Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 13512–13523.
DOI: 10.3934/math.2025607
Received: 25 February 2025
Revised: 28 May 2025
Accepted: 06 June 2025
Published: 12 June 2025

Research article

An asymptotically probabilistic method for a class of partial integrodifferential equations

Alioune Coulibaly*

Département Mathématiques, Informatique et Modélisation, UFR Sciences et Technologies Avancées, Université Amadou Mahtar Mbow, Pôle Urbain de Diamniadio - BP : 45927 Dakar NAFA VDN Dakar, Sénégal

* **Correspondence:** Email: alioune.coulibaly@uam.edu.sn.

Abstract: In this paper, we consider a nonlocal boundary condition and examine the asymptotic behavior of the solution to a family of nonlocal partial differential equations in the half-space. Our approach is fully probabilistic and builds upon the works of Huang et al. Bernoulli, 28 (2022), 1648–1674 and Diakhaby et al. Stoch. Anal. Appl., 34 (2016), 496–509. Reflected stochastic differential equations, driven by multiplicative Lévy noise and with singular coefficients, play an important role in our method.

Keywords: nonlocal PDE; Hamilton-Jacobi equations; viscosity solutions; Lévy noise
Mathematics Subject Classification: 60H30, 35K40

1. Introduction

Let $(b, c, \sigma) : \mathbf{D}^3 \times \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^d$, $(\beta, \gamma, \varrho) : \partial\mathbf{D}^3 \times \mathbb{R}^{\mathrm{d}-1} \longrightarrow \mathbb{R}^d$, and ε be small positive. Our principal focus is the limit-solution, when ε goes to zero, of the following nonlocal partial differential equation (PDE) with rapidly oscillating coefficients:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \mathcal{L}_\varepsilon^{\sigma,b,c} u^\varepsilon(t,x) + \dfrac{1}{\varepsilon} g(x_\varepsilon, u^\varepsilon(t,x)).u^\varepsilon(t,x), & x \in \mathbf{D}, \\ \mathcal{L}_\varepsilon^{\varrho,\beta,\gamma} u^\varepsilon(t,x) + \dfrac{1}{\varepsilon} h(x_\varepsilon, u^\varepsilon(t,x)).u^\varepsilon(t,x) = 0, & x \in \partial\mathbf{D}, \\ u^\varepsilon(0,x) = u_0(x), & x \in \overline{\mathbf{D}}, \end{cases} \tag{1.1}$$

where $\mathbf{D} = \{\left(x^1, \ldots, x^d\right) \in \mathbb{R}^d : x^1 > 0\}$. Definitively, the boundary $\partial\mathbf{D}$ is supposed to be homeomorphic to $\mathbb{R}^{d-1}$. Letting set $x_\varepsilon := \left(\frac{x}{\delta_\varepsilon}\right)_{\varepsilon>0}$, $\delta_\varepsilon > 0$, the family of linear integro-differential operators $\mathcal{L}_{\varepsilon}^{\varpi_i,\tau_i,\rho_i}$ $(i := 1, 2)$ are given by

$$\mathcal{L}_{\varepsilon}^{\varpi_i,\tau_i,\lambda_i} f(x) := \int_{\mathbb{R}_*^{d+1-i}} \left[f\left(x + \varepsilon\varpi_i\left(x_\varepsilon, y\right)\right) - f(x) - \varepsilon \sum_{j=1}^{d} \varpi_i^j\left(x_\varepsilon, y\right) \partial_j f(x) \right] \nu_i^{\varepsilon}(dy)$$
$$+ \left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1} \sum_{j=1}^{d} \tau_i^j\left(x_\varepsilon\right) \partial_j f(x) + \sum_{j=1}^{d} \lambda_i^j\left(x_\varepsilon\right) \partial_j f(x),\ x \in \mathbb{R}^{d+1-i},$$

with ϖ_i in $\{\sigma, \varrho\}$, τ_i in $\{b, \beta\}$ and λ_i in $\{c, \gamma\}$ respectively.

Such a problem lends itself to a natural interpretation through the lens of stochastic processes, particularly via the probabilistic representation of solutions to nonlocal partial differential equations (PDEs) and integrodifferential operators. The connection with Keller-Segel-type models emerges through the structural similarities in modeling interacting particle systems and their associated mean-field limits. These links facilitate a deeper comprehension encoded in the solution u^{ε}, especially in the context of aggregation phenomena, chemotaxis, and related collective behaviors (see, for example [2, 8, 10]).

The study conducted by Diakhaby et al. [5] addresses a homogenization problem for PDEs involving Wentzell-type boundary conditions, where the principal operator is generated by Brownian motion. Their analysis relies on classical probabilistic homogenization applied to local dynamics driven by diffusive processes. In a related but distinct direction, Huang et al. [9] investigated a class of PDEs defined on the whole space $\mathbb{R}^d$, in which the governing operator is of integrodifferential type, capturing the nonlocal behavior characteristic of jump or Lévy processes. Their methodology is grounded in nonlocal analytical techniques, tailored to address the singular nature of the operator.

The present work departs from these contributions in two significant respects. First, we integrate homogenization methods with large deviation principles within a multiscale framework that incorporates both typical fluctuations and rare events. This allows us to characterize the asymptotic behavior of solutions to PDEs with rapidly oscillating coefficients while accounting for the influence of rare but significant deviations. Second, our formulation encompasses nonlocal dynamics subject to nonlocal boundary conditions, thereby extending the scope of previous analyses. In particular, our framework accommodates integrodifferential operators not only in the domain but also at the boundary, in the spirit of the approach initiated by [5], but adapted to a broader class of nonlocal phenomena.

To do this, we first derive an action functional $S_{0,t}$ expressed in terms of effective coefficients of the reflected stochastic differential equations (RSDEs). The solution u^{ε} of (1.1) is then formulated using a type of BSDE in [1]. By the logarithm transform method, a family of Hamilton-Jacobi equations is introduced [14], whose limit helps us to determine the asymptotic behavior of u^{ε} as ε vanishes. Finally, the existence of a potential function V^* (which depends on $S_{0,t}$) is established, dictating that the solution u^{ε} converges uniformly to zero when (t, x) belongs to a negatively valued V^* set and to one when V^* is zero.

