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Research article

Algebraic and spectral properties of H-Toeplitz operators on the Bergman space

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Abstract: In this paper, we discuss the zero product problem of two H-Toeplitz operators and the commuting problem of H-Toeplitz and Hankel operators on the Bergman space. We establish necessary and sufficient conditions for the product of an H-Toeplitz operator and a Hankel operator equals another H-Toeplitz (Hankel) operator for a certain class of symbols. Moreover, we study the spectrum of the H-Toeplitz operators B_{z^N} and $B_{\bar{z}^N}$, where N is a non-negative integer.

Keywords: H-Toeplitz operator; Hankel operator; Bergman space; algebraic properties; spectral properties
Mathematics Subject Classification: 47B35

1. Introduction

Let dA denote the Lebesgue area measure on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, normalized so that the measure of $\mathbb{D}$ is 1. $L^2(\mathbb{D})$ denotes the space of the Lebesgue measurable functions f on $\mathbb{D}$ with the following norm:

$$\|f\|_2 = \left(\int_{\mathbb{D}} |f(z)|^2 dA(z) \right)^{\frac{1}{2}} < \infty.$$

The Bergman space L_a^2 consists of all analytic functions f in $L^2(\mathbb{D})$, which is a closed subspace of $L^2(\mathbb{D})$. Moreover, L_a^2 is a reproducing kernel Hilbert space and the orthonormal basis is given by $\{e_n\}_{n=0}^{+\infty}$, where $e_n(z) = \sqrt{n+1}z^n$. The collection of essentially bounded (with respect to the measure dA) functions is denoted by $L^\infty(\mathbb{D})$.

Let P be the orthogonal projection from $L^2(\mathbb{D})$ onto L_a^2. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ and the Toeplitz operator T_φ on the Bergman space are defined by

$$M_\varphi f = \varphi f, \quad T_\varphi f = PM_\varphi f,$$

respectively, where $f \in L_a^2$. Let $J : L_a^2 \to \overline{L_a^2}$ be the operator defined by $J(e_n) = \overline{e}_{n+1}$, where $\overline{L_a^2}$ denotes the complex conjugate of L_a^2. For $\varphi \in L^\infty(\mathbb{D})$, the Hankel operator H_φ with symbol φ is defined by

$$H_\varphi(f) = P(\varphi J(f)), \quad f \in L_a^2.$$

The harmonic Bergman space L_h^2 is the closed subspace of $L^2(\mathbb{D})$ consisting of the harmonic functions on $\mathbb{D}$. The operator $K : L_a^2 \to L_h^2$ is defined by

$$K(e_{2n}(z)) = e_n(z) = \sqrt{n+1}z^n,$$

and

$$K(e_{2n+1}(z)) = \overline{e_{n+1}(z)} = \sqrt{n+2}\overline{z}^{n+1},$$

for all $n \geqslant 0$ and $z \in \mathbb{D}$. It can be observed that K is bounded on L_a^2 with $\|K\| = 1$. For $\varphi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator $B_\varphi : L_a^2 \to L_a^2$ is defined by

$$B_\varphi(f) = PM_\varphi K(f),$$

for $f \in L_a^2$.

Note that H-Toeplitz operators are closely related to Toeplitz and Hankel operators. In fact, for each nonnegative integer n, we have

$$B_\varphi(e_{2n}) = PM_\varphi K(e_{2n}) = PM_\varphi(e_n) = T_\varphi(e_n),$$

and

$$B_\varphi(e_{2n+1}) = PM_\varphi K(e_{2n+1}) = PM_\varphi J(e_n) = H_\varphi(e_n).$$

Toeplitz operators and Hankel operators on the Bergman space have been widely studied. The boundedness, compactness, and Schatten ideal properties of Toeplitz and Hankel operators on the Bergman space have attracted a lot of attention (see [2, 18, 19]). For the commutativity and the hyponormality of Toeplitz operators on the Bergman space, one can consult [1, 11, 12]. In [5, 15, 16] the authors investigated the invertibility of Toeplitz operators on the Bergman space. The spectrum of Toeplitz operators on the Bergman space was also studied; see [6, 13, 17] for detailed discussions about this topic.

Besides Toeplitz and Hankel operators, researchers have also investigated other operators on various function spaces. In 2021, Gupta and Singh [7] introduced and studied the notion of H-Toeplitz operators on the Bergman space. They obtained a necessary and sufficient condition for an H-Toeplitz operator to be a co-isometry or a partial isometry, explored their invariant subspaces and kernels, and discussed the compactness, Fredholmness, and commutativity. The H-Toeplitz operator is neither a class of Toeplitz operators nor a class of Hankel operators, yet it exhibits specific associations with both Toeplitz and Hankel operators. In addition, an n-order H-Toeplitz matrix has $2n-1$ degrees of freedom instead of n^2, meaning that solving systems of linear equations with such matrices as coefficient matrices becomes comparatively easy for large n. In 2022, Liang et al. investigated the commutativity of H-Toeplitz operators with quasi-homogeneous symbols on the Bergman space [10]. Later, Kim and Lee studied the contractive and expansive H-Toeplitz operators with analytic, coanalytic, and harmonic symbols on the Bergman space [9]. In the case that one H-Toeplitz operator a bounded symbol and the other a quasi-homogeneous symbol, Ding and Chen recently characterized when their product is equal to another H-Toeplitz operator see [4]. They also obtained equivalent characterizations for the product of an H-Toeplitz operator and a Toeplitz operator equal to another H-Toeplitz operator with a harmonic symbol. In [3], Ding studied the commutativity of Toeplitz and H-Toeplitz operators on the Bergman space. In the recent paper [8], Kim et al. established necessary and sufficient conditions for H-Toeplitz operators to be contractive and expansive on weighted Bergman spaces. However, the other algebraic properties and spectral structures of H-Toeplitz operators on the Bergman space remain unknown at present.

Motivated by the above works, we will study the zero-product problem, the commuting problem, and spectral properties of H-Toeplitz operators on the Bergman space. In Section 2, we discuss the zero-product problem of H-Toeplitz operators on the Bergman space see Theorems 2.1–2.3, respectively. Section 3 is devoted to solving the problem of when the H-Toeplitz operator B_φ commutes with the Hankel operator H_ψ see Proposition 3.1 and Theorem 3.1. Furthermore, characterizations for the product of an H-Toeplitz and a Hankel operator equal to another H-Toeplitz (Hankel) operator on the Bergman space are also obtained see Theorem 3.2 and Proposition 3.3. In the final section, we investigate the point spectrum of the H-Toeplitz operators B_{z^N} and $B_{\bar{z}^N}$ on the Bergman space L_a^2 and the main result is contained in Propositions 4.1 and 4.2.

2. The zero product problem for H-Toeplitz operators

In this section, we focus on the zero-product problem for H-Toeplitz operators on the Bergman space. The first main theorem of this section shows that $B_\psi B_\varphi = 0$ holds in the trivial case, where $\varphi \in H^\infty$ and ψ is a polynomial.

Theorem 2.1. *Suppose that $\varphi \in H^\infty$ with the Taylor series $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{N} b_n z^n$, where N is a non-negative integer. Then $B_\psi B_\varphi = 0$, if and only if, $\varphi = 0$ or $\psi = 0$.*

Proof. We only need to prove the necessity. Suppose that $B_\psi B_\varphi = 0$. If $\psi = 0$, the conclusion holds trivially. Otherwise, there exists $n \in \{0, 1, \cdots, N\}$ such that $b_n \neq 0$. Without loss of generality, we can assume that $b_N \neq 0$.

