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Research article

Proinov-type relational contractions and applications to boundary value problems

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Abstract: This manuscript deals with proving certain metrical fixed-point findings for a class of generalized contractions involving a couple (ψ, φ) of test functions employing a local class of transitive relation. The outcomes investigated herein refine, modify, unify, and sharpen various existing outcomes. To attest the accuracy of our outcomes, we provide a few examples. By means of our outcomes, we impart an explanation of the reality of a solution to a boundary value problem.

Keywords: fixed points; Picard sequence; **S**-completeness; locally transitive relation; upper and lower solutions
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1. Introduction

In the sequel, the following notations and acronyms will be utilized:

- $\mathbb{N}$: the set of natural numbers
- $\mathbb{N}_0$: $\mathbb{N} \cup \{0\}$
- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}^+ := (0, \infty)$
- $\mathbb{R}_0^+ := [0, \infty)$
- MS : metric space
- BVP: boundary value problem
- BCP : Banach contraction principle
- $F(\mathbf{Q})$: fixed-point set of function $\mathbf{Q}$

- $C(S)$: the space of continuous functions on a set S
- $C'(S)$: the space of continuously differentiable functions on a set S

Fixed-point theory is one of the foremost prominent topics in nonlinear analysis, and subsequently, mathematics. Its findings can be extended to a wide range of integral equations, differential equations, and matrix equations to demonstrate the existence and uniqueness of various types of nonlinear problems. The classical BCP [1] is an extremely important conclusion of metric fixed point theory. The concept of BCP is crucial across many mathematical disciplines. It is successfully attempted to investigate solutions of Volterra as well as Fredholm integral equations, BVPs, nonlinear matrix equations, and nonlinear integro-differential equations, and to illustrate the convergence of algorithms in mathematical computing. Multiple variants of the BCP are accessible in the existing literature of metric fixed point theory; e.g., Boyd and Wong [2], Alber and Guerre-Delabriere [3], Ćirić [4], Kirk [5], Dutta and Choudhury [6], Jachymski [7] and related references. One of the noted classes of generalizations of BCP involves the contraction-inequality of the following form:

$$\psi(\sigma(\mathbf{Q}\mathsf{u}, \mathbf{Q}\mathsf{v})) \leq \varphi(\sigma(\mathsf{u}, \mathsf{v})). \tag{1.1}$$

In recent years, various authors established fixed point theorems under contraction condition of the form (1.1) employing different perspectives, e.g., Amini-Harandi and Petruşel [8], Berzig [9], Proinov [10], Górnicki [11], Popescu [12], Olaru and Secelean [13], Roldán López de Hierro et al. [14], Găvruţa and Manolescu [15], and similar others. In follow-up evaluation, Ω refers to the class of the pair (ψ, φ) of the functions $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}$ that verify:

(a) ψ is monotonic increasing;
(b) $\varphi(t) < \psi(t)$, for every $t > 0$;
(c) Any one of the subsequent circumstances is valid:
  (c_1) $\limsup\limits_{t\to\epsilon+} \varphi(t) < \psi(\epsilon+)$, for every $\epsilon > 0$;
  or,
  (c_2) $\limsup\limits_{t\to\epsilon} \varphi(t) < \liminf\limits_{t\to\epsilon+} \psi(t)$, for every $\epsilon > 0$;
  or,
  (c_3) $\limsup\limits_{t\to\epsilon+} \varphi(t) < \liminf\limits_{t\to\epsilon} \psi(t)$, for every $\epsilon > 0$.

The above family of functions is considered by Górnicki [11] in order to obtain the analogues of some outcomes of Proinov [10] in the setup of preordered metric space.

In 2015, Alam and Imdad [16] presented an inventive adaptation of the BCP to arbitrary binary relations. There are so many generalizations of this core results; however, we merely refer to the recent works due to [17], Alam and Imdad [18], Arif et al. [19], Algehyne et al. [20], and Alamer and Khan [21]. One of the most notable advantages of relational contractions is that no pair of elements is essential; the contraction inequality is enough to remain valid for comparable elements. Relational contractions remain somewhat weaker than their corresponding common contractions. Due to this, such outcomes are implemented to solve certain kinds of integral equations, BVPs and matrix equations, wherein the fixed point findings pertaining to ordinary MS are not laid down.

This paper is an attempt to address outcomes on fixed points of a contraction map of the form (1.1) under the family Ω involving a locally $\mathbf{Q}$-transitive relation. A few examples are furnished for attesting to the credibility of our findings. We entail the availability of a unique solution of a periodic BVP solution by utilizing our findings.

2. Preliminaries

This section covers some essential concepts and preliminary findings for the discussions to follow. In upcoming descriptions, let **U** be a nonempty set, σ a metric on **U**, **Q** a function on **U**, and **S** a binary relation on **U**, (i.e., $\mathbf{S} \subseteq \mathbf{U}^2$). We say that

Definition 2.1. *[16] The points u and v of $\boldsymbol{U}$ remain $\boldsymbol{S}$-comparative if $(u, v) \in \boldsymbol{S}$ or $(v, u) \in \boldsymbol{S}$. We write it as $[u, v] \in \boldsymbol{S}$.*

Definition 2.2. *[22] $\boldsymbol{S}^{-1} = \{(u, v) \in \boldsymbol{U}^2 : (v, u) \in \boldsymbol{S}\}$ is the inverse of $\boldsymbol{S}$.*

Definition 2.3. *[22] $\boldsymbol{S}^s := \boldsymbol{S} \cup \boldsymbol{S}^{-1}$ is the symmetric closure of $\boldsymbol{S}$.*

Proposition 2.1. *[16] $(u, v) \in \boldsymbol{S}^s \Longleftrightarrow [u, v] \in \boldsymbol{S}$.*

Definition 2.4. *[16] A sequence $\{u_n\} \subset \boldsymbol{U}$ is $\boldsymbol{S}$-preserving if*

$$(u_n, u_{n+1}) \in \boldsymbol{S}, \quad \forall\, n \in \mathbb{N}_0.$$

Definition 2.5. *[16] $\boldsymbol{S}$ is $\boldsymbol{Q}$-closed if*

$$(u, v) \in \boldsymbol{S} \Rightarrow (\boldsymbol{Q}u, \boldsymbol{Q}v) \in \boldsymbol{S}.$$

Proposition 2.2. *[17] If $\boldsymbol{S}$ is $\boldsymbol{Q}$-closed, then $\boldsymbol{S}^s$ is also $\boldsymbol{Q}$-closed.*

Proposition 2.3. *[18] If $\boldsymbol{S}$ is $\boldsymbol{Q}$-closed, then for each $n \in \mathbb{N}_0$, $\boldsymbol{S}$ is also $\boldsymbol{Q}^n$-closed.*

