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Research article

Application of fixed point result to the boundary value problem using the M-type generalized contraction condition for best proximity point considerations

Rajagopalan Ramaswamy[1], **Penumarthy Parvateesam Murthy**[2], **Pushplata Sahu**[2,3], **Rayan Abdulrahman Alkhowaiter**[1], **Ola Ashour Abdelnaby**[1] **and Gunaseelan Mani**[4,*]

[1] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia

[2] Department of Mathematics, Guru Ghasidas Vishwavidyalay, Bilaspur, Chhattisgarh, 495009, India

[3] Department of Mathematics, Pt Sundarlal Sharma (Open) University, Bilaspur 495009, (C.G), India

[4] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India

* **Correspondence:** Email: mathsguna@yahoo.com.

Abstract: This paper serves a dual purpose: to introduce a contraction condition and to demonstrate its application. We established a new framework for obtaining best proximity points in complete metric spaces, extending and generalizing several existing fixed point theorems. From this foundational result, multiple corollaries were derived, providing broader applicability in various mathematical settings. To validate the theoretical development, we applied our results to boundary value problems and dynamic market equilibrium models, illustrating both the mathematical robustness and real-world relevance of the proposed method.

Keywords: complete metric space; M-type generalized contraction; P-property; best proximity point
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1. Introduction

Altun et al. [1] introduced the concept of p-proximal contractions and p-proximal contractive mappings on metric spaces. Amini-Harandi [2] proved the best proximity point theorem for non-self proximal generalized contractions. Aydi et al. [3] generalized $\mathcal{MT}$- cyclic contraction mappings with respect to an auxiliary function φ and investigated the existence of a best proximity point of such mappings in the setting of metric spaces. Basha [4] proposed a best proximity point theorem for non-self-proximal contractions. Debnath [5] discussed existing theories as well as recent developments in the area of metric fixed point theory. Hussain et al. [6] proved the existence of best proximity point results in the context of b-metric spaces. Kostić et al. [7] introduced a special type of w-distance which generalizes some recent best proximity point results involving simulation functions. Nesić [8] proved fixed point theorems on metric space. Raj [9] proved a best proximity point theorem for weakly contractive non-self-mappings. Samet [10] proved the existence and uniqueness of best proximity points for two classes of non-self-contractive mappings. Todorčević [11] proved fixed point theorems on harmonic quasiconformal mappings and hyperbolic-type metrics.

To understand the need of proximal point,we will start from fixed points.

By a fixed point, we shall mean a point that is invariant under any transformation. Let Q be a nonempty set and let $\mathsf{H} : \mathsf{Q} \to \mathsf{Q}$. If $\mathsf{H}(\theta) = \theta$, then θ is a fixed point of H. To understand the concept, we shall consider a quadratic equation $\theta^2 - 8\theta + 15 = 0$. If we write $\mathsf{H}(\theta) = \frac{\theta^2+15}{8}$, then $\theta = 3$ and $\theta = 5$ are two fixed points of H.

However, an equation of the form $\mathsf{H}\theta = \theta$ does not necessarily have a solution if H happens to be a non-self-mapping. In such cases, fixed-point theorems do not apply directly, and a fixed point might not exist within the domain. In this case, we get an appropriate solution. Therefore, we can say that when we can not get the fixed point of a function, then we need a best proximity point ($\mathcal{BPP}$) so that we can get an approximate solution.

Let H be a mapping from S to T which is non-self-mapping, where S, and T are closed subsets of a metric space ($\mathcal{MS}$) $(\mathsf{Q}, \wp)$. Clearly, the set of fixed point is empty. In this case, we find an element θ in the domain S it is closest to $\mathsf{H}\theta$, which is in the co-domain T. We will get the solution of this type of problem in the $\mathcal{BPP}$ analysis as applicable.

If the distance of θ and $\mathsf{H}\theta$ is equal to the distance of the set S and T then the point θ in S is called the $\mathcal{BPP}$ of H.

Example 1.1. Let $\mathsf{Q} = \mathbb{R}^2$ be a complete $\mathcal{MS}$ with the usual metric $\wp$. Let $\mathsf{S} = \{(1, \theta) : \theta \in \mathbb{R}\}$ and $\mathsf{T} = \{(0, \theta) : \theta \in \mathbb{R}\}$ be closed subsets of Q and we have $\wp(\mathsf{S}, \mathsf{T}) = 1$. Let $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ be defined by $\mathsf{H}((1, \theta)) = (0, \theta)$. Therefore, we have no fixed point but we do have a $\mathcal{BPP}$ $(1, 0)$.

Alam et al. [12] established a fixed point for a class of interpolative contractions by introducing novel forms of interpolative generalized Gupta-Saxena-Reich-type contractions and generalized Gupta-Saxena-Kannan-type contractions within the setting of M_v^b-metric spaces. Dong et al. [13] studied a class of second-order geometric quasilinear hyperbolic partial differential equations. Shcheglov et al. [14] established the parameter identification problem in a system of time-dependent quasi-linear partial differential equations. Ye and Hofmann [15] obtained non-negative stable approximate solutions to ill-posed linear operator equations in a Hilbert space setting, and developed two novel non-negativity preserving iterative regularization methods. Lin et al. [16] developed a new numerical approach to reconstruct the locations of the discontinuity of the conduction coefficient in elliptic partial differential equations. Baravdish et al. [17] introduced a new image denoising model: the damped flow, which is a second-order nonlinear evolution equation associated with a class of energy functionals of an image. Sen and Karapinar [18] investigated some convergence properties of quasi-cyclic and cyclic Jungck modified TS-iterative schemes in complete metric spaces and Banach spaces. Naraghirad [19] introduced the notion of the BUC property, which extends the notion of the UC property. Abkar and Norouzian [20] introduced the notion of quasi-cyclic-noncyclic pair and its relevant new notion of coincidence quasi-best proximity points in a convex metric space.

2. Preliminaries

Now, we recall a few notations for the better understanding of best proximity points theory. Here $\mathcal{MS}$ means metric space, $\mathcal{BPP}$ means best proximity point, P-Pro means P-property, and UFP means unique fixed point.

Let $(\mathsf{Q}, \wp)$ be a $\mathcal{MS}$ and S, T be two non-empty subsets of a complete $\mathcal{MS}$ $(\mathsf{Q}, \wp)$.

$$\begin{aligned}\wp(\mathsf{S},\mathsf{T}) =& \inf\{\wp(\theta,\ell) : \theta \in \mathsf{S}, \ell \in \mathsf{T}\};\\ \mathsf{S}_0 =& \{\theta \in \mathsf{S} : \wp(\theta,\ell) = \wp(\mathsf{S},\mathsf{T}), \text{ for some } \ell \in \mathsf{T}\};\\ \mathsf{T}_0 =& \{\ell \in \mathsf{T} : \wp(\theta,\ell) = \wp(\mathsf{S},\mathsf{T}), \text{ for some } \theta \in \mathsf{S}\}.\end{aligned}$$

Definition 2.1. Let $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ be a non-self- mapping and an element $\theta \in \mathsf{S}$ is known to be a $\mathcal{BPP}$ of H iff it satisfies the condition

$$\wp(\theta, \mathsf{H}\theta) = \wp(\mathsf{S}, \mathsf{T}).$$

Definition 2.2. [9] Let $(\mathsf{Q}, \wp)$ be a $\mathcal{MS}$ and S, T be non-void subsets of Q with $\mathsf{S}_0 \neq \emptyset$. Then (S, T) is referred to be P-Pro if

$$\left.\begin{aligned}\wp(\theta_1,\ell_1) = \wp(\mathsf{S},\mathsf{T})\\ \wp(\theta_2,\ell_2) = \wp(\mathsf{S},\mathsf{T})\end{aligned}\right\} \text{ implies } \wp(\theta_1,\theta_2) = \wp(\ell_1,\ell_2),$$

where $\theta_1, \theta_2 \in \mathsf{S}$ and $\ell_1, \ell_2 \in \mathsf{T}$.