Since $B_\psi B_\varphi = 0$, we have

$$B_\psi B_\varphi z^{2m} = 0,$$

for any non-negative integer m. Direct calculations give us that

$$\begin{aligned} 0 = B_\psi B_\varphi z^{2m} = B_\psi B_\varphi \frac{e_{2m}(z)}{\sqrt{2m+1}} &= \frac{\sqrt{m+1}}{\sqrt{2m+1}} B_\psi \Big(\sum_{k=0}^{\infty} \frac{a_k}{\sqrt{k+m+1}} e_{k+m}(z) \Big) \\ &= \frac{\sqrt{m+1}}{\sqrt{2m+1}} B_\psi \Big[\sum_{k=0}^{\infty} \frac{a_{2k}}{\sqrt{2k+m+1}} e_{2k+m}(z) + \sum_{k=0}^{\infty} \frac{a_{2k+1}}{\sqrt{2k+m+2}} e_{2k+m+1}(z) \Big]. \end{aligned} \tag{2.1}$$

In particular,

$$
\begin{aligned}
0 = B_\psi B_\varphi z^{4N+2} &= \frac{\sqrt{2N+2}}{\sqrt{4N+3}} B_\psi \Big[\sum_{k=0}^{\infty} \frac{a_{2k}}{\sqrt{2k+2N+2}} e_{2k+2N+1}(z) + \sum_{k=0}^{\infty} \frac{a_{2k+1}}{\sqrt{2k+2N+3}} e_{2k+2N+2}(z) \Big] \\
&= \frac{\sqrt{2N+2}}{\sqrt{4N+3}} \Big[\sum_{k=0}^{\infty} \frac{a_{2k}}{\sqrt{2k+2N+2}} P(\psi \bar{e}_{k+N+1})(z) + \sum_{k=0}^{\infty} \frac{a_{2k+1}}{\sqrt{2k+2N+3}} \psi(z) e_{k+N+1}(z) \Big] \qquad (2.2) \\
&= \frac{\sqrt{2N+2}}{\sqrt{4N+3}} \sum_{k=0}^{\infty} \sum_{n=0}^{N} \frac{b_n a_{2k+1} \sqrt{k+N+2}}{\sqrt{2k+2N+3}} z^{n+k+N+1},
\end{aligned}
$$

where the last equality is due to

$$
P(z^n \bar{z}^m) = \begin{cases} 0, & \text{if } n < m, \\ \frac{n-m+1}{n+1} z^{n-m}, & \text{if } n \geqslant m. \end{cases}
$$

Considering the coefficients of $z^{n+k+N+1}$ in Eq (2.2), we can derive the following system of linear equations:

$$
\begin{cases}
\frac{\sqrt{N+2}}{\sqrt{2N+3}} a_1 b_0 & = 0, \\
\frac{\sqrt{N+3}}{\sqrt{2N+5}} a_3 b_0 + \frac{\sqrt{N+2}}{\sqrt{2N+3}} a_1 b_1 & = 0, \\
\frac{\sqrt{N+4}}{\sqrt{2N+7}} a_5 b_0 + \frac{\sqrt{N+3}}{\sqrt{2N+5}} a_3 b_1 + \frac{\sqrt{N+2}}{\sqrt{2N+3}} a_1 b_2 & = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\frac{\sqrt{2N+2}}{\sqrt{4N+3}} a_{2N+1} b_0 + \frac{\sqrt{2N+1}}{\sqrt{4N+1}} a_{2N-1} b_1 + \cdots + \frac{\sqrt{N+2}}{\sqrt{2N+3}} a_1 b_N & = 0, \\
\frac{\sqrt{2N+3}}{\sqrt{4N+5}} a_{2N+3} b_0 + \frac{\sqrt{2N+2}}{\sqrt{4N+3}} a_{2N+1} b_1 + \cdots + \frac{\sqrt{N+3}}{\sqrt{2N+5}} a_3 b_N & = 0, \\
\frac{\sqrt{2N+4}}{\sqrt{4N+7}} a_{2N+5} b_0 + \frac{\sqrt{2N+3}}{\sqrt{4N+5}} a_{2N+3} b_1 + \cdots + \frac{\sqrt{N+4}}{\sqrt{2N+7}} a_5 b_N & = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{cases} \qquad (2.3)
$$

From the relationship between the rank of the coefficient matrix of a homogeneous linear system and the existence of non-trivial solutions, we know that if $a_1 \neq 0$, then

$$
b_0 = b_1 = \cdots = b_N = 0.
$$

Based on the previous assumption that $b_N \neq 0$, we can conclude that $a_1 = 0$. Inserting this result into system (2.3) and applying the same method yields $a_3 = 0$. Continuing this process, we ultimately derive that $a_{2k+1} = 0$ for $k = 0, 1, \cdots, N$.

On the other hand, letting $m = 2N$ in (2.1) gives

$$
\begin{aligned}
0 = B_\psi B_\varphi z^{4N} &= \frac{\sqrt{2N+1}}{\sqrt{4N+1}} B_\psi \Big[\sum_{k=0}^{\infty} \frac{a_{2k}}{\sqrt{2k+2N+1}} e_{2k+2N}(z) + \sum_{k=0}^{\infty} \frac{a_{2k+1}}{\sqrt{2k+2N+2}} e_{2k+2N+1}(z) \Big] \\
&= \frac{\sqrt{2N+1}}{\sqrt{4N+1}} \Big[\sum_{k=0}^{\infty} \frac{a_{2k}}{\sqrt{2k+2N+1}} \psi(z) e_{k+N}(z) + \sum_{k=0}^{\infty} \frac{a_{2k+1}}{\sqrt{2k+2N+2}} P(\psi \overline{e}_{k+N+1})(z) \Big] \\
&= \frac{\sqrt{2N+1}}{\sqrt{4N+1}} \sum_{k=0}^{\infty} \sum_{n=0}^{N} \frac{b_n a_{2k} \sqrt{k+N+1}}{\sqrt{2k+2N+1}} z^{n+k+N}.
\end{aligned} \tag{2.4}
$$

Considering the coefficient of z^{n+k+N} in Eq (2.4) also leads to the following system of linear equations:

$$
\begin{cases}
\frac{\sqrt{N+1}}{\sqrt{2N+1}} a_0 b_0 & = 0, \\
\frac{\sqrt{N+2}}{\sqrt{2N+3}} a_2 b_0 + \frac{\sqrt{N+1}}{\sqrt{2N+1}} a_0 b_1 & = 0, \\
\frac{\sqrt{N+3}}{\sqrt{2N+5}} a_4 b_0 + \frac{\sqrt{N+2}}{\sqrt{2N+3}} a_2 b_1 + \frac{\sqrt{N+1}}{\sqrt{2N+1}} a_0 b_2 & = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\frac{\sqrt{2N+1}}{\sqrt{4N+1}} a_{2N} b_0 + \frac{\sqrt{2N}}{\sqrt{4N-1}} a_{2N-2} b_1 + \cdots + \frac{\sqrt{N+1}}{\sqrt{2N+1}} a_0 b_N & = 0, \\
\frac{\sqrt{2N+2}}{\sqrt{4N+3}} a_{2N+2} b_0 + \frac{\sqrt{2N+1}}{\sqrt{4N+1}} a_{2N} b_1 + \cdots + \frac{\sqrt{N+2}}{\sqrt{2N+3}} a_2 b_N & = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{cases}
$$

Using the same method as previously discussed, we can conclude that $a_{2k} = 0$ for $k \geqslant 0$. Therefore, if $B_\psi B_\varphi = 0$ then we have $\varphi = 0$ or $\psi = 0$. □

Let φ and ψ be bounded co-analytic functions. In the next result, we discuss when the product $B_\varphi B_\psi$ equals zero on the Bergman space.