Definition 2.6. *[17] $(\boldsymbol{U}, \sigma)$ is $\boldsymbol{S}$-complete if each $\boldsymbol{S}$-preserving Cauchy sequence in $\boldsymbol{U}$ is convergent.*

Definition 2.7. *[17] $\boldsymbol{Q}$ is $\boldsymbol{S}$-continuous at a point $u \in \boldsymbol{U}$ if for any $\boldsymbol{S}$-preserving sequence $\{u_n\}$ verifying $u_n \overset{\sigma}{\longrightarrow} u$, we have $\boldsymbol{Q}(u_n) \overset{\sigma}{\longrightarrow} \mathbf{Q}(u)$. Moreover, $\boldsymbol{Q}$ is $\boldsymbol{S}$-continuous if it is $\boldsymbol{S}$-continuous at each point of $\boldsymbol{U}$.*

Definition 2.8. *[16] $\boldsymbol{S}$ is σ-self-closed if each $\boldsymbol{S}$-preserving sequence $\{u_n\}$ verifying $u_n \overset{\sigma}{\longrightarrow} u$ contains a subsequence $\{u_{n_k}\}$* with $[u_{n_k}, u] \in \boldsymbol{S}$, $\forall\, k \in \mathbb{N}_0$.

Definition 2.9. *[23] Given a pair $u, v \in \boldsymbol{U}$, a path of length l in $\boldsymbol{S}$ from u to v is a finite ordered set $\{w_0, w_1, w_2, ..., w_l\} \subset \boldsymbol{U}$ that verifies:*

(i) $w_0 = u$ and $w_l = v$;
(ii) $(w_i, w_{i+1}) \in \mathbf{S}$, for each i $(0 \leq i \leq l - 1)$.

Definition 2.10. *[17] A subset $\boldsymbol{W}$ of $\boldsymbol{U}$ is $\boldsymbol{S}$-connected if any two elements of $\boldsymbol{W}$ enjoy a path.*

Definition 2.11. *[18] $\boldsymbol{S}$ is $\boldsymbol{Q}$-transitive if for all $u, v, w \in \boldsymbol{U}$,*

$$(\boldsymbol{Q}u, \boldsymbol{Q}v), (\boldsymbol{Q}v, \boldsymbol{Q}w) \in \boldsymbol{S} \Rightarrow (\boldsymbol{Q}u, \boldsymbol{Q}w) \in \boldsymbol{S}.$$

Definition 2.12. *[18] $\boldsymbol{S}$ is locally transitive if for every $\boldsymbol{S}$-preserving sequence $\{u_n\} \subset \boldsymbol{U}$ (with range $\boldsymbol{W} = \{u_n : n \in \mathbb{N}\}$), the restriction $\boldsymbol{S}|_{\boldsymbol{W}}$ is transitive.*

Definition 2.13. *[18] $\boldsymbol{S}$ is locally $\boldsymbol{Q}$-transitive if for every $\boldsymbol{S}$-preserving sequence $\{u_n\} \subset \boldsymbol{Q}(\boldsymbol{U})$ (with range $\boldsymbol{W} = \{u_n : n \in \mathbb{N}\}$), the restriction $\boldsymbol{S}|_{\boldsymbol{W}}$ is transitive.*

The following outcomes achieves the variation of 'locally **Q**-transitivity' over other concepts of 'transitivity':

Proposition 2.4. *[18] We have*

(i) $\mathbf{S}$ is $\mathbf{Q}$-transitive $\Leftrightarrow$ $\mathbf{S}|_{\mathbf{Q(U)}}$ is transitive;
(ii) $\mathbf{S}$ is locally $\mathbf{Q}$-transitive $\Leftrightarrow$ $\mathbf{S}|_{\mathbf{Q(U)}}$ is locally transitive;
(iii) $\mathbf{S}$ is transitive $\Rightarrow$ $\mathbf{S}$ is $\mathbf{Q}$-transitive $\Rightarrow$ $\mathbf{S}$ is locally $\mathbf{Q}$-transitive;
(iv) $\mathbf{S}$ is transitive $\Rightarrow$ $\mathbf{S}$ is locally transitive $\Rightarrow$ $\mathbf{S}$ is locally $\mathbf{Q}$-transitive.

Definition 2.14. *[24] A sequence $\{u_n\} \subset \mathbf{P}$ is semi-Cauchy if*

$$\lim_{n\to\infty} \sigma(u_n, u_{n+1}) = 0, \ \forall\, n \in \mathbb{N}.$$

Obviously, each Cauchy sequence is semi-Cauchy, but the converse is not true, as demonstrated by the following example.

Example 2.1. *Consider $\mathbf{P} = \mathbb{R}$ with metric $\sigma(u, v) = |u - v|$, for all $u, v \in \mathbf{P}$. Then, the sequence $\{u_n\} \subset \mathbf{P}$ defined by $u_n = \sum_{k=1}^{n} \frac{1}{k}$ is semi-Cauchy but not Cauchy.*

Lemma 2.1. *[12] Let $\{u_n\}$ be a semi-Cauchy sequence in a MS $(\boldsymbol{U}, \sigma)$. If $\{u_n\}$ is not Cauchy, then we can determine a couple of subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ and a positive real number ϵ with*

$$\lim_{k\to\infty} \sigma(u_{n_k+1}, u_{m_k+1}) = \lim_{k\to\infty} \sigma(u_{n_k}, u_{m_k}) = \epsilon + . \tag{2.1}$$

3. Main results

For a relation **S** and a function **Q** on a set **U**, the subsequent annotations will be utilized:

- $\mathbf{S_Q} := \{(\mathsf{u}, \mathsf{v}) \in \mathbf{S} : \mathbf{Q}(\mathsf{u}) \neq \mathbf{Q}(\mathsf{v})\}$,
- $\mathbf{U}(\mathbf{Q}, \mathbf{S}) := \{\mathsf{u} \in \mathbf{U} : (\mathsf{u}, \mathbf{Q}\mathsf{u}) \in \mathbf{S}\}$.

Obviously, $(\mathsf{u}, \mathsf{v}) \in \mathbf{S_Q} \Longrightarrow \mathbf{Q}(\mathsf{u}) \neq \mathbf{Q}(\mathsf{v}) \Longrightarrow \mathsf{u} \neq \mathsf{v}$.

Now, we shall prove the fixed point theorem via a locally **Q**-transitive relation employing the pair of functions belonging to the family Ω.

