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*Research article*

# Some generalized matrix means inequalities

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**Abstract:** In this paper, we will give some AM-GM-HM singular values inequalities and some weighted power mean inequalities, our results generalized and complete the existed ones.

**Keywords:** power mean inequalities; singular values; convex function
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## 1. Introduction

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, and let $\mathbb{B}(\mathbb{H})$ denote the algebra of all bounded linear operators acting on $\mathbb{H}$. A self adjoint operator $A$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$, while it is said to be strictly positive if $A$ is positive and invertible, denoted by $A \geq 0$ and $A > 0$, respectively. In this paper, $A - B \geq 0$ means $A \geq B$. Moreover, we identify the matrix algebra $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices with entries in the complex field $\mathbb{C}$ with the space of $\mathbb{B}(\mathbb{C}^n)$. Let $A^*$ denote the conjugate transpose of $A$. By positive definite matrices, we mean the strictly positive operators on $\mathbb{B}(\mathbb{C}^n)$, and we let $\mathbb{M}_n^+(\mathbb{C})$ denote the cone of positive $n \times n$ complex matrices. The singular values of $A$, that is, the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{\frac{1}{2}}$ , are denoted by $s_j(A)$, $j = 1, 2, \cdots, n$, and arranged in a non-increasing order. Weyl's monotonicity principle explains that $s_j(A) \geq s_j(B)$ when $A \geq B > 0$.

In this paper, we define the $v$-weighted arithmetic-geometric-harmonic means (AM-GM-HM) by

$$a\nabla_v b = (1 - v)a + vb, \quad a\sharp_v b = a^{1-v}b^v \quad \text{and} \quad a!_v b = ((1 - v)a^{-1} + vb^{-1})^{-1}$$

for $a, b > 0$ and $v \in [0, 1]$. Meanwhile, the corresponding $v$-weighted operator AM-GM-HM is

$$A\nabla_v B = (1 - v)A + vB, \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}} \quad \text{and} \quad A!_v B = ((1 - v)A^{-1} + vB^{-1})^{-1}$$

for $A, B > 0$ and $v \in [0, 1]$. We denote these by $A\nabla B$, $A\sharp B$, and $A!B$ for brevity when $v = \frac{1}{2}$, respectively. Moreover, we also define the $v$-weighted operator geometric mean

$$A\natural_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}} \text{ for } v \notin [0, 1].$$

A more generalized $v$-weighted AM-GM-HM is the weighted power mean defined by

$$M_p(a, b, v) := a\sharp_{p,v} b = ((1 - v)a^p + vb^p)^{\frac{1}{p}},$$

where $a, b > 0$, $p \neq 0$, and $v \in [0, 1]$. The following proposition explains that the weighted power mean is an increasing function.

**Proposition 1.1.** [5, p.26] For $a, b > 0$, $v \in [0, 1]$, and $p \neq 0$, let $M_p(a, b, v) = ((1 - v)a^p + vb^p)^{\frac{1}{p}}$ and $M_0(a, b, v) = a^{1-v}b^v$. Then,

$$M_s(a, b, v) \leq M_p(a, b, v) \text{ for } s \leq p. \tag{1.1}$$

Moreover, we define the weighted operator power mean as follows: If $A, B > 0$ and $v \in [0, 1]$, then

$$M_p(A, B, v) := A\sharp_{p,v} B = A^{\frac{1}{2}}((1 - v)I + v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p)^{\frac{1}{p}} A^{\frac{1}{2}}$$

for $p \neq 0$, and

$$A\sharp_{0,v} B = A\sharp_v B.$$

It is easy to see that $A\sharp_{1,v} B = A\nabla_v B$ and $A\sharp_{-1,v} B = A!_v B$. Moreover, $A\sharp_{p,v} B = B\sharp_{p,1-v} A$ is consistent with the properties of $v$-weighted operator arithmetic-geometric-harmonic means.

