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Research article

Blow-up phenomena for porous medium equation driven by the fractional p-sub-Laplacian

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Abstract: In this paper, we investigate the global existence and blow-up phenomena of the solution to the fractional nonlinear porous medium equation on stratified groups, employing the concavity method. Specifically, we establish the necessary conditions for the existence of global solutions and blow-up solutions. Our findings not only contribute to the understanding of this equation on stratified groups but also extend the existing knowledge in the classical Euclidean setting to the fractional case.

Keywords: fractional porous medium equation; fractional Poincaré inequality; fractional p-sub-Laplacian; concavity method; blow-up solution
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1. Introduction

This paper studies the fractional nonlinear porous medium equation of the form

$$u_t(x,t) + \left(-\Delta_{p,\mathbb{G}}\right)^s (u^m(x,t)) = f(u(x,t)),\ m \geq 1,\ (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

where Ω denotes an open, bounded subset of a stratified (Lie) group $\mathbb{G}$. The model involves the extension of the p-Laplacian operator by the fractional p-sub-Laplacian $\left(-\Delta_{p,\mathbb{G}}\right)^s$ with $p \in (1,\infty)$ and $s \in (0,1)$. Note that case $s = 1$ is understood as the p-sub-Laplacian operator. The source function f satisfies $f(0) = 0$, and $f(u) > 0$ when $u > 0$. We consider the initial condition

$$u(x,0) = u_0(x) \geq 0, \quad x \in \bar{\Omega}, \quad u_0 \in C_0^\infty(\Omega), \tag{1.2}$$

and the boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.3}$$

The porous medium equation is a well-known nonlinear parabolic equation with numerous applications in fluid flow, diffusion, and various other fields such as lubrication and mathematical biology. For a comprehensive mathematical analysis, see [1], and for more recent references, see [2]. The fractional porous medium equation is discussed by [3, 4]. Fractional Sobolev spaces are well described in [5], while the fractional p-sub-Laplacian is covered in [6]. For studies on blow-up phenomena and global existence in Euclidean space, the seminal reference is the book by [7]. Blow-up in the fractional setting is studied in [8, 9], and within group settings in [10].

Fractional Laplacian models have also been used in many applications instead of integer-order Laplacians [11]. For instance, when $m = 1$ in Eq (1.1) on the Euclidean space, it describes stable jump Lévy processes, anomalous diffusion, and population dynamics [12].

Stan and Vázquez [13] studied the propagation properties of the problem with $p = 2$. Their research suggests that exponential propagation is a common phenomenon, and the presence of traveling wave behavior can be reduced to the classical Fisher-KPP model. They also studied the existence of a unique mild solution to problem (1.1) using the semigroup approach. Grillo, Muratori, and Punzo [14, 15] investigated the existence and uniqueness of the solution to the problem with $p = 2$ and $f = 0$.

Blow-up and global existence phenomena have also been extensively investigated using the perturbed energy method, with notable contributions found in [16, 17] and the references therein. For additional related results concerning existence, blow-up behavior, and optimal decay estimates, we refer the reader to [18, 19] and the references cited therein.

In recent years, various researchers have investigated the global and blow-up behavior of solutions to nonlinear equations using different techniques. One approach is the so-called concavity method, which was first introduced by Levine [20, 21]. Chung and Choi [22] recently proposed a condition of the form

$$\alpha \int_0^u f(s)ds \leq uf(u) + \beta u^p + \alpha\theta, \quad u > 0, \tag{1.4}$$

for certain parameters β, α, and θ. Utilizing the concavity method, they established blow-up results for parabolic equations. Building on this work, Ruzhansky, Sabitbek, and Torebek [23] extended the analysis to a nonlinear porous medium equation by generalizing condition (1.4) to obtain similar blow-up criteria for positive solutions. We also refer to an earlier related work of Jleli, Kirane, and Samet on the Heisenberg group [24]. Nevertheless, relatively limited research has been conducted on fractional problems for the Heisenberg group and/or more general classes of stratified Lie groups.

Recently, the fractional version of this problem with $m = 1$ (the linear case) in the Euclidean space has been investigated in our papers [25, 26]. Additionally, important results related to global and blow-up solutions, with the nonlinearity function represented as $f(u) = u(u - 1)$, can be found in [27].

In this work, we aim to expand on existing ideas by investigating the global and blow-up phenomena of the fractional porous medium problem (1.1) on stratified groups. Our contribution seems new not only for the case of general stratified groups but also for the Heisenberg group case and even for the classical Euclidean case. Specifically, we prove the existence of global and blow-up solutions under certain conditions.