Theorem 3.1. *Assuming that $(\boldsymbol{U}, \sigma)$ is a MS endowed with a relation $\boldsymbol{S}$ and $\boldsymbol{Q}$ a function on $\boldsymbol{U}$. Also,*

(i) $\mathbf{U}(\mathbf{Q}, \mathbf{S}) \neq \emptyset$,
(ii) $\mathbf{S}$ is $\mathbf{Q}$-closed and locally $\mathbf{Q}$-transitive,
(iii) $(\mathbf{U}, \sigma)$ is $\mathbf{S}$-complete,
(iv) $\mathbf{Q}$ is $\mathbf{S}$-continuous or $\mathbf{S}$ is σ-self-closed,
(v) $\exists\, (\psi, \varphi) \in \Omega$ with

$$(\mathsf{u}, \mathsf{v}) \in \mathbf{S_Q} \Longrightarrow \psi(\sigma(\mathbf{Q}\mathsf{u}, \mathbf{Q}\mathsf{v})) \leq \varphi(\sigma(\mathsf{u}, \mathsf{v})),$$

Then, $\mathbf{U}$ has a fixed point.

Proof. The proof distinguishes into several steps:
Step 1. By assumption (i), $\exists\, \mathsf{u}_0 \in \mathbf{U}(\mathbf{Q}, \mathbf{S})$. Define

$$\mathsf{u}_n := \mathbf{Q}^n(\mathsf{u}_0),\ \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

so that

$$\mathsf{u}_{n+1} = \mathbf{Q}(\mathsf{u}_n),\ \forall\, n \in \mathbb{N}_0.$$

Thus, $\{\mathsf{u}_n\}$ is the Picard sequence based at the initial point u_0.
Step 2. We assert that $\{\mathsf{u}_n\}$ is an $\mathbf{S}$-preserving sequence. As $(\mathsf{u}_0, \mathbf{Q}\mathsf{u}_0) \in \mathbf{S}$, using $\mathbf{Q}$-closedness of $\mathbf{S}$ and Proposition 2.3, we get

$$(\mathbf{Q}^n\mathsf{u}_0, \mathbf{Q}^{n+1}\mathsf{u}_0) \in \mathbf{S},$$

so that

$$(\mathsf{u}_n, \mathsf{u}_{n+1}) \in \mathbf{S},\quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

Therefore, the sequence $\{\mathsf{u}_n\}$ is $\mathbf{S}$-preserving.
Step 3. We consider the case: $\sigma(\mathsf{u}_{n_0}, \mathsf{u}_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$. Then, employing (3.1), we get $\mathsf{u}_{n_0} = \mathsf{u}_{n_0+1} = \mathbf{Q}(\mathsf{u}_{n_0}) = 0$. This yields that $\mathsf{u}_{n_0} \in F(\mathbf{Q})$, and hence, our task is accomplished. In either case (when $\sigma(\mathsf{u}_n, \mathsf{u}_{n+1}) > 0$, for all $n \in \mathbb{N}_0$), we'll continue the succeeding steps.
Step 4. We prove that $\{\mathsf{u}_n\}$ is semi-Cauchy, i.e.,

$$\lim_{n\to\infty} \sigma(\mathsf{u}_n, \mathsf{u}_{n+1}) = 0. \tag{3.3}$$

Denote $\sigma_n := \sigma(\mathsf{u}_n, \mathsf{u}_{n+1}) > 0$. Clearly, $(\mathsf{u}_n, \mathsf{u}_{n+1}) \in \mathbf{S}_{\mathbf{Q}}$. Employing assumption (v), we attain, for all $n \in \mathbb{N}_0$ that

$$\psi(\sigma_{n+1}) \le \varphi(\sigma_n),$$

which by using axiom (b) of Ω reduces to

$$\psi(\sigma_{n+1}) \le \varphi(\sigma_n) < \psi(\sigma_n). \tag{3.4}$$

It follows from (3.4) and axiom (a) that $\sigma_{n+1} < \sigma_n$ for each $n \in \mathbb{N}_0$. Thus, the sequence $\{\sigma_n\} \subset \mathbb{R}^+$ is monotonically decreasing which is also bounded below. Consequently, $\exists\, \delta \ge 0$ with $\sigma_n \xrightarrow{\mathbb{R}} \delta$.

Let $\delta > 0$. Next, we use property (c) of Ω. If (ψ, φ) satisfies axiom (c_1), then, employing limit superior in (3.4), we attain

$$\psi(\delta+) = \lim_{n\to\infty} \psi(\sigma_{n+1}) \le \limsup_{n\to\infty} \varphi(\sigma_n) \leqslant \limsup_{t\to\delta+} \psi(t),$$

which contradicts axiom (c_1). Thus, we conclude that $\lim\limits_{n\to\infty} \sigma_n = 0$.

Second, assume that the pair (ψ, φ) satisfies axiom (c_2). Letting the limit inferior in (3.4), we attain

$$\liminf_{t\to\delta+} \psi(t) \leqslant \liminf_{n\to\infty} \psi(\sigma_{n+1}) \leqslant \limsup_{n\to\infty} \varphi(\sigma_n) \leqslant \limsup_{t\to\delta} \varphi(t),$$

which contradicts axiom (c_2). Hence, we conclude that $\lim\limits_{n\to\infty} \sigma_n = 0$.

Finally, assume that (ψ, φ) satisfy axiom (c_3). Letting the limit inferior as $n \to \infty$ in (3.4), we attain

$$\liminf_{t\to\delta} \psi(t) \leqslant \liminf_{n\to\infty} \psi(\sigma_{n+1}) \leqslant \limsup_{n\to\infty} \varphi(\sigma_n) \leqslant \limsup_{t\to\delta+} \varphi(t),$$

which contradicts axiom (c_3). Hence, we conclude that $\lim_{n\to\infty} \sigma_n = 0$. Thus, in each of the cases, (3.3) is verified.

Step 5. We assert that $\{\mathsf{u}_n\}$ is Cauchy. If $\{\mathsf{u}_n\}$ is not Cauchy, then using Lemma 2.1, we can determine two subsequences $\{\mathsf{u}_{n_k}\}$ and $\{\mathsf{u}_{m_k}\}$ of $\{\mathsf{u}_n\}$ and $\epsilon > 0$ for which (2.1) holds. In view of (3.1), we have $\{\mathsf{u}_n\} \subset \mathbf{Q}(\mathbf{U})$. Using locally $\mathbf{Q}$-transitivity of $\mathbf{S}$, we get $(\mathsf{u}_{n_k}, \mathsf{u}_{m_k}) \in \mathbf{S}$. From (2.1), we find $\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1}) > \epsilon$ for all $k \in \mathbb{N}$; consequently, we have $(\mathsf{u}_{n_k}, \mathsf{u}_{m_k}) \in \mathbf{S}_{\mathbf{Q}}$. Applying condition (v) for these points, we get

$$\psi(\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1})) \leq \varphi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})),\ \forall\, k \in \mathbb{N}. \tag{3.5}$$

Using axiom (b), we obtain

$$\psi(\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1})) \leq \varphi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})) < \psi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})),$$

which, using monotonicity of ψ, gives rise

$$\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1}) < \sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k}).$$

Now, we shall employ property (c) of Ω. First, assume that the pair (ψ, φ) satisfies axiom (c_1). Employing the limit superior in (3.5), we obtain

$$\psi(\epsilon+) = \lim_{k\to\infty} \psi(\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1})) \leq \limsup_{k\to\infty} \varphi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})) \leq \limsup_{t\to\epsilon+} \varphi(t),$$

which contradicts to axiom (c_1).