The paper is organized as follows. In the next section, we present some general assumptions and preliminary results. In Section 3, we derive the action functional for RSDE driven by multiplicative Lévy noise in the half-space. The last section is devoted to the convergence of the solution of (1.1).

2. Preliminaries: Assumptions and limit RSDE

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space endowed with two independent Poisson random measures $N_1^{\varepsilon^{-1}}$ on $\mathbb{R}_*^d \times \mathbb{R}_+$ and $N_2^{\varepsilon^{-1}}$ on $\mathbb{R}_*^{d-1} \times \mathbb{R}_+$ with jump intensity measures $\nu_\varepsilon^1(dy) = \frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$ and $\nu_\varepsilon^2(dy) = \frac{\varepsilon^{-1}dy}{|y|^{(d-1)+\alpha}}$ respectively, where $\alpha \in (1,2)$, $\varepsilon > 0$. We assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual $\mathbb{P}$-null conditions. Letting $\tilde{N}_1$ and $\tilde{N}_2$ be the compensated Poisson random measures in the following sense,

$$\tilde{N}_1^{\varepsilon^{-1}}(dyds) := N_1^{\varepsilon^{-1}}(dyds) - \nu_\varepsilon^1(dy)ds,$$
$$\tilde{N}_2^{\varepsilon^{-1}}(dyds) := N_2^{\varepsilon^{-1}}(dyds) - \nu_\varepsilon^2(dy)d\phi_s^\varepsilon.$$

Let $\left\{L_{i,t}^{\varepsilon^{-1}} : t \geq 0\right\}$ be the multi-dimensional Lévy process given by

$$L_{i,t}^{\varepsilon^{-1}} := \int_0^t \int_{B\backslash\{0\}} y\tilde{N}_i^{\varepsilon^{-1}}(dyds) + \int_0^t \int_{B^c} yN_i^{\varepsilon^{-1}}(dyds), \quad (i = 1,2),$$

where B is the open ball in $\mathbb{R}^{d+1-i}$ centering at the origin with radius one. Fixing α in $(1,2)$, we next consider the following RSDE :

$$\begin{cases} dX_t^\varepsilon = \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}} b\left(\frac{X_t^\varepsilon}{\delta_\varepsilon}\right)dt + c\left(\frac{X_t^\varepsilon}{\delta_\varepsilon}\right)dt + \varepsilon\sigma\left(\frac{X_{t-}^\varepsilon}{\delta_\varepsilon}, dL_{1,t}^{\varepsilon^{-1}}\right) + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\beta\left(\frac{X_t^\varepsilon}{\delta_\varepsilon}\right)d\phi_t^\varepsilon + \gamma\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}\right)d\phi_t^\varepsilon + \varepsilon\varrho\left(\frac{X_{t-}^\varepsilon}{\delta_\varepsilon}, dL_{2,t}^{\varepsilon^{-1}}\right), \\ X_0^\varepsilon = x \in \overline{\mathbf{D}}, \text{ and } \phi_0^\varepsilon = 0. \end{cases} \tag{2.1}$$

Note that $X^{1,\varepsilon}$ represents the first component of X^ε. We assume that X^ε evolves within $\overline{\mathbf{D}}$, i.e., $X^{1,\varepsilon}$ remains non-negative, and the local times ϕ^ε increases when and only when $X^{1,\varepsilon}$ is zero. We denote by C^k $\left(C_b^k\right)$ the spaces of (bounded) continuous functions whose derivatives up to order k (with integer $k \geq 0$) exist and are also (bounded) continuous. The Hölder spaces C^κ (C_b^κ) are defined as the subspaces of $C^{\lfloor\kappa\rfloor}$ $(C_b^{\lfloor\kappa\rfloor})$ consisting of functions whose $\lfloor\kappa\rfloor$-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent $\kappa - \lfloor\kappa\rfloor, \kappa > 0$. Notice that the two spaces $C^{\lfloor\kappa\rfloor}$ and $C_b^{\lfloor\kappa\rfloor}$ obviously coincide when the underlying domain is compact. The space $C_b^{\lfloor\kappa\rfloor}$ is a Banach space endowed with the norm $\|f\|_\kappa = \|f\|_{\lfloor\kappa\rfloor} + [\nabla^{\lfloor\kappa\rfloor}f]_{\kappa-\lfloor\kappa\rfloor}$, where the seminorm $[\cdot]_{\kappa'}$ with $0 < \kappa' \leq 1$ is defined as:

$$[f]_{\kappa'} := \sup_{x,y,x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\kappa'}}.$$

For the assumptions we start with the followings (For more details, see [9]).
$(\mathbf{A}_1)$

i) τ_i and λ_i are all periodic of period 1 in each component $(i := 1,2)$.
ii) $x \mapsto \varpi_i(x,\cdot)$ is 1-periodic in each component $(i := 1,2)$.
iii) τ_i and λ_i are of class C_b^κ with κ satisfying : $1 - \frac{\alpha}{2} < \kappa < 1$.
iv) γ is such that $\gamma^1(x) = 1$.

