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Research article

A study of generalized distribution series and their mapping properties in univalent function theory

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Abstract: This paper deals with the study of some classes of analytic functions in the open unit disk defined by using subordinations and connected with the distribution series, and these new classes reduces to some inequalities involving the first and second-order derivatives of the functions. For a particular choice of parameters, these new classes reduce to many others studied by different authors, and give an extension of many of these previously investigated classes. First, we determine several simple sufficient conditions for generalized distribution series that belong to the new defined classes $S(D,E;\delta,\rho)$ and $K(D,E;\delta,\rho)$, and we establish certain inclusion properties between the subclasses $\mathcal{H}(\lambda,\eta)$ and $K(D,E;\delta,\rho)$. In addition, we obtain sufficient conditions to show that some related series belong to certain subclasses of analytic functions. Each of the results are followed by some special cases that were recently obtained in different articles. Additionally, specific instances of our primary outcomes are briefly mentioned. The main tools for obtaining our new results consist of a few simple inequalities presented in the second section, while the scope of the paper is to extend and continue the studies connecting specific problems of analytic functions with generalized distribution series.

Keywords: generalized distribution series; univalent functions; uniformly convex function and uniformly starlike functions; integral convolution
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Suppose $\mathcal{A}$ is the class of analytic functions in $\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\}$ and normalized with $\omega(0) = \omega'(0) - 1 = 0$. Each $\omega \in \mathcal{A}$ has the following form:

$$\omega(\xi) = \xi + \sum_{J=2}^{\infty} c_J \xi^J, \ \xi \in \mathbb{D}. \tag{1.1}$$

Likewise, we will use $\mathcal{S}$ to denote the class of all univalent functions of $\mathcal{A}$ in $\mathbb{D}$.

The convolution of analytic functions w given by (1.1) and $g(\xi) = \xi + \sum_{J=2}^{\infty} b_J \xi^J$, $\xi \in \mathbb{D}$ is defined as

$$(\omega * g)(\xi) := \xi + \sum_{J=2}^{\infty} c_J b_J \xi^J, \ \xi \in \mathbb{D},$$

while the integral convolution is given by (see [1])

$$(\omega \circledast g)(\xi) = \xi + \sum_{J=2}^{\infty} \frac{c_J b_J}{J} \xi^J, \ \xi \in \mathbb{D}.$$

$ST(\delta)$ indicates that a function $\omega \in \mathcal{A}$ belongs to the class of starlike functions of order δ if

$$\mathrm{Re}\left[\frac{\xi\omega'(\xi)}{\omega(\xi)}\right] > \delta, \ (\xi \in \mathbb{D}, \ 0 \le \delta < 1),$$

whereas the class $CV(\delta)$ of convex functions of order δ is considered to contain the function $\omega \in \mathcal{A}$ if

$$\mathrm{Re}\left(1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)}\right) > \delta, \ (\xi \in \mathbb{D}, \ 0 \le \delta < 1).$$

The classes $ST(\delta)$ and $CV(\delta)$ were initially presented and established by Silverman [2] and Robertson [3]. Specifically, the classes of starlike and convex functions in $\mathbb{D}$ are $ST \equiv ST(0)$ and $CV \equiv CV(0)$, respectively.

Remark that

$$\omega \in CV(\delta) \Leftrightarrow \ \xi\omega' \in ST(\delta).$$

A function $\omega \in UCV(\delta, \rho)$, if [4]

$$\mathrm{Re}\left(1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)} - \delta\right) > \rho\left|\frac{\xi\omega''(\xi)}{\omega'(\xi)}\right|, \ (\xi \in \mathbb{D}, \ \omega \in \mathcal{A}),$$

where $\rho \ge 0$ and $\delta \in [0, 1)$. This class is called the class of uniformly convex functions of order δ and type ρ. Also, the function $\omega \in SP(\delta, \rho)$ if

$$\mathrm{Re}\left(\frac{\xi\omega'(\xi)}{\omega(\xi)} - \delta\right) > \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right|, \ (\xi \in \mathbb{D}, \ \omega \in \mathcal{A}),$$

where $\rho \ge 0$ and $\delta \in [0, 1)$.