Second, assume that ψ and φ satisfy axiom (c_2). Employing the limit inferior in (3.5), we get

$$\liminf_{t\to\epsilon} \psi(t) = \liminf_{k\to\infty} \psi(\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1})) \leq \limsup_{k\to\infty} \varphi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})) \leq \limsup_{t\to\epsilon+} \varphi(t),$$

which contradicts to axiom (c_2).

Finally, assume that ψ and φ satisfy axiom (c_3). Employing the limit inferior as $k \to \infty$ in (3.5), we attain

$$\liminf_{t\to\epsilon} \psi(t) = \liminf_{k\to\infty} \psi(\sigma(\mathsf{u}_{n_k+1}, \mathsf{u}_{m_k+1})) \leq \limsup_{k\to\infty} \varphi(\sigma(\mathsf{u}_{n_k}, \mathsf{u}_{m_k})) \leq \limsup_{t\to\epsilon+} \varphi(t),$$

which contradicts to axiom (c_3). Therefore, in each of the cases, $\{\mathsf{u}_n\}$ is Cauchy, which is also $\mathbf{S}$-preserving. Employing $\mathbf{S}$-completeness of $(\mathbf{U}, \sigma)$, $\exists\, \bar{\mathsf{u}} \in \mathbf{U}$ with $\mathsf{u}_n \xrightarrow{\sigma} \bar{\mathsf{u}}$.

Step 6. We verify that $\bar{\mathsf{u}}$ is a fixed point of $\mathbf{U}$ employing the hypothesis (iv). Assume that $\mathbf{Q}$ is $\mathbf{S}$-continuous. As $\{\mathsf{u}_n\}$ is $\mathbf{S}$-preserving with $\mathsf{u}_n \xrightarrow{\sigma} \bar{\mathsf{u}}$, using $\mathbf{S}$-continuity of $\mathbf{Q}$, we obtain $\mathsf{u}_{n+1} = \mathbf{Q}(\mathsf{u}_n) \xrightarrow{\sigma} \mathbf{Q}(\bar{\mathsf{u}})$. Therefore, we conclude $\mathbf{Q}(\bar{\mathsf{u}}) = \bar{\mathsf{u}}$, *i.e.*, $\bar{\mathsf{u}}$ is a fixed point of $\mathbf{U}$. Alternatively, in case $\mathbf{S}$ to be σ-self-closed, there is a subsequence $\{\mathsf{u}_{n_k}\}$ of $\{\mathsf{u}_n\}$ with $[\mathsf{u}_{n_k}, \bar{\mathsf{u}}] \in \mathbf{S}_{\mathbf{Q}}$, for all $k \in \mathbb{N}$. Now two cases arise:

Case (i): If for infinitely many values of k, $[\mathsf{u}_{n_k}, \bar{\mathsf{u}}] \notin \mathbf{S}_{\mathbf{Q}}$, then we have $\sigma(\mathbf{Q}\mathsf{u}_{n_k+1}, \mathbf{Q}\bar{\mathsf{u}}) = 0$ yielding thereby

$$\sigma(\bar{\mathsf{u}}, \mathbf{Q}\bar{\mathsf{u}}) \quad \leq \quad \sigma(\bar{\mathsf{u}}, \mathsf{u}_{n_k+1}) + \sigma(\mathsf{u}_{n_k+1}, \mathbf{Q}\bar{\mathsf{u}}) = \sigma(\bar{\mathsf{u}}, \mathsf{u}_{n_k+1})$$

$$\rightarrow \quad 0 \text{ as } k \to \infty,$$

so that $\mathbf{U}(\bar{\mathsf{u}}) = \bar{\mathsf{u}}$ and hence the proof is completed.
Case (ii): Assume that $[\mathsf{u}_{n_k}, \bar{\mathsf{u}}] \in \mathbf{S_Q}$ for infinitely many values of k. In view of the symmetric property of metric σ, the contraction condition (v) will be satisfied for all $[\mathsf{u}, \mathsf{v}] \in \mathbf{S_Q}$. Thus, we obtain

$$\psi(\mathsf{u}_{n_k+1}, \mathbf{Q}\bar{\mathsf{u}}) = \psi(\mathbf{Q}\mathsf{u}_{n_k}, \mathbf{Q}\bar{\mathsf{u}}) \leq \varphi(\mathsf{u}_{n_k}, \bar{\mathsf{u}}) < \psi(\mathsf{u}_{n_k}, \bar{\mathsf{u}}),$$

so that

$$\psi(\sigma(\mathsf{u}_{n_k+1}, \mathbf{Q}\bar{\mathsf{u}})) < \psi(\sigma(\mathsf{u}_{n_k}, \bar{\mathsf{u}})).$$

Using monotonicity of ψ above equality give rise to

$$\sigma(\mathsf{u}_{n_k+1}, \mathbf{Q}\bar{\mathsf{u}}) < \sigma(\mathsf{u}_{n_k}, \bar{\mathsf{u}}) \to 0 \text{ as } k \to \infty,$$

so that $\mathbf{Q}(\bar{\mathsf{u}}) = \bar{\mathsf{u}}$, and hence, $\bar{\mathsf{u}}$ is a fixed point of $\mathbf{Q}$. □

Theorem 3.2. *In alliance with the predictions of Theorem 3.1, if*

(u) $\mathbf{Q}(\mathbf{U})$ is $\mathbf{S}^s$-connected,

then, $\mathbf{Q}$ owns a unique fixed point.

