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Research article

Rigidity of almost Ricci solitons on compact Riemannian manifolds

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Abstract: Considering an almost Ricci soliton (ARS) (N, g, η, κ) on a compact Riemannian manifold (N, g), we use the Ricci curvature in the direction of the potential vector field η to derive necessary and sufficient conditions for (N, g) to be isometric to a sphere. This expands on several recent results regarding Ricci solitons and almost Ricci solitons by applying specific integral inequalities involving the Ricci curvature evaluated in the direction η. Furthermore, we present conditions under which η is either Killing or parallel; in particular, the ARS is trivial.

Keywords: almost Ricci solitons; compact Riemannian manifolds; Ricci and scalar curvatures
Mathematics Subject Classification: 53C24, 53C25, 53C42, 53E20, 53Z05

1. Introduction

In his famous paper on Ricci flows [1], Richard S. Hamilton introduced Ricci solitons, which have played a crucial role in deepening our comprehension of the Ricci flow's long-term dynamics and its relationship with geometric structures. Ricci solitons represent self-similar solutions that provide valuable perspectives on the geometry of manifolds.

A Ricci soliton on a Riemannian manifold (N, g) is defined by the existence of a vector field η and a constant κ such that the following equation holds:

$$Ric + \frac{1}{2}L_\eta g = \kappa g, \tag{1.1}$$

where $L_\eta g$ denotes the Lie derivative of the metric g in the direction of η, and Ric represents the Ricci curvature of (N, g). In this context, (N, g, η, κ) is referred to as a Ricci soliton, with η known as the potential vector field and κ as the Ricci soliton constant.

In [2], S. Pigola introduced the concept of an almost Ricci soliton by allowing the Ricci soliton constant κ to be a function. More specifically, a quadruple (N, g, η, κ) is defined as an almost Ricci soliton (ARS) if Eq (1.1) holds, where κ is a smooth function on N, referred to as the potential function of the ARS.

If the soliton potential vector field η is Killing, then the ARS (N, g, η, κ) is said to be trivial. We also say that the ARS (N, g, η, κ) is parallel if its potential vector field η is a parallel vector field.

An almost Ricci soliton (N, g, η, κ) is categorized as steady under the condition $\kappa = 0$, as shrinking under the condition $\kappa > 0$, and as expanding under the condition $\kappa < 0$. When the potential vector field η is gradient, i.e, $\eta = Df$ for some smooth function f on N, the almost Ricci soliton (N, g, f, κ) is called a gradient. Equation (1.1) becomes

$$Ric + \mathbf{H}^f = \kappa g, \tag{1.2}$$

where $\mathbf{H}^f$ is the Hessian of the function f.

The study of almost Ricci solitons (ARS) provides a natural generalization of Ricci solitons by allowing the soliton constant κ to vary as a smooth function rather than remaining constant. These structures naturally appear in the study of Ricci flow, where they describe shapes of spaces that evolve in a self-similar way over time. Recently, researchers have paid more attention to compact ARS spaces because they often show strong and rigid geometric behavior under certain curvature conditions.

In [3], Corollary 1 and Theorem 2, it was shown that in dimension ≥ 3, if a compact ARS has either constant scalar curvature or a nontrivial conformal vector field, then the space must be isometric to a standard Euclidean sphere. This extends earlier results about Ricci solitons to the more general ARS case, showing that symmetry and compactness strongly influence the geometry.

In [4], it has been shown that if a nontrivial ARS (N, g, η, κ) has constant scalar curvature, then the vector field η is gradient, and the space (N, g) must again be a sphere. This connects ARS spaces with classical gradient Ricci solitons, suggesting that many of the same rigid properties still apply even in this more general setting.

It is also known (see [5]) that all compact Ricci solitons are gradient solitons. While this does not always hold for ARS (because κ is not constant), adding the condition of constant scalar curvature forces the ARS to behave like a gradient soliton.

In two dimensions, Hamilton showed in [6] that all compact Ricci solitons are trivial. However, [3] gave an example of a nontrivial compact two-dimensional almost Ricci soliton, proving that ARS allow for more complex geometry than Ricci solitons, especially in low dimensions.

For a more detailed study of these topics, including classification results and more examples, readers can refer to the works (see [7–13]).

Finally, it is worth noting that many researchers have effectively applied soliton theory across various scientific and engineering disciplines. They have also devised several powerful techniques for obtaining analytic solutions, including the inverse scattering transform [14] and Hirota's bilinear method [15], among others.

The present work presents several sufficient conditions for a compact almost Ricci soliton to be isometric to a sphere and other conditions under which it becomes trivial or parallel.