Remark that

$$\omega \in UCV(\delta,\rho) \Leftrightarrow \ \xi\omega' \in SP(\delta,\rho).$$

By specializing the parameters in $SP(\delta,\rho)$ and $UCV(\delta,\rho)$, we can identify several additional subclasses, which have already been studied by other researchers.

Note that:

(i) $SP(0,\rho) =: SP(\rho)$ and $UCV(0,\rho) =: UCV(\rho)$. For the equality relation, see Definition 1.1, Definition 1.2, Theorem 2.1 of [5], and relation (3) of [6], while for the second one see Theorem 2.2 of [7] and relation (2) of [6];

(ii) $SP(0,1) =: SP$ and $UCV(0,1) =: UCV$. Details of these two subclasses can be found in [8,9];

(iii) $SP(\delta,0) =: \mathcal{S}^*(\delta)$ and $UCV(0,\delta) =: K(\delta)$, which represent the usually normalized starlike and convex functions of order δ, respectively.

A function $\omega \in \Theta_\tau(D,E)$ $(-1 \leq E < D \leq 1,\ \tau \in \mathbb{C} \setminus \{0\})$, if

$$\left|\frac{\omega'(\xi)-1}{(D-E)\tau - E\left(\omega'(\xi)-1\right)}\right| < 1,\ (\xi \in \mathbb{D},\ \omega \in \mathcal{A}),$$

and the class $\Theta_\tau(D,E)$ was defined earlier by Dixit and Pal [10].

Definition 1.1 ([11]). If ω and g are analytic in $\mathbb{D}$, we say that ω is *subordinate* to g, denoted $\omega(\xi) \prec g(\xi)$, if there exists an analytic function h, with $h(0) = 0$ and $|h(\xi)| < 1$ for all $\xi \in \mathbb{D}$, such that $\omega(\xi) = g(h(\xi))$, $\xi \in \mathbb{D}$.

Furthermore, if g is univalent in $\mathbb{D}$, then

$$\omega(\xi) \prec g(\xi) \Leftrightarrow \omega(\mathbb{D}) \subset g(\mathbb{D}) \quad \text{and} \quad \omega(0) = g(0).$$

Definition 1.2 ([12]). (i) For $\rho \geq 0$, $\delta \in [0,1)$, and $-1 \leq E < D \leq 1$, the function $\omega \in S(D,E;\delta,\rho)$ if it satisfies the subordination

$$\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right| \prec (1-\delta)\frac{1+D\xi}{1+E\xi} + \delta,$$

or equivalently

$$\left|\frac{\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right| - 1}{E\left[\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right|\right] - [E + (D-E)(1-\delta)]}\right| < 1,\ (\xi \in \mathbb{D}). \tag{1.2}$$

(ii) For $\rho \geq 0$, $\delta \in [0,1)$, and $-1 \leq E < D \leq 1$, the function $\omega(\xi) \in K(D,E;\delta,\rho)$ if it satisfies

$$1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)} - \rho\left|\frac{\xi\omega''(\xi)}{\omega'(\xi)}\right| \prec (1-\delta)\frac{1+D\xi}{1+E\xi} + \delta,$$

or equivalently

$$\left|\frac{1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)} - \rho\left|\frac{\xi\omega''(\xi)}{\omega'(\xi)}\right|}{E\left[1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)} - \rho\left|\frac{\xi\omega''(\xi)}{\omega'(\xi)}\right|\right] - [E + (D-E)(1-\delta)]}\right| < 1,\ \xi \in \mathbb{D}.$$

It is easy to see that

$$\omega \in K(D,E;\delta,\rho) \Leftrightarrow \xi\omega'(\xi) \in S(D,E;\delta,\rho). \tag{1.3}$$

Furthermore, we observe that different authors have investigated the following specific situations of the previously specified classes:

