tilde{g}', \tilde{g}''$ exist at a fixed point $u \in [0,\infty)$. Then, one has*

$$\lim_{\kappa\to\infty} \kappa(\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)) = \tilde{g}'(u)\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda\Big] + u\tilde{g}''(u).$$

Proof. In accordance with Taylor's formula for the function $\tilde{g}$, we get

$$\tilde{g}(y) = \tilde{g}(u) + (y-u)\tilde{g}'(u) + \frac{1}{2}(y-u)^2\tilde{g}''(u) + t(y,u)(y-u)^2, \tag{2.1}$$

where $t(y,u)$ is the Peano remainder and

$$\lim_{y\to u} t(y,u) = 0.$$

Applying operators on both the sides in (2.1), we yield

$$\begin{aligned}(\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)) &= \tilde{g}'(u)\mathcal{H}_\kappa^{\mathcal{A}}((y-u);u) + \frac{1}{2}\tilde{g}''(u)\mathcal{H}_\kappa^{\mathcal{A}}((y-u)^2;u) \\ &+ \mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u).\end{aligned}$$

In view of Lemma 1.4,

$$\begin{aligned}\kappa(\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u)-\tilde{g}(u)) &= \tilde{g}'(u)\Big[\lambda+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big]\\ &+ \frac{\tilde{g}''(u)}{2}\frac{1}{\kappa}\Big[2u\kappa+(2\lambda+4)\Big\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\Big\}+\\ &+ 2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h,1)\tilde{\Lambda}(1)}+\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}+\lambda^2+3\lambda+2\Big]\\ &+ \kappa\mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u).\end{aligned}$$

Applying the limits on both the sides of the above expression, we get

$$\begin{aligned}\lim_{\kappa\to\infty}\kappa(\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u)-\tilde{g}(u)) &= \tilde{g}'(u)\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\Big]+u\tilde{g}''(u)\\ &+ \lim_{\kappa\to\infty}\kappa\mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u).\end{aligned}$$

Now, we need to show that

$$\lim_{\kappa\to\infty}\kappa\mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u)=0.$$

In view of the Cauchy-Schwarz inequality, we calculate last term of the above expression can be written as

$$\kappa\mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u)\leq\sqrt{\mathcal{H}_\kappa^{\mathcal{A}}(t^2(y,u);u)}\sqrt{\kappa^2\mathcal{H}_\kappa^{\mathcal{A}}((y-u)^4;u)}. \tag{2.2}$$

We see that $t^2(u,u)=0$ and $t^2(y,u)\in C_B[0,\infty)$. Thus, we have

$$\lim_{\kappa\to\infty}\mathcal{H}_\kappa^{\mathcal{A}}(t^2(y,u);u)=t^2(u,u)=0. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\lim_{\kappa\to\infty}\kappa\mathcal{H}_\kappa^{\mathcal{A}}(t(y,u)(y-u)^2;u)=0.$$

Hence, the proof is completed. □

According to Shisha et al. [28], the order of convergence relative to the Ditzian-Totik modulus of continuity can easily be proved.

Theorem 2.3. *Consider $\tilde{g}\in C_B[0,\infty)$, and for the operators $\mathcal{H}_\kappa^{\mathcal{A}}(.;.)$ presented in Eq* (1.4)*, we acquire*

$$|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u)-\tilde{g}(u)|\leq 2\omega(\tilde{g};\delta),$$

where $\delta=\sqrt{\mathcal{H}_\kappa^{\mathcal{A}}((y-u)^2;u)}$.

Proof. In accordance with Lemmas 1.3 and 1.4, and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u)-\tilde{g}(u)| &\leq \mathcal{H}_\kappa^{\mathcal{A}}(|\tilde{g}(y)-\tilde{g}(u)|;u) \\ &\leq \mathcal{H}_\kappa^{\mathcal{A}}\left(\left(1+\frac{|y-u|}{\delta}\right)\omega(\tilde{g},\delta);u\right) \\ &\leq \left(1+\frac{1}{\delta}\mathcal{H}_\kappa^{\mathcal{A}}(|y-u|;u)\right)\omega(\tilde{g},\delta) \\ &\leq \left(1+\frac{1}{\delta}\sqrt{\mathcal{H}_\kappa^{\mathcal{A}}((y-u)^2;u)}\right)\omega(\tilde{g},\delta). \end{aligned}$$