The organization of this paper is as follows: Section 2 contains the fundamental concepts, presenting an overview of the essential principles and key equations within almost Ricci soliton theory. Section 3 determines the sufficient conditions for an almost Ricci soliton on a compact Riemannian manifold to be trivial or parallel. Theorem 3.6 states that given a non-parallel almost Ricci soliton (N, g, η, κ) on an n-dimensional compact and connected Riemannian manifold (N, g), where $n \geq 3$, then, (N, g) is isometric to a Euclidean sphere $S^n(c)$ if and only if the following inequality holds:

$$\int_N (\mathrm{Ric}(\eta, \eta) + (n-1)g(\eta, D\kappa))\, dv \leq 0. \tag{1.3}$$

Theorem 3.8 demonstrates that for an almost Ricci soliton on an n-dimensional compact Riemannian manifold (N, g), with $n \geq 2$, the condition

$$\int_N (\mathrm{Ric}(\eta, \eta) + (n-2)g(\eta, D\kappa))\, dv \leq 0 \tag{1.4}$$

leads to the potential vector field being parallel. This extends Theorem 3 in [3], which shows that an almost Ricci soliton on a compact Riemannian manifold, with $n \geq 3$, is trivial (i.e, the potential vector field is Killing) if condition (1.4) is satisfied. This also generalizes Theorem 1.1 in [16], stating that a compact Ricci soliton is trivial whenever

$$\int_N (\mathrm{Ric}(\eta, \eta))\, dv \leq 0.$$

2. Preliminaries

On an n-dimensional Riemannian manifold (N, g), where $n \geq 2$, the Riemannian curvature tensor R is given by:

$$R(U, V)Z = D_{[U,V]}Z - D_U D_V Z + D_V D_U Z, \tag{2.1}$$

for all $U, V, Z \in \mathfrak{X}(N)$, with D being the Levi-Civita connection on (N, g) and $\mathfrak{X}(N)$ representing the collection of smooth vector fields on (N, g). For more details, see [17].

If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on the manifold (N, g), the divergence of a vector field $U \in \mathfrak{X}(N)$ is described by the equation:

$$\mathrm{div}(U) = \sum_{i=1}^{n} g(D_{D_{e_i} e_i} U, e_i).$$

Similarly, the Laplacian of U is given by:

$$\Delta U = \sum_{i=1}^{n} D_{e_i} D_{e_i} U - D_{D_{e_i} e_i} U.$$

The divergence of the $(1, 1)$-tensor ϕ on (N, g) is expressed as:

$$\mathrm{div}(\phi) = \sum_{i=1}^{n} (D_{e_i}\phi)(e_i).$$

The gradient of a smooth function f on N is the vector field Df characterized by:

$$g(Df, U) = U(f), \tag{2.2}$$

for all $U \in \mathfrak{X}(N)$, and the Hessian of f is the symmetric $(0,2)$ tensor field defined by"

$$\mathbf{H}^f(U,V) = g(D_U Df, V), \tag{2.3}$$

for all $U, V \in \mathfrak{X}(N)$.

The Laplacian of a smooth function f on N is defined as:

$$\Delta f = \operatorname{div}(Df).$$

Taking the trace of (1.1) yields

$$S + div(\eta) = n\kappa, \tag{2.4}$$

where S is the scalar curvature of (N, g). Also, we have

$$\operatorname{div}(\kappa\eta) = g(\eta, D\kappa) + n\kappa^2 - \kappa S, \tag{2.5}$$

$$\operatorname{div}(S\eta) = g(\eta, DS) + n\kappa S - S^2, \tag{2.6}$$

and

$$\operatorname{div}((n\kappa - S)\eta) = (n\kappa - S)^2 + \eta(n\kappa - S). \tag{2.7}$$

For a gradient Ricci soliton (N, g, f, κ), Equation (2.4) becomes

$$S + \Delta f = n\kappa. \tag{2.8}$$

Take a Riemannian manifold (N, g), and $\eta \in \mathfrak{X}(N)$. Let θ_η be the 1-form dual to η (i.e, $\theta_\eta(U) = g(U, \eta)$, $U \in \mathfrak{X}(N)$), then the Lie derivative L_η can be written as

$$L_\eta g(U,V) + d\theta_\eta(U,V) = 2g(D_U\eta, V), \tag{2.9}$$

for all $U, V \in \mathfrak{X}(N)$.