3. Main result

Now, we are going to introduce a new type of contraction condition called the M-type generalized contraction for obtaining $\mathcal{BPP}$s in a $\mathcal{MS}$ $(\mathsf{Q}, \wp)$ which follows:

Definition 3.1. Let $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ be a non-self-mapping, and it is said to be a M-type generalized contraction if it satisfies the conditions

$$\wp(\theta_1, \mathsf{H}\ell_1) = \wp(\mathsf{S},\mathsf{T}) \;\; \textit{and} \;\; \wp(\theta_2, \mathsf{H}\ell_2) = \wp(\mathsf{S},\mathsf{T})$$

implying

$$\begin{aligned}[1 + \vartheta\wp(\ell_1,\ell_2)]\,\wp(\theta_1,\theta_2) \;\leq\;& \vartheta\,[\wp(\ell_1,\theta_1)\wp(\ell_2,\theta_2) + \wp(\ell_2,\theta_1)\wp(\ell_1,\theta_2)]\\ &+ \mathsf{g}\max\left\{\wp(\ell_1,\ell_2), \wp(\ell_1,\theta_1), \wp(\ell_2,\theta_2), \frac{1}{2}\,[\wp(\ell_2,\theta_1) + \wp(\ell_1,\theta_2)]\right\},\end{aligned}$$

where $\theta_1, \theta_2, \ell_1, \ell_2 \in \mathsf{S}$ with $\theta_1 \neq \theta_2 (\wp(\theta_1,\theta_2) > 0)$, $\vartheta \geq 0$, $0 < \mathsf{g} < 1$.

Example 3.1. Let $\mathsf{S} = [0,2]$, $\mathsf{T} = [1]$, and define a non-self-mapping $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ as

$$\mathsf{H}(\theta) = 1 \in [1,3], \quad \forall \quad \theta \in [0,2].$$

Also define the metric $\wp(\theta,\ell) = |\theta - \ell|$.

Take $\theta_1 = 0, \theta_2 = 2, \ell_1 = 2,$ *and* $\ell_2 = 0$.

$$\wp(\theta_1, \mathsf{H}\ell_1) = \wp(\mathsf{S}, \mathsf{T}) = 1 \ \ and \ \ \wp(\theta_2, \mathsf{H}\ell_2) = \wp(\mathsf{S}, \mathsf{T}) = 1.$$

Choose $\vartheta = 1$ and $\mathsf{g} = 1$.

$$\begin{aligned}[1 + \vartheta\wp(\ell_1, \ell_2)]\,\wp(\theta_1, \theta_2) \ &\leq \ \vartheta\,[\wp(\ell_1, \theta_1)\wp(\ell_2, \theta_2) + \wp(\ell_2, \theta_1)\wp(\ell_1, \theta_2)] \\ &\quad + \mathsf{g}\max\left\{\wp(\ell_1, \ell_2), \wp(\ell_1, \theta_1), \wp(\ell_2, \theta_2), \frac{1}{2}\,[\wp(\ell_2, \theta_1) + \wp(\ell_1, \theta_2)]\right\}\end{aligned}$$

where $\theta_1, \theta_2, \ell_1, \ell_2 \in \mathsf{S}$ with $\theta_1 \neq \theta_2, \vartheta \geq 0,\ 0 < \mathsf{g} < 1$.

Now, we are ready to establish a proximal point theorem using Definition (3.1).

Theorem 3.1. Let S and T be closed subsets of a complete $\mathcal{MS}$ $(\mathsf{Q}, \wp)$ implying that S_0 and T_0 both are non-empty. Assume that $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ satisfies the following:

(i) H is continuous and satisfies the M-type generalized contraction;

(ii) $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$;

(iii) the pair (S, T) satisfies the P-Pro.

Then, there exists a unique element θ in S implying that

$$\wp(\theta, \mathsf{H}\theta) = \wp(\mathsf{S}, \mathsf{T}).$$

Proof. Let $\theta_0 \in \mathsf{S}_0$. Since $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$, we have $\mathsf{H}\theta_0 \in \mathsf{T}_0$. By the definition of T_0, we can find an element $\theta_1 \in \mathsf{S}_0$ implying that

$$\wp(\theta_1, \mathsf{H}\theta_0) = \wp(\mathsf{S}, \mathsf{T}).$$

Again, since $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$, guaranteeing that we can find an element $\theta_2 \in \mathsf{S}_0$ implying that

$$\wp(\theta_2, \mathsf{H}\theta_1) = \wp(\mathsf{S}, \mathsf{T}).$$

Proceeding in this way, we can construct a sequence $\{\theta_{\mathsf{v}}\}$ in S_0 implying that

$$\wp(\theta_{\mathsf{v}+1}, \mathsf{H}\theta_{\mathsf{v}}) = \wp(\mathsf{S}, \mathsf{T}), \text{ for all } \mathsf{v} \geq 0. \tag{3.1}$$

Since (S, T) satisfies the P-Pro, we have

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) = \wp(\mathsf{H}\theta_{\mathsf{v}-1}, \mathsf{H}\theta_{\mathsf{v}}), \text{ for all } \mathsf{v} \in \mathbb{N}. \tag{3.2}$$

If for some $\mathsf{v}_0 \in \mathbb{N}$, $\wp(\theta_{\mathsf{v}_0}, \theta_{\mathsf{v}_0+1}) = 0$, consequently $\wp(\mathsf{H}\theta_{\mathsf{v}_0-1}, \mathsf{H}\theta_{\mathsf{v}_0}) = 0$ implying $\mathsf{H}\theta_{\mathsf{v}_0-1} = \mathsf{H}\theta_{\mathsf{v}_0}$ implies $\wp(\theta_{\mathsf{v}_0}, \mathsf{H}\theta_{\mathsf{v}_0}) = \wp(\mathsf{S}, \mathsf{T})$. Thus θ_{v_0} is the $\mathcal{BPP}$ of H.

Let us consider $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) > 0$, for all $\mathsf{v} \geq 0$. By the M-type generalized contraction, we have

$$\begin{aligned}[1 + \vartheta\wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}})]\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) &\leq \vartheta\Big[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}}) \\ &\quad \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}-1})\Big] + \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \\ &\quad \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}), \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}), \frac{1}{2}\Big[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}})+ \\ &\quad \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}-1})\Big]\Big\}, \\ &\leq \vartheta[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}})] + \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \\ &\quad \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}), \frac{1}{2}\left[\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\right]\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) &\leq \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}), \\ &\quad \frac{1}{2}\left[\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\right]\Big\}, \end{aligned} \tag{3.3}$$

for all $\mathsf{v} \in \mathbb{N}$.

If $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}) < \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}})$, from (3.3), we have

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}).$$

But $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1})$ is never less than 0. So, $(1 - \mathsf{g}) \leq 0$. But g is in $(0, 1)$. So, we get a contradiction.