To structure this paper, we organize our discussion as follows: Section 2 provides preliminaries on stratified groups, fractional Sobolev spaces, and a blow-up property. Section 3 establishes the main results on the global existence and blow-up phenomena of the solution to the fractional nonlinear porous medium equation on stratified groups by using the concavity method. In Section 4, we briefly discuss consequences of our results in the context of the Euclidean setting.

2. Preliminaries

Extensive research has been conducted in the field of fractional Sobolev spaces on stratified groups, with contributions from recent works such as [28, 29]. In this section, we aim to facilitate an understanding of the subsequent sections by introducing essential notations that hold importance.

2.1. Fractional Sobolev spaces and stratified Lie groups

Definition 2.1. A Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is said to be a stratified group (or a homogeneous Carnot group) if it satisfies the following conditions:

- The space $\mathbb{R}^N$ can be decomposed as $\mathbb{R}^N = \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_r}$ for some natural numbers $N_1, N_2, \ldots, N_r$ such that $N = N_1 + N_2 + \cdots + N_r$, and the dilation $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is provided by

$$\delta_\lambda(x) \equiv \sigma_\lambda(x^{(1)}, x^{(2)}, ...x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, ...\lambda^r x^{(r)})$$

  is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$, where $x' \equiv x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 1, 2, ...r$.
- Let N be defined as above, and let $X_1, \ldots, X_N$ be the left-invariant vector fields on $\mathbb{G}$ satisfying $X_k(0) = \partial/\partial x_k|_0$ for $k = 1, \ldots, N$. Then the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of $\mathbb{G}$. That is, we have the Hörmander condition

$$\text{rank}(\text{Lie}\{X_1, X_2, ...X_N\}) = N$$

  for every $x \in \mathbb{R}^N$.

Note that the left-invariant vector fields $X_1, X_2, ...X_N$ are called the Jacobian generators of a stratified group $\mathbb{G}$, and r is called the step of $\mathbb{G}$. Additionally, the left-invariant vector fields X_k can be written in the explicit form

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)} \left(x', x^2, \ldots, x^{l-1}\right) \frac{\partial}{\partial x_m^{(l)}},$$

where $a_{k,m}^{(l)}$ is a homogeneous polynomial with degree $l - 1$.

The homogeneous dimension of $\mathbb{G}$ is given by

$$Q = \sum_{k=1}^{r} kN_k, \quad N = N_1.$$

Remark 2.2. The (classical) Euclidean group $(\mathbb{R}^N, +)$ with the dilation

$$\delta_\lambda(x) = \lambda x, \quad \lambda > 0,$$

where Jacobian generators are given by $\partial_{x_1}, ..., \partial_{x_N}$ with step $r = 1$, is obviously a (Abelian) stratified group.

In this paper, to simplify the notations, $u(x,t)$ can be written simply as u. However, if u is part of an integrand with the differential $d\tau$, then it is understood as $u := u(x,\tau)$. The same idea applies for $u(x) := u(x,t)$, $u(y) := u(y,t)$, and $f(u) := f(u(x,t))$.

Definition 2.3. Let $\Omega \subset \mathbb{G}$ be an open subset. Let $u : \Omega \to \mathbb{R}$ be a measurable function. The Gagliardo semi-norm $[u]_{s,p,\Omega}$ is defined as follows:

$$[u]_{s,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{\left|y^{-1}\circ x\right|^{Q+ps}} dxdy \right)^{\frac{1}{p}}.$$

Definition 2.4. Let $\Omega \subset \mathbb{G}$ be an open subset. For $p > 1$, and $0 < s < 1$, the space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty \right\}, \tag{2.1}$$

and is equipped with the norm

$$||u||_{W^{s,p}(\Omega)} = \left(||u||^p_{L^p(\Omega)} + [u]^p_{s,p,\Omega} \right)^{\frac{1}{p}}.$$

This space is called the *fractional Sobolev space* on $\mathbb{G}$.

We define the space $W^{s,p}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm $||u||_{L^p(\Omega)} + [u]_{s,p,\mathbb{G}}$. It can be represented in the form

$$W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\mathbb{G}) : u = 0 \text{ in } \mathbb{G}\backslash\Omega \},$$

where Ω is an open bounded set (with at least a continuous boundary $\partial\Omega$).

Definition 2.5. Let $1 < p < \infty$ and $0 < s < 1$. The fractional p-sub-Laplacian on $\mathbb{G}$ is defined as follows:

$$\left(-\Delta_{p,\mathbb{G}}\right)^s u(x) := C_{Q,s,p} P.V. \int_{\mathbb{G}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{\left|y^{-1}\circ x\right|^{Q+ps}} dy, \quad x \in \mathbb{G}, \tag{2.2}$$

where $P.V.$ denotes the Cauchy principal value, and $C_{Q,s,p} > 0$ is independent of u.