Now, consider B a symmetric tensor field, and ϕ a skew-symmetric tensor field on (N, g) with the following equations:

$$L_\eta g(U,V) = 2g(B(U), V), \tag{2.10}$$

and

$$d\theta_\eta(U,V) = 2g(\phi(U), V), \tag{2.11}$$

where $U, V \in \mathfrak{X}(N)$. We get from (2.9) that

$$D_U\eta = B(U) + \phi(U), \tag{2.12}$$

for each $U \in \mathfrak{X}(N)$.

A vector field η is called closed conformal if

$$D_U\eta = \Psi U, \tag{2.13}$$

for each $U \in \mathfrak{X}(N)$, where Ψ is a smooth function on (N, g).

3. Almost Ricci solitons on compact Riemannian manifolds

3.1. Useful formulas

In this subsection, we prove several key lemmas that will be crucial for the rest of the paper.

Assume (N, g, η, κ) is an almost Ricci soliton on a compact Riemannian manifold (N, g). By substituting (2.10) in (1.1), we deduce that

$$B = \kappa I - Q, \tag{3.1}$$

where Q is the Ricci operator, that is, the operator satisfying

$$Ric(U, V) = g(Q(U), V),$$

for all $U, V \in \mathfrak{X}(N)$. Also, by substituting (3.1) in (2.12), it follows that

$$D_U \eta = \kappa U - Q(U) + \phi(U), \tag{3.2}$$

for all $U \in \mathfrak{X}(N)$.

The following lemma gives the value of the Ricci curvature in the direction of η.

Lemma 3.1. *Let (N, g, η, κ) be an almost Ricci soliton on an n-dimensional Riemannian manifold (N, g). Then*

$$Ric(\eta, V) = (1 - n)V(\kappa) + \frac{1}{2}V(S) - g(div(\phi), V), \tag{3.3}$$

for all $V \in \mathfrak{X}(N)$.

Proof. From (2.1), we obtain

$$R(U, V)\eta = U(\kappa)V - V(\kappa)U - (D_U Q)(V) + (D_V Q)(U) + (D_U \phi)(V) - (D_V \phi)(U),$$

for all $U, V \in \mathfrak{X}(N)$.

Assume $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on (N, g) such that it is parallel. Considering equation (1.1) and Corollary 54 in [17], p 88, we obtain

$$\begin{aligned} Ric(V, \eta) &= \sum_{i=1}^{n} g(R(e_i, V)\eta, e_i) \\ &= \sum_{i=1}^{n} g(e_i(\kappa)V - V(\kappa)e_i - (D_{e_i} Q)(V) + (D_V Q)(e_i) \\ &+ (D_{e_i}\phi)(V) - (D_V \phi)(e_i), e_i) \\ &= V(\kappa) - nV(\kappa) + g(\mathrm{div}(Q), V) - g(\mathrm{div}(\phi), V) \\ &= (1 - n)V(\kappa) + \frac{1}{2}V(S) - g(\mathrm{div}(\phi), V). \end{aligned}$$

□

Remark 1. *From the above lemma, we see that*

$$Q(\eta) = (1-n)D\kappa + \frac{1}{2}DS - div(\phi). \tag{3.4}$$

In the following lemma, we compute the divergence of the image of η by the Ricci tensor Q.

Lemma 3.2. *Let* (N, g, η, κ) *be an almost Ricci soliton on an n-dimensional Riemannian manifold* (N, g)*. Then*

$$div(Q(\eta)) = \frac{1}{2}g(DS, \eta) + \kappa S - \|Q\|^2. \tag{3.5}$$

Proof. Consider $\{e_1, \ldots, e_n\}$ as a local orthonormal frame on (N, g). By considering (3.4), we deduce

$$\begin{aligned}
\mathrm{div}(Q(\eta)) &= \sum_{i=1}^{n} g(D_{e_i}Q(\eta), e_i) \\
&= \sum_{i=1}^{n} g((D_{e_i}Q)(\eta), e_i) + \sum_{i=1}^{n} g(Q(D_{e_i}\eta), e_i) \\
&= \sum_{i=1}^{n} g(\eta, (D_{e_i}Q)e_i) + \sum_{i=1}^{n} g(Q(D_{e_i}\eta), e_i) \\
&= g(\eta, \sum_{i=1}^{n}(D_{e_i}Q)e_i) + \sum_{i=1}^{n} g(D_{e_i}\eta, Q(e_i)) \\
&= g(\eta, \frac{1}{2}DS) + \sum_{i=1}^{n} g(\kappa e_i - Q(e_i) + \phi(e_i), Q(e_i)) \\
&= g(\eta, \frac{1}{2}DS) + \kappa\sum_{i=1}^{n} g(e_i, Q(e_i)) - \sum_{i=1}^{n} g(Q(e_i), Q(e_i)) \\
&+ \sum_{i=1}^{n} g(Q(e_i), \phi(e_i)) \\
&= g(\eta, \frac{1}{2}DS) + \kappa S - \|Q\|^2.
\end{aligned}$$

□

The following formula is well known in the case of Ricci solitons. We extend it here to the case of almost Ricci solitons.