(i)

$$S(D,E;\delta,0) =: S(D,E;\delta) = \left\{\omega \in \mathcal{A} : \left|\frac{\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1}{E\frac{\xi\omega'(\xi)}{\omega(\xi)} - [E + (D-E)(1-\delta)]}\right| < 1,\ \xi \in \mathbb{D}\right\},$$

and

$$S(D,E;0,0) =: S(D,E) = \left\{\omega \in \mathcal{A} : \left|\frac{\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1}{E\frac{\xi\omega'(\xi)}{\omega(\xi)} - D}\right| < 1,\ \xi \in \mathbb{D}\right\},$$

see [13];

(ii)

$$K(D,E;\delta,0) =: K(D,E;\delta) = \left\{\omega \in \mathcal{A} : \left|\frac{1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)}}{E\left(1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)}\right) - [E + (D-E)(1-\delta)]}\right| < 1,\ \xi \in \mathbb{D}\right\},$$

while

$$K(D,E;0,0) =: K(D,E) = \left\{\omega \in \mathcal{A} : \left|\frac{1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)}}{E\left(1 + \frac{\xi\omega''(\xi)}{\omega'(\xi)}\right) - D}\right| < 1,\ \xi \in \mathbb{D}\right\},$$

see [13];

(iii) $S(\gamma,-\gamma;0,0) =: S(\gamma)$ and $K(\gamma,-\gamma;0,0) =: K(\gamma)$ (see [14]);

(iv) $S(D,E;0,0) =: S(D,E)$ and $K(D,E;0,0) =: K(D,E)$ (see [13]);

(v) $S(1,-1;0,\rho) =: SP(\rho)$ and $K(1,-1;0,\rho) =: UCV(\rho)$ (see [5,7]);

(vi) $S(1,-1;\delta,0) =: \mathcal{S}^*(\delta)$ and $K(1,-1;\delta,0) =: K(\delta)$ (see [2,3]).

Definition 1.3. [6, p. 123] A function $\omega \in \mathcal{H}(\lambda,\eta)$ if it satisfies the inequality

$$\mathrm{Re}\left[e^{i\lambda}\left(\omega'(z) - \eta\right)\right] > 0,\ (\xi \in \mathbb{D},\ \eta < 1,\ |\lambda| \le \frac{\pi}{2},\ \omega \in \mathcal{A}).$$

Geometric function theory relies heavily on the applications of hypergeometric functions [15, 16], generalized hypergeometric functions [17, 18], confluent hypergeometric functions [19], the Fox–Wright function [20], the Wright function [21], and generalized Bessel functions. The year 2014 saw the introduction of Poisson distribution series by Porwal [22] (see also [23]), who also obtained different properties for a few classes of univalent functions connected with the probability density function and geometric function theory. Following the publication of this paper, a number of researchers developed Mittag–Leffler type Poisson distribution series [24], hypergeometric distribution series [25], Pascal distribution series [26], confluent hypergeometric distribution series [27], hypergeometric distribution type series [28], Borel distribution series [29], and binomial distribution series [30], and discovered some intriguing characteristics of various classes of univalent functions. Generalized distribution series were recently proposed by Porwal [31], who also obtained some sufficient and necessary criteria for functions to belong to different subclasses of univalent functions.

Because this distribution is a generalization of practically all discrete probability distributions, the study of generalized distribution series is of particular interest (see, e.g., [32,33]). We also mention the new papers connected with the q-version distribution like [34–36].

Let $\sum\limits_{\jmath=0}^{\infty} t_\jmath$, where $t_\jmath \geq 0$ for all $\jmath \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, be convergent and

$$S := \sum_{\jmath=0}^{\infty} t_\jmath.$$

We define the generalized discrete probability distribution by

$$p(\jmath) := \frac{t_\jmath}{S}, \ (\jmath \in \mathbb{N}_0). \tag{1.4}$$

Obviously, $p(\jmath)$ is a probability mass function where $p(\jmath) \geq 0$ and $\sum\limits_{\jmath=0}^{\infty} p_\jmath = 1$. Further, we define

$$\phi(x) := \sum_{\jmath=0}^{\infty} t_\jmath x^k, \tag{1.5}$$

and from (1.4) it is obvious that the series given by (1.5) is convergent for $|x| < 1$ and $x = 1$.