In the sequel, the d-vectors-valued functions ϱ and β are defined with the convention that $\varrho^1 = \beta^1 = 0$. Moreover, we impose that all jumps from $\overline{\mathbf{D}}$ are inside $\overline{\mathbf{D}}$. In other words, $\forall x \in \overline{\mathbf{D}},\ x + \varpi_i(x,\cdot) \in \overline{\mathbf{D}}$.
$(\mathbf{A}_2)$

i) $y \mapsto \varpi_i(\cdot, y)$ is of class C^2 and there exists $\alpha - 1 < \lambda \leq 1$ such that:

$$|\varpi_i(x_1, y) - \varpi_i(x_1, y)| \leq C\,|x_1 - x_2|\,|y|\,, \quad x_1', x_2' \in \mathbb{R}^{d+1-i}.$$

ii) $\forall y \in \mathbb{R}^{d+1-i}$, $\varpi_i(\cdot, -y) = -\varpi_i(\cdot, y)$, $(i := 1, 2)$.

iii) $\nabla_y \varpi_i(\cdot, y)$ is non-degenerate $(i := 1, 2)$.

iv) There exists $\xi_i \in C_b\left(\mathbb{R}^{d+1-i}, \mathbb{R}_+^*\right)$ such that $(i := 1, 2)$:

$$\xi_i(\cdot)^{-1}|y| \leq \varpi_i(\cdot, y) \leq \xi_i(\cdot)|y|.$$

Owing to the oddness condition $\mathbf{A}_2$. ii) and the inherent symmetry of the jump intensity measure ν_i, let us introduce the linear operator $\mathcal{A}^{\varpi_i,\nu_i}$ defined as [9]:

$$\mathcal{A}^{\varpi_i,\nu_i} f(x) = \int_{\mathbb{R}_*^{d+1-i}} \left[f\,(x+y) - f(x) - \sum_{j=1}^{d} y^j \partial_j f(x) \mathbf{1}_B(y) \right] \nu^{\varpi_i,i}(x, dy),$$

where the kernel $\nu^{\varpi,\alpha,i}$ is given by

$$\nu^{\varpi_i,i}(x, A) = \int_{\mathbb{R}_*^{d+1-i}} \mathbf{1}_A\,(\varpi_i(x,y))\,\nu_i(dy), \qquad x \in \mathbb{R}^{d+1-i}, A \in \mathcal{B}\left(\mathbb{R}_*^{d+1-i}\right).$$

By identifying the periodic function on $\mathbb{R}^d$ of period 1 with its restriction on the d-dimensional torus $\mathbb{T}^d$, we define the quotient process $\tilde{X}_t^\varepsilon := \frac{1}{\delta_\varepsilon} X^\varepsilon_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}$, via the canonical quotient map $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d/\mathbb{Z}^d$. It is not hard to check that the corresponding local time is $\tilde{\phi}_t^\varepsilon := \frac{1}{\delta_\varepsilon} \phi^\varepsilon_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}$. By virtue of self-similarity, we have

$$\left\{ \frac{\varepsilon}{\delta} L^{\varepsilon^{-1}}_{i,(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t} \right\} \equiv \left\{ \frac{\varepsilon}{\delta_\varepsilon} L_{i,(\delta_\varepsilon/\varepsilon)^\alpha t} \right\} \overset{d}{:=} \{L_{i,t}\}\,.$$

$(\mathbf{A}_3)$ The sequence (δ_ε) decreases at a rate greater than (ε).

Fix $x \in \mathbb{T}^d$, we next consider the following RSDE $\tilde{X}$, defined as:

$$\tilde{X}_t = x + \int_0^t b\left(\tilde{X}_s\right) ds + \int_0^t \beta\left(\tilde{X}_s\right) d\tilde{\phi}_s + \sum_{i=1}^{2} \int_0^t \overline{\varpi_i}\left(\tilde{X}_{s-}, dL_{i,s}\right),$$

where $\overline{\varpi_i}(\cdot, y) := \left\langle \nabla_y \varpi_i(\cdot, 0), y \right\rangle$ is the point-wise limit as $\varepsilon \downarrow 0$ of $\frac{\varepsilon}{\delta_\varepsilon} \varpi_i\left(\cdot, \frac{\delta_\varepsilon}{\varepsilon} y\right)$. A stronger form of convergence is required, as detailed below. Explicitly, we need the following to be in force.

$(\mathbf{A}_4)$ $\frac{1}{\eta} \varpi_i(\cdot, \eta y) \longrightarrow \overline{\varpi_i}(\cdot, y)$ uniformly, as $\eta \to 0$, $\forall y \in \mathbb{R}^{d+1-i}$ $(i := 1, 2)$.

Let's define $\mathcal{L}_i$ $(i := 1, 2)$ as the linear integro-partial differential operators given by

$$\mathcal{L}_i := \mathcal{A}^{\overline{\varpi_i},\nu_i} + \tau_i \cdot \nabla.$$

By assumptions, there exists a Feller process on $\mathbb{R}^{d+1-i}$ with generator $\mathcal{L}_i$. Furthermore, due to the periodicity of the coefficients, this process induces a process $\tilde{X}$ on $\mathbb{T}^{d+1-i}$, which is a strong Markov process. Moreover, such a process is ergodic (see, [9]). We denote by μ_i the unique invariant measure on $\left(\mathbb{T}^{d+1-i}, \mathcal{B}\left(\mathbb{T}^{d+1-i}\right)\right)$. For the limit of the RSDE (2.1), the centering condition must hold.

($\mathbf{A}_5$) $\displaystyle\int_{\mathbb{T}^{d+1-i}} \tau_i(x)\mu_i(dx) = 0$ $(i := 1, 2)$ for τ_i in $\{b, \beta\}$.

From [9, Proposition 4.11], there exists a unique periodic $\hat{b}_i \in C^{\alpha+\kappa}\left(\overline{\mathbf{D}}, \mathbb{R}^d\right)$ solution of equation:

$$\mathcal{L}_i\hat{b}_i + \tau_i = 0 \quad \text{such that} \quad \int_{\mathbb{T}^{d+1-i}} \hat{b}_i(x)\mu_i(dx) = 0. \tag{2.2}$$

We now define $(i := 1, 2)$:

$$\begin{aligned}
\overline{\lambda}_i &:= \int_{\mathbb{T}^{d+1-i}} \left(I + \nabla\hat{b}\right)(x)\lambda_i(x)\mu_i(dx),\\
\overline{\nu}_i(A) &:= \int_{\mathbb{R}_*^{d+1-i}} \int_{\mathbb{T}^{d+1-i}} \mathbf{1}_A\left(\varpi_i(x, y)\right)\mu_i(dx)\nu_i(dy),\ A \in \mathcal{B}\left(\mathbb{R}_*^{d+1-i}\right),\\
\overline{\mathcal{L}}_i f(x) &:= \int_{\mathbb{R}_*^{d+1-i}} \left[f(x + y) - f(x) - \sum_{j=1}^{d} y^j\partial_j f(x)\mathbf{1}_B(y)\right]\overline{\nu}_i(dy) + \sum_{j=1}^{d} \overline{\lambda}_i^j \cdot \partial_j f(x).
\end{aligned}$$

Now we are in a position to state the limit of the RSDE (2.1) with effective coefficients.