Theorem 2.2. *Suppose that $\overline{\varphi}, \overline{\psi} \in H^\infty$ with the Taylor series $\overline{\varphi}(z) = \sum\limits_{n=0}^{\infty} \overline{a}_n z^n$ and $\overline{\psi}(z) = \sum\limits_{n=0}^{\infty} \overline{b}_n z^n$. Then $B_\varphi B_\psi = 0$, if and only if, $\varphi = 0$ or $\psi = 0$.*

Proof. We only need to show the necessity. Suppose that $B_\varphi B_\psi = 0$. If $\varphi = 0$, the result follows immediately. Otherwise, there exists $M \geqslant 0$ such that $a_M \neq 0$. It follows from $B_\varphi B_\psi = 0$ that

$$
\begin{aligned}
0 = B_\varphi B_\psi z^{4m} &= B_\varphi B_\psi \frac{e_{4m}(z)}{\sqrt{4m+1}} = \frac{1}{\sqrt{(4m+1)(2m+1)}} B_\varphi \Big(\sum_{k=0}^{2m} b_{2m-k} \sqrt{k+1} e_k(z) \Big) \\
&= \frac{1}{\sqrt{(4m+1)(2m+1)}} \Big[B_\varphi \Big(\sum_{k=0}^{m} b_{2m-2k} \sqrt{2k+1} e_{2k}(z) \Big) + B_\varphi \Big(\sum_{k=0}^{m-1} b_{2m-2k-1} \sqrt{2k+2} e_{2k+1}(z) \Big) \Big] \\
&= \frac{1}{\sqrt{(4m+1)(2m+1)}} \Big[\sum_{k=0}^{m} b_{2m-2k} \sqrt{2k+1} \sqrt{k+1} P\Big(\sum_{n=0}^{\infty} a_n \overline{z}^n z^k \Big) + \sum_{k=0}^{m-1} b_{2m-2k-1} \sqrt{2k+2} P(\varphi \overline{e}_{k+1})(z) \Big] \\
&= \frac{1}{\sqrt{(4m+1)(2m+1)}} \sum_{k=0}^{m} \sum_{n=0}^{k} \frac{\sqrt{2k+1}(n+1)}{\sqrt{k+1}} b_{2m-2k} a_{k-n} z^n,
\end{aligned} \tag{2.5}
$$

for any nonnegative integer m. Considering the constant term in (2.5), it can be observed that

$$\sum_{k=0}^{m} \frac{\sqrt{2k+1}}{\sqrt{k+1}} b_{2m-2k} a_k = 0.$$

Then we obtain the following system of linear equations:

$$\begin{cases} b_0 a_0 = 0, & (m=0), \\ b_2 a_0 + \frac{\sqrt{3}}{\sqrt{2}} b_0 a_1 = 0, & (m=1), \\ b_4 a_0 + \frac{\sqrt{3}}{\sqrt{2}} b_2 a_1 + \frac{\sqrt{5}}{\sqrt{3}} b_0 a_2 = 0, & (m=2), \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ b_{2M} a_0 + \frac{\sqrt{3}}{\sqrt{2}} b_{2M-2} a_1 + \cdots + \frac{\sqrt{2M-1}}{\sqrt{M}} b_2 a_{M-1} + \frac{\sqrt{2M+1}}{\sqrt{M+1}} b_0 a_M = 0, & (m=M), \\ b_{2M+2} a_0 + \frac{\sqrt{3}}{\sqrt{2}} b_{2M} a_1 + \cdots + \frac{\sqrt{2M-1}}{\sqrt{M}} b_4 a_{M-1} \frac{\sqrt{2M+1}}{\sqrt{M+1}} b_2 a_M + \frac{\sqrt{2M+3}}{\sqrt{M+2}} b_0 a_{M+1} = 0, & (m=M+1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . & \end{cases} \tag{2.6}$$

Since $a_M \neq 0$, it follows from system (2.6) that $b_0 = 0$. Substituting $b_0 = 0$ back into (2.6) and using $a_M \neq 0$, we obtain $b_2 = 0$. Continuing this process, we finally conclude that $b_{2k} = 0$ for $k \geqslant 0$.

Furthermore,

$$\begin{aligned} 0 = B_\varphi B_\psi z^{4m+2} &= \frac{1}{\sqrt{(4m+3)(2m+2)}} B_\varphi \Big(\sum_{k=0}^{2m+1} b_{2m+1-k} \sqrt{k+1} e_k(z) \Big) \\ &= \frac{1}{\sqrt{(4m+3)(2m+2)}} B_\varphi \Big[\sum_{k=0}^{m} b_{2m+1-2k} \sqrt{2k+1} e_{2k}(z) + \sum_{k=0}^{m} b_{2m-2k} \sqrt{2k+2} e_{2k+1}(z) \Big] \\ &= \frac{1}{\sqrt{(4m+3)(2m+2)}} \sum_{k=0}^{m} \sum_{n=0}^{k} \frac{\sqrt{2k+1}(n+1)}{\sqrt{k+1}} b_{2m+1-2k} a_{k-n} z^n. \end{aligned}$$

Then using the same method as the one used in solving Eq (2.5), we get that $b_{2k+1} = 0$ for $k \geqslant 0$. Therefore, if $a_M \neq 0$ then we have $\varphi = 0$. This finishes the proof. □

From Theorem 2.2, we get the following corollary.

Corollary 2.1. *Suppose that $\varphi(z) = f_1(z) + \overline{f}_2(z)$ is a bounded harmonic function and the analytic functions f_1 and f_2 with the Taylor series $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} \overline{b}_n z^n$ and $\psi(z) = \sum_{n=0}^{N} \overline{c}_n z^n$, where N is a non-negative integer. Then $B_\varphi B_{\overline{\psi}} = 0$, if and only if, $\varphi = 0$ or $\psi = 0$.*

Proof. It suffices to show the necessity. Suppose that $B_\varphi B_{\overline{\psi}} = 0$. If $\psi = 0$, the conclusion holds trivially. Otherwise, there exists $n \in \{0, 1, \cdots, N\}$, such that $c_n \neq 0$. We may assume $c_N \neq 0$. Since

$$\begin{aligned} 0 = B_\varphi B_{\overline{\psi}} z^{2N} &= \frac{1}{\sqrt{2N+1}} B_\varphi B_{\overline{\psi}} e_{2N}(z) = \frac{1}{\sqrt{2N+1}} B_\varphi P(\overline{\psi} e_N)(z) \\ &= \frac{c_N}{\sqrt{(2N+1)(N+1)}} P(\varphi e_0)(z) = \frac{c_N}{\sqrt{(2N+1)(N+1)}} f_1(z), \end{aligned}$$

we have $f_1 = 0$. This implies that $\varphi = \overline{f}_2$ and hence $B_{\overline{f}_2} B_{\overline{\psi}} = 0$. By Theorem 2.5, we have $\varphi = f_2 = 0$. □

We are now ready to prove the third main theorem of this section.

Theorem 2.3. *Suppose that $\varphi(z) = f_1(z) + \overline{f}_2(z)$ is a bounded harmonic function and the analytic functions f_1 and f_2 with the Taylor series $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=1}^{\infty} \overline{b}_n z^n$ and $\psi(z) = \sum_{n=0}^{N} c_n z^n$, where N is a non-negative integer. Then $B_\varphi B_\psi = 0$, if and only if, $\varphi = 0$ or $\psi = 0$.*

Proof. We only need to prove the necessity. Suppose that $B_\psi B_\varphi = 0$. If $\psi = 0$, the conclusion holds trivially. Otherwise, there exists $n \in \{0, 1, \cdots, N\}$, such that $c_n \neq 0$. In this case, we may assume that $c_N \neq 0$ without loss of generality. Elementary computation yields that

$$B_\psi z^{2m} = \frac{\sqrt{m+1}}{\sqrt{2m+1}} P(\psi(z) z^m) = \frac{\sqrt{m+1}}{\sqrt{2m+1}} \sum_{n=0}^{N} c_n z^{n+m}, \tag{2.7}$$

and

$$\begin{aligned} B_\psi z^{2m+1} &= \frac{1}{\sqrt{2m+2}} P(\psi \overline{e}_{m+1})(z) = \frac{\sqrt{m+2}}{\sqrt{2m+2}} P\Big(\sum_{n=0}^{N} c_n z^n \overline{z}^{m+1}\Big) \\ &= \begin{cases} 0, & \text{if } m \geqslant N, \\ \frac{\sqrt{m+2}}{\sqrt{2m+2}} \sum_{n=0}^{N-m-1} \frac{n+1}{n+m+2} c_{n+m+1} z^n, & \text{if } m \leqslant N-1, \end{cases} \end{aligned} \tag{2.8}$$

for any non-negative integer m. By (2.8), it follows from

$$0 = B_\varphi B_\psi z^{2N-1} = B_\varphi \frac{c_N}{N+1} = \frac{c_N}{N+1} f_1(z),$$

that $f_1 = 0$.