Porwal [31] introduced a generalized distribution series by

$$M_\phi(\xi) := \xi + \sum_{\jmath=2}^{\infty} \frac{t_{\jmath-1}}{S} \xi^\jmath, \ (\xi \in \mathbb{D}), \tag{1.6}$$

and we define $\mathcal{P}_\phi(\omega) : \mathcal{A} \to \mathcal{A}$ by the convolution and $\mathcal{R}_\phi(\omega) : \mathcal{A} \to \mathcal{A}$ by the integral convolution as follows:

$$\mathcal{P}_\phi(\omega)(\xi) := M_\phi(\xi) * \omega(\xi) = \xi + \sum_{\jmath=2}^{\infty} c_\jmath \frac{t_{\jmath-1}}{S} \xi^\jmath, \ (\xi \in \mathbb{D}),$$

and

$$\mathcal{R}_\phi(\omega)(\xi) := M_\phi(\xi) \circledast \omega(\xi) = \xi + \sum_{\jmath=2}^{\infty} \frac{c_\jmath}{\jmath} \frac{t_{\jmath-1}}{S} \xi^\jmath, \ (\xi \in \mathbb{D}).$$

Also, we define the function

$$\Phi_\phi(\mu;\xi) := (1-\mu) M_\phi(\xi) + \mu\xi \left(M_\phi(\xi)\right)' = \xi + \sum_{\jmath=2}^{\infty} \left[1 + \mu(\jmath - 1)\right] \frac{t_{\jmath-1}}{S} \xi^\jmath, \ (\xi \in \mathbb{D}, \ \mu \geq 0). \tag{1.7}$$

As in the above manner, we will define $\mathcal{Q}_\phi(\omega) : \mathcal{A} \to \mathcal{A}$ by the convolution and $\mathcal{G}_\phi(\omega) : \mathcal{A} \to \mathcal{A}$ by the integral convolution with the following relations:

$$\mathcal{Q}_\phi(\omega)(\xi) := \Phi_\phi(\mu;\xi) * \omega(\xi) = \xi + \sum_{\jmath=2}^{\infty} \left[1 + \mu(\jmath - 1)\right] c_\jmath \frac{t_{\jmath-1}}{S} \xi^\jmath, \ (\xi \in \mathbb{D})$$

and

$$\mathcal{G}_\phi(\omega)(\xi) := \Phi_\phi(\mu;\xi) \circledast \omega(\xi) = \xi + \sum_{J=2}^{\infty} [1 + \mu(J-1)] \frac{c_J}{J} \frac{t_{J-1}}{S} \xi^J, \ (\xi \in \mathbb{D}).$$

In the current paper, motivated by the above mentioned works, first we found sufficient conditions for generalized distribution series such that an analytic function ω belongs to $K(D, E; \delta, \rho)$ and $S(D, E; \delta, \rho)$. Since these new classes extend many previously defined and studied, we found simple conditions that imply the inclusion relation between $\mathcal{H}(\lambda, \eta)$ and $K(D, E; \delta, \rho)$. Another objective was to find sufficient conditions such that the analytic functions M_ϕ, $\Phi_\phi(\mu; \cdot)$, $\mathcal{P}_\phi(\omega)$, $\mathcal{R}_\phi$, and $\mathcal{G}_\phi(\omega)$ defined by convolution or integral convolution or combinations of these belong to some of these new classes. The goal is to underline and continue the studies of some properties of the generalized distribution series defined with these type of convolutions that seems to not be sufficiently investigated till now.