We define the following product:

$$\left\langle \left(-\Delta_{p,\mathbb{G}}\right)^s u, \varphi \right\rangle = \iint_{\mathbb{G}\times\mathbb{G}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{\left|y^{-1}\circ x\right|^{Q+ps}} dxdy \tag{2.3}$$

for all $\varphi \in W^{s,p}_0(\Omega)$.

2.2. *Fractional p-sub-Laplacian eigenvalue problem*

The equation on a stratified group $\mathbb{G}$ of the form

$$\begin{cases} \left(-\Delta_{p,\mathbb{G}}\right)^s u = \lambda |u|^{p-2} u, \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{G}\backslash\Omega \end{cases} \tag{2.4}$$

is called the fractional p-sub-Laplacian eigenvalue problem.

We recall the Poincaré inequality for the Gagliardo seminorm [28]:

$$\lambda_{Q,p,s}(\Omega)\int_{\Omega}|u(x)|^{p}dx \le \int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^{p}}{\left|y^{-1}\circ x\right|^{Q+ps}}dxdy. \tag{2.5}$$

where $p > 1$, $0 < s < 1$, and $\lambda_{Q,p,s}(\Omega) = C(Q, p, s)^{-p}|\Omega|^{-\frac{ps}{Q}} > 0$.

Definition 2.6. $u^m \in L^\infty\left(0, T; W_0^{s,p}(\Omega)\right)$, with $u_t \in L^\infty\left(0, T; L^2(\Omega)\right)$, is called a weak (local) solution of (1.1)–(1.3) if it satisfies the following equation:

$$\begin{aligned}&\int_{\Omega}u_t(x,t)\varphi(x)dx + \iint_{\mathbb{G}\times\mathbb{G}}\frac{|u^m(x)-u^m(y)|^{p-2}(u^m(x)-u^m(y))(\varphi(x)-\varphi(y))}{\left|y^{-1}\circ x\right|^{Q+ps}}dxdy\\ &= \int_{\Omega}f(u)\varphi(x)dx \end{aligned}\tag{2.6}$$

for all $\varphi \in W_0^{s,p}(\Omega)$ and a.e. time $0 \le t \le T$ with

$$u^m(x,0) = u_0^m(x) \quad \text{in } W_0^{1,p}(\Omega).$$

2.3. Blow up property

Here we provide a definition of the blow-up phenomena and recall an important lemma that is used in our proofs.

Definition 2.7. Assume that $u(x, t)$ is a weak solution to problem (1.1). We say that $u(x, t)$ blows up in finite time T if it satisfies the estimate

$$\lim_{t\to T^-}\int_0^t\int_{\Omega}u^{m+1}(x,\tau)dxd\tau = +\infty.$$

Lemma 2.8. *[21,30,31] Let $\mathcal{E}(t)$ be a twice-differentiable function satisfying the following inequalities for some constant $\rho > 0$:*

$$\begin{cases}\mathcal{E}''(t)\mathcal{E}(t) - (1+\rho)\mathcal{E}'^2(t) \ge 0, & t > 0,\\ \mathcal{E}(0) > 0, \quad \textit{and} \quad \mathcal{E}'(0) > 0.\end{cases}\tag{2.7}$$

Then there exists

$$0 < T^* \le \frac{\mathcal{E}(0)}{\rho\mathcal{E}'(0)},$$

such that $\lim_{t\to T^*}\mathcal{E}(t) = +\infty$.

3. Global existence and blow-up solutions

Throughout this paper, we refer to the function u as a weak solution in the sense of Definition 2.6. Furthermore, we assume the existence of a local weak solution for the problem under consideration.

Theorem 3.1. *Let Ω be a bounded open set of a stratified group $\mathbb{G}$. Assume that*

$$\alpha F(u) \geq u^m f(u) + \beta u^{pm} + \alpha\theta, \quad u > 0, \tag{3.1}$$

where

$$F(u) = \frac{pm}{m+1} \int_0^u s^{m-1} f(s) ds,$$

for $m \geq 1$, and

$$0 > \beta \geq \frac{(\alpha - (m+1))\lambda_{Q,p,s}(\Omega)}{(m+1)}, \text{ and } \alpha \leq 0, \tag{3.2}$$

where $1 < p < \infty$, and $\lambda_{Q,p,s}(\Omega) > 0$ is the constant given in (2.5). *For the initial data $u_0 \in C_0^\infty(\Omega)$, if we have a constant $\theta > 0$ such that*

$$[u_0]^p_{s,p,\mathbb{G}} \leq (m+1) \int_\Omega (F(u_0) - \theta) dx, \tag{3.3}$$

then a non-negative solution u of problem (1.1)–(1.3) *is global and satisfies the estimate*

$$\|u(x,t)\|^{m+1}_{L^{m+1}(\Omega)} \leq \|u_0(x)\|^{m+1}_{L^{m+1}(\Omega)}.$$

Proof. Assume that $u \geq 0$ is a solution of the fractional porous medium Eq (1.1), and

$$E(t) := \int_\Omega u^{m+1}(x,t) dx.$$

Then

$$\begin{aligned} E'(t) &= (m+1) \int_\Omega u^m(x,t) u_t(x,t) dx \\ &= (m+1)\Big(- \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u^m(x) - u^m(y)|^{p-2}(u^m(x) - u^m(y))^2}{\left|y^{-1} \circ x\right|^{Q+ps}} dxdy \\ &\quad + \int_\Omega u^m(x,t) f(u(x,t)) dx\Big). \end{aligned}$$