By selecting $\delta=\sqrt{\mathcal{H}_\kappa^{\mathcal{A}}((y-u)^2;u)}$, we obtained the desired proof. □

3. Locally approximation results

We recall a few functional spaces and functional relations in this part. As $C_B[0,\infty)$ denotes a real-valued functional space which acquires bounded and continuous functions. Now, Peetre's K-functional [26] is defined as

$$K_2(\tilde{g},\delta)=\inf_{\tilde{h}\in C_B^2[0,\infty)}\left\{\|\tilde{g}-\tilde{h}\|_{C_B[0,\infty)}+\delta\|\tilde{h}''\|_{C_B^2[0,\infty)}\right\},$$

where $C_B^2[0,\infty)=\{\tilde{h}\in C_B[0,\infty):\tilde{h}',\tilde{h}''\in C_B[0,\infty)\}$ is associated with norm $\|\tilde{h}\|=\sup_{0\leq y<\infty}|\tilde{h}(y)|$ and second-order Ditzian-Totik modulus of smoothness is presented by

$$\omega_2(\tilde{g};\sqrt{\delta})=\sup_{0<k\leq\sqrt{\delta}}\ \sup_{y\in[0,\infty)}|\tilde{g}(y+2k)-2\tilde{g}(y+k)+\tilde{g}(y)|.$$

As described in ([26] by DeVore and Lorentz on page no. 177, Theorem 2.4), as

$$K_2(\tilde{g};\delta)\leq\tilde{C}\omega_2(\tilde{g};\sqrt{\delta}), \tag{3.1}$$

where $\tilde{C}$ is an absolute constant. To establish the next result, we consider the auxiliary operator defined as:

$$\widehat{\mathcal{H}}_\kappa^{A}(\tilde{g};u)=\mathcal{H}_\kappa^{A}(\tilde{g};u)+\tilde{g}(u)-\tilde{g}\left(\frac{1}{\kappa}\left[\kappa u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\right]\right), \tag{3.2}$$

where $\tilde{g}\in C_B[0,\infty)$, $u\geq 0$, and $\kappa>1$. From Eq (3.2), we have

$$\widehat{\mathcal{H}}_\kappa^{A}(1;u)=1,\ \widehat{\mathcal{H}}_\kappa^{A}(\gamma_u^1(y);u)=0 \text{ and } |\widehat{\mathcal{H}}_\kappa^{A}(\tilde{g};u)|\leq 3\|\tilde{g}\|. \tag{3.3}$$

Lemma 3.1. *If $u\geq 0$, one has*

$$|\widehat{\mathcal{H}}_\kappa^{A}(\tilde{g};u)-\tilde{g}(u)|\leq\Theta(u)\|\tilde{g}''\|,$$

where $\tilde{g}\in C_B^2[0,\infty)$ and $\Theta(u)=\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2(y);u)+(\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^1(y);u)^2$.

Proof. For $\tilde{g} \in C_B^2[0,\infty)$ and by Taylor expansion, we get

$$\tilde{g}(y) = \tilde{g}(u) + (y-u)\tilde{g}'(u) + \int_u^y (y-v)\tilde{g}''(v)dv. \tag{3.4}$$

Implementing the auxiliary operators $\widehat{\mathcal{H}}_\kappa^A(.;.)$ introduced in Eq (3.2) to Eq (3.4), we get

$$\widehat{\mathcal{H}}_\kappa^A(\tilde{g};u) - \tilde{g}(u) = \tilde{g}'(u)\widehat{\mathcal{H}}_\kappa^A(\gamma_u^1(y);u) + \widehat{\mathcal{H}}_\kappa^A\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big).$$

Using the Eqs (3.3) and (3.4), one yield

$$\begin{aligned}\widehat{\mathcal{H}}_\kappa^A(\tilde{g};u) - \tilde{g}(u) &= \widehat{\mathcal{H}}_\kappa^{\mathcal{A}}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)\\ &= \mathcal{H}_\kappa^A\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)\\ &- \int_u^{\frac{1}{\kappa}\left[\kappa u+\lambda+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\right]} \left(\frac{1}{\kappa}\Big[\kappa u+\lambda+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1\Big]-v\right)\tilde{g}''(v)dv,\end{aligned}$$

$$\begin{aligned}|\widehat{\mathcal{H}}_\kappa^A(\tilde{g};u) - \tilde{g}(u)| &\leq \left|\mathcal{H}_\kappa^{\mathcal{A}}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)\right|\\ &+\left|\int_u^{\frac{1}{\kappa}\left[\kappa u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\right]} \left(\frac{1}{\kappa}\Big[\kappa u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\Big]-v\right)\tilde{g}''(v)dv\right|.\end{aligned} \tag{3.5}$$