Theorem 2.1. *Suppose assumptions ($\mathbf{A}_1$)–($\mathbf{A}_5$) hold true. The process X^ε weakly converges to the process X as $\varepsilon \downarrow 0$. Moreover, on the space $\mathcal{D}\left([0, T], \mathbb{T}^d\right)$ equipped with the sup-norm topology,*

$$X^\varepsilon \Longrightarrow X_t = x + \overline{c}t + \overline{L}_{1,t} + \left(\overline{\gamma} + \overline{L}_{2,t}\right)\phi_t\mathbf{1}_{\{X_t^1=0\}},$$

where $\overline{L}_{i,t}$ is a symmetric Lévy process with jump intensity measure $\overline{\nu}_i(A)$, $A \in \mathcal{B}\left(\mathbb{R}_*^{d+1-i}\right)$, $i := 1, 2$.

Proof. This is the same as the combination of the proof of [9, Proposition 5.2], which applies in the absence of a boundary, and the proof of [11, Theorem 1] (see, also [13]), which accounts for the presence of a boundary. □

For the coefficients of PDE (1.1), we impose $u_0 \in C_b\left(\mathbb{R}^d\right)$ and $\sup_{x\in\mathbb{R}^d} u_0(x) < \infty$. Set $U_0 := \left\{x \in \overline{\mathbf{D}} : u_0(x) > 0\right\}$, then we have $\overline{\overset{\circ}{U}_0} = \overline{U}_0$. Let ζ_i in the set $\{g, h\}$ $(i := 1, 2)$. The function $\zeta_i : \mathbb{R}^{d+i-1} \times \mathbb{R} \to \mathbb{R}$ is periodic in each direction with respect to the first argument. We also need the following.

($\mathbf{A}_6$) $\zeta_i \in C_b^\kappa(\mathbb{R}^{d+1-i} \times \mathbb{R}, \mathbb{R})$ and satisfies:

i) $\zeta_i(x, y) > 0, \forall x \in \mathbb{R}^{d+1-i},\ y \in [0, 1) \cup \mathbb{R}_-^*$;
ii) $\zeta_i(x, y) \leq 0, \forall x \in \mathbb{R}^{d+1-i},\ y > 1$;
iii) $\max \zeta_i(x, y) = \zeta_i(x) = \zeta_i(x, 0) > 0,\ \forall x \in \mathbb{R}^{d+1-i}$.

Before concluding this section, it is interesting to make a heuristic study of (1.1). Consider a version of the BSDE that was introduced in [1].

For each fixed $(t,x) \in [0,T] \times \overline{\mathbf{D}}$, let $(Y^\varepsilon, U_i^\varepsilon)$ $(i := 1,2)$ be a solution of :

$$\begin{cases} Y_t^\varepsilon = u_0(X_t^\varepsilon) + \frac{1}{\varepsilon}\int_s^t g\left(\frac{X_r^\varepsilon}{\delta_\varepsilon}, Y_r^\varepsilon\right) \cdot Y_r^\varepsilon dr + \frac{1}{\varepsilon}\int_s^t h\left(\frac{X_r^\varepsilon}{\delta_\varepsilon}, Y_r^\varepsilon\right) \cdot Y_r^\varepsilon d\phi_r^\varepsilon - \sum_{i=1}^{2}\int_s^t U_{i,r}^\varepsilon dL_{i,r}\,, \\ \sqrt{\mathbb{E}\int_s^t \int_{\mathbb{R}_*^d} U_{1,r}^\varepsilon(y)^2 \nu_1(dy)dr} + \sqrt{\mathbb{E}\int_s^t \int_{\mathbb{R}_*^{d-1}} U_{2,r}^\varepsilon(y)^2 \nu_2(dy)d\phi_r^\varepsilon} < \infty. \end{cases}$$

We have for all $(t,x) \in [0,+\infty[\times \overline{\mathbf{D}}$, the solution $u^\varepsilon(t,x)$ of the PDE (1.1) is of the form

$$Y_0^\varepsilon = u^\varepsilon(t,x).$$

Remark 2.2.

- If $\overline{u}_0 \le 1$, then $\forall \varepsilon > 0,\ 0 \le Y_s^\varepsilon \le 1, d\mathbb{P} \times ds\ a.s.$.
- If $\zeta_i(\cdot,y) \le \kappa_i(y) < 0\ y \in]1,+\infty[$, where κ_i $(i = 1,2)$ is Lipschitz continuous, then $\limsup_{\varepsilon\to 0} Y_t^\varepsilon \le 1$ uniformly in any compact set of $]0,+\infty[\times \mathbb{R}^d$.

To demonstrate this, we will follow by combining similar results established in [12, 14].