Lemma 3.3. *Let* (N, g) *be an n-Riemannian manifold. If* (N, g, η, κ) *is an almost Ricci soliton on* (N, g)*, then*

$$\|Ric - \frac{S}{n}g\|^2 = \|Q\|^2 - \frac{S^2}{n}. \tag{3.6}$$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the Ricci operator Q. Then

$$\|Ric - \frac{S}{n}g\|^2 = \sum_{i=1}^{n}\left(\lambda_i - \frac{S}{n}\right)^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\lambda_i^2 + \frac{S^2}{n^2} - 2\frac{S}{n}\lambda_i\right)\\
&= \sum_{i=1}^{n}\lambda_i^2 + \sum_{i=1}^{n}\frac{S^2}{n^2} - 2\frac{S}{n}\sum_{i=1}^{n}\lambda_i\\
&= \|Q\|^2 + \frac{S^2}{n} - 2\frac{S^2}{n}\\
&= \|Q\|^2 - \frac{S^2}{n}.
\end{aligned}
$$

□

The following lemma provides a useful formula for the Laplacian of the potential vector field for an almost Ricci soliton.

Lemma 3.4. *Let (N, g, η, κ) be an almost Ricci soliton on an n-Riemannian manifold (N, g). Then*

$$
\Delta\eta = (2-n)D\kappa - Q(\eta). \tag{3.7}
$$

Proof. Consider $\{e_1, \dots, e_n\}$ as a local orthonormal frame on (N, g). Considering Eq (1.1) and Corollary 54 in [17], p 88, we obtain

$$
\begin{aligned}
\Delta\eta &= \sum_{i=1}^{n} D_{e_i}D_{e_i}\eta - D_{D_{e_i}e_i}\eta\\
&= \sum_{i=1}^{n} D_{e_i}(\kappa e_i - Q(e_i) + \phi(e_i))\\
&= \sum_{i=1}^{n}\left(g(\kappa, e_i)e_i - D_{e_i}Q(e_i) + D_{e_i}\phi(e_i)\right)\\
&= \sum_{i=1}^{n}\left(g(\kappa, e_i)e_i - (D_{e_i}Q)(e_i) + (D_{e_i}\phi)(e_i)\right)\\
&= D\kappa - \frac{1}{2}DS + \operatorname{div}(\phi).
\end{aligned}
$$

Inserting (3.4) in the preceding equation yields (3.7). □

3.2. Rigidity results

Theorem 3.5. *Let (N, g, η, κ) be an almost Ricci soliton on an n-dimensional compact Riemannian manifold (N, g), where $n \geq 2$. If the inequality*

$$
\int_N \left(Ric(\eta, \eta) + n(\kappa^2 + g(\eta, D\kappa))\right) dv \leq 0 \tag{3.8}
$$

is satisfied, then the manifold (N, g) is Ricci flat, and η is a closed conformal vector field.

Proof. By (3.3) and (3.5), we have

$$\begin{aligned}Ric(\eta,\eta) &= (1-n)g(\eta, D\kappa) + \frac{1}{2}g(\eta, DS) - g(\mathrm{div}(\phi),\eta)\\ &= (1-n)g(\eta, D\kappa) + \frac{1}{2}g(\eta, DS) + \|\phi\|^2 + \mathrm{div}(\phi(\eta))\\ &= (1-n)g(\eta, D\kappa) + \mathrm{div}(Q(\eta)) - \kappa S + \|Q\|^2 + \|\phi\|^2 + \mathrm{div}(\phi(\eta)).\end{aligned}$$

Consequently, it follows that

$$Ric(\eta,\eta) + (n-1)g(\eta, D\kappa) = \mathrm{div}(Q(\eta)) - \kappa S + \|Q\|^2 + \|\phi\|^2 + \mathrm{div}(\phi(\eta)). \tag{3.9}$$

By substituting the value of κS from (2.5) in (3.9), we deduce that

$$Ric(\eta,\eta) = -ng(\eta, D\kappa) + \mathrm{div}(Q(\eta)) - \mathrm{div}(\kappa\eta) - n\kappa^2 + \|Q\|^2 + \|\phi\|^2 + \mathrm{div}(\phi(\eta)). \tag{3.10}$$

By integrating (3.10), one obtains

$$\int_N \Big(Ric(\eta,\eta) + n(\kappa^2 + g(\eta, D\kappa))\Big)\,dv = \int_N \Big(\|Q\|^2 + \|\phi\|^2\Big)\,dv. \tag{3.11}$$

Now, it is clear that condition (3.8) in (3.11) yields $Q = \phi = 0$. We deduce from (2.12) that $D\eta = \kappa I$, which means that η is a closed conformal vector field.