Applying the Poincaré inequality and the assumptions (3.1) and (3.2), we have

$$\begin{aligned} E'(t) &\leq (m+1)\Big(- \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u^m(x) - u^m(y)|^p}{\left|y^{-1} \circ x\right|^{Q+ps}} dxdy + \int_\Omega (\alpha F(u) - \beta u^{pm} - \alpha\theta) dx\Big) \\ &\leq (m+1)\alpha \left(-\frac{1}{m+1} \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u^m(x,t) - u^m(y,t)|^p}{|y^{-1} \circ x|^{Q+ps}} dxdy + \int_\Omega (F(u) - \theta) dx \right) \\ &\quad + \Big[(\alpha - (m+1))\lambda_{Q,p,s}(\Omega) - (m+1)\beta\Big] \int_\Omega |u(x,t)|^{pm} dx \\ &\leq (m+1)\alpha \left(-\frac{1}{m+1} \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u^m(x,t) - u^m(y,t)|^p}{|y^{-1} \circ x|^{Q+ps}} dxdy + \int_\Omega (F(u) - \theta) dx \right) \\ &\leq (m+1)\alpha \mathcal{I}(t), \end{aligned} \tag{3.4}$$

where we set

$$\mathcal{I}(t) := -\frac{1}{m+1}\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x,t)-u^m(y,t)|^p}{|y^{-1}\circ x|^{Q+ps}}dxdy + \int_{\Omega}(F(u)-\theta)dx.$$

We can further derive that

$$\mathcal{I}(t) = \mathcal{I}(0) + \int_0^t \frac{d}{d\tau}\mathcal{I}(\tau)d\tau = \mathcal{I}(0) + \frac{pm}{m+1}\int_0^t\int_{\Omega}[u_\tau(x,\tau)]^2 u^{m-1}(x,\tau)dxd\tau. \tag{3.5}$$

To see it, we compute

$$\begin{aligned}
&\int_0^t \frac{d}{d\tau}\mathcal{I}(\tau)d\tau \\
&= -\frac{1}{m+1}\int_0^t \frac{d}{d\tau}\left(\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x,\tau)-u^m(y,\tau)|^p}{|y^{-1}\circ x|^{Q+ps}}dxdy\right)d\tau \\
&\quad + \int_0^t\int_{\Omega}\frac{d}{d\tau}(F(u(x,\tau))-\theta)dxd\tau \\
&= -\frac{p}{m+1}\int_0^t\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x)-u^m(y)|^{p-2}(u^m(x)-u^m(y))([u^m(x)]_\tau-[u^m(y)]_\tau)}{|y^{-1}\circ x|^{Q+ps}}dxdyd\tau \\
&\quad + \int_0^t\int_{\Omega}(F_u(u(x,\tau))u_\tau(x,\tau)dxd\tau \\
&= -\frac{p}{m+1}\int_0^t\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x)-u^m(y)|^{p-2}(u^m(x)-u^m(y))([u^m(x)]_\tau-[u^m(y)]_\tau)}{|y^{-1}\circ x|^{Q+ps}}dxdyd\tau \\
&\quad + \frac{pm}{m+1}\int_0^t\int_{\Omega}u^{m-1}(x,\tau)f(u)\,u_\tau(x,\tau)dxd\tau,
\end{aligned}$$

From the definition of the weak solution, it implies that

$$\begin{aligned}
\int_0^t \frac{d}{d\tau}\mathcal{I}(\tau)d\tau &= \frac{p}{m+1}\int_0^t\int_{\Omega}u_\tau(x,\tau)u^m_\tau(x,\tau)dxd\tau \\
&\quad - \frac{p}{m+1}\int_0^t\int_{\Omega}f(u)\,u^m_\tau(x,\tau)dxd\tau \\
&\quad + \frac{pm}{m+1}\int_0^t\int_{\Omega}u^{m-1}(x,\tau)f(u)\,u_\tau(x,\tau)dxd\tau \\
&= \frac{pm}{m+1}\int_0^t\int_{\Omega}[u_\tau(x,\tau)]^2u^{m-1}(x,\tau)dxd\tau \\
&\quad - \frac{pm}{m+1}\int_0^t\int_{\Omega}f(u)\,u_\tau(x,\tau)u^{m-1}_\tau(\tau,x)dxd\tau \\
&\quad + \frac{pm}{m+1}\int_0^t\int_{\Omega}u^{m-1}(x,\tau)f(u)\,u_\tau(x,\tau)dxd\tau \\
&= \frac{pm}{m+1}\int_0^t\int_{\Omega}[u_\tau(x,\tau)]^2\,u^{m-1}(x,\tau)dxd\tau.
\end{aligned}$$