Since,

$$\left|\int_u^y (y-v)\tilde{g}''(v)dv\right| \leq (y-u)^2 \parallel \tilde{g}'' \parallel, \tag{3.6}$$

then

$$\left|\int_u^{\frac{1}{\kappa}\left[\kappa u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\right]} \left(\frac{1}{\kappa}\Big[\kappa u+\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)}+\frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}+1+\lambda\Big]-v\right)\tilde{g}''(v)dv\right|$$

$$\leq \left(\frac{1}{\kappa}\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda\Big]\right)^2 \parallel \tilde{g}'' \parallel . \tag{3.7}$$

In accordance with (3.5)–(3.7), we acquire

$$\begin{aligned}|\widehat{\mathcal{H}}_\kappa^A(\tilde{g};u) - \tilde{g}(u)| &\leq \left\{\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2(y);u) + \left(\frac{1}{\kappa}\Big[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda\Big]\right)^2\right\}\|\tilde{g}''\| \\ &= \Theta(u)\|\tilde{g}''\|.\end{aligned}$$

This proves the required result. □

Theorem 3.1. *For $\tilde{g} \in C_B[0,\infty)$, there corresponds a non-negative constant $\tilde{C} > 0$ such that*

$$|\ \mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)\ | \leq 4\tilde{C}\omega_2(\tilde{g};\frac{1}{2}\sqrt{\Theta(u)}) + \omega(\tilde{g};\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^1(y);u),$$

where $\Theta(u)$ is given by in Lemma 3.1.

Proof. Let $\tilde{g} \in C_B[0,\infty)$ and $\tilde{h} \in C_B^2[0,\infty)$. Then, by the definition of $\widehat{\mathcal{H}}_\kappa^A(.;.)$ given in (3.2), we get

$$\begin{aligned}|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| &\leq |\widehat{\mathcal{H}}_\kappa^A(\tilde{g}-\tilde{h};u)| + |(\tilde{g}-\tilde{h})(u)| + |\widehat{\mathcal{H}}_\kappa^A(\tilde{h};u) - \tilde{h}(u)| \\ &+ \left|\tilde{g}\Big(\frac{1}{\kappa}\Big[\kappa u + \lambda + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1\Big]\Big) - \tilde{g}(u)\right|.\end{aligned}$$

According to Lemma 3.1 and the inequalities mentioned in Eq (3.3), we acquire

$$\begin{aligned}|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| &\leq 4\|\tilde{g}-\tilde{h}\| + |\widehat{\mathcal{H}}_\kappa^A(\tilde{h};u) - \tilde{h}(u)| \\ &+ \left|\tilde{g}\left(\frac{1}{\kappa}\Big[\kappa u + \lambda + \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1\Big]\right) - \tilde{g}(u)\right| \\ &\leq 4\|\tilde{g}-\tilde{h}\| + \Theta(u)\|\tilde{h}''\| + \omega\Big(\tilde{g};\mathcal{H}_\kappa^{\mathcal{A}}(y-u);u)\Big).\end{aligned}$$

Using Eq (3.1), we established the required result. □

Further, we address the next result in the Lipschitz type space (see [29]):

$$Lip_{\tilde{M}}^{\zeta_1,\zeta_2}(\eta) := \Big\{\tilde{g} \in C_B[0,\infty) : |\tilde{g}(t) - \tilde{g}(u)| \leq \tilde{M}\frac{|t-u|^\eta}{(t+\zeta_1 u+\zeta_2 u^2)^{\frac{\eta}{2}}} : u, t \in (0,\infty)\Big\},$$

where $\tilde{M} > 0$, $0 < \eta \leq 1$ and $\zeta_1, \zeta_2 > 0$.