Also, since $Q = 0$ we get $Ric = 0$, meaning that (N, g) is Ricci flat. □

Theorem 3.6. *Let (N, g) be an n-dimensional connected and compact Riemannian manifold, where $n \geq 3$. If (N, g, η, κ) is a non-parallel almost Ricci soliton, then (N, g) is isometric to a Euclidean sphere $S^n(c)$ if and only if the subsequent inequality is satisfied*

$$\int_N (Ric(\eta,\eta) + (n-1)g(\eta, D\kappa))\,dv \leq 0. \tag{3.12}$$

Proof. From (2.6), we have

$$\kappa S = \frac{1}{n}\Big(\mathrm{div}(S\eta) + S^2 - g(\eta, DS)\Big). \tag{3.13}$$

By substituting the value of κS in (3.9) and then utilizing (3.6), we obtain

$$\begin{aligned}Ric(\eta,\eta) + (n-1)g(\eta, D\kappa) &= \|Ric - \frac{S}{n}g\|^2 + \mathrm{div}\left(Q(\eta) - \frac{1}{n}\mathrm{div}(S\eta) + \phi(\eta)\right)\\ &\quad + \frac{1}{n}g(\eta, DS) + \|\phi\|^2.\end{aligned} \tag{3.14}$$

On the other hand, Theorem 1 in [4] states that

$$\int_N \|Ric - \frac{S}{n}g\|^2 dv = \frac{n-2}{2n}\int_N g(\eta, DS)dv. \tag{3.15}$$

By integrating Eq (3.14), followed by the substitution of equation (3.15) in the result, we derive

$$\int_N (\mathrm{Ric}(\eta,\eta) + (n-1)g(\eta, D\kappa))\,dv = \frac{1}{2}\int_N \|Ric - \frac{S}{n}g\|^2 dv + \int_N \|\phi\|^2 dv.$$

Taking into account the hypothesis (3.12), we deduce that $Ric = \frac{S}{n}g$ and $\phi = 0$.

Given that $n \geq 3$ and $Ric = \frac{S}{n}g$, the Schur Lemma (refer to [17]) can be applied, implying that S must be constant. Also, because $\phi = 0$, η is a closed conformal vector field with conformal function $\kappa - \frac{S}{n}$, that is

$$D_U\eta = \left(\kappa - \frac{S}{n}\right)U, \tag{3.16}$$

for all $U \in \mathfrak{X}(N)$.

Given that η is not parallel, it follows that $\kappa \neq \frac{S}{n}$ meaning that η is a non-Killing conformal vector field. From the fact that $Ric = \frac{S}{n}g$ and equation (3.16), we obtain

$$L_\eta Ric = \frac{2S}{n}\left(\kappa - \frac{S}{n}\right)g. \tag{3.17}$$

This relation allows us to apply Theorem 4.2 from [18], p. 54, leading to the conclusion that (N, g) is isometric to the standard Euclidean sphere $S^n(c)$ with radius $\frac{1}{\sqrt{c}}$, where $c = \frac{S}{n(n-1)}$.

Conversely, assume that (N, g, η, κ) is a non-parallel almost Ricci soliton on a connected, compact Riemannian manifold (N, g), where (N, g) is isometric to a Euclidean sphere $S^n(c)$, with $c > 0$. In other words, the Ricci curvature is expressed as $Ric = (n-1)cg$, whereas the scalar curvature is defined by $S = n(n-1)c$. From (1.1), it follows that

$$\frac{1}{2}L_\eta g = (\kappa - (n-1)c)g. \tag{3.18}$$

But, by Corollary 2 in [4], η is gradient, i.e, $\eta = Df$ for some smooth function f on N. Then (3.18) becomes

$$\mathbf{H}^f = (\kappa - (n-1)c)g.$$

It necessarily follows that $\kappa \neq (n-1)c$, as otherwise $D^2 f = 0$, implying that $Df = \eta$ is a constant vector field, indicating that η is parallel.

As demonstrated by Lemma 2.3 in [18] on page 52, Eq (3.18) shows that

$$\Delta\kappa = -nc(\kappa - (n-1)).$$

Additionally, as indicated by (2.4), it follows that

$$\Delta f = n\kappa - S = n(\kappa - (n-1)).$$

We deduce that $\Delta(\kappa + cf) = 0$. According to the well-known Hopf Lemma, this implies that $\kappa + cf$ is a constant.