It is well-known that AM-GM-HM has the basic inequalities

$$A!_v B \leq A\sharp_v B \leq A\nabla_v B, \tag{1.2}$$

where $A, B \in \mathbb{M}_n^+(\mathbb{C})$ and $v \in [0, 1]$. Furthermore, if $A \leq B$ under the conditions in (1.2), then

$$A \leq A!_v B,\ A\sharp_v B,\ A\nabla_v B \leq B. \tag{1.3}$$

Due to the computation of $A\sharp_v B$ not being as easy as $A!_v B$ nor $A\nabla_v B$, it is of great interest to find better and sharper bounds of (1.2), we refer the readers to [2,4] and references therein for some related investigation. Recently, some singular values inequalities were given to describe the what difference between such matrices means. For example, it was shown in [3] that if $A, B \in \mathbb{M}_n^+(\mathbb{C})$ are such that $B \leq A$, then

$$\frac{1}{8}s_j(A^{-\frac{1}{2}}(A - B)^2 A^{-\frac{1}{2}}) \leq s_j(A\nabla B - A\sharp B) \leq \frac{1}{8}s_j(B^{-\frac{1}{2}}(A - B)^2 B^{-\frac{1}{2}}), \tag{1.4}$$

where $j = 1, 2, \cdots, n$ and $s_j(X)$ represents the $j^{th}$ singular value of the matrix $X$. The authors [6, Corollary 1] and [9, Corollary 2.4], show a generalization of (1.4) as follows: If $A, B \in \mathbb{M}_n^+(\mathbb{C})$ are such that $B \leq A$, then

$$\frac{v(1 - v)}{2}s_j(A^{-\frac{1}{2}}(A - B)^2 A^{-\frac{1}{2}}) \leq s_j(A\nabla_v B - A\sharp_v B) \leq \frac{v(1 - v)}{2}s_j(B^{-\frac{1}{2}}(A - B)^2 B^{-\frac{1}{2}}) \tag{1.5}$$

for $v \in [0,1]$.

In 2022, Sababheh et al. [9] showed the following AM-HM singular values inequalities: If $A, B \in \mathbb{M}_n^+(\mathbb{C})$ are such that $A \le B$, then

$$\frac{1}{4}s_j(B^{-\frac{1}{2}}(A-B)^2B^{-\frac{1}{2}}) \le s_j(A\nabla B - A!B) \le \frac{1}{4}s_j(A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}). \tag{1.6}$$

Furthermore, they [9] also obtained the following GM-HM singular values inequalities: If $A, B \in \mathbb{M}_n^+(\mathbb{C})$ are such that $A \le B$, then

$$s_j(A\sharp_v B - A!_v B) \le \frac{v(1-v)}{2}s_j((A-B)(A\sharp_v B)^{-1}(A-B)). \tag{1.7}$$

The paper is organized in the following way: In Section 2, we shall give some AM-GM-HM singular values inequalities, which generalize and complete (1.5) and (1.6). Moreover, we also show a further refinement of (1.7). In Section 3, we will present some refinements and the reverse of weighted power mean inequalities using a convex approach, our results generalizing some existing conclusions.

## 2. Some generalized singular values inequalities

We begin this section with an alternative proof of (1.5). First, we provide a lemma.

**Lemma 2.1.** *Let* $v \in [0,1]$. *We have the following:*
*If* $x \ge 1$*, then*

$$\frac{v(1-v)}{2}(x-1)^2 \ge (1-v) + vx - x^v; \tag{2.1}$$

*if* $0 < x \le 1$*, then*

$$\frac{v(1-v)}{2}(x-1)^2 \le (1-v) + vx - x^v. \tag{2.2}$$

*Proof.* Let $f(x) = (1-v) + vx - x^v - \frac{v(1-v)}{2}(x-1)^2$. Then,

$$f'(x) = v - vx^{v-1} - v(1-v)(x-1) \text{ and } f''(x) = v(1-v)(x^{v-2}-1).$$

– When $x \ge 1$, then $f''(x) \le 0$, and so $f'(x) \le f'(1) = 0$, that is, $f(x) \le f(1) = 0$, which means $\frac{v(1-v)}{2}(x-1)^2 \ge (1-v) + vx - x^v$.