Now, if $\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) < \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})$, from (3.3), we have

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}).$$

Similarly, we have

$$\wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-2}, \theta_{\mathsf{v}-1}).$$

Proceeding in this way, we have

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}) \leq \mathsf{g}^2\ \wp(\theta_{\mathsf{v}-2}, \theta_{\mathsf{v}-1}) \leq ... \tag{3.4}$$

$$;\ \mathsf{g}^{\mathsf{v}}\ \wp(\theta_0, \theta_1),\ \text{for all}\ \mathsf{v} \in \mathbb{N}. \tag{3.5}$$

Letting $\mathsf{v} \to +\infty$, we have

$$\lim_{\mathsf{v}\to+\infty} \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) = 0. \tag{3.6}$$

Now, we will show that the sequence $\{\theta_{\mathsf{v}}\}$ is a Cauchy sequence. To obtain this, we have the following: For each $\mathsf{v} > \mathfrak{m}$,

$$\wp(\theta_{\mathfrak{m}}, \theta_{\mathsf{v}}) \leq \wp(\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}+1}) + \wp(\theta_{\mathfrak{m}+1}, \theta_{\mathfrak{m}+2}) + ... + \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}})$$

$$\begin{aligned} &\leq \mathsf{g}^{\mathfrak{m}}\wp(\theta_0,\theta_1) + \mathsf{g}^{\mathfrak{m}+1}\wp(\theta_0,\theta_1) + ... + \mathsf{g}^{\mathsf{v}-1}\wp(\theta_0,\theta_1) \\ &= \mathsf{g}^{\mathfrak{m}}[1 + \mathsf{g} + \mathsf{g}^2 + ... + \mathsf{g}^{(\mathsf{v}-1)-\mathfrak{m}}]\wp(\theta_0,\theta_1) \\ &\leq \wp^{\mathfrak{m}}[1 + \mathsf{g} + \mathsf{g}^2 + ...]\wp(\theta_0,\theta_1) \\ &= \mathsf{g}^{\mathfrak{m}}\left(\frac{1}{1-\mathsf{g}}\right)\wp(\theta_0,\theta_1). \end{aligned}$$

Let $\epsilon > 0$ be an arbitrary small positive number, and since $\mathsf{g} \in (0, 1)$, we can find a large $\mathcal{N} \in \mathbb{N}$ implying that

$$\mathsf{g}^{\mathcal{N}} < \frac{\epsilon(1-\mathsf{g})}{\wp(\theta_0,\theta_1)}.$$

Then by choosing $\mathfrak{m}$ and v sufficiently larger than $\mathcal{N}$, we write

$$\wp(\theta_{\mathfrak{m}},\theta_{\mathsf{v}}) < \frac{\epsilon(1-\mathsf{g})}{\wp(\theta_0,\theta_1)}\left(\frac{1}{1-\mathsf{g}}\right)\wp(\theta_0,\theta_1).$$

This implies,

$$\wp(\theta_{\mathfrak{m}},\theta_{\mathsf{v}}) < \epsilon.$$

which concludes that $\{\theta_{\mathsf{v}}\}$ is a Cauchy sequence in Q.

Since S is a closed subset of a complete $\mathcal{MS}$ $(\mathsf{Q},\wp)$, then we can find a point θ in S implying that

$$\lim_{\mathsf{v}\to+\infty} \wp(\theta_{\mathsf{v}},\theta) = 0. \tag{3.7}$$

Suppose H is continuous, so (3.7) implies $\lim_{\mathsf{v}\to+\infty} \wp(\mathsf{H}\theta_{\mathsf{v}},\mathsf{H}\theta) = 0$. Therefore, the continuity of the metric $\wp$ implies that

$$\wp(\theta_{\mathsf{v}+1},\mathsf{H}\theta_{\mathsf{v}}) = \wp(\theta,\mathsf{H}\theta). \tag{3.8}$$

Using (3.1) and (3.8), we have

$$\wp(\theta,\mathsf{H}\theta) = \wp(\mathsf{S},\mathsf{T}).$$

This shows that θ is a $\mathcal{BPP}$ of H.

To prove the uniqueness of the $\mathcal{BPP}$, suppose κ is another $\mathcal{BPP}$ of H, that is,

$$\wp(\kappa,\mathsf{H}\kappa) = \wp(\mathsf{S},\mathsf{T}).$$

Replacing $\theta_1 = \theta, \theta_2 = \kappa$, $\ell_1 = \theta$, and $\ell_2 = \kappa$ in the M-type generalized contraction, we have

$$\begin{aligned} [1 + \vartheta\wp(\theta,\kappa)]\wp(\theta,\kappa) \leq\ & \vartheta[\wp(\theta,\theta)\wp(\kappa,\kappa) + \wp(\theta,\kappa)\wp(\kappa,\theta)] + \mathsf{g}\max\Big\{\wp(\theta,\kappa), \\ & \wp(\theta,\theta), \wp(\kappa,\kappa), \frac{1}{2}[\wp(\theta,\kappa) + \wp(\kappa,\theta)]\Big\}. \end{aligned}$$

This implies,

$$\begin{aligned} \wp(\theta,\kappa) &\leq \mathsf{g}\,\wp(\theta,\kappa) \\ (1-\mathsf{g})\wp(\theta,\kappa) &\leq 0. \end{aligned}$$

This is a contradiction because $1 - \mathsf{g} > 0$.

Therefore, θ and κ must be identical. Hence, H has a unique $\mathcal{BPP}$. □

Next, the following is a $\mathcal{BPP}$ theorem for a non-self-mapping satisfying the M-type generalized contraction condition without assuming continuity of H.

Theorem 3.2. Let S and T be closed subsets of a complete $\mathcal{MS}$ $(\mathsf{Q}, \wp)$ implying that S_0 and T_0 both are non-empty. Assume that $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ satisfies the following:

(i) H is an M-type generalized contraction;

(ii) $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$;

(iii) the pair (S, T) satisfies the P-Pro.

Then, there exists a unique element θ in S implying that

$$\wp(\theta, \mathsf{H}\theta) = \wp(\mathsf{S}, \mathsf{T}).$$

Proof. Similar to Theorem 3.1, we can construct a sequence $\{\theta_{\mathsf{v}}\}$ in S_0 implying that

$$\wp(\theta_{\mathsf{v}+1}, \mathsf{H}\theta_{\mathsf{v}}) = \wp(\mathsf{S}, \mathsf{T}), \text{ for } \quad \text{all } \mathsf{v} \geq 0. \tag{3.9}$$

The sequence $\{\theta_{\mathsf{v}}\}$ is a Cauchy sequence, and we can prove this part the same way as in Theorem 3.1.

Since S is a closed subset of a complete $\mathcal{MS}$ $(\mathsf{Q}, \wp)$, then there exists a point θ in S implying that

$$\lim_{\mathsf{v}\to+\infty} \wp(\theta_{\mathsf{v}}, \theta) = 0. \tag{3.10}$$

Since (S, T) satisfies the P-Pro, we have

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) = \wp(\mathsf{H}\theta_{\mathsf{v}-1}, \mathsf{H}\theta_{\mathsf{v}}), \text{ for all } \mathsf{v} \in \mathbb{N}. \tag{3.11}$$

If for some $\mathsf{v}_0 \in \mathbb{N}$, $\wp(\theta_{\mathsf{v}_0}, \theta_{\mathsf{v}_0+1}) = 0$, consequently $\wp(\mathsf{H}\theta_{\mathsf{v}_0-1}, \mathsf{H}\theta_{\mathsf{v}_0}) = 0$ implies $\mathsf{H}\theta_{\mathsf{v}_0-1} = \mathsf{H}\theta_{\mathsf{v}_0}$ implying $\wp(\theta_{\mathsf{v}_0}, \mathsf{H}\theta_{\mathsf{v}_0}) = \wp(\mathsf{S}, \mathsf{T})$. Thus θ_{v_0} is the $\mathcal{BPP}$ of H.

Let us consider $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) > 0$, for all $\mathsf{v} \geq 0$. By the M-type generalized contraction, we have,

$$\begin{aligned}
[1 + \vartheta\wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}})]\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) &\leq \vartheta\Big[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}}) \\
&\quad \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}-1})\Big] + \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}) \\
&\quad \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}), \frac{1}{2}\Big[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}-1})\Big]\Big\}, \\
&\leq \vartheta[\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}})] + \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \\
&\quad \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}), \frac{1}{2}[\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})]\Big\}.
\end{aligned}$$

This implies,

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\max\Big\{\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}), \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}), \frac{1}{2}[\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})]\Big\}, \tag{3.12}$$

for all $\mathsf{v} \in \mathbb{N}$.