Consequently, $D\kappa = -cDf = -c\eta$, and therefore

$$\mathrm{Ric}(\eta, \eta) + (n-1)g(\eta, D\kappa) = c(n-1)|\eta|^2 - c(n-1)|\eta|^2 = 0.$$

This demonstrates that the inequality (3.12) holds true as an equality. □

Next, we shall give an example of an almost Ricci soliton on the Euclidean sphere $S^n(c)$, where $c > 0$ satisfies (3.12).

Example 1. *Consider $(S^n(c), \langle , \rangle)$, the Euclidean sphere in the Euclidean space $(\mathbb{R}^{n+1}, \langle , \rangle)$ of radius $\frac{1}{\sqrt{c}}$, where $c > 0$.*

Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections of $S^n(c)$ and $\mathbb{R}^{n+1}$, respectively.

Let $\bar{Z} \in \mathfrak{X}(\mathbb{R}^{n+1})$ be a constant vector field, with Z as its restriction to $S^n(c)$.

If $\psi : S^n(c) \longrightarrow \mathbb{R}^{n+1}$ represents the position vector field, then consider the unit normal vector field on $S^n(c)$, defined by $N = \sqrt{c}\psi$.

Write

$$Z = Z^T + \theta N, \tag{3.19}$$

where Z^T is the tangential component of Z, and $\theta = \langle Z, N\rangle$.

Employing the Gauss and Weingarten formulas, we obtain

$$\nabla_U Z^T = -\sqrt{c}\theta U, \tag{3.20}$$

and

$$\nabla\theta = \sqrt{c}Z^T. \tag{3.21}$$

Consequently, Z^T serves as a closed conformal vector field associated with the conformal function $-\sqrt{c}\theta$.

Assume that $(S^n(c), \langle , \rangle, Z^T, \kappa)$ is an almost Ricci soliton on $(S^n(c), \langle , \rangle)$. Given that for the Euclidean sphere $(S^n(c)$, we have $Ric = (n-1)cg$, and from (3.20) we know that $\frac{1}{2}L_{Z^T} g = -\sqrt{c}\theta g$, by substituting these expressions into equation (1.1), we deduce that

$$-\sqrt{c}\theta I + (n-1)cI = \kappa I,$$

indicating that the function κ is given explicitely by

$$\kappa = (n-1)c - \theta\sqrt{c}.$$

Since $Ric(Z^T, Z^T) = (n-1)c\langle Z^T, Z^T\rangle$ and $\langle Z^T, D\kappa\rangle = -c\langle Z^T, Z^T\rangle$, it follows that

$$\int_{S^n(c)} \Big(Ric(Z^T, Z^T) + (n-1)\langle Z^T, D\kappa\rangle\Big)\, dv = 0,$$

meaning that the inequality (3.12) is satisfied as an equality.

We close this subsection with the following result, which characterizes Euclidean spheres in terms of the constancy of the scalar curvature along the integral curves of the potential vector field.

Theorem 3.7. *Let (N, g) be an n-dimensional compact oriented Riemannian manifold, $n \geq 3$. If (N, g, η, κ) is a non-trivial almost Ricci soliton on (N, g), then (N, g) is isometric to a Euclidean sphere if and only if η leaves the scalar curvature S constant (i.e, S is constant along the integral curves of η).*

Proof. Assume that η leaves the scalar curvature S constant. This means that $g(\eta, DS) = 0$.

Substituting this in Eq (3.15), we obtain

$$\int_N \|Ric - \frac{S}{n}g\|^2 dv = 0.$$

This implies that $Ric = \frac{S}{n}g$. With $n \geq 3$, Schur's Lemma implies that S must be constant. According to (1.1), we have

$$\frac{1}{2}L_\eta g = (\kappa - \frac{S}{n})g = \frac{1}{n}(n\kappa - S)g = \frac{\text{div}(\eta)}{n}g.$$

Given that (N, g, η, κ) is non-trivial, it follows that $\text{div}(\eta) \neq 0$, indicating that η is a non-trivial conformal vector field. It follows that η is a non-homothetic conformal vector field on a compact Einstein Riemannian manifold. Following a similar argument as in the proof of Theorem 3.6 above, and applying Theorem 4.2 from [18], p. 54, we conclude that (N, g) is isometric to a Euclidean sphere. The converse is trivial, since if (N, g) is isometric to a Euclidean sphere, then the scalar curvature is constant. □

3.3. Characterization of Ricci solitons

The next result extends Theorem 3 from [3], shown when $n \geq 3$ and with the conclusion that η is a Killing vector field. It also generalizes Theorem 1.1 from [16], which proved for a Ricci soliton on a compact Riemannian manifold that satisfies $\int_N (\text{Ric}(\eta, \eta))\, dv \leq 0$.