Section 2 contains two new lemmas that give sufficient conditions for a function $\omega \in \mathcal{A}$ to belongs to $S(D, E; \delta, \rho)$ and $K(D, E; \delta, \rho)$, respectively, and another one regarding the bounds for the coefficients of the functions of the class $\mathcal{H}(\lambda, \eta)$. Section 3 contains our main results, where with the aid of these lemmas, we determine sufficient conditions such that the analytic functions M_ϕ, $\Phi_\phi(\mu; \cdot)$, $\mathcal{P}_\phi(\omega)$, and $\mathcal{R}_\phi$ belong to each of the classes $S(D, E; \delta, \rho)$ and $K(D, E; \delta, \rho)$, emphasizing some special cases that were already obtained by different authors. Section 4 contains a similar result for the image of the function M_ϕ by the well-known Alexander–type integral operator, and the paper ends with the conclusions regarding the obtained results.

2. Preliminaries

The next two lemmas give sufficient conditions for the parameters such that $\omega \in S(D, E; \delta, \rho)$ and $\omega \in K(D, E; \delta, \rho)$, and will be needed to establish our results.

Lemma 2.1. *Let $\omega \in \mathcal{A}$. If*

$$\sum_{J=2}^{\infty} \Big[(1 + |E|)(1 + \rho)(J - 1) + (D - E)(1 - \delta)\Big] \left|c_J\right| \leq (D - E)(1 - \delta), \tag{2.1}$$

$$(0 \leq \delta < 1, \ \rho \geq 0, \ -1 \leq E < D \leq 1)$$

holds, then $\omega \in S(D, E; \delta, \rho)$.

Proof. First, we emphasize that, according to inequality (1.2), we have $\omega \in S(D, E; \delta, \rho)$ if and only if the function

$$\hbar(\xi) := \frac{\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right| - 1}{E\left(\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right|\right) - [E + (D - E)(1 - \delta)]}$$

is analytic in $\mathbb{D}$ and satisfies $|\hbar(\xi)| < 1$, $\xi \in \mathbb{D}$.

Without using the generality, we could assume the $\hbar$ is continuous on the closed disc $\overline{\mathbb{D}}$. If not, then we could replace in our proof the function $\hbar$ by $\hbar_r$, where $\hbar_r(\xi) := \hbar(r\xi)$, with $0 < r < 1$, and this new function is analytic in $\overline{\mathbb{D}}$, hence it is continuous on $\overline{\mathbb{D}}$ for all $0 < r < 1$. After making a proof similar to the next one, by letting $r \to 1^-$, we get the required result.

According to the principle of the maximum of the modulus for an analytic non-constant function on a domain, we have that $\hbar$ attained the maximum of its modulus on the boundary of $\partial\mathbb{D}$. Therefore, to prove that inequality (1.2) holds, it is sufficient to prove that

$$\begin{aligned}W(\xi) &:= |\omega(\xi)|\left\{\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right| - 1\right|\right.\\ &\left.- \left|E\left(\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right|\right) - [E + (D-E)(1-\delta)]\right|\right\} \leq 0,\ \xi = \mathrm{e}^{it},\ t \in [0, 2\pi],\end{aligned} \tag{2.2}$$

where we used the fact that $W(0) < 0$. Consequently, we will prove in the following that the assumption inequality (2.1) implies that (2.2) holds for all $z = \mathrm{e}^{it}$, $t \in [0, 2\pi]$.

A simple computation shows that, for $\xi = \mathrm{e}^{i\theta}$, $\theta \in [0, 2\pi]$, we have

$$\begin{aligned}W(z) :=& |\omega(\xi)|\left\{\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right| - 1\right|\right.\\ &\left.- \left|E\left(\frac{\xi\omega'(\xi)}{\omega(\xi)} - \rho\left|\frac{\xi\omega'(\xi)}{\omega(\xi)} - 1\right|\right) - [E + (D-E)(1-\delta)]\right|\right\}\\ =& \left|\xi\omega'(\xi) - \omega(\xi) - \rho\mathrm{e}^{i\theta}\left|\xi\omega'(\xi) - \omega(\xi)\right|\right|\\ &- \left|E\Big(\xi\omega'(\xi) - \omega(\xi) - \rho\mathrm{e}^{i\theta}\left|\xi\omega'(\xi) - \omega(\xi)\right|\Big) - (D-E)(1-\delta)\omega(\xi)\right|.\end{aligned} \tag{2.3}$$