Theorem 3.2. *Considering the sequence of linear positive operators in (1.4) and $\tilde{g} \in Lip_M^{\zeta_1,\zeta_2}(\eta)$, one obtains*

$$|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \tilde{M}\left(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2}\right)^{\frac{\eta}{2}}, \tag{3.8}$$

where $0 < \eta < 2$, $\zeta_1, \zeta_2 \in (0,\infty)$ and $\lambda(u) = \mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2(y);u)$.

Proof. For $0 < \eta < 2$, $u \geq 0$, and in accordance with Hölder's inequality by selecting $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$, we obtain

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq (|\mathcal{H}_{\kappa}^{\mathcal{A}}(|\tilde{g}(y) - \tilde{g}(u)|^{\frac{2}{\eta}};u))^{\frac{\eta}{2}}$$
$$\leq \tilde{M}\left(\mathcal{H}_{\kappa}^{\mathcal{A}}\left(\frac{|y-u|^2}{(y+\zeta_1 u+\zeta_2 u^2)};u\right)\right)^{\frac{\eta}{2}}.$$

Since $\dfrac{1}{y+\zeta_1 u+\zeta_2 u^2} < \dfrac{1}{\zeta_1 u+\zeta_2 u^2}$, for all $u \in (0,\infty)$, we get

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \tilde{M}\left(\frac{\mathcal{H}_{\kappa}^{\mathcal{A}}(|y-u|^2;u)}{\zeta_1 u+\zeta_2 u^2}\right)^{\frac{\eta}{2}} \leq \tilde{M}\Big(\frac{\lambda(u)}{\zeta_1 u+\zeta_2 u^2}\Big)^{\frac{\eta}{2}}.$$

Thus, we have the desired result. □

Further, we address the local approximation in terms of the r^{th}-order modulus of smoothness, followed by the Lipschitz-type function introduced by Lenze [30] as

$$\widetilde{\omega}_r(\tilde{g};u) = \sup_{y\neq u, y\in(0,\infty)} \frac{|\tilde{g}(y)-\tilde{g}(u)|}{|y-u|^r},\ u \in [0,\infty) \text{ and } r \in (0,1]. \tag{3.9}$$

Theorem 3.3. *Assume $\tilde{g} \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for every $u \in [0,\infty)$, we have*

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \widetilde{\omega}_r(\tilde{g};u)\big(\lambda(u)\big)^{\frac{r}{2}},$$

where $\lambda(u)$ is the same as in the above Theorem 3.2.

Proof. It can be observed that

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \mathcal{H}_{\kappa}^{\mathcal{A}}(|\tilde{g}(y)-\tilde{g}(u)|;u).$$

Using Eq (3.9), one gets

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \widetilde{\omega}_r(\tilde{g};u)\mathcal{H}_{\kappa}^{\mathcal{A}}(|y-u|^r;u).$$

Then, by employing Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we obtain

$$|\mathcal{H}_{\kappa}^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)| \leq \widetilde{\omega}_r(\tilde{g};u)(\mathcal{H}_{\kappa}^{\mathcal{A}}(|y-u|^2;u))^{\frac{r}{2}}.$$

Thus, we conclude the proof. □

4. Global approximation properties

Consider $\nu(u) = 1+u^2, 0 \leq u < \infty$ as the weight function. Then, $B_\nu[0,\infty) = \{\tilde{g}(u) : |\tilde{g}(u)| \leq \tilde{M}_{\tilde{g}}(1+u^2)$, here the constant $\tilde{M}_{\tilde{g}}$ depends on $\tilde{g}$ and $C_\nu[0,\infty)$ represents the continuous functional space in $B_\nu[0,\infty)$ along with the norm $\|\tilde{g}\|_\nu = \sup\limits_{u\in[0,\infty)} \frac{|\tilde{g}(u)|}{\nu(u)}$ and $C_\nu^{\tilde{k}}[0,\infty) = \{\tilde{g} \in C_\nu[0,\infty) : \lim\limits_{u\to\infty} \frac{\tilde{g}(u)}{\nu(u)} = \tilde{k}$, where the constant $\tilde{k}$ depends on $\tilde{g}$.

If $\tilde{g}$ is a function defined on $[0, b]$ where $b > 0$, then the modulus of continuity is given by

$$\omega_b(\tilde{g}, \delta) = \sup_{|y-u|\leq\delta} \sup_{u,y\in[0,b]} |\tilde{g}(y) - \tilde{g}(u)|. \tag{4.1}$$

It is straightforward to observe that for $\tilde{g} \in C_\nu[0, \infty)$, the modulus of continuity defined in Eq (4.1) tends to zero.