Theorem 3.8. *Let (N, g) be an n-dimensional compact Riemannian manifold, $n \geq 2$. If (N, g, η, κ) is an almost Ricci soliton on (N, g) satisfying the following inequality*

$$\int_N (Ric(\eta, \eta) + (n-2)g(\eta, D\kappa))\, dv \leq 0, \tag{3.22}$$

then η is a parallel vector field.

Proof. By Lemma 2 in [3], we have

$$\frac{1}{2}\Delta|\eta|^2 = |D\eta|^2 - Ric(\eta, \eta) - (n-2)g(\eta, D\kappa). \tag{3.23}$$

By integrating the above equation, we obtain

$$\int_N (\text{Ric}(\eta, \eta) + (n-2)g(\eta, D\kappa))\, dv = \int_N |D\eta|^2 dv, \tag{3.24}$$

and from the hypothesis (3.22), it follows that $D\eta = 0$, which means that η is parallel.

□

Theorem 3.9. *Let (N, g, η, κ) be an almost Ricci soliton on an n-dimensional compact Riemannian manifold (N, g), where $n \geq 2$. If (N, g, η, κ) satisfies the following inequality*

$$\int_N (Ric(Q(\eta), \eta) + (n-1)g(DS, D\kappa))\, dv \leq 0, \tag{3.25}$$

then η is a parallel vector field, and (N, g) is Einstein.

Proof. By (3.3) and (3.4), we have

$$Ric(Q(\eta), \eta) + Ric(\text{div}(\phi), \eta) = \frac{1}{2}Ric(\eta, DS) - (n-1)Ric(\eta, D\kappa)$$

$$= \frac{1}{2}\left(-(n-1)g(DS, D\kappa) + \frac{1}{2}|DS|^2 - g(\mathrm{div}(\phi, DS)\right)$$
$$-(n-1)\left(-(n-1)g(D\kappa, \eta) + \frac{1}{2}g(DS, S\kappa) - g(\mathrm{div}(\phi), D\kappa)\right)$$
$$= -(n-1)g(DS, D\kappa) + \frac{1}{2}\mathrm{div}(\phi(DS))$$
$$+ \frac{1}{4}|DS|^2 + (n-1)^2|D\kappa|^2 - (n-1)\mathrm{div}(\phi(D\kappa)).$$

So, we have

$$Ric(Q(\eta), \eta) + Ric(\mathrm{div}(\phi), \eta) = -(n-1)g(DS, D\kappa) + \frac{1}{2}\mathrm{div}(\phi(DS)) + \frac{1}{4}|DS|^2 + (n-1)^2|D\kappa|^2 - (n-1)\mathrm{div}(\phi(D\kappa)). \tag{3.26}$$

Also, from (3.3), one obtains

$$Ric(\mathrm{div}(\phi), \eta) = (n-1)div(\phi(D\kappa) - \frac{1}{2}\mathrm{div}(\phi(DS)) - |\mathrm{div}(\phi)|^2. \tag{3.27}$$

Substituting (3.27) in (3.26) and integrating the resulting equation yields

$$\int_N (\mathrm{Ric}(Q(\eta), \eta) + (n-1)g(DS, D\kappa))\, dv = \int_N \left(\frac{1}{4}|DS|^2 + (n-1)^2|D\kappa|^2 + |\mathrm{div}(\phi)|^2\right) dv. \tag{3.28}$$

Assuming the hypothesis (3.25), it follows that the scalar curvature S and the potential function κ are constant, and $\mathrm{div}(\phi) = 0$. From equation (2.4), it is deduced that $n\kappa = S$.

From (3.5), it follows

$$||Q||^2 = \kappa S.$$

It follows that Eq (3.6) becomes

$$||Ric - \frac{S}{n}g||^2 = 0.$$

This implies that (N, g) is an Einstein manifold, and that η is a Killing vector field.

Furthermore, given that $\int_N Ric(\eta, \eta)dv = 0$ from (3.28), then (3.24) implies that

$$\int_N |D\eta|^2 dv = 0,$$

which means that $D\eta = 0$, and so η is a parallel vector field. □

Remark According to [19], it is proved that an n-dimensional compact almost Ricci soliton (N, g, η, κ), where $n \geq 3$, becomes a trivial Ricci soliton precisely when $n\kappa - S$ remains constant along the integral curve of the potential field η. This condition, essentially, is equivalent to $\eta.\mathrm{div}(\eta) = 0$. We aim to extend this result as follows.