Furthermore,

$$\begin{aligned} 0 &= B_\varphi B_\psi z^{2N-5} = B_{\overline{f}_2} B_\psi z^{2N-5} = B_{\overline{f}_2}\Big(\sum_{n=0}^{2} \frac{n+1}{n+N-1} c_{n+N-2} z^n\Big) \\ &= B_{\overline{f}_2}\Big[\frac{c_{N-2}}{N-1} e_0(z) + \frac{2c_{N-1}}{N}\frac{e_1(z)}{\sqrt{2}} + \frac{3c_N}{N+1}\frac{e_2(z)}{\sqrt{3}}\Big] \\ &= \frac{\sqrt{6}c_N}{N+1} P(\overline{f}_2(z) z) = \frac{\sqrt{6}c_N}{N+1} P\Big(\sum_{n=1}^{\infty} b_n \overline{z}^n z\Big) = \frac{\sqrt{6}c_N}{2(N+1)} b_1. \end{aligned}$$

This indicates that $b_1 = 0$. Using the same method, further calculation yields

$$\begin{aligned} 0 &= B_{\overline{f}_2} B_\psi z^{2N-9} = B_{\overline{f}_2} \sum_{n=0}^{4} \frac{n+1}{n+N-3} c_{n+N-4} z^n \\ &= \frac{\sqrt{15}c_N}{N+1} P(\overline{f}_2(z) z^2) = \frac{\sqrt{15}c_N}{3(N+1)} P\Big(\sum_{n=2}^{\infty} b_n \overline{z}^n z^2\Big) = \frac{\sqrt{15}c_N}{3(N+1)} b_2, \end{aligned}$$

and hence $b_2 = 0$. Now we need to consider the following two cases:

Case 1. If N is odd, then there exists a non-negative integer M such that $N = 2M + 1$. Continuing the above process, after M steps, we obtain

$$\begin{aligned} 0 = B_{\overline{f}_2} B_\psi z = B_{\overline{f}_2} \sum_{n=0}^{2M} \frac{n+1}{n+2} c_{n+1} z^n &= \frac{\sqrt{(M+1)(2M+1)}c_{2M+1}}{2M+2} P(\overline{f}_2(z) z^M) \\ = \frac{\sqrt{(M+1)(2M+1)}c_{2M+1}}{2M+2} P\Big(\sum_{n=M}^{\infty} b_n \overline{z}^n z^M \Big) &= \frac{\sqrt{(M+1)(2M+1)}c_{2M+1}}{2(M+1)^2} b_M. \end{aligned}$$

Therefore, we get that $b_1 = b_2 = \cdots = b_M = 0$.

In addition, by (2.7), we have

$$\begin{aligned} 0 = B_{\overline{f}_2} B_\psi z^{4j+2} &= B_{\overline{f}_2} \Big(\sum_{n=0}^{2M+1} c_n z^{n+2j+1} \Big) \\ &= B_{\overline{f}_2} \Big(\sum_{n=0}^{M} c_{2n} \frac{e_{2n+2j+1}(z)}{\sqrt{2n+2j+1}} \Big) + B_{\overline{f}_2} \Big(\sum_{n=0}^{M} c_{2n+1} \frac{e_{2n+2j+2}(z)}{\sqrt{2n+2j+3}} \Big) \\ &= P\Big(\sum_{k=M+1}^{\infty} b_k \overline{z}^k \sum_{n=0}^{M} \frac{\sqrt{n+j+2}c_{2n+1}}{\sqrt{2n+2j+3}} z^{n+j+1} \Big). \end{aligned} \tag{2.9}$$

Let $j = 0$ in (2.9); then we have

$$\frac{c_{2M+1} b_{M+1}}{\sqrt{M+2}\sqrt{2M+3}} = 0.$$

This implies that $b_{M+1} = 0$. Similarly, let $j = 1$ in (2.9); we have that

$$0 = P\Big(\sum_{k=M+2}^{\infty} b_k \overline{z}^k \sum_{n=0}^{M} \frac{\sqrt{n+3}c_{2n+1}}{\sqrt{2n+5}} z^{n+2} \Big) = \frac{c_{2M+1} b_{M+2}}{\sqrt{M+3}\sqrt{2M+5}},$$

to obtain $b_{M+2} = 0$. Repeating the discussion as above, we conclude that $b_{M+k} = 0$ for $k \geqslant 1$.

Case 2. If N is even, then there exists a non-negative integer M such that $N = 2M$. Similarly to the odd case, continuing the above process for $M - 1$ steps yields

$$\begin{aligned} 0 = B_{\overline{f}_2} B_\psi z^3 = B_{\overline{f}_2} \sum_{n=0}^{2M-2} \frac{n+1}{n+3} c_{n+2} z^n &= \frac{\sqrt{M(2M-1)}c_{2M}}{2M+1} P(\overline{f}_2(z) z^{M-1}) \\ = \frac{\sqrt{M(2M-1)}c_{2M}}{2M+1} P\Big(\sum_{n=M-1}^{\infty} b_n \overline{z}^n z^{M-1} \Big) &= \frac{\sqrt{2M-1}}{\sqrt{M}(2M+1)} c_{2M} b_{M-1}. \end{aligned}$$

Hence, we have $b_1 = b_2 = \cdots = b_{M-1} = 0$.

Moreover, by (2.7), we have

$$\begin{aligned} 0 = B_{\overline{f}_2} B_\psi z^{4j} &= B_{\overline{f}_2} \Big(\sum_{n=0}^{2M} c_n z^{n+2j} \Big) \\ &= B_{\overline{f}_2} \Big(\sum_{n=0}^{M} c_{2n} \frac{e_{2n+2j}(z)}{\sqrt{2n+2j}} \Big) + B_{\overline{f}_2} \Big(\sum_{n=0}^{M-1} c_{2n+1} \frac{e_{2n+2j+1}(z)}{\sqrt{2n+2j+2}} \Big) \\ &= P\Big(\sum_{k=M}^{\infty} b_k \overline{z}^k \sum_{n=0}^{M} \frac{\sqrt{n+j+1}c_{2n}}{\sqrt{2n+2j+1}} z^{n+j} \Big). \end{aligned}$$

Using the same method as in (2.9), we can similarly obtain $b_{M-1+k} = 0$ for $k \geqslant 1$. By combining the preceding two cases, we derive $f_2 = 0$, and hence $\varphi = 0$. This completes the proof of Theorem 2.3. □

3. The product of H-Toeplitz and Hankel operators

In this section, we first study the commutativity of an H-Toeplitz operator and a Hankel operator on the Bergman space L_a^2. In addition, we discuss when the product of an H-Toeplitz operator and a Hankel operator is equal to another H-Toeplitz operator or a Hankel operator.

The following proposition characterizes the commutativity problem for H-Toeplitz and Hankel operators with analytic polynomial symbols.

Proposition 3.1. *Suppose that* $\varphi(z) = \sum_{n=0}^{N} a_n z^n$ *and* $\psi(z) = \sum_{n=0}^{M} b_n z^n$.
(1) *If* $2M \geqslant N$ *and* $b_M \neq 0$, *then* $H_\varphi B_\psi = B_\psi H_\varphi$, *if and only if,* φ *is constant;*
(2) *If* $N = M = 1$, *then* $H_\varphi B_\psi = B_\psi H_\varphi$, *if and only if,* φ *is constant or* ψ *is constant.*

Proof. Since $H_{a_0} z^m = 0$ for any non-negative integer m, we only need to prove the necessity.

To prove (1), if

$$H_\varphi B_\psi = B_\psi H_\varphi,$$

we have

$$H_\varphi B_\psi z^{2M-1} = B_\psi H_\varphi z^{2M-1}. \tag{3.1}$$

For $2M > N$, noting that

$$B_\psi H_\varphi z^{2M-1} = \frac{1}{\sqrt{2M}} B_\psi P(\varphi \bar{e}_{2M})(z) = \frac{\sqrt{2M+1}}{\sqrt{2M}} B_\psi P\Big(\sum_{n=0}^{N} a_n z^n \bar{z}^{2M}\Big) = 0,$$

and

$$H_\varphi B_\psi z^{2M-1} = H_\varphi B_\psi \frac{e_{2M-1}(z)}{\sqrt{2M}} = \frac{1}{\sqrt{2M}} H_\varphi P(\psi \bar{e}_M)(z) = \frac{b_M}{\sqrt{M}\sqrt{M+1}} \sum_{n=0}^{N-1} \frac{(n+1)a_{n+1}}{n+2} z^n.$$

It follows from (3.1) and $b_M \neq 0$ that $a_k = 0$ for $k \in \{1, 2, \cdots, N\}$.