3. Action functional

The corresponding action functional is obtained by the Legendre transform of the limit (if it exists, [6, 7]) of the logarithmic moment generating function defined as :

$$\lim_{\varepsilon\to 0} \Lambda_t^\varepsilon(\theta) := \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left(\frac{1}{\varepsilon}\langle \theta, X_t^\varepsilon\rangle\right)\right\}.$$

Let $\hat{b} \in C^{\alpha+\kappa}\left(\overline{\mathbf{D}}, \mathbb{R}^d\right)$ be as in (2.2), and let $\eta \in C^{\alpha+\kappa}\left(\overline{\mathbf{D}}, \mathbb{R}^d\right)$ be the solution of the system

$$\begin{cases} \mathcal{L}_1\eta = 0 & \text{in } \mathbb{R}^d, \\ \mathcal{L}_2\eta := \overline{\gamma} - \left(I + \nabla\hat{b}\right)\gamma & \text{on } \partial\mathbf{D}. \end{cases} \tag{3.1}$$

In view of the fact that $\nu_i(\varepsilon A) = \varepsilon^{-\alpha}\nu_i^\varepsilon(A)$ $(i := 1,2)$, let us introduce for all $\varphi \in C^{\alpha+\kappa}$

$$\begin{aligned} H_i^{\varepsilon,\varphi}(x,\cdot) &:= \varphi\Big(x + \frac{\varepsilon}{\delta_\varepsilon}\varpi_i(x, \frac{\delta_\varepsilon}{\varepsilon}\cdot)\Big) - \varphi(x), & x \in \mathbb{R}^{d+1-i}, \\ Q_i^{\varepsilon,\varphi}(x) &:= \mathcal{A}^{\frac{\varepsilon}{\delta_\varepsilon}\varpi_i(\cdot,(\delta_\varepsilon/\varepsilon)\cdot),\nu^{\alpha,i}}\varphi(x) - \mathcal{A}^{\overline{\varpi}_i,\nu^{\alpha,i}}\varphi(x), & x \in \mathbb{R}^{d+1-i}. \end{aligned}$$

From [9, Lemma 5.3] we notice that: for all $\varphi \in C^{\alpha+\kappa}$,

$$\sup_{0\le t\le T}\left|\delta_\varepsilon \int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} Q_1^{\varepsilon,\varphi}\left(\tilde{X}_s^\varepsilon\right) ds + \delta_\varepsilon \int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} Q_2^{\varepsilon,\varphi}\left(\tilde{X}_s^\varepsilon\right) d\tilde{\phi}_s^\varepsilon\right| \longrightarrow 0, \tag{3.2}$$

$$\sup_{0\le t\le T}\left|\delta_\varepsilon \int_{\mathbb{R}_*^{d+1-i}} \int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} H_i^{\varepsilon,\varphi}\left(\tilde{X}_s^\varepsilon, y\right) \tilde{N}_i(dyds)\right| \longrightarrow 0. \tag{3.3}$$

In addition, as $\int_{\mathbb{T}^{d-1}} \mathcal{L}_2\eta(x)\mu_2(dx) = 0$, we have (see, for example [3,4]):

$$\lim_{\varepsilon\to 0}\mathbb{E}\left\{\max_{0\le t\le T}\left|\Xi_t^\varepsilon + \delta_\varepsilon\left[\eta\left(\tilde{X}^\varepsilon_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right) - \eta\left(x_\varepsilon\right)\right]\right|\right\} = 0,$$

where

$$\Xi_t^\varepsilon := -\delta_\varepsilon\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \mathcal{A}^{(\varepsilon/\delta_\varepsilon)\varrho\left(\cdot,\frac{\delta_\varepsilon}{\varepsilon}\cdot\right),\nu_2}\hat{b}\left(\tilde{X}_s^\varepsilon\right)d\tilde{\phi}_s^\varepsilon - \int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\nabla\eta b\left(\tilde{X}_s^\varepsilon\right)ds.$$

Using Itô's formula along with the correctors defined in Eqs (2.2) and (3.1), we have

$$\begin{aligned}\hat{X}_t^{\varepsilon,\delta_\varepsilon} &= X_t^\varepsilon + \delta_\varepsilon\left[\hat{b}\left(\frac{X_t^\varepsilon}{\delta_\varepsilon}\right) - \hat{b}\left(x_\varepsilon\right)\right] + \delta_\varepsilon\left[\eta\left(\frac{X_t^\varepsilon}{\delta_\varepsilon}\right) - \eta\left(x_\varepsilon\right)\right] + \Xi_t^\varepsilon \\ &= x + \int_0^t\left(I + \nabla\hat{b}_\varepsilon\right)c_\varepsilon\left(X_s^\varepsilon\right)ds + \overline{\gamma}\int_0^t \mathbf{1}_{\{X_s^\varepsilon\in\partial\mathbf{D}\}}d\phi_s^\varepsilon + \int_0^t\varepsilon\sigma_\varepsilon\left(X_{s-}^\varepsilon, dL_{1,s}^{\varepsilon^{-1}}\right) + \int_0^t\varepsilon\varrho_\varepsilon\left(X_{s-}^\varepsilon, dL_{2,s}^{\varepsilon^{-1}}\right) + o(1),\end{aligned}$$

where $f_\varepsilon(x) = f\left(\frac{x}{\delta_\varepsilon}\right)$ for $f(x)$ in $\left\{c, \nabla\hat{b}(x), \sigma(x,\cdot), \varrho(x,\cdot)\right\}$. Please note that $\hat{X}^\varepsilon$ and X^ε have the same limit, when ε goes to zero. For $\theta\in\mathbb{R}^d$, let $\mathcal{H}_i^{\varpi_i(.,y),\nu_i}$ be defined by:

$$\mathcal{H}^{\varpi_i(.,y),\nu_i}(\theta) := \int_{\mathbb{R}_*^{d+1-i}}\left\{e^{\langle\theta,\varpi_i(.,y)\rangle} - 1 - \langle\theta,\varpi_i(.,y)\rangle\,\mathbf{1}_B(y)\right\}\nu_i(dy).$$