If $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1}) < (\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}})$, from (3.12), we have.

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}).$$

But $\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1})$ is never less than 0, and so, $(1 - \mathsf{g}) \leq 0$. But g is in $(0, 1)$. So, we get a contradiction.

Now, if $\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}) < \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}-1})$, from (3.12), we have,

$$\wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}).$$

Similarly, we have

$$\wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}) \leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-2}, \theta_{\mathsf{v}-1}).$$

Proceeding in this way, we have

$$\begin{aligned} \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) &\leq \mathsf{g}\ \wp(\theta_{\mathsf{v}-1}, \theta_{\mathsf{v}}) \leq \mathsf{g}^2\ \wp(\theta_{\mathsf{v}-2}, \theta_{\mathsf{v}-1}) \leq ... \\ &\leq \mathsf{g}^{\mathsf{v}}\ \wp(\theta_0, \theta_1),\ \text{for all } \mathsf{v} \in \mathbb{N}. \end{aligned} \tag{3.13}$$

On the other hand, from (3.11) and (3.13), we have

$$\wp(\mathsf{H}\theta_{\mathsf{v}-1}, \mathsf{H}\theta_{\mathsf{v}}) = \wp(\theta_{\mathsf{v}}, \theta_{\mathsf{v}+1}) \leq \mathsf{g}^{\mathsf{v}}\wp(\theta_0, \theta_1),\ \text{for all } \mathsf{v} \in \mathbb{N}. \tag{3.14}$$

To obtain this, we have the following: For each $\mathsf{v} > \mathfrak{m}$,

$$\begin{aligned} \wp(\mathsf{H}\theta_{\mathfrak{m}-1}, \mathsf{H}\theta_{\mathsf{v}-1}) &\leq \wp(\mathsf{H}\theta_{\mathfrak{m}-1}, \mathsf{H}\theta_{\mathfrak{m}}) + \wp(\mathsf{H}\theta_{\mathfrak{m}}, \theta\mathsf{H}_{\mathfrak{m}+1}) + ... + \wp(\mathsf{H}\theta_{\mathsf{v}-2}, \theta_{\mathsf{v}-1}), \\ &\leq \mathsf{g}^{\mathfrak{m}}\wp(\theta_0, \theta_1) + \mathsf{g}^{\mathfrak{m}+1}\wp(\theta_0, \theta_1) + ... + \mathsf{g}^{\mathsf{v}-1}\wp(\theta_0, \theta_1), \\ &= \mathsf{g}^{\mathfrak{m}}[1 + \mathsf{g} + \mathsf{g}^2 + ... + \mathsf{g}^{(\mathsf{v}-1)-\mathfrak{m}}]\wp(\theta_0, \theta_1), \\ &\leq \mathsf{g}^{\mathfrak{m}}[1 + \mathsf{g} + \mathsf{g}^2 + ...]\wp(\theta_0, \theta_1), \\ &= \mathsf{g}^{\mathfrak{m}}\left(\frac{1}{1 - \mathsf{g}}\right)\wp(\theta_0, \theta_1). \end{aligned}$$

Let $\epsilon > 0$ be an arbitrary small positive number, and since $\mathsf{g} \in (0, 1)$, we can find a large $\mathcal{N} \in \mathbb{N}$ implying that

$$\mathsf{g}^{\mathcal{N}} < \frac{\epsilon(1 - \mathsf{g})}{\wp(\theta_0, \theta_1)}.$$

Then by choosing $\mathfrak{m}$ and v sufficiently larger than $\mathcal{N}$, we write

$$\wp(\mathsf{H}\theta_{\mathfrak{m}-1}, \mathsf{H}\theta_{\mathsf{v}-1}) < \frac{\epsilon(1 - \mathsf{g})}{\wp(\theta_0, \theta_1)}\left(\frac{1}{1 - \mathsf{g}}\right)\wp(\theta_0, \theta_1).$$

This implies,

$$\wp(\mathsf{H}\theta_{\mathfrak{m}-1}, \mathsf{H}\theta_{\mathsf{v}-1}) < \epsilon.$$

This concludes that $\{\mathsf{H}\theta_{\mathsf{v}}\}$ is a Cauchy sequence in Q.

Since T is a closed subset of a complete $\mathcal{MS}$ $(\mathsf{Q}, \wp)$, then we can find a point θ^* in T implying that

$$\lim_{\mathsf{v}\to+\infty} \wp(\mathsf{H}\theta_{\mathsf{v}}, \theta^*) = 0. \tag{3.15}$$

From (3.9), (3.10), and (3.15), we have

$$\wp(\theta, \theta^*) = \lim_{\mathsf{v}\to+\infty} \wp(\theta_{\mathsf{v}+1}, \mathsf{H}\theta_{\mathsf{v}}) = \wp(\mathsf{S}, \mathsf{T}). \tag{3.16}$$

Thus θ must be an element of S_0. Since $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$, we have

$$\wp(\kappa, \mathsf{H}\theta) = \wp(\mathsf{S}, \mathsf{T}) \tag{3.17}$$

for some element κ in S.

Using the P-Pro, (3.16), and (3.17), we have

$$\wp(\theta_{\mathsf{v}+1}, \kappa) = \wp(\mathsf{H}\theta_{\mathsf{v}}, \mathsf{H}\theta), \text{ for } \text{ all } \mathsf{v} \in \mathbb{N}.$$

If for some v_0, $\wp(\theta_{\mathsf{v}_0+1}, \kappa) = 0$, consequently $\wp(\mathsf{H}\theta_{\mathsf{v}_0}, \mathsf{H}\theta) = 0$ implies $\mathsf{H}\theta_{\mathsf{v}_0} = \mathsf{H}\theta$ implying $\wp(\theta_{\mathsf{v}_0}, \mathsf{H}\theta_{\mathsf{v}_0}) = \wp(\mathsf{S}, \mathsf{T})$. Thus θ_{v_0} is the $\mathcal{BPP}$ of H.

So,for all $\mathsf{v} \geq 0$, $\wp(\theta_{\mathsf{v}+1}, \kappa) > 0$, by M-type generalized contraction, we have

$$\begin{aligned}[1 + \vartheta\wp(\theta_{\mathsf{v}+1}, \kappa)]\wp(\theta_{\mathsf{v}}, \theta) \;\leq\; & \vartheta[\wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}})\wp(\kappa, \theta) + \wp(\kappa, \theta_{\mathsf{v}})\wp(\theta_{\mathsf{v}+1}, \theta)] \\ & + \mathsf{g}\max\left\{\wp(\theta_{\mathsf{v}+1}, \kappa), \wp(\theta_{\mathsf{v}+1}, \theta_{\mathsf{v}}), \wp(\kappa, \theta), \frac{1}{2}[\wp(\kappa, \theta_{\mathsf{v}}) + \wp(\theta_{\mathsf{v}+1}, \theta)]\right\}.\end{aligned}$$

Letting $\mathsf{v} \to +\infty$, we obtain

$$\begin{aligned}[1 + \vartheta\wp(\theta, \kappa)]\wp(\theta, \theta) \;\leq\; & \vartheta[\wp(\theta, \theta)\wp(\kappa, \theta) + \wp(\kappa, \theta)\wp(\theta, \theta)] + \mathsf{g}\max\Big\{\wp(\theta, \kappa), \\ & \wp(\theta, \theta), \wp(\kappa, \theta), \frac{1}{2}[\wp(\kappa, \theta) + \wp(\theta, \theta)]\Big\}.\end{aligned}$$

This implies,

$$\mathsf{g}\,\wp(\theta, \kappa) \leq 0,$$

which implies that θ and κ must be identical. From (3.17), we have

$$\wp(\theta, \mathsf{H}\theta) = \wp(\mathsf{S}, \mathsf{T}).$$

This shows that θ is a $\mathcal{BPP}$ of H.