For $2M = N$, we first observe that

$$B_\psi H_\varphi z^{2M-1} = B_\psi H_\varphi z^{N-1} = \frac{1}{\sqrt{N}} B_\psi P(\varphi \bar{e}_N)(z) = \frac{a_N}{\sqrt{N}\sqrt{N+1}} \psi(z),$$

and

$$H_\varphi B_\psi z^{2M-1} = \frac{b_M}{\sqrt{M}\sqrt{M+1}} \sum_{n=0}^{N-1} \frac{(n+1)a_{n+1}}{n+2} z^n.$$

If

$$H_\varphi B_\psi = B_\psi H_\varphi,$$

then we have

$$H_\varphi B_\psi z^{2M-1} = B_\psi H_\varphi z^{2M-1}.$$

This yields that

$$\frac{b_M}{\sqrt{M}\sqrt{M+1}}\sum_{n=0}^{N-1}\frac{(n+1)a_{n+1}}{n+2}z^n = \frac{a_N}{\sqrt{N}\sqrt{N+1}}\sum_{n=0}^{M} b_n z^n. \tag{3.2}$$

Since $N = 2M > M$, we obtain that $b_M a_N = 0$, which gives that $a_N = 0$. It follows from (3.2) that φ is constant.

The proof of (2) is similar to that of (1). In fact, using

$$H_{a_0+a_1z}B_{b_0+b_1z}z^m = \begin{cases} \frac{a_1b_0}{\sqrt{2}}, & \text{if } m = 0, \\ \frac{a_1b_1}{2\sqrt{2}}, & \text{if } m = 1, \\ 0, & \text{if } m \geqslant 2, \end{cases}$$

and

$$B_{b_0+b_1z}H_{a_0+a_1z}z^m = \begin{cases} \frac{a_1b_0}{\sqrt{2}} + \frac{a_1b_1}{\sqrt{2}}z, & \text{if } m = 0, \\ 0, & \text{if } m \geqslant 1, \end{cases}$$

we obtain that $H_{a_0+a_1z}B_{b_0+b_1z} = B_{b_0+b_1z}H_{a_0+a_1z}$, if and only if, $a_1b_1 = 0$. □

For the case that φ is co-analytic and ψ is an analytic polynomial, we obtain the necessary and sufficient condition for the commutativity of B_φ and H_ψ on the Bergman space.

Theorem 3.1. *Suppose that $\overline{\varphi} \in H^\infty$ with the Taylor series $\overline{\varphi}(z) = \sum_{n=0}^{\infty} \overline{a}_n z^n$ and $\psi(z) = \sum_{n=0}^{N} b_n z^n$. Then*

$$B_\varphi H_\psi = H_\psi B_\varphi,$$

if and only if, φ is constant or ψ is constant.

Proof. If

$$B_\varphi H_\psi = H_\psi B_\varphi,$$

then we have

$$B_\varphi H_\psi z^{2N+2m} = H_\psi B_\varphi z^{2N+2m},$$

for any non-negative integer m. Since

$$B_\varphi H_\psi z^{2N+2m} = \frac{\sqrt{2N+2m+2}}{2N+2m+1}B_\varphi P\Big(\sum_{n=0}^{N} b_n z^n \overline{z}^{2N+2m+1}\Big) = 0,$$

and

$$\begin{aligned}
H_\psi B_\varphi z^{2N+2m} &= \frac{\sqrt{N+m+1}}{\sqrt{2N+2m+1}} H_\psi P\Big(\sum_{k=0}^{\infty} a_k \bar{z}^k z^{N+m}\Big)\\
&= \frac{1}{\sqrt{(N+m+1)(2N+2m+1)}} H_\psi \Big(\sum_{k=0}^{N+m} a_{N+m-k}(k+1) z^k\Big)\\
&= \frac{1}{\sqrt{(N+m+1)(2N+2m+1)}} P\Big(\sum_{n=0}^{N} b_n z^n \sum_{k=0}^{N+m} a_{N+m-k}\sqrt{k+1}\,\bar{e}_{k+1}(z)\Big)\\
&= \frac{1}{\sqrt{(N+m+1)(2N+2m+1)}} \sum_{k=0}^{N-1}\sum_{n=0}^{N-k-1} a_{N+m-k} b_{n+k+1} \frac{\sqrt{(k+1)(k+2)}(n+1)}{n+k+2} z^n,
\end{aligned}$$

we obtain

$$\sum_{k=0}^{N-1}\sum_{n=0}^{N-k-1} a_{N+m-k} b_{n+k+1} \frac{\sqrt{(k+1)(k+2)}(n+1)}{n+k+2} z^n = 0. \tag{3.3}$$

Considering the coefficients of z^n in equations in (3.3) yields that

$$\begin{cases}
\frac{\sqrt{2}}{2} a_{N+m} b_1 + \frac{\sqrt{6}}{3} a_{N+m-1} b_2 + \cdots + \frac{\sqrt{N(N+1)}}{N+1} a_{m+1} b_N & = 0,\\
\frac{2\sqrt{2}}{3} a_{N+m} b_2 + \frac{2\sqrt{6}}{4} a_{N+m-1} b_3 + \cdots + \frac{2\sqrt{(N-1)N}}{N+1} a_{m+2} b_N & = 0,\\
\frac{3\sqrt{2}}{4} a_{N+m} b_3 + \frac{3\sqrt{6}}{5} a_{N+m-1} b_4 + \cdots + \frac{3\sqrt{(N-2)(N-1)}}{N+1} a_{m+3} b_N & = 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\frac{\sqrt{2}(N-1)}{N} a_{N+m} b_{N-1} + \frac{\sqrt{6}(N-1)}{N+1} a_{N+m-1} b_N & = 0,\\
\frac{\sqrt{2}N}{N+1} a_{N+m} b_N & = 0.
\end{cases}$$

If $b_N \neq 0$ $(N > 0)$, then $a_{m+1} = a_{m+2} = \cdots = a_{N+m} = 0$ for any non-negative integer m. This implies that $a_k = 0$ for $k \geqslant 1$. Thus we have $\varphi = 0$. □

In the following theorem, we establish necessary and sufficient conditions for the product of an H-Toeplitz operator and a Hankel operator to equals another H-Toeplitz operator or a Hankel operator on the Bergman space.

Theorem 3.2. *Suppose that $f, h \in H^\infty$ with the Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $h(z) = \sum_{n=0}^{\infty} c_n z^n$, $g_1(z) = \sum_{n=0}^{N} b_n z^n$ and $g_2(z) = z^M$, where N is a non-negative integer and M is a positive integer. Then we have*
(1) $H_f B_{g_1} = B_h$, if and only if, $h = g_1 = 0$, or $h = 0$ and f is constant;
(2) $B_f H_{g_1} = H_h$, if and only if, g_1 and h are constants, or $f = 0$ and h is constant;
(3) $H_{g_2} B_f = B_h$, if and only if, $f = h = 0$.

Proof. Let us prove (1) first. we first note that

$$H_f B_{g_1} z^{2N+1} = H_f B_{g_1} \frac{e_{2N+1}(z)}{\sqrt{2N+2}} = \frac{1}{\sqrt{2N+2}} H_f P(g_1 \bar{e}_{N+1})(z) = \frac{\sqrt{N+2}}{\sqrt{2N+2}} H_f P\Big(\sum_{n=0}^{N} b_n z^n \bar{z}^{N+1}\Big) = H_f 0 = 0,$$

and

$$B_h z^{2N+1} = B_h \frac{e_{2N+1}(z)}{\sqrt{2N+2}} = \frac{1}{\sqrt{2N+2}} P(h\overline{e}_{N+1})(z) = \frac{\sqrt{N+2}}{\sqrt{2N+2}} P\Big(\sum_{n=0}^{\infty} c_n z^n \overline{z}^{N+1}\Big) = \frac{\sqrt{N+2}}{\sqrt{2N+2}} \sum_{n=0}^{\infty} \frac{(n+1)c_{n+N+1}}{n+N+2} z^n.$$

Using

$$H_f B_{g_1} z^{2N+1} = B_h z^{2N+1},$$

we derive that $c_{n+N+1} = 0$ for any $n \geqslant 0$. This implies that $h(z) = \sum_{n=0}^{N} c_n z^n$.