Proof. Due to Theorem 3.1, $F(\mathbf{Q}) \neq \emptyset$. Choose $\mathsf{u}^*, \mathsf{v}^* \in F(\mathbf{Q})$, then for each $n \in \mathbb{N}_0$, we arrive at

$$\mathbf{Q}^n(\mathsf{u}^*) = \mathsf{u}^* \text{ and } \mathbf{Q}^n(\mathsf{v}^*) = \mathsf{v}^*.$$

Clearly $\mathsf{u}^*, \mathsf{v}^* \in \mathbf{Q}(\mathbf{U})$. By $\mathbf{S}^s$-connectedness of $\mathbf{Q}(\mathbf{U})$, we determine a path $\mathsf{w}_0, \mathsf{w}_1, \mathsf{w}_2, ..., \mathsf{w}_l$ between u^* and v^*; so,

$$\mathsf{w}_0 = \mathsf{u}^*, \mathsf{w}_l = \mathsf{v}^* \text{ and } [\mathsf{w}_i, \mathsf{w}_{i+1}] \in \mathbf{S}, \ \forall\, i = 0, 1, ..., l-1.$$

As $\mathbf{S}$ is $\mathbf{Q}$-closed, we have

$$[\mathbf{Q}^n \mathsf{w}_i, \mathbf{Q}^n \mathsf{w}_{i+1}] \in \mathbf{S}, \ \forall\, n \in \mathbb{N}_0 \text{ and } \ \forall\, i = 0, 1, ..., l-1.$$

Denote

$$\delta_i^n := \sigma(\mathbf{Q}^n \mathsf{w}_i, \mathbf{Q}^n \mathsf{w}_{i+1}) \ \forall\, n \in \mathbb{N}_0 \text{ and } \ \forall\, i = 0, 1, ..., l-1.$$

We show that

$$\lim_{n\to\infty} \delta_n^i = 0. \tag{3.6}$$

For every fixed i, consider the two possible cases:
Case (i): Assume that

$$\delta_{n_0}^i = \sigma(\mathbf{Q}^{n_0} \mathsf{w}_i, \mathbf{Q}^{n_0} \mathsf{w}_{i+1}) = 0, \text{ for some } n_0 \in \mathbb{N}_0,$$

thereby implying $\mathbf{Q}^{n_0}(\mathsf{w}_i) = \mathbf{Q}^{n_0}(\mathsf{w}_{i+1})$. By (3.1), we attain $\mathbf{U}^{n_0+1}(\mathsf{w}_i) = \mathbf{Q}^{n_0+1}(\mathsf{w}_{i+1})$; so, $\delta_{n_0+1}^i = 0$. Using induction, we get $\delta_n^i = 0 \ \forall\, n \geq n_0$ so that $\lim_{n\to\infty} \delta_n^i = 0$.
Case (ii): If for every $n \in \mathbb{N}_0$, $\delta_n^i > 0$, then we have $(\mathbf{Q}^n \mathsf{w}_i, \mathbf{Q}^n \mathsf{w}_{i+1}) \in \mathbf{S}^s_{\mathbf{Q}}$. From (v), we attain

$$\psi(\delta_{n+1}^i) \quad = \quad \psi(\sigma(\mathbf{Q}^{n+1} \mathsf{w}_i, \mathbf{Q}^{n+1} \mathsf{w}_{i+1}))$$

$$
\begin{aligned}
&= \psi(\sigma(\mathbf{Q}(\mathbf{Q}^n \mathsf{w}_i), \mathbf{Q}(\mathbf{Q}^n \mathsf{w}_{i+1}))) \\
&\leq \varphi(\sigma(\mathbf{Q}^n \mathsf{w}_i, \mathbf{Q}^n \mathsf{w}_{i+1})) \\
&= \varphi(\delta_n^i),
\end{aligned}
$$

so that

$$\psi(\delta_{n+1}^i) \leq \varphi(\delta_n^i). \tag{3.7}$$

Using axiom (b) of Ω, (3.7) reduces to

$$\psi(\delta_{n+1}^i) \leq \varphi(\delta_n^i) < \psi(\delta_n^i), \ \forall \ n \in \mathbb{N}_0,$$

which, in view of axiom (a), reduces to $\delta_{n+1}^i < \delta_n^i$ for all $n \in \mathbb{N}_0$. Hence, proceeding with the proof of Theorem 3.1, we can determine $\delta^i \geq 0$ satisfying $\delta_n^i \xrightarrow{\mathbb{R}} \delta^i$.

In view of property (c), let us assume that (ψ, φ) satisfies axiom (c_1). Employing limit superior in (3.7), we obtain

$$\psi(\delta^i+) = \lim_{n\to\infty} \psi(\delta_{n+1}^i) \leq \limsup_{n\to\infty} \varphi(\delta_n^i) \leqslant \limsup_{\delta_n^i \to \delta^i+} \psi(\delta_n^i),$$

which is a contradiction to axiom (c_1). Hence, we conclude that $\lim_{n\to\infty} \delta_n^i = 0$.

Second, assume that the pair (ψ, φ) satisfies axiom (c_2). Letting the limit inferior in (3.7), we obtain

$$\liminf_{t\to\delta^i+} \psi(t) \leqslant \liminf_{n\to\infty} \psi(\delta_{n+1}^i) \leqslant \limsup_{n\to\infty} \varphi(\delta_n^i) \leqslant \limsup_{t\to\delta^i} \varphi(t),$$

which is a contradiction to axiom (c_2). Hence, we conclude that $\lim_{n\to\infty} \delta_n^i = 0$.

Finally, assume that (ψ, φ) satisfies axiom (c_3). Letting the limit inferior in (3.7), we attain

$$\liminf_{t\to\delta^i} \psi(t) \leqslant \liminf_{n\to\infty} \psi(\delta_{n+1}^i) \leqslant \limsup_{n\to\infty} \varphi(\delta_n^i) \leqslant \limsup_{t\to\delta^i+} \varphi(t),$$

which is a contradiction to axiom (c_3). Hence, we conclude that $\lim_{n\to\infty} \delta_n^i = 0$.

Hence, (3.6) is proved. Using the triangle inequality, we find

$$
\begin{aligned}
\sigma(\mathsf{u}^*, \mathsf{v}^*) &= \sigma(\mathbf{Q}^n \mathsf{w}_0, \mathbf{Q}^n \mathsf{w}_k) \\
&\leq \delta_n^0 + \delta_n^1 + \cdots + \delta_n^{k-1} \\
&\to 0 \text{ as } n \to \infty;
\end{aligned}
$$

so, $\mathsf{u}^* = \mathsf{v}^*$. Thus, $\mathbf{Q}$ admits a unique fixed point. □

4. Illustrative examples

In the following, we provide two instances to substantiate the relevance of Theorems 3.1 and 3.2.