Theorem 4.1. *Let $\tilde{g} \in C_\nu[0, \infty)$ and $\omega_{b+1}(\tilde{g}; \delta)$ denote the modulus of smoothness defined on $[0, b+1] \subset [0, \infty)$. Then, for $y \in [0, b]$, we obtain*

$$||\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u) - \tilde{g}||_{C[0,b]} \leq 4\tilde{M}_{\tilde{g}}(1 + b^2)\Delta_s(b) + 2\omega_{b+1}(\tilde{g}; \sqrt{\Delta_s(b)}),$$

where $\Delta_s(b) = \max_{u\in[0,b]} \mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2; u)$.

Proof. From [31], for any $u \in [0, b]$ and $y \in [0, \infty)$, we have

$$|\tilde{g}(y) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1 + b^2) + \left(1 + \frac{|y - u|}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta).$$

Implementing the operator $\mathcal{H}_\kappa^{\mathcal{A}}(.; .)$ on both the sides, we acquire

$$\begin{aligned} |\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u) - \tilde{g}(u)| \leq\ & 4\tilde{M}_{\tilde{g}}(1 + b^2)\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2; u) \\ & + \left(1 + \frac{\mathcal{H}_\kappa^{\mathcal{A}}(|y - u|; u)}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta). \end{aligned}$$

Now, in accordance with Lemma 1.4 and $x \in [0, b]$, one has

$$|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1 + b^2)\Delta_s(b) + \left(1 + \frac{\sqrt{\Delta_s(b)}}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta).$$

By selecting $\delta = \sqrt{\Delta_s(b)}$, the desired result can easily be obtained. □

Remark 4.1. *In this article, we employ the test function defined by $\tilde{g}_\theta(y) = y^\theta$, $\theta \in \{0, 1, 2\}$.*

Theorem 4.2. *([32, 33]) Assume that the sequence of linear positive operators $(L_\kappa)_{\kappa\geq1}$ mapping from $C_\nu[0, \infty)$ to $B_\nu[0, \infty)$ satisfies the conditions*

$$\lim_{\kappa\to\infty} ||L_\kappa(\tilde{g}_\theta; .) - \tilde{g}_\theta||_\nu = 0, \quad \textit{where } \theta = 0, 1, 2,$$

and thus, for $\tilde{g} \in C_\nu^{\tilde{k}}[0, \infty)$, we get

$$\lim_{\kappa\to\infty} ||L_\kappa(\tilde{g}; .) - \tilde{g}||_\nu = 0.$$

Theorem 4.3. *Let $\tilde{g} \in C_\nu^{\tilde{k}}[0, \infty)$. Then, we obtain*

$$\lim_{\kappa\to\infty} ||\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; .) - \tilde{g}||_\nu = 0.$$

Proof. To prove the result of Theorem 4.3, it is enough to verify that

$$\lim_{\kappa\to\infty} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_\theta;.) - \tilde{g}_\theta\|_\nu = 0, for\ \theta = 0,1,2.$$

Considering the Lemma 1.3, one can see $\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_0;.) - 1\|_\nu = 0$, where $\kappa \to \infty$, and also

$$\begin{aligned}\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1;.) - \tilde{g}_1\|_\nu &= \sup_{u\in[0,\infty)} \frac{1}{\nu(u)}\left|\frac{1}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda\right]\right| \\ &= \frac{1}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda\right] \sup_{u\in[0,\infty)} \frac{1}{1+u^2}.\end{aligned}$$

For $\kappa \to \infty$, we get $\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1;.) - \tilde{g}_1\|_\nu \to 0$.
Also,

$$\begin{aligned}\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2;.) - \tilde{g}_2\|_\nu \leq& \left(\frac{2}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + \lambda + 2\right]\right) \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\ &+ \left(\frac{1}{\kappa^2}\left[(2\lambda+4)\left\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right\} + \left\{2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)}\right.\right.\right. \\ &+ \left.\left.\left.\frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}\right\} + \lambda^2 + 3\lambda + 2\right]\right) \sup_{y\in[0,\infty)} \frac{1}{1+u^2}.\end{aligned}$$

Which implies $\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2;.) - \tilde{g}_2\|_\nu \to 0$ as $\kappa \to \infty$. Thus, we conclude the proof of the Theorem 4.3. □