– When $0 < x \le 1$, then $f''(x) \ge 0$, and so $f'(x) \le f'(1) = 0$, that is, $f(x) \ge f(1) = 0$, which means $\frac{v(1-v)}{2}(x-1)^2 \le (1-v) + vx - x^v$. □

*Alternative proof of (1.5).* By applying functional calculus for the operator $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ in (2.1), we have

$$\begin{aligned}&(1-v)I + vB^{-\frac{1}{2}}AB^{-\frac{1}{2}} - (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^v\\ &\le \frac{v(1-v)}{2}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - I)B^{\frac{1}{2}}B^{-1}B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - I).\end{aligned} \tag{2.3}$$

Replacing $v$ with $1-v$ in (2.3), we get

$$(1-v)B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + vI - (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{1-v}$$

$$\leq \frac{v(1-v)}{2}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}-I)B^{\frac{1}{2}}B^{-1}B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}-I), \tag{2.4}$$

and then, multiplying $B^{\frac{1}{2}}$ on both sides of inequality (2.4), we obtain

$$A\nabla_v B - A\sharp_v B \leq \frac{v(1-v)}{2}(A-B)B^{-1}(A-B). \tag{2.5}$$

Combining inequality (2.5) and Weyl's monotonicity principle, we have

$$s_j(A\nabla_v B - A\sharp_v B) \leq s_j\Big(\frac{v(1-v)}{2}(A-B)B^{-1}(A-B)\Big).$$

Since $s_j(X^*X) = s_j(XX^*)$ for $j = 1, 2, \cdots, n$, it can be seen that

$$s_j\Big(\frac{v(1-v)}{2}(A-B)B^{-1}(A-B)\Big) = \frac{v(1-v)}{2}s_j(B^{-\frac{1}{2}}(A-B)^2B^{-\frac{1}{2}}).$$

We complete the proof of the second inequality in (1.5).

On the other hand, putting $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (2.2), and using the same technique as above, then we can get the first inequality in (1.5). □

Next, we show a generalization of (1.6).

**Theorem 2.2.** *Let $A, B \in \mathbb{M}_n^+(\mathbb{C})$ be such that $A \leq B$. Then,*

$$v(1-v)s_j(B^{-\frac{1}{2}}(A-B)^2B^{-\frac{1}{2}}) \leq s_j(A\nabla_v B - A!_v B) \leq v(1-v)s_j(A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}).$$

*Proof.* Noting the equality

$$1 - v + vx - (1 - v + vx^{-1})^{-1} = \frac{v(1-v)(1-x)^2}{(1-v)x+v},$$

we have

$$s_j(A\nabla_v B - A!_v B) = v(1-v)s_j((A-B)(B\nabla_v A)^{-1}(A-B)).$$

The condition $A \leq B$ implies $A \leq B\nabla_v A \leq B$, that is, $A^{-1} \geq (B\nabla_v A)^{-1} \geq B^{-1}$. So,

$$\begin{aligned} v(1-v)s_j(B^{-\frac{1}{2}}(A-B)^2B^{-\frac{1}{2}}) &= v(1-v)s_j((A-B)B^{-1}(A-B)) \\ &\leq v(1-v)s_j\big((A-B)(B\nabla_v A)^{-1}(A-B)\big) \\ &\leq v(1-v)s_j((A-B)A^{-1}(A-B)) \\ &= v(1-v)s_j(A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}), \end{aligned}$$

as desired. □

At the end of this section, we will show a further refinement of (1.7).

**Lemma 2.3.** *Let $x \geq 1$ and $v \in [0, \frac{1}{2}]$. Then,*

$$x^v - ((1-v) + vx^{-1})^{-1} \leq \frac{v(1-v)}{2}(x-1)^2(1-v+vx)^{-1}. \tag{2.6}$$

*Proof.* Let

$$f(x) = x^{v} - ((1-v) + vx^{-1})^{-1} - \frac{v(1-v)}{2}(x-1)^{2}(1-v+vx)^{-1}.$$

Then,

$$f'(x) = vx^{v-1} - \frac{v}{((1-v)x+v)^{2}} - \frac{v(1-v)(v-2vx+vx^{2}+2x-2)}{2(1-v+vx)^{2}},$$

and

$$f''(x) = v(1-v)\Big(((1-v)x+v)^{-3}f_1(x) + f_2(x)\Big),$$

where $f_1(x) = 1 - x^{v-2}((1-v)x+v)^{3}$ and $f_2(x) = ((1-v)x+v)^{-3} - (1-v+vx)^{-3}$. So,

$$f_1'(x) = x^{v-3}((1-v)x+v)^{2}h(x) \ \text{ for } \ h(x) = (v^{2}-1)x + 2v - v^{2}.$$