To prove the uniqueness of the $\mathcal{BPP}$, suppose υ be another $\mathcal{BPP}$ of H, that is,

$$\wp(\upsilon, \mathsf{H}\upsilon) = \wp(\mathsf{S}, \mathsf{T}).$$

Replacing $\theta_1 = \theta, \theta_2 = \upsilon$, $\ell_1 = \theta$, and $\ell_2 = \upsilon$ in the M-type generalized contraction, we have

$$[1 + \vartheta\wp(\theta, \upsilon)]\wp(\theta, \upsilon) \leq \vartheta[\wp(\theta, \theta)\wp(\upsilon, \upsilon) + \wp(\theta, \upsilon)\wp(\upsilon, \theta)] + \mathsf{g}\max\Big\{\wp(\theta, \upsilon), \wp(\theta, \theta),$$

$$\wp(\upsilon,\upsilon), \frac{1}{2}\left[\wp(\theta,\upsilon)+\wp(\upsilon,\theta)\right]\Big\}.$$

This implies ,

$$\begin{aligned}\wp(\theta,\upsilon) \leq& \mathsf{g}\,\wp(\theta,\upsilon)\\ (1-\mathsf{g})\wp(\theta,\upsilon) \leq& 0.\end{aligned}$$

This is a contradiction because $1-\mathsf{g}>0$.

Therefore, θ and υ must be identical. Hence, H has a unique $\mathcal{BPP}$. □

If $\vartheta = 0$ in Theorem 3.2, we have the following corollary:

Corollary 3.1. Let S and T be closed subsets of a complete $\mathcal{MS}$ $(\mathsf{Q},\wp)$ implying that S_0 and T_0 both are non-empty. Suppose that $\mathsf{H}:\mathsf{S}\to\mathsf{T}$ satisfies the following conditions:

(i)

$$\wp(\theta_1,\mathsf{H}\ell_1)=\wp(\mathsf{S},\mathsf{T}) \;\; \textit{and} \;\; \wp(\theta_2,\mathsf{H}\ell_2)=\wp(\mathsf{S},\mathsf{T})$$

which implies

$$\wp(\theta_1,\theta_2)\leq \mathsf{g}\max\Big\{\wp(\ell_1,\ell_2),\wp(\ell_1,\theta_1),\wp(\ell_2,\theta_2),\frac{1}{2}[\wp(\ell_2,\theta_1)+\wp(\ell_1,\theta_2)]\Big\}$$

where $\theta_1,\theta_2,\ell_1,\ell_2\in\mathsf{S}$ with $\theta_1\neq\theta_2$, $0<\mathsf{g}<1$;

(ii) $\mathsf{H}(\mathsf{S}_0)\subseteq\mathsf{T}_0$;

(iii) the pair (S,T) satisfies the P-Pro.

Then, there exists a unique element θ in S implying that

$$\wp(\theta,\mathsf{H}\theta)=\wp(\mathsf{S},\mathsf{T}).$$

Let $(\mathsf{Q},\wp)$ be a $\mathcal{MS}$ and S,T are subsets of Q. If $\mathsf{S}\cap\mathsf{T}\neq\emptyset$, then $\wp(\mathsf{S},\mathsf{T})=0$. In this case, the $\mathcal{BPP}$ result is converted into a fixed point result as given below.

Theorem 3.3. Let S and T be closed subsets of a complete $\mathcal{MS}$ $(\mathsf{Q},\wp)$ implying that S_0 and T_0 both are non-empty. Suppose that $\mathsf{H}:\mathsf{S}\to\mathsf{T}$ satisfies the following conditions:

$$\begin{aligned}[1+\vartheta\wp(\ell_1,\ell_2)]\,\wp(\mathsf{H}\ell_1,\mathsf{H}\ell_2) \;\leq\;& \vartheta[\wp(\ell_1,\mathsf{H}\ell_1)\wp(\ell_2,\mathsf{H}\ell_2)+\wp(\ell_2,\mathsf{H}\ell_1)\\ &\wp(\ell_1,\mathsf{H}\ell_2)]+\mathsf{g}\max\Big\{\wp(\ell_1,\ell_2),\wp(\ell_1,\mathsf{H}\ell_1),\\ &\wp(\ell_2,\mathsf{H}\ell_2),\frac{1}{2}[\wp(\ell_2,\mathsf{H}\ell_1)+\wp(\ell_1,\mathsf{H}\ell_2)]\Big\}\end{aligned}$$

for all $\ell_1,\ell_2\in\mathsf{S}$, where $\vartheta\geq 0, 0<\mathsf{g}<1$, and $\mathsf{H}(\mathsf{S}_0)\subseteq\mathsf{T}_0$. Then, H has a UFP.

If we take $\mathsf{S}=\mathsf{T}=\mathsf{Q}$, then the Theorem 3.2 converts into Nesič [8] result, which is as follows:

Theorem 3.4. Let $(\mathsf{Q}, \wp)$ be a complete $\mathcal{MS}$ and suppose that $\mathsf{H} : \mathsf{Q} \to \mathsf{Q}$ is the mapping satisfying the following conditions:

$$\begin{aligned}[1 + \vartheta\wp(\ell_1, \ell_2)]\,\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \;\le\; & \vartheta[\wp(\ell_1, \mathsf{H}\ell_1)\wp(\ell_2, \mathsf{H}\ell_2) + \wp(\ell_2, \mathsf{H}\ell_1) \\ & \wp(\ell_1, \mathsf{H}\ell_2)] + \mathsf{g}\max\Big\{\wp(\ell_1, \ell_2), \wp(\ell_1, \mathsf{H}\ell_1), \\ & \wp(\ell_2, \mathsf{H}\ell_2), \frac{1}{2}[\wp(\ell_2, \mathsf{H}\ell_1) + \wp(\ell_1, \mathsf{H}\ell_2)]\Big\}\end{aligned}$$

for all $\ell_1, \ell_2 \in \mathsf{Q}$, where $\vartheta \ge 0, 0 < \mathsf{g} < 1$. Then, H has a UFP.

If we take $\vartheta = 0$, then the Theorem 3.4 is as follows:

Theorem 3.5. Let $(\mathsf{Q}, \wp)$ be a complete $\mathcal{MS}$ and suppose that $\mathsf{H} : \mathsf{Q} \to \mathsf{Q}$ is the mapping satisfying the following conditions:

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \le \mathsf{g}\max\Big\{\wp(\ell_1, \ell_2), \wp(\ell_1, \mathsf{H}\ell_1), \wp(\ell_2, \mathsf{H}\ell_2), \frac{1}{2}[\wp(\ell_2, \mathsf{H}\ell_1) + \wp(\ell_1, \mathsf{H}\ell_2)]\Big\}$$

for all $\ell_1, \ell_2 \in \mathsf{Q}$, where $0 < \mathsf{g} < 1$. Then, H has a UFP.

Now, we shall give an example to support our main theorem:

Example 3.2. Let $\mathsf{Q} = \mathbb{R}^2$ be a complete $\mathcal{MS}$ with the usual metric $\wp$ defined by

$$\wp((\theta_1, \theta_2), (\ell_1, \ell_2)) = \sqrt{(\theta_1 - \ell_1)^2 + (\theta_2 - \ell_2)^2},$$

for all $(\theta_1, \theta_2), (\ell_1, \ell_2) \in \mathbb{R}^2$.