Example 4.1. *Let $\boldsymbol{U} = [2, 4]$ be a MS with usual metric σ. On $\boldsymbol{U}$, consider the relation $\boldsymbol{S} = \{(2, 2), (2, 3), (3, 2), (3, 3), (0, 4)\}$. Then, $(\boldsymbol{U}, \sigma)$ is a complete MS. Define a function $\boldsymbol{Q}$ on $\boldsymbol{U}$*

$$\mathbf{Q}(\mathsf{u}) = \begin{cases} 2 \text{ if } 2 \leq \mathsf{u} \leq 3, \\ 3 \text{ if } 3 \leq \mathsf{u} \leq 4. \end{cases}$$

Thus, $\mathbf{S}$ is $\mathbf{Q}$-closed. Assuming that $\{\mathsf{u}_n\} \subset \mathbf{U}$ is $\mathbf{S}$-preserving sequence and $\mathsf{u}_n \xrightarrow{\sigma} \mathsf{u}$. Consequently, we conclude $(\mathsf{u}_n, \mathsf{u}_{n+1}) \in \mathbf{S}$, for every $n \in \mathbb{N}$. Note that $(\mathsf{u}_n, \mathsf{u}_{n+1}) \notin \{(2,4)\}$, implying thereby $(\mathsf{u}_n, \mathsf{u}_{n+1}) \in \{(2,2),(2,3),(3,2),(3,3)\}$, $\forall\ n \in N$; so, $\{\mathsf{u}_n\} \subset \{2,3\}$. Closedness of $\{2,3\}$ yields that $[\mathsf{u}_n, \mathsf{u}] \in \mathbf{S}$. Hence, $\mathbf{S}$ is σ-self-closed. Define the functions $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\psi(t) = t^2 \text{ and } \varphi(t) = \frac{t^2}{(t^2+1)}.$$

Then, $(\psi, \varphi) \in \Omega$ and the contraction-inequality (v) of Theorem 3.1 is verified for (ψ, φ). Moreover, the remaining hypotheses of Theorems 3.1 and 3.2 are also verified. This concludes that $\mathbf{Q}$ owns a unique fixed point (namely: $\bar{\mathsf{u}} = 2$).

Example 4.2. *Take $\mathbf{U} = \mathbb{R}^+$ with Euclidean metric σ. Construct a relation $\mathbf{S}$ on $\mathbf{U}$ by*

$$\mathbf{S} := \{(\mathsf{u}, \mathsf{v}) \in \mathbf{U}^2 : \mathsf{u}^2 + 2\mathsf{u} = \mathsf{v}^2 + 2\mathsf{v}\}.$$

Clearly, the MS $(\mathbf{U}, \sigma)$ forms an $\mathbf{S}$-complete. Define a function $\mathbf{Q}$ on $\mathbf{U}$ by

$$\mathbf{Q}(\mathsf{u}) = \ln(\mathsf{u}^2 + 2\mathsf{u} + 1), \forall \mathsf{u} \in \mathbf{U}.$$

Then, $\mathbf{S}$ is a locally finitely $\mathbf{Q}$-transitive and $\mathbf{Q}$-closed relation, while $\mathbf{Q}$ is $\mathbf{S}$-continuous. Also, $\mathbf{U}(\mathbf{Q}, \mathbf{S}) \neq \emptyset$ as $(0, \mathbf{Q}0) \in \mathbf{S}$.

Take $(\mathsf{u}, \mathsf{v}) \in \mathbf{S}$. Then, we have

$$\mathbf{Q}(\mathsf{u}) = \ln(\mathsf{u}^2 + 2\mathsf{u} + 1) = \ln(\mathsf{v}^2 + 2\mathsf{v} + 1) = \mathbf{Q}(\mathsf{v})$$

yielding thereby

$$(\mathbf{Q}\mathsf{u})^2 + 2\mathbf{Q}\mathsf{u} = (\mathbf{Q}\mathsf{v})^2 + 2\mathbf{Q}\mathsf{v}.$$

This implies that $(\mathbf{Q}\mathsf{u}, \mathbf{Q}\mathsf{v}) \in \mathbf{S}$, and hence, $\mathbf{S}$ is $\mathbf{Q}$-closed. Define the pair $(\psi, \varphi) \in \Omega$ such that

$$\psi(t) = \begin{cases} \ln(t+1), & \text{if } t \leq 1, \\ \dfrac{3t}{4}, & \text{if } t > 1, \end{cases} \tag{4.1}$$

and $\varphi(t) = 2t/3$. Then, for all $(\mathsf{u}, \mathsf{v}) \in \mathbf{S}$, we can easily verify the following condition:

$$\psi(\sigma(\mathbf{Q}\mathsf{u}, \mathbf{Q}\mathsf{v})) \leq \varphi(\sigma(\mathsf{u}, \mathsf{v})).$$

Thus far, the requirements of Theorems 3.1 and 3.2 are all fulfilled. Thus, $\mathbf{Q}$ owns a unique fixed point (namely: $\bar{\mathsf{u}} = 0$).

5. An application to BVP

Consider the following first-order periodic BVP:

$$\begin{cases} \omega'(\xi) = \hbar(\xi, \omega(\xi)), & \text{for each } \xi \in [0, L], \\ \omega(0) = \omega(L), \end{cases} \tag{5.1}$$

where $\hbar : [0, L] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Definition 5.1. *[25] $\underline{\omega} \in C^{(1)}[0, L]$ is named as a lower solution of (5.1) if*

$$\begin{cases} \underline{\omega}'(\xi) \leq \hbar(\xi, \underline{\omega}(\xi)), & \text{for each } \xi \in [0, L], \\ \underline{\omega}(0) \leq \underline{\omega}(L). \end{cases}$$

Definition 5.2. *[25] $\overline{\omega} \in C^{(1)}[0, L]$ is named as an upper solution of (5.1) if*

$$\begin{cases} \overline{\omega}'(\xi) \geq \hbar(\xi, \overline{\omega}(\xi)), & \text{for each } \xi \in [0, L], \\ \overline{\omega}(0) \geq \overline{\omega}(L). \end{cases}$$

We now present the outcome, insuring a solution to Problem (5.1).

Theorem 5.1. *Along with the problem (5.1), if $\exists\ \lambda, \alpha > 0$ with*

$$\alpha \leq \left(\frac{2\lambda(e^{\lambda L} - 1)}{L(e^{\lambda L} + 1)} \right)^{\frac{1}{2}},$$

such that for $l, m \in \mathbb{R}$ *with* $l \geq m$,

$$0 \leq \hbar(\xi, l) + \lambda l - [\hbar(\xi, m) + \lambda m] \leq \alpha \sqrt{\ln[(l–m)^2 + 1]}. \tag{5.2}$$

If (5.1) admits a lower solution, then it possesses a unique solution.