Observe that $\mathcal{I}(t) > 0$, due to the assumption $\mathcal{I}(0) > 0$ (given in (3.3)) and the positive integrand in (3.5).

Finally, we conclude

$$E'(t) \leq (m+1)\alpha\left(\mathcal{I}(0) + \frac{pm}{m+1}\int_0^t \int_\Omega [u_\tau(x,\tau)]^2 u^{m-1}(x,\tau)dxd\tau\right) \leq 0,$$

since $\alpha \leq 0$. It yields that

$$E(t) - E(0) \leq 0,$$

and

$$\|u(x,t)\|^{m+1}_{L^{m+1}(\Omega)} \leq \|u_0(x)\|^{m+1}_{L^{m+1}(\Omega)}$$

for all $t \in (0, \infty)$. □

Theorem 3.2. *Assume Ω is a bounded open set of a stratified group $\mathbb{G}$, and let f be a function that satisfies the inequality*

$$\alpha F(u) \leq u^m f(u) + \beta u^{pm} + \alpha\theta, \quad u > 0,$$

where $1 < p < \infty$, $\theta > 0$ is a constant, and $m \geq 1$. Here we have

$$F(u) := \frac{pm}{m+1}\int_0^u s^{m-1} f(s)ds.$$

Furthermore, suppose that $\alpha > m+1$ and $0 < \beta \leq \frac{(\alpha-(m+1))\lambda_{Q,p,s}(\Omega)}{(m+1)}$, where $\lambda_{Q,p,s}(\Omega) > 0$ is the constant given in (2.5).

If the initial condition $u_0 \in C_0^\infty(\Omega)$ satisfies

$$[u_0]^p_{s,p,\mathbb{G}} \leq (m+1)\int_\Omega (F(u_0) - \theta)dx, \tag{3.6}$$

for a constant θ, then a solution u of (1.1)–(1.3) *blows up in finite time T^*, that is,*

$$0 < T^* \leq \frac{M}{\rho \int_\Omega u_0^{m+1}(x)dx},$$

where

$$M = \frac{pm\left(\int_\Omega u_0^{m+1}(x)dx\right)^2}{(m+1)^2(\sqrt{\alpha pm} - (m+1))\mathcal{I}(0)}$$

with $\rho = \varepsilon = \frac{\sqrt{\alpha pm}}{m+1} - 1 > 0$.

Proof. Let $u \geq 0$ be a solution of the fractional porous medium Eq (1.1). Assume that

$$\mathcal{E}(t) := \int_\Omega \int_0^t u^{m+1}(x,\tau)d\tau dx + M,$$

where $M > 0$ is a constant that will be defined later.

According to Lemma 2.8, we need to prove that

$$\mathcal{E}''(t)\mathcal{E}(t) - (1+\rho)\left(\mathcal{E}'(t)\right)^2 > 0. \tag{3.7}$$

Applying the fundamental calculus, we have

$$\begin{aligned}\mathcal{E}'(t) &= \int_\Omega u^{m+1}(x,t)dx \\ &= (m+1)\int_\Omega\int_0^t u^m(x,\tau)u_\tau(x,\tau)d\tau dx + \int_\Omega u_0^{m+1}(x)dx.\end{aligned}$$

Now we compute

$$\begin{aligned}\mathcal{E}''(t) &= (m+1)\int_\Omega u^m(x,t)u_t(x,t)dx \\ &\geq (m+1)\Big(-\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x)-u^m(y)|^p}{\left|y^{-1}\circ x\right|^{Q+ps}}dxdy + \int_\Omega(\alpha F(u)-\beta u^{pm}-\alpha\theta)dx\Big) \\ &\geq (m+1)\alpha\left(-\frac{1}{m+1}\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x,t)-u^m(y,t)|^p}{|y^{-1}\circ x|^{Q+ps}}dxdy + \int_\Omega(F(u(x,t))-\theta)dx\right) \\ &\quad + \Big[(\alpha-(m+1))\lambda_{Q,p,s}(\Omega)-(m+1)\beta\Big]\int_\Omega|u(x,t)|^{pm}dx \\ &\geq (m+1)\alpha\left(-\frac{1}{m+1}\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x,t)-u^m(y,t)|^p}{|y^{-1}\circ x|^{Q+ps}}dxdy + \int_\Omega(F(u(x,t))-\theta)dx\right) \\ &\geq (m+1)\alpha\mathcal{I}(t),\end{aligned}$$