Let $\mathsf{S} = \{(1, \mathfrak{a}) : \mathfrak{a} \in \mathbb{R}\}$ and $\mathsf{T} = \{(0, \mathfrak{a}) : \mathfrak{a} \in \mathbb{R}\}$ are closed subsets of Q, then we have $\wp(\mathsf{S}, \mathsf{T}) = 1$, $\mathsf{S}_0 = \mathsf{S}$, and $\mathsf{T}_0 = \mathsf{T}$.

Let $\mathsf{H} : \mathsf{S} \to \mathsf{T}$ is defined by

$$\mathsf{H}((1, \mathfrak{a})) = \left(0, \frac{\mathfrak{a}}{2}\right).$$

Here, $\mathsf{H}((1, 0)) = (0, 0), \mathsf{H}((1, 1)) = (0, 0.5), \mathsf{H}((1, 2)) = (0, 1)$, and so on. This implies that $\mathsf{H}(\mathsf{S}_0) \subseteq \mathsf{T}_0$.

For all $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \in \mathbb{R}$, we have,

$$\wp((1, \mathfrak{a}_1), \mathsf{H}(1, \mathfrak{a}_3)) = 1 \text{ implies } \mathfrak{a}_1 = \frac{\mathfrak{a}_3}{2}$$

and

$$\wp((1, \mathfrak{a}_2), \mathsf{H}(1, \mathfrak{a}_4)) = 1 \text{ implies } \mathfrak{a}_2 = \frac{\mathfrak{a}_4}{2},$$

which implies that the pair (S, T) satisfies the *P*-Pro.

Now,

$$\begin{aligned}[1 + \vartheta|\varrho_3 - \varrho_4|]|\varrho_1 - \varrho_2| \;\le\; & \vartheta[|\varrho_1 - \varrho_3||\varrho_2 - \varrho_4| + |\varrho_1 - \varrho_4||\varrho_2 - \varrho_3|] \\ & + \mathsf{g}\max\Big\{|\varrho_1 - \varrho_2|, |\varrho_1 - \varrho_3|, |\varrho_2 - \varrho_4|, \frac{1}{2}[|\varrho_1 - \varrho_4| + |\varrho_2 - \varrho_3|]\Big\},\end{aligned}$$

$$[1+\vartheta|\varrho_3-\varrho_4|]\frac{|\varrho_3-\varrho_4|}{2} \leq \vartheta\left[\frac{|\varrho_3||\varrho_4|}{4}+\left|\frac{\varrho_3}{2}-\varrho_4\right|\left|\frac{\varrho_4}{2}-\varrho_3\right|\right] + \mathsf{g}\max\left\{\frac{|\varrho_3-\varrho_4|}{2},\frac{|\varrho_3|}{2},\frac{|\varrho_4|}{2},\frac{1}{2}\left[\left|\frac{\varrho_3}{2}-\varrho_4\right|+\left|\frac{\varrho_4}{2}-\varrho_3\right|\right]\right\}.$$

Therefore,

$$\frac{|\varrho_3-\varrho_4|}{2} \leq \mathfrak{a}\left[\frac{|\varrho_3||\varrho_4|}{4}+\left|\frac{\varrho_3}{2}-\varrho_4\right|\left|\frac{\varrho_4}{2}-\varrho_3\right|-\frac{|\varrho_3-\varrho_4|^2}{2}\right] + \mathfrak{b}\max\left\{\frac{|\varrho_3-\varrho_4|}{2},\frac{|\varrho_3|}{2},\frac{|\varrho_4|}{2},\frac{1}{2}\left[\left|\frac{\varrho_3}{2}-\varrho_4\right|+\left|\frac{\varrho_4}{2}-\varrho_3\right|\right]\right\}, \tag{3.18}$$

for all $\varrho_3, \varrho_4 \in \mathbb{R}^+$ and $\mathfrak{a} \geq 0$, $0 < \mathfrak{b} < 1$. We can see that the right hand side of (3.18) is greater than the left hand side as shown in Figure 1.

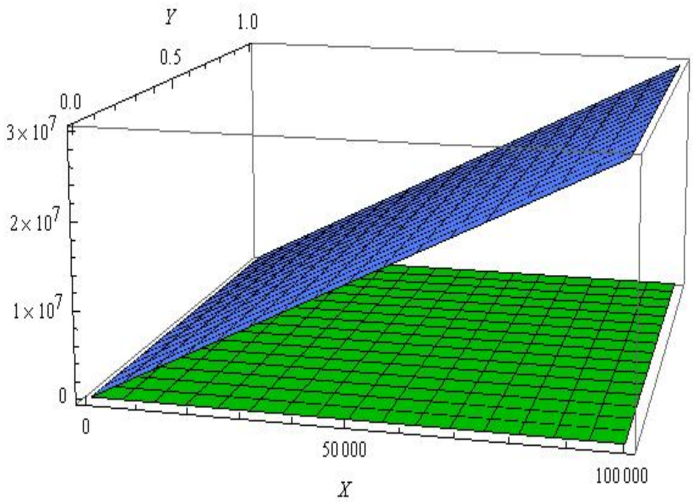

Figure 1. The right hand side is greater than the left hand side

Hence H is a M-type generalized contraction. Finally, it is easy to see that H has a unique $\mathcal{BPP}$ $(1,0)$. Therefore, all hypotheses of Theorem 3.2 are satisfied.

Now that we have the opportunity, let us talk about an application for Theorem 3.5. We will develop a solution for the boundary value problem.

4. An application for the boundary value problem

As an application of the above result, we are trying to solve the boundary value problem for the third-order differential equation:

$$\begin{cases} \ell'''(\theta) = \mathcal{F}(\theta, \ell(\theta)), \theta \in [0,1], \\ \ell(0) = 0 = \ell'(1), \\ \ell'(0) = \ell(1), \end{cases} \tag{4.1}$$

where $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This problem is equivalent to the integral equation

$$\ell(\theta) = \int_0^1 \mathcal{G}(\theta, \upsilon)\mathcal{F}(\upsilon, \ell(\upsilon))d\upsilon, \text{ for } \theta \in [0,1], \tag{4.2}$$

where

$$\mathcal{G}(\theta,\upsilon) = \begin{cases} -\frac{\upsilon^2}{2}(\theta^2 - 2\theta - 1) + \upsilon(\theta^2 - 2\theta); & 0 \le \upsilon \le \theta \le 1, \\ -\frac{\theta^2}{2}(\upsilon^2 - 2\upsilon + 1) + \theta(\upsilon^2 - \upsilon); & 0 \le \theta \le \upsilon \le 1. \end{cases}$$

First, consider the space $\mathsf{Q} = C([0,1], \mathbb{R}^+)$ as the space of non-negative continuous real-valued functions defined on $[0,1]$. Now, we define the metric $\wp$ on Q, that is,

$$\wp(\ell_1, \ell_2) = \sup_{\theta \in [0,1]} |\ell_1(\theta) - \ell_2(\theta)|,$$

for $\ell_1, \ell_2 \in \mathsf{Q}$. Then $(\mathsf{Q}, \wp)$ is a complete $\mathcal{MS}$.

Consider the self-mapping $\mathsf{H} : \mathsf{Q} \to \mathsf{Q}$ defined by

$$\mathsf{H}(\ell(\theta)) = \int_0^1 \mathcal{G}(\theta,\upsilon)\mathcal{F}(\upsilon, \ell(\upsilon))d\upsilon, \theta \in [0,1].$$

Suppose the following condition holds:

$$|\mathcal{F}(\upsilon, \ell_1(\upsilon)) - \mathcal{F}(\upsilon, \ell_2(\upsilon))| \le |\ell_1 - \ell_2|, \text{ for all } \upsilon \in [0,1] \text{ and } \ell_1, \ell_2 \in \mathsf{Q}.$$

Then the system of third-order differential equations given by (4.2) has a unique solution $\ell^* \in \mathsf{Q}$.