Proof. Rewrite Problem (5.1) as

$$\begin{cases} \omega'(\xi) + \lambda\omega(\xi) = \hbar(\xi, \omega(\xi)) + \lambda\omega(\xi), \text{ for } \xi \in [0, L], \\ \omega(0) = \omega(L), \end{cases} \tag{5.3}$$

Equation (5.3) is equivalent to the integral equation

$$\omega(\xi) = \int_0^L F(\xi, \tau)[\hbar(\tau, \omega(\tau)) + \lambda\omega(\tau)]d\tau, \tag{5.4}$$

where the Green function is

$$F(\xi, \tau) = \begin{cases} \dfrac{e^{\lambda(L+\tau-\xi)}}{e^{\lambda L} - 1}, \ 0 \leq \tau < \xi \leq L; \\ \dfrac{e^{\lambda(\tau-\xi)}}{e^{\lambda L} - 1}, \ 0 \leq \xi < \tau \leq L. \end{cases}$$

Denote $\mathbf{U} := C[0, L]$. Define a function $\mathbf{Q} : \mathbf{U} \longrightarrow \mathbf{U}$ by

$$(\mathbf{Q}\omega(\xi) = \int_0^L F(\xi, \tau)[\hbar(\tau, \omega(\tau)) + \lambda\omega(\tau)]d\tau, \tag{5.5}$$

Thus, $\theta \in \mathbf{U}$ is a fixed point of $\mathbf{Q}$ if and only if, $\theta \in C^1[0, L]$ forms a solution of (5.4), and hence, of (5.1). On $\mathbf{U}$, endow a relation

$$\mathbf{S} = \{(\omega, \nu) \in \mathbf{U} \times \mathbf{U} : \omega(\xi) \leq \nu(\xi),\ \forall\ \xi \in [0, L]\}; \tag{5.6}$$

and a metric

$$\sigma(\omega, \nu) = \sup_{\xi \in [0,L]} |(\omega(\xi) - \nu(\xi)|,\ \forall\ \omega, \nu \in \mathbf{U}. \tag{5.7}$$

Now, we check all the presumptions of Theorem 3.2.

(i) Assuming that $\underline{\omega}(\xi)$ is a lower solution for (5.1). We conclude

$$\underline{\omega}'(\xi) + \lambda\underline{\omega}(\xi) \leq \hbar(\xi, \underline{\omega}(\xi)) + \lambda\underline{\omega}(\xi),\ \text{ for } \xi \in [0, L].$$

Taking the product with $e^{\lambda\xi}$, we attain

$$(\underline{\omega}(\xi)e^{\lambda\xi})' \leq [\hbar(\xi, \underline{\omega}(\xi)) + \lambda\underline{\omega}(\xi)]e^{\lambda\xi},\ \text{ for } \xi \in [0, L],$$

or

$$\underline{\omega}(\xi)e^{\lambda\xi} \leq \underline{\omega}(0) + \int_0^\xi [\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]e^{\lambda\tau}d\tau,\ \text{ for } \xi \in [0, L].$$

As $\underline{\omega}(0) \leq \underline{\omega}(L)$, the last inequality gives us

$$\underline{\omega}(0)e^{\lambda\xi} \leq \underline{\omega}(L)e^{\lambda L} \leq \underline{\omega}(0) + \int_0^L [\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]e^{\lambda\tau}d\tau,$$

so that

$$\underline{\omega}(0) \leq \int_0^L \frac{e^{\lambda\tau}}{e^{\lambda L} - 1}[\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]d\tau,$$

which, using (5.6), gives rise

$$\underline{\omega}(\xi)e^{\lambda\xi} \leq \int_0^\xi \frac{e^{\lambda(L+\tau)}}{e^{\lambda L} - 1}[\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]d\tau + \int_\xi^L \frac{e^{\lambda\tau}}{e^{\lambda L} - 1}[\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]d\tau,$$

and consequently,

$$\begin{aligned}
\underline{\omega}(\xi) &\leq \int_0^\xi \frac{e^{\lambda(L+\tau-\xi)}}{e^{\lambda L} - 1}d\tau + \int_0^\xi \frac{e^{\lambda(\tau-\xi)}}{e^{\lambda L} - 1}[\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]d\tau \\
&= \int_0^L F(\xi, \tau)[\hbar(\tau, \underline{\omega}(\tau)) + \lambda\underline{\omega}(\tau)]d\tau \\
&= (\mathbf{Q}\underline{\omega})(\xi),\ \text{ for } \xi \in [0, L].
\end{aligned}$$

(ii) Take $(\omega, \nu) \in \mathbf{S}$. Then, for each $\tau \in [0, L]$, we have $\omega(\tau) \leq \nu(\tau)$. Consequently, using (5.2), we obtain

$$\hbar(\tau, \omega(\tau) + \lambda\omega(\tau) \leq \hbar(\tau, \nu(\tau)) + \lambda\nu(\tau),$$

which yields that

$$\begin{aligned}(\mathbf{Q}\omega(\xi) &= \int_0^L F(\xi,\tau)[\hbar(\tau,\omega(\tau)) + \lambda\omega(\tau)]d\tau \\ &\leq \int_0^L F(\xi,\tau)[\hbar(\tau,\omega(\tau)) + \lambda\omega(\tau)]d\tau \\ &= (\mathbf{Q}\nu(\xi).\end{aligned}$$

It follows that $(\mathbf{Q}\omega, \mathbf{Q}\nu) \in \mathbf{S}$ so that $\mathbf{S}$ is $\mathbf{Q}$-closed. Also, $\mathbf{S}$ being transitive is locally $\mathbf{Q}$-transitive.
(iii) The MS $(\mathbf{U}, \sigma)$ being complete is $\mathbf{S}$-complete.
(iv) Let $\{\omega_n\} \subset \mathbf{U}$ be $\mathbf{S}$-preserving sequence converging to $\omega \in \mathbf{U}$. Hence, for every $\xi \in [0, L], \{\omega_n(\xi)\}$ is an increasing sequence in $\mathbb{R}$ converging to $\omega(\xi)$, and so, $\forall\ n \in \mathbb{N}$ and $\tau \in [0, L]$, we conclude $\omega_n(\xi) \leq \omega(\xi)$. Again, due to (5.6), it follows that $(\omega_n, \omega) \in \mathbf{S},\ \forall\ n \in \mathbb{N}$. Thus, $\mathbf{S}$ is σ-self-closed.
(v) Take $(\omega, \nu) \in \mathbf{S_Q}$. Then, for each $\tau \in [0, L]$, we attain $\omega(\tau) \leq \nu(\tau)$. Consequently, using (5.2), we obtain

$$\begin{aligned}\sigma(\omega, \mathbf{Q}\nu) &= \sup_{\xi\in[0,L]} |(\mathbf{Q}\omega(\xi) - (\mathbf{Q}\nu(\xi)| \\ &= \sup_{\xi\in[0,L]} ((\mathbf{Q}\nu(\xi) - (\mathbf{Q}\omega(\xi)) \\ &= \sup_{\xi\in[0,L]} \int_0^L F(\xi,\tau)[\hbar(\tau,\nu(\tau)) + \lambda\nu(\tau) - \hbar(\tau,\omega(\tau)) - \lambda\omega(\tau)]d\tau \\ &\leq \sup_{\xi\in[0,L]} \int_0^L F(\xi,\tau)\alpha\sqrt{\ln[(\omega(\tau) - \nu(\tau))^2 + 1]}d\tau.\end{aligned}$$