Now, we will find a majorant for the first term of the RHS of (2.3), and a minorant for the second term of the RHS of (2.3). Thus, for the majorant of the first term, we have

$$\begin{aligned}&\left|\xi\omega'(\xi) - \omega(\xi) - \rho\mathrm{e}^{i\theta}\left|\xi\omega'(\xi) - \omega(\xi)\right|\right|\\ &= |\xi\omega'(\xi) - \omega(\xi)|\left|1 - \rho\mathrm{e}^{i(\theta+\phi)}\right| \leq (1+\rho)\,|\xi\omega'(\xi) - \omega(\xi)|\,,\ |\xi| = 1,\end{aligned} \tag{2.4}$$

while for the minorant of the second term, we get

$$\begin{aligned}&\left|E\Big(\xi\omega'(\xi) - \omega(\xi) - \rho\mathrm{e}^{i\theta}\left|\xi\omega'(\xi) - \omega(\xi)\right|\Big) - (D-E)(1-\delta)\omega(\xi)\right|\\ &= \left|E\Big(\mathrm{e}^{i\varphi}\,|\xi\omega'(\xi) - \omega(\xi)| - \rho\mathrm{e}^{i\theta}\,|\xi\omega'(\xi) - \omega(\xi)|\Big) - (D-E)(1-\delta)\omega(\xi)\right|\\ &= \left|E\mathrm{e}^{i\varphi}\Big(1 - \rho\mathrm{e}^{i(\theta-\varphi)}\Big)\,|\xi\omega'(\xi) - \omega(\xi)| - (D-E)(1-\delta)\omega(\xi)\right|\\ &\geq |(D-E)(1-\delta)\omega(\xi)| - \left|E\mathrm{e}^{i\varphi}\Big(1 - \rho\mathrm{e}^{i(\theta-\varphi)}\Big)\,|\xi\omega'(\xi) - \omega(\xi)|\right|\\ &= (D-E)(1-\delta)|\omega(\xi)| - |E|\left|1 - \rho\mathrm{e}^{i(\theta-\varphi)}\right|\,|\xi\omega'(\xi) - \omega(\xi)|\\ &= (D-E)(1-\delta)\left|\xi + \sum_{J=2}^{\infty} c_J\,\xi^J\right| - |E|\,(1+\rho)\,|\xi\omega'(\xi) - \omega(\xi)|\\ &\geq (D-E)(1-\delta)\left(|\xi| - \left|\sum_{J=2}^{\infty} c_J\,\xi^J\right|\right) - |E|\,(1+\rho)\,|\xi\omega'(\xi) - \omega(\xi)|\end{aligned}$$

$$
\begin{aligned}
&= (D-E)(1-\delta)|\xi| - (D-E)(1-\delta)\left|\sum_{\jmath=2}^{\infty} c_\jmath \xi^\jmath\right| - |E|(1+\rho)|\xi\omega'(\xi)-\omega(\xi)| \\
&\geq (D-E)(1-\delta)|\xi| - (D-E)(1-\delta)\sum_{\jmath=2}^{\infty} |c_\jmath||\xi|^\jmath - |E|(1+\rho)|\xi\omega'(\xi)-\omega(\xi)| \\
&\geq (D-E)(1-\delta) - (D-E)(1-\delta)\sum_{\jmath=2}^{\infty} |c_\jmath| - |E|(1+\rho)|\xi\omega'(\xi)-\omega(\xi)|, \ |\xi|=1. \qquad (2.5)
\end{aligned}
$$

Therefore, from relation (2.3) together with inequalities (2.4) and (2.5), we get

$$
\begin{aligned}
W(\xi) \leq &(1+\rho)\ |\xi\omega'(\xi)-\omega(\xi)| \\
&- \left[(D-E)(1-\delta) - (D-E)(1-\delta)\sum_{\jmath=2}^{\infty}|c_\jmath| - |E|(1+\rho)|\xi\omega'(\xi)-\omega(\xi)|\right] \\
\leq &(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)|c_\jmath||\xi|^\jmath - (D-E)(1-\delta) + (D-E)(1-\delta)\sum_{\jmath=2}^{\infty}|c_\jmath| \\
\leq &(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)|c_\jmath| - (D-E)(1-\delta) + (D-E)(1-\delta)\sum_{\jmath=2}^{\infty}|c_\jmath| \\
= &\sum_{\jmath=2}^{\infty}\Big[(1+|E|)(1+\rho)(\jmath-1) + (D-E)(1-\delta)\Big]\left|c_\jmath\right| - (D-E)(1-\delta), \ |\xi|=1. \qquad (2.6)
\end{aligned}
$$