Corollary 4.1. *Let* $\tilde{g} \in C_\nu^{\tilde{k}}[0,\infty)$ *and* $\zeta > 0$. *Then,*

$$\lim_{\kappa\to\infty} \sup_{u\in[0,\infty)} \frac{|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)|}{(1+u^2)^{1+\zeta}} = 0.$$

Proof. From the result of above Theorem 4.3,

$$\lim_{\kappa\to\infty} \sup_{u\in[0,\infty)} \frac{|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)|}{1+u^2} = 0.$$

Then, the inequality

$$\lim_{\kappa\to\infty} \sup_{u\in[0,\infty)} \frac{|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};u) - \tilde{g}(u)|}{(1+u^2)^{1+\zeta}} = 0,$$

is immediate, for $\zeta > 0$. □

5. A-statistical approximation

We revisit some notation from [34]. Suppose that $B = (b_{\kappa\mu})$ is an infinite, non-negative summability matrix. A sequence $u := (u_\mu)$ is A-statistically convergent to L, denoted as $st_B - \lim u = L$, if for each $\epsilon > 0$,

$$\lim_\kappa \sum_{\mu:|u_\mu - L|\geq\epsilon} b_{\kappa\mu} = 0.$$

Let $q = (q_\kappa)$ be a sequence such that the following assertions are true:

$$st_B - \lim_{\kappa} q_\kappa = 1 \text{ and } st_B - \lim_{\kappa} q_\kappa^\kappa = b,\ \ 0 \le b < 1. \tag{5.1}$$

In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit.

Theorem 5.1. *Consider $B = (b_{\kappa\mu})$ be a non-negative regular summability matrix and the sequence $q = (q_\kappa)$ along with condition (5.1), $q_\kappa \in (0, 1)$, $\kappa \in \mathbb{N}$. Then, for each $\tilde{g} \in C_\nu^0[0, \infty)$, $st_B - \lim_{\kappa} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; .) - \tilde{g}\|_\nu = 0$.*

Proof. In accordance with Lemma 1.3, one has

$$st_B - \lim_{\kappa} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_0; .) - \tilde{g}_0\|_\nu = 0,$$

and

$$\begin{aligned} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1; .) - \tilde{g}_1\|_\nu &= \sup_{u\in[0,\infty)} \frac{1}{1+u^2} \left| \frac{1}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda \right] \right| \\ &= \frac{1}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda \right] \sup_{u\in[0,\infty)} \frac{1}{1+u^2}. \end{aligned}$$

Now,

$$\begin{aligned} \tilde{K}_1 &:= \left\{ \kappa : \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1; .) - \tilde{g}_1\| \ge \epsilon \right\}, \\ \tilde{K}_2 &:= \left\{ \kappa : \frac{1}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 + \lambda \right] \ge \epsilon \right\}, \end{aligned}$$

which implies that $\tilde{K}_1 \subseteq \tilde{K}_2$, this showing that $\sum_{\mu\in\tilde{K}_1} b_{\kappa\mu} \le \sum_{\mu\in\tilde{K}_2} b_{\kappa\mu}$. Therefore, we get

$$st_B - \lim_{\kappa} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1; .) - \tilde{g}_1\|_\nu = 0. \tag{5.2}$$

Now, by using Lemma 1.3, we have

$$\begin{aligned} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2; .) - \tilde{g}_2\|_\nu \le \sup_{u\in[0,\infty)} \frac{1}{\nu(u)} \Bigg| &\left(\frac{2}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + \lambda + 2 \right] u \right) \\ &+ \left(\frac{1}{\kappa^2}\left[(2\lambda+4)\left\{ \frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right\} + \left\{ 2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)} \right. \right. \right. \\ &\left. \left. \left. + \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right\} + \lambda^2 + 3\lambda + 2 \right] \right) \Bigg|. \end{aligned}$$

For a given $\varepsilon > 0$, we have the following sets:

$$\tilde{M}_1 := \left\{ \kappa : \left\| \mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2; .) - \tilde{g}_2 \right\|_\nu \ge \epsilon \right\},$$

$$\tilde{M}_2 := \left\{\kappa : \frac{2}{\kappa}\left[\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + \lambda + 2\right] \geq \frac{\epsilon}{2}\right\},$$

$$\tilde{M}_3 := \left\{\kappa : \frac{1}{\kappa^2}\left[(2\lambda+4)\left\{\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right\} + \left\{2\frac{\tilde{\xi}'(h,1)\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)\tilde{\xi}(h,1)} + \frac{\tilde{\xi}''(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)}\right\} + \lambda^2 + 3\lambda + 2\right] \geq \frac{\epsilon}{2}\right\}.$$