Theorem 3.10. *An almost Ricci soliton (N, g, η, κ) on an n-dimensional compact Riemannian manifold with $n \geq 3$ is trivial if and only if the vector field η is incompressible, meaning that $div(\eta) = 0$.*

Proof. Assume that (N, g, η, κ) is trivial. Then, η is a Killing vector field, meaning that $L_\eta g = 0$. Consequently, $\mathrm{div}(\eta) = 0$.

Conversely, by assuming that $\mathrm{div}(\eta) = 0$, we deduce that $n\kappa = S$. By integrating Eq (2.6), we obtain

$$\int_N g(\eta, DS)dv = 0.$$

Substituting this in Eq (3.15), it follows that

$$\int_N \|Ric - \frac{S}{n}g\|^2 dv = 0.$$

This implies that $Ric = \frac{S}{n}g$. With $n \geq 3$, Schur's Lemma indicates that S must be constant. Since $n\kappa = S$, it follows that κ is also constant. Therefore, (N, g, η, κ) is trivial.

□

Upon reviewing the previous outcome, we derive the following result:

Corollary 1. *A compact almost Ricci soliton of dimension $n \geq 3$ with an affine potential vector field is trivial.*

Proof. By [20], any affine vector field on a compact Riemannian manifold is incompressible. □

Theorem 3.11. *Let (N, g, η, κ) be an almost Ricci soliton on an n-dimensional compact Riemannian manifold (N, g), where $n \geq 2$. If (N, g, η, κ) satisfies the following inequality*

$$\int_N (g(\eta, DS) - 2g(\eta, D\kappa))\, dv \leq 0, \tag{3.29}$$

then, η is a Killing vector field, and (N, g, η, κ) is trivial.

Proof. By taking the inner product of (3.7) with η and subsequently integrating, it follows

$$\int_N (Ric(\eta, \eta) + g(\eta, \Delta\eta) + (n-2)g(\eta, D\kappa))\, dv = 0. \tag{3.30}$$

According to Proposition 5.10 in [21], it follows that

$$\int_N \left(Ric(\eta, \eta) + g(\eta, \Delta\eta) + \frac{1}{2}|L_\eta g|^2 - (\mathrm{div}(\eta))^2 \right) dv = 0. \tag{3.31}$$

Using Eq (2.12), we obtain

$$|D\eta|^2 = |B|^2 + |\phi|^2,$$

and

$$tr(D\eta)^2 = |B|^2 - |\phi|^2,$$

where $tr(D\eta)^2$ represents the trace of $(D\eta)^2$.

Referencing Lemma 5.9 in [21], we have

$$|L_\eta g|^2 = 4|B|^2. \tag{3.32}$$

Utilizing (3.31) and considering Eqs (2.7) and (3.32), we obtain

$$\int_N (g(\eta, DS) - 2g(\eta, D\kappa))\, dv = 2 \int_N |B|^2 dv. \tag{3.33}$$

Assuming that $\int_N (g(\eta, S) - 2g(\eta, D\kappa))\, dv \leq 0$, we deduce that $B = 0$, implying that $Ric = \kappa g$. According to (1.1), η becomes a Killing vector field, and the almost Ricci soliton is trivial. □

4. Conclusions

In this paper, we extend and refine recent works about Ricci solitons and almost Ricci solitons by using methods based on integral inequalities that involve Ricci curvature and the potential vector field. Unlike previous studies that often focused on specific curvature assumptions or geometric structures, we use integral conditions to set clear criteria for when a compact Riemannian manifold with an almost Ricci soliton is isometric to a sphere. We also explore the potential vector field in more detail, identifying conditions where it becomes either Killing or parallel, making the almost Ricci soliton trivial. By focusing on both the geometry and the properties of the vector field, our work not only broadens the scope of earlier results but also provides a more unified approach for studying solitons with fewer assumptions. Our methods and findings show how effective integral techniques are in analyzing Ricci-type structures and contribute to the ongoing development of Ricci soliton theory. We expect that future research will focus on exploring the properties of almost Ricci solitons in important contexts, like spacelike submanifolds of Lorentzian manifolds, especially within generalized Robertson-Walker (GRW) spacetimes and general Lorentzian warped products.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

Author contributions

Conceptualization, M.G.; Investigation, N.A. and M.G.; Methodology, N.A. and M.G.; Resources, N.A.; Validation, M.G.; Writing-original draft, N.A.; Writing-review and editing, N.A. and M.G.

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