From Girsanov's formula:

$$\begin{aligned}\Lambda_t^\varepsilon(\theta) = \langle\theta,x\rangle + \varepsilon\log\tilde{\mathbb{E}}\Bigg\{&\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\left(I+\nabla\hat{b}\right)c\left(\tilde{X}_s^\varepsilon\right)\rangle ds\right) \\ &\times\exp\left(\frac{1}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\overline{\gamma}\cdot\mathbf{1}_{\{\tilde{X}_s^{1,\varepsilon}=0\}}\rangle d\phi^\varepsilon_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})s}\right) \\ &\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{H}^{\sigma\left(\tilde{X}_s^\varepsilon,y\right),\nu_1}ds + \frac{1}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{H}^{\varrho\left(\tilde{X}_s^\varepsilon,y\right),\nu_2}d\phi^\varepsilon_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})s}\right)\Bigg\},\end{aligned} \tag{3.4}$$

where $\tilde{\mathbb{E}}$ is the expectation operator with respect to the probability $\tilde{\mathbb{P}}$ defined as

$$\begin{aligned}d\tilde{\mathbb{P}} := &\exp\left(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{H}^{\sigma\left(\tilde{X}_s^\varepsilon,y\right),\nu_1}(\theta)ds - \frac{1}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{H}^{\varrho\left(\tilde{X}_s^\varepsilon,y\right),\nu_2}(\theta)d\phi^\varepsilon_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})s}\right) \\ &\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\sigma\left(\tilde{X}_{s-}^\varepsilon, dL_{1,s}\right)\rangle + \frac{1}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta,\varrho\left(\tilde{X}_{s-}^\varepsilon, dL_{2,(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})s}\right)\rangle\right)\times d\mathbb{P}.\end{aligned}$$

For all $\theta\in\mathbb{R}^d$, let Φ_i $(i := 1,2)$ be as follows:

$$\Phi_1(z,\theta) := \langle\theta,\left(I+\nabla\hat{b}\right)c\left(z\right)\rangle + \mathcal{H}^{\sigma(z,y),\nu_1}(\theta)\ \text{ and }\ \Phi_2(z,\theta) := \mathcal{H}^{\varrho(z,y),\nu_2}(\theta).$$

Let $\Psi_{i,\theta}\in C^{\alpha+\kappa}$ $(i := 1,2)$ be (respectively) the unique solution of:

$$\mathcal{L}_1\Psi_{1,\theta}(z) + \Phi_1(z,\theta) = \int_{\mathbb{T}^d}\Phi_1(z,\theta)\mu_1(dz),$$

and

$$\begin{cases}\mathcal{L}_1\Psi_{2,\theta}(z)=0,\\ \mathcal{L}_2\Psi_{2,\theta}(z)=\int_{\mathbb{T}^{d-1}}\Phi_2(z,\theta)\mu_2(dz)-\Phi_2(z,\theta).\end{cases}$$

Such that $\int_{\mathbb{T}^{d+1-i}}\Psi_{i,\theta}(x)\mu_i(dx)=0$. Such a solution $\Psi_{i,\theta}$ must exist once more due to the Fredholm alternative. Using Itô's formula to $\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left[\sum_{i=1}^2\Psi_{i,\theta}\right]\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}\right)$, substituting the result into Eq (3.4), and considering the two limits in (3.2) and (3.3), we then obtain:

$$\begin{aligned}\Lambda_t^\varepsilon(\theta)=&\langle\theta,x\rangle+t\cdot\int_{\mathbb{T}^d}\Phi_1(z,\theta)\mu(dz)\\&+\varepsilon\log\tilde{\mathbb{E}}\Bigg\{\exp\left(\frac{1}{\varepsilon}\left[\langle\theta,\overline{\gamma}\rangle+\int_{\mathbb{T}^{d-1}}\Phi_2(z,\theta)\tilde{\mu}(dz)\right]\mathbf{1}_{\{\tilde{X}_t^{1,\varepsilon}=0\}}\tilde{\phi}_t^\varepsilon\right)\\&\times\exp\left(-\frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\left[\sum_{i=1}^2\nabla\Psi_{i,\theta}\right]c\left(\tilde{X}_s^\varepsilon\right)ds\right)\\&\times\exp\left(-\frac{\delta_\varepsilon^{\alpha+1}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\left[\sum_{i=1}^2\nabla\Psi_{i,\theta}\right]\gamma\left(\tilde{X}_s^\varepsilon\right)d\tilde{\phi}_s^\varepsilon\right)\\&\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\left[\sum_{i=1}^2\Psi_{i,\theta}\left(\tilde{X}^\varepsilon_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right)-\sum_{i=1}^2\Psi_{i,\theta}\big(x_\varepsilon\big)\right]+o(1)\right)\Bigg\}.\end{aligned}$$

Given that the coefficients are bounded and considering the exponential integrability of the local time ϕ_t and a certain analytical property of its logarithmic moment-generating function (see, for example [3]), we observe that

$$\begin{aligned}&\lim_{\varepsilon\to0}\varepsilon\log\tilde{\mathbb{E}}\Bigg\{\exp\Bigg(\frac{1}{\varepsilon}\Big[\langle\theta,\overline{\gamma}\rangle+\int_{\mathbb{T}^{d-1}}\Phi_2(z,\theta)\mu_2(dz)\Big]\mathbf{1}_{\{\tilde{X}_t^{1,\varepsilon,\delta_\varepsilon}=0\}}\tilde{\phi}_t^\varepsilon\\&\quad+\Lambda_1^{\nabla\Psi_{\cdot,\theta}b_1}+\Lambda_2^{\nabla\Psi_{\cdot,\theta}\gamma}+\Lambda_3^{\sum_i\Psi_{i,\theta}}\Bigg)\Bigg\}\longrightarrow t\left(\langle\theta,\overline{\gamma}\rangle+\int_{\mathbb{T}^{d-1}}\Phi_2(z,\theta)\mu_2(dz)\right).\end{aligned}$$

Now, we have

$$\lim_{\varepsilon\to0}\Lambda_t^\varepsilon(\theta)=\langle x,\theta\rangle+t\left(\langle\theta,\overline{\gamma}\rangle+\sum_{i=1}^2\int_{\mathbb{T}^{d+1-i}}\Phi_i(z,\theta)\mu_i(dz)\right):=\mathcal{J}(\theta).$$

Letting $\overline{\mathcal{J}}$ be the Legendre-Fenchel transform of $\mathcal{J}$. Similarly as in [3], we put $S_{0,T}(\varphi):=\int_0^T\overline{\mathcal{J}}(\dot{\varphi}(s))\,ds$ if $\varphi\in\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ and $\varphi(0)=x$; $S_{0,T}(\varphi):=+\infty$ if else. Then, we have