where

$$\mathcal{I}(t) := -\frac{1}{m+1}\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u^m(x,t)-u^m(y,t)|^p}{|y^{-1}\circ x|^{Q+ps}}dxdy + \int_\Omega(F(u(x,t))-\theta)dx,$$

and $0 < \beta \leq \frac{(\alpha-(m+1))\lambda_{Q,p,s}(\Omega)}{(m+1)}$ with $\lambda_{Q,p,s}(\Omega) > 0$ is the constant given in (2.5). Moreover, we have $\mathcal{I}(0) > 0$ from the assumption (3.6), and estimate

$$\mathcal{I}(t) = \mathcal{I}(0) + \frac{pm}{m+1}\int_0^t\int_\Omega[u_\tau(x,\tau)]^2u^{m-1}(x,\tau)dxd\tau > 0$$

obtained in the proof of Theorem 3.1. Therefore, it yields

$$\begin{aligned}\mathcal{E}''(t) &\geq (m+1)\alpha\mathcal{I}(t) \\ &\geq (m+1)\alpha\mathcal{I}(0) + \alpha pm\int_0^t\int_\Omega[u_\tau(x,\tau)]^2u^{m-1}(\tau,x)dxd\tau,\end{aligned}$$

where $\alpha > m+1$. For arbitrary $\varepsilon > 0$, we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}[\mathcal{E}'(t)]^2 &= \Big((m+1)\int_\Omega\int_0^t u^m(x,\tau)u_\tau(x,\tau)d\tau dx + \int_\Omega u_0^{m+1}(x)dx\Big)^2 \\ &\leq (m+1)^2(1+\varepsilon)\left(\int_\Omega\int_0^t u^m(x,\tau)u_\tau(x,\tau)dxd\tau\right)^2 \\ &\quad + \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2.\end{aligned}$$

Now we apply the Hölder inequality to obtain

$$
\begin{aligned}
[\mathcal{E}'(t)]^2 = &(m+1)^2(1+\varepsilon)\left(\int_\Omega\int_0^t u^{(m+1)/2+(m-1)/2}(x,\tau)u_\tau(x,\tau)dxd\tau\right)^2\\
&+\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2\\
\le&(m+1)^2(1+\varepsilon)\left(\int_\Omega\left(\int_0^t u^{m+1}d\tau\right)^{1/2}\left(\int_0^t u^{m-1}[u_\tau(x,\tau)]^2d\tau\right)^{1/2}dx\right)^2\\
&+\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2\\
\le&(m+1)^2(1+\varepsilon)\left(\int_0^t\int_\Omega u^{m+1}dxd\tau\right)\left(\int_0^t\int_\Omega u^{m-1}[u_\tau(x,\tau)]^2dxd\tau\right)\\
&+\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2.
\end{aligned}
$$

Using the computations above, we have

$$
\begin{aligned}
\mathcal{E}''(t)\mathcal{E}(t)\ge&\left[(m+1)\alpha\mathcal{I}(0)+\alpha pm\int_0^t\int_\Omega[u_\tau(x,\tau)]^2u^{m-1}(\tau,x)dxd\tau\right]\\
&\times\Big(\int_\Omega\int_0^t u^{m+1}(x,\tau)d\tau dx+M\Big)\\
\ge&(m+1)\alpha\mathcal{I}(0)M\\
&+\alpha pm\int_0^t\int_\Omega[u_\tau(x,\tau)]^2u^{m-1}(x,\tau)dxd\tau\int_\Omega\int_0^t u^{m+1}(x,\tau)d\tau dx.
\end{aligned}
$$

Finally, using the estimates above, we obtain

$$
\begin{aligned}
&\mathcal{E}''(t)\mathcal{E}(t)-(1+\rho)\left(\mathcal{E}'(t)\right)^2\\
&\ge\alpha(m+1)\mathcal{I}(0)M+\alpha pm\left(\int_0^t\int_\Omega[u_\tau(x,\tau)]^2u^{m-1}(x,\tau)dxd\tau\right)\left(\int_\Omega\int_0^t u^{m+1}(x,\tau)d\tau dx\right)\\
&-(1+\rho)(1+\varepsilon)(m+1)^2\left(\int_0^t\int_\Omega u^{m+1}(x,\tau)dxd\tau\right)\left(\int_0^t\int_\Omega u^{m-1}(x,\tau)[u_\tau(x,\tau)]^2dxd\tau\right)\\
&\quad-(1+\rho)\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2.
\end{aligned}
$$