By computations, we have that $h'(x) = v^{2} - 1 \le 0$ implies $h(x) \le h(1) = 2v - 1 \le 0$, so $f_1'(x) \le 0 \Rightarrow f_1(x) \le f_1(1) = 0$. On the other hand, $(1-v)x + v \ge 1 - v + vx$ implies $f_2(x) \le 0$. Therefore, we have $f''(x) \le 0$. That is,

$$f'(x) \le f'(1) = 0 \Rightarrow f(x) \le f(1) = 0,$$

as desired. □

**Theorem 2.4.** *Let $A, B \in \mathbb{M}_n^{+}(\mathbb{C})$ be such that $A \le B$. If $v \in [0, \frac{1}{2}]$, then*

$$\begin{aligned} s_j(A\sharp_v B - A!_v B) &\le \tfrac{v(1-v)}{2} s_j\big((A-B)(A\nabla_v B)^{-1}(A-B)\big) \\ &\le \tfrac{v(1-v)}{2} s_j\big((A-B)(A\sharp_v B)^{-1}(A-B)\big). \end{aligned}$$

*Proof.* By applying functional calculus for the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (2.6), we have

$$\begin{aligned} &(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{v} - \big((1-v)I + v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}\big)^{-1} \\ &\le \frac{v(1-v)}{2}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})((1-v)I + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}). \end{aligned} \tag{2.7}$$

Multiplying $A^{\frac{1}{2}}$ on both sides of (2.7), by the Weyl's monotonicity principle, we can complete the proof. □

## 3. Some generalized power mean inequalities

We begin this section with a convex argument that implies some refinements and the reverse of the weighted power mean inequalities. We refer the reader to [10] for general treatment of convex functions. First, we list a result obtained by Sababheh, Furuichi, Heydarbeygi, and Moradi.

**Proposition 3.1.** *[8] Let $f$ be an increasing function on $[0,\infty)$ with $f(0)=0$ such that $f(\sqrt{x})$ is convex. If $r_0=\min\{v,1-v\}$ and $v\in[0,1]$. Then,*

$$f\left(\sqrt{(1-v)a^2+vb^2}\right)+2r_0\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\le(1-v)f(a)+vf(b).$$

Next, we give a further refinement and reverse of Proposition 3.1 using the following lemma.

**Lemma 3.2.** *[1, 7] Let $f$ be a convex function satisfying $f:[0,\infty)\to[0,\infty)$. If $a,b\ge 0$, $r_0=\min\{v,1-v\}$ for $v\in[0,1]$, then,*

$$f((1-v)a+vb)+2r_0\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\le(1-v)f(a)+vf(b).$$

**Theorem 3.3.** *Let $f$ be an increasing function on $[0,\infty)$ with $f(0)=0$ such that $f(\sqrt{x})$ is convex. If $r=\min\{v,1-v\}$, $R=\max\{v,1-v\}$, and $r_0=\min\{2r,1-2r\}$ for $v\in[0,1]$, then we obtain the following conclusions:*
*(i) When $0\le v\le\frac{1}{2}$, we have*

$$\begin{aligned}
&f\left(\sqrt{(1-v)a^2+vb^2}\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&+2r_0\left(\frac{f(a)+f\left(\sqrt{\frac{a^2+b^2}{2}}\right)}{2}-f\left(\sqrt{\frac{3a^2+b^2}{4}}\right)\right)\\
&\le(1-v)f(a)+vf(b) \qquad (3.1)\\
&\le f\left(\sqrt{(1-v)a^2+vb^2}\right)+2R\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&-2r_0\left(\frac{f(b)+f\left(\sqrt{\frac{a^2+b^2}{2}}\right)}{2}-f\left(\sqrt{\frac{a^2+3b^2}{4}}\right)\right);
\end{aligned}$$