Theorem 3.1. *Fix $x\in\overline{\boldsymbol{D}}$ and suppose ($\boldsymbol{A}_1$)–($\boldsymbol{A}_5$) hold. Then, the path-family of $\{X_t^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviations principle with good rate function $S_{0,T}(\varphi)$ for all $\varphi\in\mathcal{D}\left([0,T];\overline{\boldsymbol{D}}\right)$.*

Let us define

$$\overline{\Pi} := \sum_{i=1}^{2} \int_{\mathbb{T}^{d+1-1}} \zeta_i(x)\mu_i(dx).$$

We have

Corollary 3.2. *Let K be a Borel subset on $\mathcal{D}\Big([0,t];\overline{\boldsymbol{D}}\Big)$ and h, g be elements of $C^{\alpha+\kappa}\Big(\mathbb{R}^d\Big)$ and $C^{\alpha+\kappa}\Big(\mathbb{R}^{d-1}\Big)$, respectively. Then, we have*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\mathbf{1}_K e^{\left\{\frac{1}{\varepsilon}\int_0^t g\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}\right)ds+\frac{1}{\varepsilon}\int_0^t h\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}\right)d\phi_s^\varepsilon\right\}}\right] \leq \overline{\Pi}t - \inf_{\phi\in\mathring{K}} S_{0,t}(\phi),$$

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\mathbf{1}_K e^{\left\{\frac{1}{\varepsilon}\int_0^t g\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}\right)ds+\frac{1}{\varepsilon}\int_0^t h\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}\right)d\phi_s^\varepsilon\right\}}\right] \leq \overline{\Pi}t - \inf_{\phi\in\overline{K}} S_{0,t}(\phi).$$

4. Convergence

The solution of the PDE (1.1) has the following representation:

$$u^\varepsilon(t,x) = \mathbb{E}\left\{u_0\left(X_t^\varepsilon\right) e^{\left(\frac{1}{\varepsilon}\int_0^t g\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}, Y_s^\varepsilon\right)ds + \frac{1}{\varepsilon}\int_0^t h\left(\frac{X_s^\varepsilon}{\delta_\varepsilon}, Y_s^\varepsilon\right)d\phi_s^\varepsilon\right)}\right\}.$$

Before proceeding, let us introduce the Hamiltonian $\mathcal{H}^{\varepsilon,\varpi,\nu_i}$

$$\mathcal{H}^{\varepsilon,\varpi,\nu_i} w^\varepsilon(t,x) := \int_{\mathbb{R}_*^{d+1-i}} \Bigg[e^{\left\{\frac{1}{\varepsilon}\left[w^\varepsilon(t,x+\varepsilon\varpi(x_\varepsilon,y)) - w^\varepsilon(t,x)\right]\right\}} - 1 - \varpi\left(x_\varepsilon,y\right)\partial_j w^\varepsilon\left(t,x\right)\mathbf{1}_B(y)\Bigg]\nu_i(dy),\ x\in\mathbb{R}^{d+1-i}.$$

To summarize, at least informally, $w^\varepsilon(t,x) = \varepsilon\log u^\varepsilon(t,x)$ is a viscosity solution of (see, for example [14]):

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t}(t,x) = \mathcal{L}_\varepsilon^{\sigma,b,c} w^\varepsilon(t,x) + \mathcal{H}^{\varepsilon,\sigma,\nu_1} w^\varepsilon(t,x) - \mathcal{A}^{\varepsilon\sigma,\nu^{\alpha,1}} w^\varepsilon(t,x) + h\Big(x_\varepsilon, e^{\left\{\frac{1}{\varepsilon}w^\varepsilon(t,x)\right\}}\Big), & x\in\mathbf{D}, \\ \mathcal{L}_\varepsilon^{\varrho,\beta,\gamma} w^{\varepsilon,\delta}(t,x) + \mathcal{H}^{\varepsilon,\varrho,\nu_2} w^\varepsilon(t,x) - \mathcal{A}^{\varepsilon\varrho,\nu_2} w^\varepsilon(t,x) + g\Big(x_\varepsilon, e^{\left\{\frac{1}{\varepsilon}w^\varepsilon(t,x)\right\}}\Big) = 0, & x\in\partial\mathbf{D}, \\ w^\varepsilon(0,x) = \varepsilon\log\left(u_0(x)\right), & x\in U_0, \\ \lim\limits_{t\to 0} w^\varepsilon(t,x) = -\infty, & x\in\overline{\mathbf{D}}\backslash U_0. \end{cases}$$

Let us define

$$\overline{w}^*(t,x) = \limsup_{\eta\to 0}\Big\{w^\varepsilon(s,y):\ \varepsilon\leq\eta,\ (s,y)\in B_\eta(t,x)\Big\},$$

$$\underline{w}_*(t,x) = \liminf_{\eta\to 0}\Big\{w^\varepsilon(s,y):\ \varepsilon\leq\eta,\ (s,y)\in B_\eta(t,x)\Big\}.$$

Remark 4.1. [3] $\overline{w}^*$ and $\underline{w}_*$ are sub- and super-viscosity solutions of :

$$\begin{cases} \max_w \left(\dfrac{\partial w}{\partial t}(t,x) - \mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}_1} \nabla w(t,x) - \overline{c} \cdot \nabla w(t,x) - \overline{\Pi} \right) = 0, & x \in \mathbf{D}, \\ \left(\mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}_2} + \overline{\gamma} \right) \nabla w(t,x) = 0, & x \in \partial \mathbf{D}, \\ w(0,x) \ = 0, & x \in U_0, \\ \lim_{t \to 0} w(t,x) \ = \ -\infty, & x \in \overline{\mathbf{D}} \backslash U_0, \end{cases}$$

where

$$\mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}_i} w := \int_{\mathbb{R}_*^{d+i-1}} \left\{ e^{\langle w,y \rangle} - 1 - \langle w,y \rangle \, \mathbf{1}_B(y) \right\} \overline{\nu}_i(dy).$$

Set $\rho^2(t,x,y) := \inf \left\{ S_{0,t}(\varphi) : \varphi_0 = x, \varphi_t = y \right\}$ and $\rho^2(t,x,U_0) := \inf_{y \in U_0} \rho^2(t,x,y)$. From this, we have

Remark 4.2. [12] For all $(t,x) \in \,]0,\infty[\, \times \overline{\mathbf{D}}$,

$$-\rho^2(t,x,U_0) \le \underline{w}_*(t,x) \le \overline{w}^*(t,x) \le \min \left(\overline{\Pi} t - \rho^2(t,x,U_0); 0 \right).$$

Then we have $\underline{w}_* \ge \overline{w}^*$.