Consequently, it follows that if the inequality (2.1) holds, then the RHS of (2.6) will be less or equal to zero, hence $W(\xi) \leq 0$ for all $|\xi| = 1$. Thus, according to (2.2), we conclude that $|\hbar(\xi)| < 1$, $\xi \in \mathbb{D}$, i.e., $\omega \in S(D, E; \delta, \rho)$. □

Using the equivalence (1.3), from Lemma 2.1 we obtain the following necessary condition for a function $\omega \in \mathcal{A}$ of the form (1.1) to be of the class $S(D, E; \delta, \rho)$.

Lemma 2.2. *Let $\omega \in \mathcal{A}$. If*

$$
\sum_{\jmath=2}^{\infty} \jmath\Big[(1+|E|)(1+\rho)(\jmath-1) + (D-E)(1-\delta)\Big]\left|c_\jmath\right| \leq (D-E)(1-\delta),
$$

$$
(0 \leq \delta < 1,\ \rho \geq 0,\ -1 \leq E < D \leq 1)
$$

holds, then $\omega \in K(D, E; \delta, \rho)$.

The last lemma of this section gives us the upper bounds of the modules of the coefficients of functions that belong to the classes $\mathcal{H}(\lambda, \eta)$.

Lemma 2.3. *[6, p. 123] If $\omega \in \mathcal{H}(\lambda, \eta)$ is of the form* (1.1)*, then*

$$
\left|c_\jmath\right| \leq \frac{2(1-\eta)\cos\lambda}{\jmath},\ \jmath \in \mathbb{N}.
$$

3. Main results

Throughout the paper, we will assume, unless otherwise noted, that $0 \le \delta < 1$, $\rho \ge 0$, $0 \le \eta < 1$, $|\lambda| \le \frac{\pi}{2}$, and $-1 \le E < D \le 1$. We derive sufficient conditions in our first two results for the distribution series function M_ϕ defined by (1.6) to belong to $S(D, E; \delta, \rho)$ and $K(D, E; \delta, \rho)$, respectively.

Theorem 3.1. *If the inequality*

$$(1 + |E|)(1 + \rho)\phi'(1) + (D - E)(1 - \delta)\phi(1) \le (D - E)(1 - \delta)(S + \phi(0)) \tag{3.1}$$

is satisfied, then $M_\phi \in S(D, E; \delta, \rho)$.

Proof. To prove that $M_\phi \in S(D, E; \delta, \rho)$, we will use the sufficient condition given by Lemma 2.1. Thus, using assumption (3.1), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty} [(1 + |E|)(1 + \rho)(j - 1) + (D - E)(1 - \delta)] \frac{t_{j-1}}{S} \\
&= \frac{1}{S} \left[(1 + |E|)(1 + \rho) \sum_{j=2}^{\infty} (j - 1)t_{j-1} + (D - E)(1 - \delta) \sum_{j=2}^{\infty} t_{j-1} \right] \\
&= \frac{1}{S} \left[(1 + |E|)(1 + \rho) \sum_{j=1}^{\infty} j t_j + (D - E)(1 - \delta) \sum_{j=1}^{\infty} t_j \right] \\
&= \frac{1}{S} [(1 + |E|)(1 + \rho)\phi'(1) + (D - E)(1 - \delta)(\phi(1) - \phi(0))] \le (D - E)(1 - \delta),
\end{aligned}$$

and according to Lemma 2.1, it follows that $M_\phi \in S(D, E; \delta, \rho)$. □

We emphasize the next two special cases that generalize some results previously obtained by different authors.