It can be observed that $\tilde{M}_1 \subseteq \tilde{M}_2 \bigcup \tilde{M}_3$. Therefore, we acquire

$$\sum_{\mu\in\tilde{M}_1} b_{\kappa\mu} \leq \sum_{\mu\in\tilde{M}_2} b_{\kappa\mu} + \sum_{\mu\in\tilde{M}_3} b_{\kappa\mu}.$$

As $\kappa \to \infty$, we have

$$st_B - \lim_{\kappa} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_2; u) - \tilde{g}_2\|_\nu = 0. \tag{5.3}$$

Thus, we concludes the proof of the Theorem 5.1. □

Next, we will examine the convergence rate of A-statistical approximation with respect to Peetre's K-functional for the operators $\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u)$.

The Peetre's K-functional of the function $f \in C_B[0,\infty)$ is defined by

$$K(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B^2[0,\infty)}\right\},$$

where $\delta > 0$ and

$$C_B^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\},$$

endowed with the norm

$$\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}.$$

Theorem 5.2. *Let* $\tilde{g} \in C_B^2[0,\infty)$*. Then,*

$$st_B - \lim_{\kappa} \|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0,\infty)} = 0.$$

Proof. Considering Taylor's result, we have

$$\tilde{g}(y) = \tilde{g}(u) + \tilde{g}'(u)(y-u) + \frac{1}{2}\tilde{g}''(\eta)(y-u)^2,$$

where $y \leq \eta \leq u$. Operating $\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u)$, on both sides in above equation, one get

$$\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; u) - \tilde{g}(u) = \tilde{g}'(u)\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^1(y); u) + \frac{1}{2}\tilde{g}''(\eta)\mathcal{H}_\kappa^{\mathcal{A}}(\gamma_u^2(y); u),$$

which yields that

$$\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0,\infty)} \leq \|\tilde{g}'\|_{C_B[0,\infty)}\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1-, .)\|_{C_B[0,\infty)}$$

$$+\frac{1}{2}\|\tilde{g}''\|_{C_B[0,\infty)}\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g}_1-,.)^2\|_{C_B[0,\infty)}$$
$$=\tilde{W}_1+\tilde{W}_2, \quad say. \tag{5.4}$$

Based on Eqs (5.2) and (5.3), it follows that

$$\lim_{\kappa}\sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} b_{\kappa\mu} = 0,$$
$$\lim_{\kappa}\sum_{\mu\in\mathbb{N}:\tilde{W}_2\geq\frac{\epsilon}{2}} b_{\kappa\mu} = 0.$$

From Eq (5.4), we have

$$\lim_{\kappa}\sum_{\mu\in\mathbb{N}:\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};\cdot)-\tilde{g}\|_{C_B[0,\infty)}\geq\epsilon} b_{\kappa\mu}\leq\lim_{\kappa}\sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} b_{\kappa\mu}+\lim_{\kappa}\sum_{\mu\in N:\tilde{W}_2\geq\frac{\epsilon}{2}} b_{\kappa\mu}.$$

Thus, $st_B-\lim_{\kappa}\|\mathcal{H}_\kappa^{\mathcal{A}}(\tilde{g};\cdot)-\tilde{g}\|_{C_B[0,\infty)}\to 0.$ as $\kappa\to\infty$.

Hence, we arrive the proof. □

6. Conclusions

In this article, we introduce a sequence of positive linear operators using generalized Appell polynomials in integral form. These operators are designed to approximate functions defined on a Lebesgue measurable space and are known as Szász-Integral type operators introduced in (1.4). Moreover, we derive estimates crucial for establishing the rate of convergence and accuracy of approximation. Further, we explore various aspects of approximation, including local and global results, as well as A-statistical approximation, utilizing these operators to obtain enhanced approximations across different functional spaces.

Author contributions

Nadeem Rao: Conceptualization, writing - original draft; Mohammad Farid: Writing - review and editing, methodology; Nand Kishor Jha: Review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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