*(ii) when $\frac{1}{2}\le v\le 1$, we have*

$$\begin{aligned}
&f\left(\sqrt{(1-v)a^2+vb^2}\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&+2r_0\left(\frac{f(b)+f\left(\sqrt{\frac{a^2+b^2}{2}}\right)}{2}-f\left(\sqrt{\frac{a^2+3b^2}{4}}\right)\right)\\
&\le(1-v)f(a)+vf(b) \qquad (3.2)\\
&\le f\left(\sqrt{(1-v)a^2+vb^2}\right)+2R\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&-2r_0\left(\frac{f(a)+f\left(\sqrt{\frac{a^2+b^2}{2}}\right)}{2}-f\left(\sqrt{\frac{3a^2+b^2}{4}}\right)\right).
\end{aligned}$$

*Proof.* Let $g(x) = f(\sqrt{x})$ for $x \in [0, \infty)$. Then, $g$ is an increasing convex function. If $0 \le v \le \frac{1}{2}$, then we have

$$\begin{aligned}
&(1-v)g(a^2) + vg(b^2) - 2r\Big(\frac{g(a^2)+g(b^2)}{2} - g\Big(\frac{a^2+b^2}{2}\Big)\Big) \\
&= (1-2v)g(a^2) + 2vg\Big(\frac{a^2+b^2}{2}\Big) \\
&\ge g\big((1-2v)a^2 + v(a^2+b^2)\big) + 2\min\{2v, 1-2v\}\Big(\frac{g(a^2)+g(\frac{a^2+b^2}{2})}{2} - g\Big(\frac{a^2+\frac{a^2+b^2}{2}}{2}\Big)\Big) \\
&= g\big((1-v)a^2 + vb^2\big) + 2\min\{2v, 1-2v\}\Big(\frac{g(a^2)+g(\frac{a^2+b^2}{2})}{2} - g\Big(\frac{3a^2+b^2}{4}\Big)\Big).
\end{aligned}$$

Here we complete the first inequality of (3.1). Next, we prove the second one of (3.1).

$$\begin{aligned}
&2R\Big(\frac{g(a^2)+g(b^2)}{2} - g\Big(\frac{a^2+b^2}{2}\Big)\Big) - (1-v)g(a^2) - vg(b^2) \\
&= (1-2v)g(b^2) + 2vg\Big(\frac{a^2+b^2}{2}\Big) - 2g\Big(\frac{a^2+b^2}{2}\Big) \\
&\ge g\big((1-2v)b^2 + v(a^2+b^2)\big) + 2\min\{2v, 1-2v\}\Big(\frac{g(b^2)+g(\frac{a^2+b^2}{2})}{2} - g\Big(\frac{b^2+\frac{a^2+b^2}{2}}{2}\Big)\Big) \\
&- 2g\Big(\frac{a^2+b^2}{2}\Big) \\
&= g\big(va^2 + (1-v)b^2\big) + 2\min\{2v, 1-2v\}\Big(\frac{g(b^2)+g(\frac{a^2+b^2}{2})}{2} - g\Big(\frac{a^2+3b^2}{4}\Big)\Big) \\
&- 2g\Big(\frac{a^2+b^2}{2}\Big),
\end{aligned}$$

that is,

$$\begin{aligned}
&2R\Big(\frac{g(a^2)+g(b^2)}{2} - g\Big(\frac{a^2+b^2}{2}\Big)\Big) - g\big(va^2 + (1-v)b^2\big) + 2g\Big(\frac{a^2+b^2}{2}\Big) \\
&- 2\min\{2v, 1-2v\}\Big(\frac{g(b^2)+g(\frac{a^2+b^2}{2})}{2} - g\Big(\frac{a^2+3b^2}{4}\Big)\Big) \\
&\ge (1-v)g(a^2) + vg(b^2).
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
&g\big(va^2 + (1-v)b^2\big) + g\big((1-v)a^2 + vb^2\big) \\
&= 2\Big(\frac{1}{2}g\big(va^2 + (1-v)b^2\big) + \frac{1}{2}g\big((1-v)a^2 + vb^2\big)\Big) \\
&\ge 2g\Big(\frac{1}{2}\big(va^2 + (1-v)b^2\big) + \frac{1}{2}\big((1-v)a^2 + vb^2\big)\Big) \\
&= 2g\Big(\frac{a^2+b^2}{2}\Big).
\end{aligned} \tag{3.4}$$

Combining inequalities (3.3) and (3.4), we get

$$\begin{aligned}
&2R\left(\frac{g(a^2)+g(b^2)}{2}-g\left(\frac{a^2+b^2}{2}\right)\right)+g((1-v)a^2+vb^2)\\
&-2\min\{2v,1-2v\}\left(\frac{g(b^2)+g(\frac{a^2+b^2}{2})}{2}-g\left(\frac{a^2+3b^2}{4}\right)\right)\\
&\geq (1-v)g(a^2)+vg(b^2).
\end{aligned}$$

Therefore, we complete the proof of (3.1).