We choose $\rho=\varepsilon=\frac{\sqrt{\alpha pm}}{m+1}-1>0$, and

$$
M:=\frac{pm\left(\int_\Omega u_0^{m+1}(x)dx\right)^2}{(m+1)^2(\sqrt{\alpha pm}-(m+1))\mathcal{I}(0)}.
$$

Thus, we arrive at

$$
\mathcal{E}''(t)\mathcal{E}(t)-(1+\rho)\left(\mathcal{E}'(t)\right)^2
$$

$$\geq \alpha(m+1)\mathcal{I}(0)M - (1+\rho)\left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^{m+1}(x)dx\right)^2 \geq 0.$$

Consequently, the blow-up time is given by

$$0 < T^* \leq \frac{M}{\rho \int_\Omega u_0^{m+1}(x)dx}.$$

It completes the proof. □

Remark 3.3. Note that the value of θ in Theorems 3.1–3.2 can be chosen to satisfy the condition $F(u) > \theta$ due to the conditions $\mathcal{I}(t) > 0$ and the assumption $\mathcal{I}(0) > 0$ in the theorems.

4. Conclusions

Let us briefly discuss some special cases, that is, the consequences in the Euclidean setting. Specifically, for an open, bounded subset $\Omega \subset \mathbb{R}^N$, consider the problem

$$\begin{cases} u_t(x,t) + \left(-\Delta_p\right)^s (u^m(x,t)) = f(u(x,t)), \ m \geq 1, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \bar{\Omega}, \\ u(x,t) = 0, \ \text{in } \mathbb{R}^N \backslash \Omega, \quad t > 0, \end{cases} \tag{4.1}$$

where the fractional operator is denoted by

$$\left(-\Delta_p\right)^s u(x) := C_{N,s,p} P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \quad x \in \mathbb{R}^N, \tag{4.2}$$

with $p \in (1,\infty)$ and $s \in (0,1)$. A function f satisfies $f(0) = 0$, and $f(u) > 0$ when $u > 0$.

According to Remark 2.2, (4.1) is a particular case of Eq (1.1) with initial-boundary conditions given by (1.2) and (1.3). Thus, the results presented in Theorems 3.1 and 3.2 imply new insights in the Euclidean case.

Specifically, in the context of problem (4.1), a global solution exists when

$$\alpha F(u) \geq u^m f(u) + \beta u^{pm} + \alpha\theta, \quad u > 0,$$

with

$$F(u) = \frac{pm}{m+1} \int_0^u z^{m-1} f(z)\, dz, \quad m \geq 1,$$

and certain conditions on the parameters α, β, and θ given in Theorem 3.1. Additionally, under other conditions on the parameters α, β, and θ provided in Theorem 3.2, blow-up solutions exist when

$$\alpha F(u) \leq u^m f(u) + \beta u^{pm} + \alpha\theta, \quad u > 0.$$

Author contributions

Khumoyun Jabbarkhanov: Conceptualization, investigation, original draft preparation, writing-review and editing; Amankeldy Toleukhanov: Investigation, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. L. Vazquez, *The porous medium equation: Mathematical theory*, Oxford: Oxford University Press, 2006.
2. B. Sabitbek, B. Torebek, Global existence and blow-up of solutions to the nonlinear porous medium equation, *arxiv Preprint*, 2022. https://doi.org/10.48550/arXiv.2104.06896
3. A. D. Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A general fractional porous medium equation, *Comm. Pure Appl. Math.*, **65** (2012), 1242–1284. https://doi.org/10.1002/cpa.21408
4. M. Bonforte, Y. Sire, J. L. Vazquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, *Discrete Cont. Dyn.*, **35** (2015), 5725–5767. https://doi.org/10.3934/dcds.2015.35.5725
5. E. D. Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *B. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
6. A. Kassymov, D. Surgan, Some functional inequalities for the fractional p-sub-Laplacian, *arxiv Preprint*, 2018. https://doi.org/10.48550/arXiv.1804.01415
7. P. Quittner, P. Souplet, *Superlinear parabolic problems: Blow-up, global existence and steady states*, Birkhäuser, 2007.
8. S. Sugitani, On nonexistence of global solutions for some nonlinear integral equations, *Osaka Math. J.*, **12** (1975), 45–51.
9. E. T. Kolkovska, J. A. L. Mimbela, A. Pérez, Blow-up and life span bounds for a reaction-diffusion equation with a time-dependent generator, *Electron. J. Differ. Eq.*, 2008, 1–18.
10. A. Pascucci, Semilinear equations on nilpotent Lie groups: Global existence and blow-up of solutions, *Le Mat.*, **53** (1998), 345–357.
11. A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, et al., What is the fractional Laplacian? A comparative review with new results, *J. Comput. Phys.*, **404** (2020), 109009. https://doi.org/10.1016/j.jcp.2019.109009