For $N = 0$, we have

$$H_f B_{b_0} e_{2m} = B_{c_0} e_{2m}, \tag{3.4}$$

for any non-negative integer m. Letting $m = 0$ and $m = 2$ in (3.4) respectively, we obtain that

$$\sqrt{2} b_0 \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}}{n+2} z^n = c_0,$$

and

$$\sqrt{3} b_0 \sum_{n=0}^{\infty} \frac{(n+1)a_{n+2}}{n+3} z^n = \sqrt{2} c_0 z.$$

Thus we have $\frac{\sqrt{2}}{2} b_0 a_1 = c_0$, $\frac{\sqrt{3}}{2} b_0 a_3 = \sqrt{2} c_0$ and $a_k = 0$ for any $k > 1$. This gives that $g_1 = h = 0$, or $h = 0$ and f is constant.

Moreover, if $N > 0$ and $b_N \neq 0$, then we have $H_f B_{g_1} z^{2N-1} = B_h z^{2N-1}$. Since $H_f B_{g_1} z^{2N-1} = \frac{b_N}{\sqrt{N}\sqrt{N+1}} \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}}{n+2} z^n$ and $B_h z^{2N-1} = \frac{c_N}{\sqrt{2N}\sqrt{N+1}}$, we obtain that $f = a_0 + a_1 z$ and $\frac{b_N a_1}{\sqrt{2}} = c_N$. From

$$\begin{aligned} H_f B_{g_1} z^{2N} &= H_f B_{g_1} \frac{e_{2N}(z)}{\sqrt{2N+1}} = H_f T_{g_1} \frac{e_N(z)}{\sqrt{2N+1}} = \frac{\sqrt{N+1}}{\sqrt{2N+1}} H_f \sum_{n=0}^{N} b_n z^{n+N} \\ &= \frac{\sqrt{N+1}}{\sqrt{2N+1}} H_f \sum_{n=0}^{N} b_n \frac{e_{n+N}(z)}{\sqrt{n+N+1}} = \frac{\sqrt{N+1}}{\sqrt{2N+1}} P\Big(f(z) \sum_{n=0}^{N} \frac{b_n \overline{e}_{n+N+1}(z)}{\sqrt{n+N+1}}\Big) \\ &= \frac{\sqrt{N+1}}{\sqrt{2N+1}} P\Big[(a_0 + a_1 z) \sum_{n=0}^{N} \frac{b_n \sqrt{n+N+2}}{\sqrt{n+N+1}} \overline{z}^{n+N+1}\Big] = 0, \end{aligned}$$

and

$$B_h z^{2N} = \frac{\sqrt{N+1}}{\sqrt{2N+1}} \sum_{n=0}^{N} c_n z^{n+N},$$

we get that $h = 0$ and hence $a_1 = 0$. Therefore, we have $g_1 = h = 0$, or $h = 0$ and f is constant.

Now we turn to the proof of (2). Suppose that $B_f H_{g_1} z^m = H_h z^m$ for any $m \geqslant N$. Then we have

$$B_f H_{g_1} z^m = B_f H_{g_1} \frac{e_m(z)}{\sqrt{m+1}} = \frac{1}{\sqrt{m+1}} B_f P(g_1 \overline{e}_{m+1})(z) = \frac{\sqrt{m+2}}{\sqrt{m+1}} B_f P\Big(\sum_{n=0}^{N} b_n z^n \overline{z}^{m+1}\Big) = 0,$$

and

$$H_h z^m = H_h \frac{e_m(z)}{\sqrt{m+1}} = \frac{1}{\sqrt{m+1}} P(h\overline{e}_{m+1})(z) = \frac{\sqrt{m+2}}{\sqrt{m+1}} \sum_{n=0}^{\infty} \frac{(n+1)c_{n+m+1}}{m+n+2} z^n.$$

This implies that $c_{n+m+1} = 0$ for any $n \geqslant 0$ and $m \geqslant N$, which yields $h(z) = \sum_{n=0}^{N} c_n z^n$.

If $N > 0$ and $b_N \neq 0$, then it follows from

$$B_f H_{g_1} z^{N-1} = H_h z^{N-1},$$

that $b_N f(z) = c_N$. Hence, $f = a_0$ and $b_N a_0 = c_N$. Similarly, using

$$B_f H_{g_1} z^{N-2} = H_h z^{N-2},$$

we have that

$$\frac{b_{N-1} a_0}{N} = \frac{c_{N-1}}{N} + \frac{2c_N}{N+1} z.$$

This shows that $c_N = 0$ and hence $a_0 = c_{N-1} = 0$. So we have

$$0 = B_f H_{g_1} z^{N-3} = H_h z^{N-3} = \frac{c_{N-2}}{N-1} + \frac{2c_{N-1}}{N} z.$$

This gives that $c_{N-2} = 0$. Repeating the discussion as above, we conclude that $f = a_0 = 0$ and $c_k = 0$ for $k \in \{1, 2, \cdots, N\}$. This implies that g_1 and h are constants, or $f = 0$ and h is constant.

To prove (3), for each non-negative integer $m \geqslant N$, elementary computations give us that

$$\begin{aligned} H_{g_2} B_f z^{2m} &= \frac{\sqrt{m+1}}{\sqrt{2m+1}} H_{g_2} \Big(\sum_{n=0}^{\infty} a_n z^{n+m} \Big) = \frac{\sqrt{m+1}}{\sqrt{2m+1}} H_{g_2} \Big(\sum_{n=0}^{\infty} \frac{a_n e_{n+m}(z)}{\sqrt{n+m+1}} \Big) \\ &= \frac{\sqrt{m+1}}{\sqrt{2m+1}} P\Big(g_2(z) \sum_{n=0}^{\infty} \frac{a_n \overline{e}_{n+m+1}(z)}{\sqrt{n+m+1}} \Big) \\ &= \frac{\sqrt{m+1}}{\sqrt{2m+1}} P\Big(z^N \sum_{n=0}^{\infty} \frac{a_n \sqrt{n+m+2}}{\sqrt{n+m+1}} \overline{z}^{n+m+1} \Big) = 0, \end{aligned}$$

and

$$B_h z^{2m} = \frac{\sqrt{m+1}}{\sqrt{2m+1}} \sum_{n=0}^{\infty} c_n z^{n+m}.$$

Combining this with

$$H_{g_2} B_f z^{2m} = B_h z^{2m}$$

yields that

$$c_k = 0, \quad k \geqslant 0,$$

to obtain that $h = 0$.

Moreover, for any integer $m > 0$, we have

$$H_{g_2} B_f z^{2m-1} = H_{g_2} B_f \frac{e_{2m-1}(z)}{\sqrt{2m}} = \frac{1}{\sqrt{2m}} H_{g_2} P(f\overline{e}_m)(z)$$

$$= \frac{\sqrt{m+1}}{\sqrt{2m}} H_{g_2} \sum_{n=0}^{\infty} \frac{(n+1)a_{n+m}}{n+m+1} z^n$$
$$= \frac{\sqrt{m+1}}{\sqrt{2m}} P\Big(g_2(z) \sum_{n=0}^{\infty} \frac{(n+1)a_{n+m}}{n+m+1} \frac{\overline{e}_{n+1}(z)}{\sqrt{n+1}}\Big)$$
$$= \frac{\sqrt{m+1}}{\sqrt{2m}} \sum_{n=0}^{N-1} \frac{a_{n+m}(N-n)\sqrt{n+1}\sqrt{n+2}}{(N+1)(n+m+1)} z^{N-n-1}.$$

This implies that $a_{n+m} = 0$ for any $n \in \{0, 1, \cdots, N-1\}$ and $m > 0$. Thus we have $f = a_0$. It follows from

$$H_{g_2} B_f 1 = H_{z^N} B_{a_0} 1 = \frac{\sqrt{2} a_0 N}{N+1} z^{N-1} = 0,$$

that $f = a_0 = 0$. Therefore, we have $f = h = 0$. This finishes the proof of Theorem 3.2. □

Let $\varphi(z)$ and $\psi(z)$ be bounded harmonic functions, and g be an analytic polynomial. We end this section with the following result, which characterizes when $B_\varphi H_g = B_\psi$ on the Bergman space.