Employing Cauchy-Schwarz inequality, we attain

$$\begin{aligned}\int_o^L F(\xi,\tau)\alpha\sqrt{\ln[(\omega(\tau) - \nu(\tau))^2 + 1]}d\tau &\leq \left(\int_0^L F(\xi,\tau)^2 d\tau\right)^{\frac{1}{2}} \\ &\left(\int_0^L \alpha^2 \ln[(\omega(\tau) - \nu(\tau))^2 + 1]\right)^{\frac{1}{2}}.\end{aligned}$$

The first integral reduces to

$$\begin{aligned}\int_0^L F(\xi,\tau)^2 d\tau &= \int_0^\xi F(\xi,\tau)^2 d\tau + \int_\xi^L F(\xi,\tau)^2 d\tau \\ &= \int_0^\xi \frac{e^{2\lambda(L+\tau-\xi)}}{(e^{\lambda L} - 1)^2}d\tau + \int_\xi^L \frac{e^{2\lambda(\tau-\xi)}}{(e^{\lambda L} - 1)^2}d\tau \\ &= \frac{1}{2\lambda(e^{\lambda L} - 1)^2}e^{2\lambda L - 1}\end{aligned}$$

$$= \frac{e^{\lambda L+1}}{2\lambda(e^{\lambda L}-1)}.$$

The second integral can be estimated as

$$\begin{aligned}\int_0^L \alpha^2 \ln[(\omega(\tau)-\nu(\tau))^2] &\leq \alpha^2 \ln[\|\omega-\nu\|^2+1]\cdot L\\ &= \alpha^2 \ln[\sigma(\omega,\nu)^2+1]\cdot L.\end{aligned}$$

Taking into account, we conclude

$$\begin{aligned}\sigma(\mathbf{Q}\omega,\mathbf{Q}\nu) &\leq \sup_{\xi\in[0,L]}\left(\frac{e^{\lambda L}+1}{2\lambda(e^{\lambda L}-1)}\right)^{\frac{1}{2}}\cdot(\alpha^2\ln[\sigma(\omega,\nu)^2+1]\cdot L)^{\frac{1}{2}}\\ &= \left(\frac{e^{\lambda L}+1}{2\lambda(e^{\lambda L}-1)}\right)^{\frac{1}{2}}\cdot\alpha\cdot\sqrt{L}(\ln[\sigma(\omega,\nu)^2+1])^{\frac{1}{2}},\end{aligned}$$

and from the last inequality, we obtain

$$\sigma(\mathbf{Q}\omega,\mathbf{Q}\nu^2 \leq \left(\frac{e^{\lambda L}+1}{2\lambda(e^{\lambda L}-1)}\right)\cdot\alpha^2\cdot L\cdot\ln[\sigma(\omega,\nu)^2+1],$$

or equivalently,

$$2\lambda(e^{\lambda L}-1)\sigma(\mathbf{Q}\omega,\mathbf{Q}\nu^2 \leq (e^{\lambda L}+1)\cdot\alpha^2\cdot L\cdot\ln[\sigma(\omega,\nu)^2+1].$$

Using the hypothesis:

$$L \leq \left(\frac{2\lambda(e^{\lambda L}-1)}{\mathbf{Q}e^{\lambda L}+1}\right)^{\frac{1}{2}},$$

the last inequality reduces to

$$2\lambda(e^{\lambda L}-1)\sigma(\mathbf{Q}\omega,\mathbf{Q}\nu^2 \leq 2\lambda(e^{\lambda L}-1)\cdot\ln[\sigma(\omega,\nu)^2+1],$$

and hence,

$$\sigma(\mathbf{Q}\omega,\mathbf{Q}\nu^2 \leq \ln[\sigma(\omega,\nu)^2+1].$$

Put $\psi(\xi)=\xi^2$ and $\varphi(\xi)=\ln(\xi^2+1)$. Then, we have $(\psi,\varphi)\in\Omega$. Thus, (5.7) reduces to

$$\psi(d(\mathbf{Q}\omega,\mathbf{Q}\nu)\leq\varphi(\sigma(\omega,\nu)),\quad \forall\,(\omega,\nu)\in\mathbf{S_Q}.$$

Let $\omega,\nu\in\mathbf{U}$ be arbitrary. Then, one has $\vartheta:=\max\{\mathbf{Q}\omega,\mathbf{Q}\nu\}\in\mathbf{U}$. As $(\mathbf{Q}\omega,\vartheta)\in\mathbf{S}$ and $(\mathbf{Q}\nu,\vartheta)\in\mathbf{S}$, $\{\mathbf{Q}\omega,\vartheta,\mathbf{Q}\nu\}$ is a path in $\mathbf{S}^s$ between $\mathbf{Q}(\omega)$ and $\mathbf{Q}(\nu)$. Thus, $\mathbf{Q}(\mathbf{U})$ is $\mathbf{S}^s$-connected, and so by Theorem 3.2, $\mathbf{Q}$ owns a unique fixed point, which forms the unique solution of Problem (5.1).

Intending to illustrate Theorem 5.1, we consider the following numerical example.

Example 5.1. *Let $\hbar(\xi,\omega(\xi))=\cos\xi$ for $0\leq\xi\leq\pi$; then $\hbar$ is a continuous function. Note that $\underline{\omega}=0$ is a lower solution for $\omega'(\xi)=\cos\xi$. Therefore, Theorem 5.1 can be applied for the given problem, and hence, $\omega(\xi)=\sin\xi$ forms the unique solution.*

□

6. Conclusions

We investigated metrical fixed-point findings for a relational contraction map under generalized contraction via a pair of test functions, which, under the preordered (reflexive and transitive) relation, deduce the corresponding outcomes of Górnicki [11]. To demonstrate our outcomes, we furnished a few examples. From an application point of view, we discussed an existence and uniqueness theorem for certain BVP under the availability of a lower solution. Analogously, we can also study the existence and uniqueness of the BVPs whenever an upper solution exists. As a future plane, we can improve our outcomes to a couple of self-maps by establishing coincidence and common fixed point theorems.

Author contributions

Abdul Wasey: Methodology, original draft writing, conceptualization, editing, funding; Wan Ainun Mior Othman: Review, supervision, funding; Esmail Alshaban: Funding, reviewing, and editing; Kok Bin Wong: Review, supervision, funding; Adel Alatawi: Funding, reviewing and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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