12. X. Cabré, J. M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, *Commun. Math. Phys.*, **320** (2013), 679–722. https://doi.org/10.1007/s00220-013-1682-5

13. D. Stan, J. L. Vázquez, The Fisher-KPP equation with nonlinear fractional diffusion, *SIAM J. Math. Anal.*, **46** (2014), 3241–3276. https://doi.org/10.1137/130918289

14. G. Grillo, M. Muratori, F. Punzo, Fractional porous media equations: Existence and uniqueness of weak solutions with measure data, *Calc. Var. Partial Dif.*, **54** (2015), 3303–3335. https://doi.org/10.1007/s00526-015-0904-4

15. G. Grillo, M. Muratori, F. Punzo, On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density, *Discrete Cont. Dyn.*, **35** (2015), 5927–5962. https://doi.org/10.3934/dcds.2015.35.5927

16. Z. Zhang, Q. Ouyang, Global existence, blow-up and optimal decay for a nonlinear viscoelastic equation with nonlinear damping and source term, *Discrete Cont. Dyn.-B*, **28** (2023), 4735–4760. https://doi.org/10.3934/dcdsb.2023038

17. B. Feng, General decay rates for a viscoelastic wave equation with dynamic boundary conditions and past history, *Mediterr. J. Math.*, **15** (2018), 103. https://doi.org/10.1007/s00009-018-1154-4

18. A. Boudiaf, S. Drabla, F. Boulanouar, General decay rate for nonlinear thermoviscoelastic system with a weak damping and nonlinear source term, *Mediterr. J. Math.*, **13** (2016), 3101–3120. https://doi.org/10.1007/s00009-015-0674-4

19. Q. Dai, Z. Yang, Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay, *Z. Angew. Math. Phys.*, **65** (2014), 885–903. https://doi.org/10.1007/s00033-013-0365-6

20. H. A. Levine, Some nonexistence and instability theorems for formally parabolic equations of the form $Pu_t = Au + J(u)$, *Arch. Ration. Mech. An.*, **51** (1973), 277–284. https://doi.org/10.1007/BF00263041

21. H. A. Levine, L. E. Payne, Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time, *J. Differ. Equations*, **16** (1974), 319–334. https://doi.org/10.1016/0022-0396(74)90018-7

22. S. Y. Chung, M. J. Choi, A new condition for the concavity method of blow-up solutions to p-Laplacian parabolic equations, *J. Differ. Equations*, **265** (2018), 6384–6399. https://doi.org/10.1016/j.jde.2018.07.032

23. M. Ruzhansky, B. Sabitbek, B. Torebek, Global existence and blow-up of solutions to porous medium equation and pseudo-parabolic equation, I. Stratified groups, *Manuscripta Math.*, **171** (2021), 377–395. https://doi.org/10.1007/s00229-022-01390-2

24. M. Jleli, M. Kirane, B. Samet, Non existence results for a class of evolution equations in the Heisenberg group, *Fract. Calc. Appl. Anal.*, **18** (2015), 717–734. https://doi.org/10.1515/fca-2015-0044

25. K. Jabbarkhanov, D. Suragan, On Fisher's equation with the fractional p-Laplacian, *Math. Method. Appl. Sci.*, **46** (2023), 12886–12894. https://doi.org/10.1002/mma.9219

26. K. Jabbarkhanov, D. Suragan, Dynamics of nonlinear anomalous reaction-diffusion models: Global existence and blow-up of solutions, *Evol. Equ. Control The.*, **14** (2025), 1128–1140. https://doi.org/10.3934/eect.2025028

27. K. Jabbarkhanov, J. E. Restrepo, D. Suragan, A reaction-diffusion equation on stratified groups, *J. Math. Sci.*, **266** (2022), 593–602. https://doi.org/10.1007/s10958-022-05965-y

28. S. Ghosh, V. Kumar, M. Ruzhansky, Compact embeddings, eigenvalue problems, and subelliptic Brezis-Nirenberg equations involving singularity on stratified Lie groups, *Math. Ann.*, **388** (2024), 4201–4249. https://doi.org/10.1007/s00208-023-02609-7

29. M. Ruzhansky, D. Suragan, *Hardy inequalities on homogeneous groups*, Birkhäuser, **327** (2019).

30. V. K. Kalantarov, O. A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types, *J. Math. Sci.*, **10** (1978), 53–70. https://doi.org/10.1007/BF01109723

31. A. Khelghati, K. Baghaei, Blow-up phenomena for a nonlocal semilinear parabolic equation with positive initial energy, *Comput. Math. Appl.*, **70** (2015), 896–902. https://doi.org/10.1016/j.camwa.2015.06.003

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