Now, let $\mathcal{O}$ be an open subset in $\mathbb{R} \times \overline{\mathbf{D}}$, and τ defined as:

$$\tau := \tau_{\mathcal{O}}(t,\phi) = \inf \left\{ s : (t-s, \phi(s)) \in \mathcal{O} \right\}.$$

Take Θ, the set of Markov functions τ. Let $t > 0, x \in \overline{\mathbf{D}}$, and $V^*(t,x)$ be the function:

$$V^*(t,x) = \inf_{\tau \in \Theta} \sup_{\left\{ \phi \in \mathcal{D}\left([0,t], \overline{\mathbf{D}} \right), \phi(0) = x, \phi(t) \in U_0 \right\}} \left\{ \overline{\Pi} \tau - S_{0,\tau}(\phi) \right\}. \tag{4.1}$$

Finally, our main result is:

Theorem 4.3. *Suppose assumptions* $(\boldsymbol{A}_1)$*–*$(\boldsymbol{A}_6)$ *hold true. Then, for every* $(t,x) \in \mathbb{R}_+^* \times \overline{\boldsymbol{D}}$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log u^\varepsilon(t,x) = V^*(t,x).$$

We have

Corollary 4.4. *[7]*

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = \begin{cases} 0 \textit{ from any compact of the set } \left\{ (t,x) \in \mathbb{R}_+^* \times \overline{\boldsymbol{D}};\ V^*(t,x) < 0 \right\}, \\ 1 \textit{ from any compact of the set } \left\{ (t,x) \in \mathbb{R}_+^* \times \overline{\boldsymbol{D}};\ V^*(t,x) = 0 \right\}. \end{cases}$$

5. Conclusions

In this work, we analyze the asymptotic behaviour of the solution to a family of nonlocal PDEs in the half-space. The principal operator of these equations is the generator of a Lévy process. We characterize such systems when the first-order moment and local time at the boundary exist. When direct control over the first-order moment (when $\alpha \in (0,1)$) and local time is not possible, we will demonstrate a certain dexterity in our approach.

Use of Generative-AI tools declaration

The authors declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author thanks the management of the UFR STA at UAM for partially covering the publication costs.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. G. Barles, R. Buckdahn, E. Pardoux, Backward stochastic differential equations and integral-partial differential equations, *Stoch. Stoch. Rep.*, **60** (1997), 57–83. https://doi.org/10.1080/17442509708834099
2. A. Columbu, R. D. Fuentes, S. Frassu, Uniform-in-time boundedness in a class of local and nonlocal nonlinear attraction-repulsion chemotaxis models with logistics, *Nonlinear Anal.-Real*, **79** (2024), 104135. https://doi.org/10.1016/j.nonrwa.2024.104135
3. A. Coulibaly, Large deviations and homogenization of semilinear nonlocal PDE with the Neumann boundary condition, *J. Adv. Math. Stud.*, **16** (2023), 459–473.
4. A. Coulibaly, A. Diedhiou, I. Sané, On a Skorokhod problem with small double limit, *Afr. Mat.*, **30** (2019), 959–972. https://doi.org/10.1007/s13370-019-00694-z
5. A. Diakhaby, Y. Ouknine, Generalized BSDEs, weak convergence, and homogenization of semilinear PDEs with the Wentzell-type boundary condition, *Stoch. Anal. Appl.*, **34** (2016), 496–509. https://doi.org/10.1080/07362994.2016.1153427
6. A. Dembo, O. Zeitouni, *Large deviations techniques and applications*, Springer Science & Business Media, **38** (2009).
7. M. I. Freidlin, R. B. Sowers, A comparison of homogenization and large deviations, with applications to wavefront propagation, *Stoch. Proc. Appl.*, **82** (1999), 23–52. https://doi.org/10.1016/S0304-4149(99)00003-4
8. S. Gnanasekaran, A. Columbu, R. D. Fuentes, N. Nithyadevi, Global existence and lower bounds in a class of tumor-immune cell interactions chemotaxis systems, *Discrete Cont. Dyn.-S*, **18** (2025), 1636–1659. https://doi.org/10.3934/dcdss.2024174
9. Q. Huang, J. Duan, R. Song, Homogenization of nonlocal partial differential equations related to stochastic differential equations with Lévy noise, *Bernoulli*, **28** (2022), 1648–1674. https://doi.org/10.3150/21-BEJ1365
10. T. Li, D. A. Soba, A. Columbu, G. Viglialoro, Dissipative gradient nonlinearities prevent δ-formations in local and nonlocal attraction-repulsion chemotaxis models, *Stud. Appl. Math.*, **154** (2025), e70018. https://doi.org/10.1111/sapm.70018

11. Y. Ouknine, E. Pardoux, *Seminar on stochastic analysis, random fields and applications, III: Homogenization of PDEs with non linearboundary condition*, Basel: Birkhäuser, **52** (2002), 229–242.

12. F. Pradeilles, Une méthode probabiliste pour l'étude de fronts d'onde dans les équations et systèmes d'équation de réaction-diffusion, *Aix-Marseille 1*, 1995.

13. H. Tanaka, *Stochastic Analysis and Applications: Homogenization of diffusion processes with boundary conditions*, CRC Press, 1984.

14. S. Yuan, R. Schilling, J. Duan, Large deviations for stochastic nonlinear systems of slow-fast diffusions with non-Gaussian Lévy noises, *Int. J. Nonlin. Mech*, **148** (2023), 104–304. https://doi.org/10.1016/j.ijnonlinmec.2022.104304