Finally, we will show that, for each $\ell_1, \ell_2 \in \mathsf{Q}$, we have

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \le \vartheta \max\left\{\wp(\ell_1,\ell_2), \wp(\mathsf{H}\ell_1, \ell_1), \wp(\mathsf{H}\ell_2, \ell_2), \frac{1}{2}\left[\wp(\mathsf{H}\ell_1, \ell_2) + \wp(\mathsf{H}\ell_2, \ell_1)\right]\right\}.$$

To this end, let $\ell_1, \ell_2 \in \mathsf{Q}$. Therefore, for each $\theta \in [0,1]$, we have

$$\begin{aligned} \wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) &= \sup_{\theta\in[0,1]} |\mathsf{H}\ell_1(\theta) - \mathsf{H}\ell_2(\theta)|, \\ &= \sup_{\theta\in[0,1]} \left| \int_0^1 \mathcal{G}(\theta,\upsilon)\mathcal{F}(\upsilon,\ell_1(\upsilon))d\upsilon - \int_0^1 \mathcal{G}(\theta,\upsilon)\mathcal{F}(\upsilon,\ell_2(\upsilon))d\upsilon \right|, \\ &= \sup_{\theta\in[0,1]} \left| \int_0^1 \mathcal{G}(\theta,\upsilon)\left[\mathcal{F}(\upsilon,\ell_1(\upsilon)) - \mathcal{F}(\upsilon,\ell_2(\upsilon))\right] d\upsilon \right|, \\ &\le \sup_{\theta\in[0,1]} \int_0^1 |\mathcal{G}(\theta,\upsilon)|\,|\mathcal{F}(\upsilon,\ell_1(\upsilon)) - \mathcal{F}(\upsilon,\ell_2(\upsilon))|\,d\upsilon, \\ &\le \sup_{\theta\in[0,1]} \int_0^1 |\mathcal{G}(\theta,\upsilon)|\,|\ell_1(\upsilon) - \ell_2(\upsilon)|\,d\upsilon, \\ &\le \sup_{\theta\in[0,1]} \int_0^1 |\mathcal{G}(\theta,\upsilon)| \sup_{\upsilon\in[0,1]} |\ell_1(\upsilon) - \ell_2(\upsilon)|\,d\upsilon, \end{aligned}$$

$$= \wp(\ell_1, \ell_2) \sup_{\theta\in[0,1]} \int_0^1 |\mathcal{G}(\theta, \upsilon)|\, d\upsilon. \tag{4.3}$$

It is easy to verify that $\int_0^1 |\mathcal{G}(\theta, \upsilon)|\, d\upsilon = -\frac{\theta^3}{6} + \frac{\theta^2}{6} + \frac{\theta}{6}$, and we can see that $\sup_{\theta\in[0,1]} \int_0^1 |\mathcal{G}(\theta, \upsilon)|\, d\upsilon \le \frac{1}{6}$.

In Figure (2), we see that the value of $\mathcal{G}(\theta, \upsilon)$ is negative for $\upsilon < \theta$ and $\theta < \upsilon$ both.

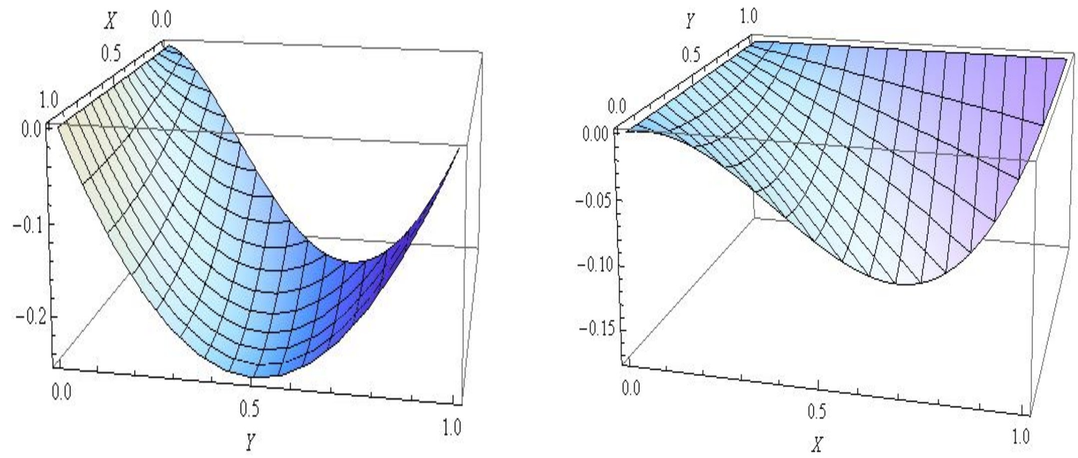

Figure 2. Graph of $\mathcal{G}(\theta, \upsilon)$.

Considering the above facts, the inequality (4.3) gives us

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \le \frac{1}{6}\wp(\ell_1, \ell_2),$$

which implies that

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \le \frac{1}{6} \max\left\{\wp(\ell_1, \ell_2), \wp(\ell_1, \mathsf{H}\ell_1), \wp(\ell_2, \mathsf{H}\ell_2), \frac{1}{2}\left[\wp(\ell_2, \mathsf{H}\ell_1) + \wp(\ell_1, \mathsf{H}\ell_2)\right]\right\}.$$

Thus, H satisfies all the conditions of Theorem 3.5.

Hence, by Theorem 3.5, H has a UFP $\ell^* \in \mathsf{Q}$. Hence, the integral equation (4.2) has a unique solution.

Example 4.1. Let $\mathsf{S} = [0, 1]$, $\mathsf{T} = [2, 3]$, and define a mapping $\mathsf{H} : \mathsf{S} \bigcup \mathsf{T} \to \mathsf{S} \bigcup \mathsf{T}$ as

$$\mathsf{H}(\theta) = \begin{cases} 2, & if \theta \in [0, 1], \\ 1, & if \theta \in [2, 3]. \end{cases}$$

Since $\mathsf{S} \bigcap \mathsf{T} = \phi$, H cannot have a fixed point.

Also define the metric $\wp(\theta, \ell) = |\theta - \ell|$.

$$dis(\mathsf{S}, \mathsf{T}) = \inf\{|\theta - \ell| : \theta \in \mathsf{S} \quad and \quad \ell \in \mathsf{T}\} = 1.$$

Now,

$$\wp(\theta, \mathsf{H}(\theta)) = dis(\mathsf{S}, \mathsf{T}) = 1.$$

Therefore $\theta = 1 \in \mathsf{S}$ and $\theta = 2 \in \mathsf{T}$ are best proximity points.

Hence, H has no fixed point but best proximity points exist.

5. Application in production-consumption equilibrium

For production $\theta_{\rm p}$ and consumption θ_τ, whether prices are rising or decreasing, daily pricing patterns and prices have a significant influence on markets. As a result, the economists are interested in the present cost $\ell(\theta)$. Now, assume

$$\theta_{\rm p} = \sigma_1 + \eta_1\ell(\theta) + \tau_1\frac{d\ell(\theta)}{d\theta} + \zeta_1\frac{d^2\ell(\theta)}{d\theta^2},$$
$$\theta_\tau = \sigma_2 + \eta_2\ell(\theta) + \tau_2\frac{d\ell(\theta)}{d\theta} + \zeta_2\frac{d^2\ell(\theta)}{d\theta^2},$$

initially $\ell(0) = 0, \frac{d\ell}{d\theta}(0) = 0$, where $\sigma_1, \sigma_2, \eta_1, \eta_2, \tau_1, \tau_2, \zeta_1$ and ζ_2 are constants. A state of dynamic economic equilibrium occurs when market forces are in balance, indicating that the current gap between production and consumption stabilizes, that is, $\theta_{\rm p} = \theta_\tau$. Thus,