Exchanging $a$ and $b$ and $v$ and $1-v$, respectively, we can get (3.2) by (3.1). □

Let $a = a^{\frac{1}{p}}$, $b = b^{\frac{1}{p}}$, and $f(x) = x^p$ for $p \geq 2$ in Theorem 3.3. Then, we obtain the following corollary.

**Corollary 3.4.** *Let* $a, b \geq 0$, $p \geq 2$, $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$, *and* $r_0 = \min\{2r, 1-2r\}$ *for* $v \in [0, 1]$.
*(i) If* $0 \leq v \leq \frac{1}{2}$, *then*

$$\begin{aligned}
&\left((1-v)a^{\frac{2}{p}}+vb^{\frac{2}{p}}\right)^{\frac{p}{2}}+2r\left(\frac{a+b}{2}-\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}\right)\\
&+2r_0\left(\frac{a+\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}}{2}-\left(\frac{3a^{\frac{2}{p}}+b^{\frac{2}{p}}}{4}\right)^{\frac{p}{2}}\right)\\
&\leq (1-v)a+vb \qquad (3.5)\\
&\leq \left((1-v)a^{\frac{2}{p}}+vb^{\frac{2}{p}}\right)^{\frac{p}{2}}+2R\left(\frac{a+b}{2}-\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}\right)\\
&-2r_0\left(\frac{b+\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}}{2}-\left(\frac{a^{\frac{2}{p}}+3b^{\frac{2}{p}}}{4}\right)^{\frac{p}{2}}\right);
\end{aligned}$$

*(ii) if* $\frac{1}{2} \leq v \leq 1$, *then*

$$\begin{aligned}
&\left((1-v)a^{\frac{2}{p}}+vb^{\frac{2}{p}}\right)^{\frac{p}{2}}+2r\left(\frac{a+b}{2}-\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}\right)\\
&+2r_0\left(\frac{b+\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}}{2}-\left(\frac{a^{\frac{2}{p}}+3b^{\frac{2}{p}}}{4}\right)^{\frac{p}{2}}\right)\\
&\leq (1-v)a+vb \qquad (3.6)\\
&\leq \left((1-v)a^{\frac{2}{p}}+vb^{\frac{2}{p}}\right)^{\frac{p}{2}}+2R\left(\frac{a+b}{2}-\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}\right)\\
&-2r_0\left(\frac{a+\left(\frac{a^{\frac{2}{p}}+b^{\frac{2}{p}}}{2}\right)^{\frac{p}{2}}}{2}-\left(\frac{3a^{\frac{2}{p}}+b^{\frac{2}{p}}}{4}\right)^{\frac{p}{2}}\right).
\end{aligned}$$

**Remark 3.5.** With the proof in Theorem 3.3, we can find that Corollary 3.4 provided some further refinements and reverses of (1.1) when $s = 2$. Moreover, we will point out that Corollary 3.4 implies the main results of [11] obtained by Zhao and Wu.

**Corollary 3.6.** *Let $a, b \geq 0$, $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$ for $v \in [0, 1]$.*
*(i) If $0 \leq v \leq \frac{1}{2}$, then*

$$\begin{aligned} &a^{1-v}b^v + r(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{a})^2 \\ &\leq (1-v)a + vb \\ &\leq a^{1-v}b^v + R(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{b})^2; \end{aligned}$$

*(ii) if $\frac{1}{2} \leq v \leq 1$, then*

$$\begin{aligned} &a^{1-v}b^v + r(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{b})^2 \\ &\leq (1-v)a + vb \\ &\leq a^{1-v}b^v + R(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{a})^2. \end{aligned}$$

*Proof.* Direct calculus computations with the aid of L'Hopital's rule, we have

$$\lim_{p\to\infty} \left((1-v)a^{\frac{2}{p}} + vb^{\frac{2}{p}}\right)^{\frac{p}{2}} = a^{1-v}b^v.$$

Then, we can complete the proof with Corollary 3.4. □

At the end of this paper, we give some operator inequalities of Corollary 3.4.