Theorem 3.3. *Suppose that $\varphi(z) = f_1(z) + \overline{f}_2(z)$ and $\psi(z) = h_1(z) + \overline{h}_2(z)$ are bounded harmonic functions and the analytic functions f_1, f_2, h_1, and h_2 with the Taylor series $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{n=1}^{\infty} \overline{b}_n z^n$, $h_1(z) = \sum_{n=0}^{\infty} d_n z^n$, $h_2(z) = \sum_{n=1}^{\infty} \overline{\xi}_n z^n$, and $g(z) = \sum_{n=0}^{N} c_n z^n$, where N is a non-negative integer. Then*

$$B_\varphi H_g = B_\psi,$$

if and only if, one of the following holds:

(1) $\psi = 0$ *and g is constant;*

(2) $\psi = 0$ *and* $\varphi(z) = \sum_{n=M}^{\infty} b_n \overline{z}^n$ *when $N = 2M$ for some positive integer M;*

(3) $\psi = 0$ *and* $\varphi(z) = \sum_{n=M+1}^{\infty} b_n \overline{z}^n$ *when $N = 2M+1$ for some positive integer M.*

Proof. If $B_\varphi H_g = B_\psi$, then

$$B_\varphi H_g z^{2m} = B_\psi z^{2m}, \tag{3.5}$$

for any non-negative integer m. For $m \geqslant N$, we have

$$B_\varphi H_g z^{2m} = \frac{1}{\sqrt{2m+1}} B_\varphi P(g \overline{e}_{2m+1})(z) = 0.$$

Moreover, the right-hand side of (3.5) can be expanded as

$$B_\psi z^{2m} = B_\psi \frac{e_{2m}(z)}{\sqrt{2m+1}} = \frac{\sqrt{m+1}}{\sqrt{2m+1}} [P(h_1(z) z^m) + P(\overline{h}_2(z) z^m)]$$
$$= \frac{\sqrt{m+1}}{\sqrt{2m+1}} \Big(\sum_{k=0}^{\infty} d_k z^{k+m} + \sum_{k=0}^{m-1} \frac{k+1}{m+1} \xi_{m-k} z^k \Big).$$

It induces that

$$\sum_{k=0}^{\infty} d_k z^{k+m} + \sum_{k=0}^{m-1} \frac{k+1}{m+1} \xi_{m-k} z^k = 0, \tag{3.6}$$

for any $m \geqslant N$. It follows from (3.6) that $d_k = 0$ for any $k \geqslant 0$ and $\xi_k = 0$ for any $k \geqslant 1$. Thus, we obtain $\psi = 0$. Combining this with $B_\varphi H_g = B_\psi$ yields that

$$\begin{aligned} 0 = B_\varphi H_g z^{N-1} &= B_\varphi H_g \frac{e_{N-1}(z)}{\sqrt{N}} = \frac{1}{\sqrt{N}} B_\varphi P(g\overline{e}_N)(z) \\ &= \frac{c_N}{\sqrt{N}\sqrt{N+1}} P(f_1 + \overline{f}_2)(z) = \frac{c_N}{\sqrt{N}\sqrt{N+1}} f_1(z). \end{aligned}$$

If $c_N \neq 0$ $(N > 0)$, then we have $f_1 = 0$.

- If $N = 2M$ for some positive integer M, then

$$\begin{aligned} 0 = B_{\overline{f}_2} H_g z &= B_{\overline{f}_2}\Big(\frac{\sqrt{3}}{\sqrt{2}} \sum_{k=0}^{2M-2} \frac{\sqrt{k+1}}{k+3} c_{k+2} e_k(z) \Big) \\ &= \frac{\sqrt{3}}{\sqrt{2}} B_{\overline{f}_2}\Big(\sum_{k=0}^{M-1} \frac{\sqrt{2k+1}}{2k+3} c_{2k+2} e_{2k}(z) + \sum_{k=0}^{M-2} \frac{\sqrt{2k+2}}{2k+4} c_{2k+3} e_{2k+1}(z) \Big) \\ &= \frac{\sqrt{3}}{\sqrt{2}} \sum_{k=1}^{M-1} \sum_{n=0}^{k-1} \frac{\sqrt{2k+1}(n+1)}{(2k+3)\sqrt{k+1}} c_{2k+2} b_{k-n} z^n \\ &= \frac{\sqrt{3}}{\sqrt{2}} \Big[\Big(\sum_{k=1}^{M-1} \frac{\sqrt{2k+1}}{(2k+3)\sqrt{k+1}} c_{2k+2} b_k \Big) + \Big(\sum_{k=2}^{M-1} \frac{2\sqrt{2k+1}}{(2k+3)\sqrt{k+1}} c_{2k+2} b_{k-1} \Big) z \\ &\quad + \Big(\sum_{k=3}^{M-1} \frac{3\sqrt{2k+1}}{(2k+3)\sqrt{k+1}} c_{2k+2} b_{k-2} \Big) z^2 + \cdots + \Big(\frac{(M-2)\sqrt{2M-3}}{(2M-1)\sqrt{M-1}} c_{2M-2} b_1 \\ &\quad + \frac{(M-2)\sqrt{2M-1}}{(2M+1)\sqrt{M}} c_{2M} b_2 \Big) z^{M-3} + \Big(\frac{(M-1)\sqrt{2M-1}}{(2M+1)\sqrt{M}} c_{2M} b_1 \Big) z^{M-2} \Big]. \end{aligned}$$

  It follows from $c_N = c_{2M} \neq 0$ that $b_k = 0$ for $k \in \{1, 2, \cdots, M-1\}$.
- If $N = 2M + 1$ for some non-negative integer M, then

$$\begin{aligned} 0 = B_{\overline{f}_2} H_g 1 &= B_{\overline{f}_2}\Big(\sqrt{2} \sum_{k=0}^{2M} \frac{\sqrt{k+1}}{k+2} c_{k+1} e_k(z) \Big) \\ &= \sqrt{2} B_{\overline{f}_2}\Big(\sum_{k=0}^{M} \frac{\sqrt{2k+1}}{2k+2} c_{2k+1} e_{2k}(z) + \sum_{k=0}^{M-1} \frac{\sqrt{2k+2}}{2k+3} c_{2k+2} e_{2k+1}(z) \Big) \\ &= \sqrt{2} \sum_{k=1}^{M} \sum_{n=0}^{k-1} \frac{\sqrt{2k+1}(n+1)}{(2k+2)\sqrt{k+1}} c_{2k+1} b_{k-n} z^n \\ &= \sqrt{2} \Big[\Big(\sum_{k=1}^{M} \frac{\sqrt{2k+1}}{(2k+2)\sqrt{k+1}} c_{2k+1} b_k \Big) + \Big(\sum_{k=2}^{M} \frac{2\sqrt{2k+1}}{(2k+2)\sqrt{k+1}} c_{2k+1} b_{k-1} \Big) z \end{aligned}$$

$$+\Big(\sum_{k=3}^{M}\frac{3\sqrt{2k+1}}{(2k+2)\sqrt{k+1}}c_{2k+1}b_{k-2}\Big)z^2+\cdots+\Big(\frac{(M-1)\sqrt{2M-1}}{2M\sqrt{M}}c_{2M-1}b_1$$
$$+\frac{(M-1)\sqrt{2M+1}}{(2M+2)\sqrt{M+1}}c_{2M+1}b_2\Big)z^{M-2}+\Big(\frac{M\sqrt{2M+1}}{(2M+2)\sqrt{M+1}}c_{2M+1}b_1\Big)z^{M-1}\Big].$$

Combing with $c_N = c_{2M+1} \neq 0$ gives that $b_k = 0$ for $k \in \{1, 2, \cdots, M\}$.