$$\sigma_1 + \eta_1\ell(\theta) + \tau_1\frac{d\ell(\theta)}{d\theta} + \zeta_1\frac{d^2\ell(\theta)}{d\theta^2} = \sigma_2 + \eta_2\ell(\theta) + \tau_2\frac{d\ell(\theta)}{d\theta} + \zeta_2\frac{d^2\ell(\theta)}{d\theta^2},$$
$$(\sigma_1 - \sigma_2) + (\eta_1 - \eta_2)\ell(\theta) + (\tau_1 - \tau_2)\frac{d\ell(\theta)}{d\theta} + (\zeta_1 - \zeta_2)\frac{d^2\ell(\theta)}{d\theta^2} = 0,$$
$$\zeta\frac{d^2\ell(\theta)}{d\theta^2} + \tau\frac{d\ell(\theta)}{d\theta} + \eta\ell(\theta) = -\sigma,$$
$$\frac{d^2\ell(\theta)}{d\theta^2} + \frac{\tau}{\zeta}\frac{d\ell(\theta)}{d\theta} + \frac{\eta}{\zeta}\ell(\theta) = \frac{-\sigma}{\zeta},$$

where $\sigma = \sigma_1 - \sigma_2, \eta = \eta_1 - \eta_2, \tau = \tau_1 - \tau_2, \zeta = \zeta_1 - \zeta_2$. Now, our initial value problem is modeled as

$$\ell''(\theta) + \frac{\tau}{\zeta}\ell'(\theta) + \frac{\eta}{\zeta}\ell(\theta) = \frac{-\sigma}{\zeta}, \quad \text{with} \quad \ell(0) = 0 \quad \text{and} \quad \ell'(0) = 0. \tag{5.1}$$

In the study of production and consumption of the duration time Γ, problem (5.1) is equivalent to

$$\ell(\theta) = \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b})\mathcal{F}(\mathsf{b}, \theta, \ell(\theta))d\theta, \tag{5.2}$$

where the Green function $\mathcal{G}(\theta, \mathsf{b})$ is

$$\mathcal{G}(\theta, \mathsf{b}) = \begin{cases} \theta e^{\frac{\eta}{2\tau}}(\mathsf{b} - \theta), & 0 \le \theta \le \ell \le \Gamma, \\ \ell e^{\frac{\eta}{2\tau}}(\mathsf{b} - \theta), & 0 \le \ell \le \theta \le \Gamma, \end{cases}$$

and $\mathcal{F}\colon [0, \Gamma] \times \mathsf{Q}^2 \to \mathbb{R}$ is a continuous function.

Let an operator $\mathsf{H}\colon \mathsf{Q} \to \mathsf{Q}$ be described as

$$\mathsf{H}\ell(\theta) = \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b})\mathcal{F}(\mathsf{b}, \theta, \ell(\theta))d\theta. \tag{5.3}$$

Then, the solution of the dynamic market equilibrium problem is expressed as (5.1) being a fixed point of H. (5.1) controls the current price $\ell(\theta)$. Let $C[0, \Gamma]$ symbolize the family of real continuous functions on $[0, \Gamma]$ and assume $\mathsf{Q} = C[0, \Gamma]$. Define $\wp\colon \mathsf{Q}\times\mathsf{Q} \to \mathbb{R}^+$ as $\wp(\ell_1, \ell_2) = \sup_{\theta\in[0,\Gamma]} |\ell_1(\theta) - \ell_2(\theta)|, \ell_1, \ell_2 \in \mathsf{Q}$. Then $(\mathsf{Q}, \wp)$ is a complete $\mathcal{MS}$.

Theorem 5.1. *Let us assume the map* $\mathsf{H}\colon \mathsf{Q} \to \mathsf{Q}$ *is a complete* $\mathcal{MS}$ $(\mathsf{Q}, \wp)$*, such that*

(1) there is a continuous function $\mathcal{F}\colon [0, \Gamma] \times \mathsf{Q}^2 \to \mathbb{R}$ *such that*

$$|\mathcal{F}(\mathsf{b}, \theta, \ell_1(\theta)) - \mathcal{F}(\mathsf{b}, \theta, \ell_2(\theta))| \leq |\ell_1(\theta) - \ell_2(\theta)|,$$

(2) $\sup_{\theta \in [0,\Gamma]} \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b}) d\theta \leq \frac{1}{7}$.

Then, the dynamic market equilibrium problem (5.1) *has a unique solution.*

Proof.

$$\begin{aligned}
|\mathsf{H}\ell_1(\theta) - \mathsf{H}\ell_2(\theta)| &= \left| \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b}) \mathcal{F}(\mathsf{b}, \theta, \ell_1(\theta)) d\theta - \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b}) \mathcal{F}(\mathsf{b}, \theta, \ell_2(\theta)) d\theta \right| \\
&\leq \int_0^{\Gamma} \left| \mathcal{G}(\theta, \mathsf{b}) \mathcal{F}(\mathsf{b}, \theta, \ell_1(\theta)) d\theta - \mathcal{G}(\theta, \mathsf{b}) \mathcal{F}(\mathsf{b}, \theta, \ell_2(\theta)) d\theta \right| \\
&\leq \int_0^{\Gamma} \left| \mathcal{G}(\theta, \mathsf{b}) (\mathcal{F}(\mathsf{b}, \theta, \ell_1(\theta)) - \mathcal{F}(\mathsf{b}, \theta, \ell_2(\theta))) d\theta \right| \\
&\leq \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b}) |(\mathcal{F}(\mathsf{b}, \theta, \ell_1(\theta)) - \mathcal{F}(\mathsf{b}, \theta, \ell_2(\theta)))| d\theta \\
&\leq \int_0^{\Gamma} \mathcal{G}(\theta, \mathsf{b}) |\ell_1(\theta) - \ell_2(\theta)| d\theta.
\end{aligned}$$

Taking the supremum on both sides, we have

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \leq \frac{1}{7} \wp(\ell_1, \ell_2),$$

which implies that

$$\wp(\mathsf{H}\ell_1, \mathsf{H}\ell_2) \leq \frac{1}{7} \max\left\{ \wp(\ell_1, \ell_2), \wp(\ell_1, \mathsf{H}\ell_1), \wp(\ell_2, \mathsf{H}\ell_2), \frac{1}{2} \left[\wp(\ell_2, \mathsf{H}\ell_1) + \wp(\ell_1, \mathsf{H}\ell_2) \right] \right\}.$$

Thus, H satisfies all the conditions of Theorem 3.5. Hence, the Eq (5.1) has a unique solution. □

6. Conclusions

In this paper, we introduced a generalized contraction condition designed to obtain best proximity points in complete metric spaces. Our results extend and unify several classical fixed point theorems, demonstrating the versatility of the proposed framework. Through the derivation of multiple corollaries, we showed the wide applicability of our method across various mathematical contexts. Furthermore, the effectiveness of our approach was illustrated by its application to boundary value problems and dynamic market equilibrium models. These examples highlight not only the theoretical significance of our findings but also their potential for addressing real-world problems. Future research may explore further generalizations and additional applications in diverse areas such as differential equations, optimization, and economic modeling.

Author contributions

R. Ramaswamy—Conceptualization, Methodology, Supervision, Funding Acquisition, Writing-Orignal Draft, Writing-Review and editing; P. P. Murthy—Conceptualisation, Methodology, Wrting-Original Draft, Writing-Review and editing; P. Sahu—Investigation, Software, Formal Analysis, Validation, Writing-Orignal Draft; R. A. Alkhowaiter—Investigation, Formal Analysis, Software, Wrting-Original Draft; O. A. Abdelnaby—Investigation, Resources, Wrting-Review and Editing; G. Mani—Methodology, Investigation, Validation, Writing-Original Draft, Writing-Reveiw and editing. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interests

The authors declare no conflicts of interest.

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