**Theorem 3.7.** *Let $A, B > 0$, $p \geq 2$, $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$, and $r_0 = \min\{2r, 1-2r\}$ for $v \in [0, 1]$.*
*(i) If $0 \leq v \leq \frac{1}{2}$, then*

$$\begin{aligned} &A\natural_{\frac{p}{2}}\Big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\Big) + r\Big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\Big) \\ &+ r_0\Big(A + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{1}{4}}(A\sharp_{\frac{2}{p}}B))\Big) \\ &\leq A\nabla_v B \\ &\leq A\natural_{\frac{p}{2}}\Big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\Big) + R\Big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\Big) \\ &- r_0\Big(B + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{3}{4}}(A\sharp_{\frac{2}{p}}B))\Big). \end{aligned} \tag{3.7}$$

*(ii) If $\frac{1}{2} \leq v \leq 1$, then*

$$\begin{aligned} &A\natural_{\frac{p}{2}}\Big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\Big) + r\Big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\Big) \\ &+ r_0\Big(B + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{3}{4}}(A\sharp_{\frac{2}{p}}B))\Big) \\ &\leq A\nabla_v B \\ &\leq A\natural_{\frac{p}{2}}\Big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\Big) + R\Big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\Big) \\ &- r_0\Big(A + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{1}{4}}(A\sharp_{\frac{2}{p}}B))\Big). \end{aligned} \tag{3.8}$$

*Proof.* Applying functional calculus with $a = I$ and $b = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.5), and then multiplying $A^{\frac{1}{2}}$ from both sides, we have

$$
\begin{aligned}
&A^{\frac{1}{2}}\Big((1-v)I + v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}\Big)^{\frac{p}{2}}A^{\frac{1}{2}} + 2r\Big(\frac{A+B}{2} - A^{\frac{1}{2}}\Big(\frac{I+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}}{2}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}\Big)\\
&+ 2r_0\Bigg(\frac{A + A^{\frac{1}{2}}\Big(\frac{I+\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{\frac{2}{p}}}{2}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}}{2} - A^{\frac{1}{2}}\Big(\frac{3I+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}}{4}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}\Bigg)\\
&= A\natural_{\frac{p}{2}}\big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\big) + r\big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\big)\\
&+ r_0\big(A + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{1}{4}}(A\sharp_{\frac{2}{p}}B))\big)\\
&\le (1-v)A + vB\\
&= A\nabla_v B\\
&\le A^{\frac{1}{2}}\Big((1-v)I + v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}\Big)^{\frac{p}{2}}A^{\frac{1}{2}} + 2R\Big(\frac{A+B}{2} - A^{\frac{1}{2}}\Big(\frac{I+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}}{2}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}\Big)\\
&- 2r_0\Bigg(\frac{B + A^{\frac{1}{2}}\Big(\frac{I+\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{\frac{2}{p}}}{2}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}}{2} - A^{\frac{1}{2}}\Big(\frac{I+3(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2}{p}}}{4}\Big)^{\frac{p}{2}}A^{\frac{1}{2}}\Bigg)\\
&= A\natural_{\frac{p}{2}}\big(A\nabla_v(A\sharp_{\frac{2}{p}}B)\big) + R\big(A + B - 2A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B))\big)\\
&- r_0\big(B + A\natural_{\frac{p}{2}}(A\nabla(A\sharp_{\frac{2}{p}}B)) - 2A\natural_{\frac{p}{2}}(A\nabla_{\frac{3}{4}}(A\sharp_{\frac{2}{p}}B))\big).
\end{aligned}
$$

Here, we complete the proof of (3.7).

Using the same method in (3.6), we can get (3.8). □

## 4. Conclusions

Among this paper, we mainly present some generalized singular values inequalities in Section 2, which improve and extend some results from earlier publications in literature. In Section 3, we give some further refinement and reversed inequalities of convex function, as an affiliated result, we present some refinement and reverse of weighted power mean inequalities, and the obtained results generalized some conclusion of Young's inequalities.

## Author contributions

All authors contributed almost the same amount of work to manuscript. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no competing interests.

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