Therefore, we have

$$\varphi(z)=\begin{cases}\sum\limits_{n=M}^{\infty} b_n\overline{z}^n, & \text{if } N=2M,\\ \sum\limits_{n=M+1}^{\infty} b_n\overline{z}^n, & \text{if } N=2M+1.\end{cases}$$

Conversely, if $N = 2M$, $\psi = 0$, and $\varphi(z) = \sum\limits_{n=M}^{\infty} b_n\overline{z}^n$, then for any non-negative integer m we derive that

$$B_\varphi H_g z^m=\begin{cases}0=B_\psi z^m, & \text{if } m\geqslant N,\\ \frac{\sqrt{m+2}}{\sqrt{m+1}}B_{\overline{f}_2}\Big(\sum\limits_{n=0}^{2M-m-1}\frac{(n+1)c_{n+m+1}}{n+m+2}z^n\Big)=0=B_\psi z^m, & \text{if } m\leqslant N-1.\end{cases}$$

Furthermore, if $N = 2M + 1$, $\psi = 0$, and $\varphi(z) = \sum\limits_{n=M+1}^{\infty} b_n\overline{z}^n$, then for any non-negative integer m we obtain that

$$B_\varphi H_g z^m=\begin{cases}0=B_\psi z^m, & \text{if } m\geqslant N,\\ \frac{\sqrt{m+2}}{\sqrt{m+1}}B_{\overline{f}_2}\Big(\sum\limits_{n=0}^{2M-m}\frac{(n+1)c_{n+m+1}}{n+m+2}z^n\Big)=0=B_\psi z^m, & \text{if } m\leqslant N-1.\end{cases}$$

This completes the proof of Theorem 3.3. □

4. The point spectra of H-Toeplitz operators

In this short section, we study the property of the point spectrum of the H-Toeplitz operators B_{z^N} and the spectrum of the H-Toeplitz operators $B_{\overline{z}^N}$ on the Bergman space.

Proposition 4.1. *Suppose that $\varphi(z) = z^N$, where N is a non-negative integer. Then the point spectrum $\sigma_p(B_\varphi)$ is contained in the closed unit disk $\overline{\mathbb{D}}$.*

Proof. Let f be a function in $\ker(\lambda I - B_\varphi)$, where I is the identity operator on L^2_a. Suppose that $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$. Then direct computations give us that

$$0=(\lambda I-B_\varphi)f(z)=\lambda\sum_{k=0}^{\infty}a_kz^k-B_\varphi\sum_{k=0}^{\infty}a_kz^k$$
$$=\lambda\sum_{k=0}^{\infty}a_kz^k-\Big(B_\varphi\sum_{k=0}^{\infty}a_{2k}z^{2k}+B_\varphi\sum_{k=1}^{\infty}a_{2k-1}z^{2k-1}\Big)$$

$$= \lambda \sum_{k=0}^{\infty} a_k z^k - \Big[\sum_{k=0}^{\infty} \frac{a_{2k}\sqrt{k+1}}{\sqrt{2k+1}} z^{k+N} + P\Big(\varphi(z) \sum_{k=1}^{\infty} \frac{a_{2k-1}\overline{e}_k(z)}{\sqrt{2k}}\Big)\Big]$$

$$= \lambda \sum_{k=0}^{\infty} a_k z^k - \Big(\sum_{k=0}^{\infty} \frac{a_{2k}\sqrt{k+1}}{\sqrt{2k+1}} z^{k+N} + \sum_{k=0}^{N-1} \frac{a_{2N-2k-1}\sqrt{N-k+1}(k+1)}{\sqrt{2N-2k}(N+1)} z^k \Big).$$

This gives that λ is an eigenvalue of B_φ, if and only if,

$$\begin{cases} \lambda a_0 = a_{2N-1}\frac{\sqrt{N+1}}{\sqrt{2N}(N+1)}, \\ \lambda a_1 = a_{2N-3}\frac{\sqrt{N}2}{\sqrt{2N-2}(N+1)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda a_{N-2} = a_3\frac{\sqrt{3}(N-1)}{\sqrt{4}(N+1)}, \\ \lambda a_{N-1} = a_1\frac{N}{N+1}, \end{cases}$$

and

$$\begin{cases} \lambda a_N = a_0, \\ \lambda a_{N+1} = a_2\frac{\sqrt{2}}{\sqrt{3}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda a_{2N-1} = a_{2N-2}\frac{\sqrt{N}}{\sqrt{2N-1}}, \\ \lambda a_{2N} = a_{2N}\frac{\sqrt{N+1}}{\sqrt{2N+1}}, \\ \lambda a_{2N+1} = a_{2N+2}\frac{\sqrt{N+2}}{\sqrt{2N+3}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{cases}$$

If $|\lambda| > 1$, then we have

$$a_{2N+k} = a_{2N+2k}\frac{\sqrt{N+k+1}}{\lambda\sqrt{2N+2k+1}}, \quad k \geqslant 0.$$

It follows that

$$a_{2N+k} = \frac{1}{\lambda}\frac{\sqrt{N+k+1}}{\sqrt{2N+2k+1}}a_{2N+2k} = \frac{1}{\lambda^2}\frac{\sqrt{N+k+1}\sqrt{N+2k+1}}{\sqrt{2N+2k+1}\sqrt{2N+4k+1}}a_{2N+4k} = \frac{1}{\lambda^{m-1}}c(N,k)a_{2N+2^m k},$$

for any non-negative integer m, where $c(N,k) \in (0,1)$ is a constant depending only on N and k. As $\lim_{n\to\infty} a_n = 0$, $a_{2N+k} = 0$ for any non-negative integer k.

Letting $G = \{a_0, a_1, \cdots, a_{2N-1}\}$, we define a bijection on G by

$$\Phi(a_k) = \begin{cases} a_{2N-1-2k}\frac{\sqrt{N+1-k}(k+1)}{\lambda\sqrt{2N-2k}(N+1)}, & \text{if } 0 \leqslant k \leqslant N-1, \\ a_{2k-2N}\frac{\sqrt{k-N+1}}{\lambda(2k-2N+1)}, & \text{if } N \leqslant k \leqslant 2N-1. \end{cases}$$

Let

$$\sigma = \begin{bmatrix} 0 & 1 & \cdots & N-1 & N & N+1 & \cdots & 2N-1 \\ 2N-1 & 2N-3 & \cdots & 1 & 0 & 2 & \cdots & 2N-2 \end{bmatrix}$$

be a permutation. Recall that every nonidentity permutation in S_n (the symmetric group of all permutations on $\{1, 2, \cdots, n\}$) is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2. See [14, Theorem 6.3] if needed. It follows that

$$|a_k| = \xi(k, \lambda)|a_k|,$$

where $k \in \{0, 1, \cdots, 2N - 1\}$ and $\xi(k, \lambda)$ is a constant depending only on k and λ and $0 < \xi(k, \lambda) < 1$. This implies that $a_k = 0$ for any $k \in \{0, 1, \cdots, 2N - 1\}$. Thus $f = 0$. This means that $\lambda \notin \sigma_p(B_\varphi)$ and hence $\sigma_p(B_\varphi) \subset \overline{\mathbb{D}}$, completing the proof. □

Proposition 4.2. *Suppose that* $\varphi(z) = \overline{z}^N$*, where* N *is a non-negative integer. Then the spectrum* $\sigma(B_\varphi)$ *is contained in the closed unit disk* $\overline{\mathbb{D}}$*.*

Proof. Since $H_{\overline{z}^N} = 0$, $B_{\overline{z}^N} = PM_{\overline{z}^N}K = T_{\overline{z}^N}PK$. It follows that

$$\sup\{|\lambda| : \lambda \in \sigma(B_{\overline{z}^N})\} \leqslant \|B_{\overline{z}^N}\| = \|T_{\overline{z}^N}PK\| \leqslant 1,$$

and subsequently $\sigma(B_{\overline{z}^N}) \subset \overline{\mathbb{D}}$. This finishes the proof. □

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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