Corollary 3.1. *Letting $\rho = 0$ in the above theorem, then $M_\phi \in S(D, E; \delta)$ if the inequality*

$$(1 + |E|)\phi'(1) + (D - E)(1 - \delta)\phi(1) \le (D - E)(1 - \delta)(S + \phi(0)) \tag{3.2}$$

holds.

Corollary 3.2. *If we put $\rho = \delta = 0$ in the above theorem, then $M_\phi \in S(D, E)$ if the inequality*

$$(1 + |E|)\phi'(1) + (D - E)\phi(1) \le (D - E)(S + \phi(0)) \tag{3.3}$$

is satisfied.

Remark 3.1. (i) Letting $D = 1$ and $E = -1$ in Theorem 3.1, we get the outcome of Porwal and Murugusundaramoorthy [33, Theorem 8, p. 228];

(ii) Putting $D = 1$ and $E = -1$ in Corollary 3.1, we get the outcome of Kota and El-Ashwah [32, Corollary 2.2] (see also Porwal [31, Theorem 10 with $\lambda = 0$]).

Theorem 3.2. *If the inequality*

$$(1+|E|)(1+\rho)(\phi''(1)+2\phi'(1))+(D-E)(1-\delta)(\phi'(1)+\phi(1)) \leq (D-E)(1-\delta)(S+\phi(0)) \tag{3.4}$$

holds, then $M_\phi \in K(D,E;\delta,\rho)$.

Proof. To prove that $M_\phi \in K(D,E;\delta,\rho)$, we will use Lemma 2.2. Thus, from assumption (3.4), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty} j[(1+|E|)(1+\rho)(j-1)+(D-E)(1-\delta)]\frac{t_{j-1}}{S} \\
&= \frac{1}{S}\sum_{j=2}^{\infty}\left[(1+|E|)(1+\rho)j(j-1)t_{j-1}+(D-E)(1-\delta)(j-1)t_{j-1}+(D-E)(1-\delta)t_{j-1}\right] \\
&= \frac{1}{S}\left[(1+|E|)(1+\rho)\sum_{j=2}^{\infty} j(j-1)t_{j-1}+(D-E)(1-\delta)\sum_{j=2}^{\infty}(j-1)t_{j-1}+(D-E)(1-\delta)\sum_{j=2}^{\infty} t_{j-1}\right] \\
&= \frac{1}{S}\left[(1+|E|)(1+\rho)\sum_{j=2}^{\infty}(j-1)(j-2)t_{j-1}+2(1+|E|)(1+\rho)\sum_{j=2}^{\infty}(j-1)t_{j-1}\right. \\
&\qquad \left.+(D-E)(1-\delta)\sum_{j=2}^{\infty}(j-1)t_{j-1}+(D-E)(1-\delta)\sum_{j=2}^{\infty} t_{j-1}\right] \\
&= \frac{1}{S}\left[(1+|E|)(1+\rho)\sum_{j=1}^{\infty} j(j-1)t_j+2(1+|E|)(1+\rho)\sum_{j=1}^{\infty} jt_j\right. \\
&\qquad \left.+(D-E)(1-\delta)\sum_{j=1}^{\infty} jt_j+(D-E)(1-\delta)\sum_{j=1}^{\infty} t_j\right] \\
&= \frac{1}{S}\Big[(1+|E|)(1+\rho)\phi''(1)+2(1+|E|)(1+\rho)\phi'(1) \\
&\qquad +(D-E)(1-\delta)(\phi'(1)+\phi(1)-\phi(0))\Big] \leq (D-E)(1-\delta)
\end{aligned}$$

and from Lemma 2.2, our conclusion follows. □

For special choices of the parameters we get the following particular cases.

Corollary 3.3. *Putting* $\rho = 0$ *in the above theorem, then* $M_\phi \in K(D,E;\delta)$ *if the inequality*

$$(1+|E|)(\phi''(1)+2\phi'(1))+(D-E)(1-\delta)(\phi'(1)+\phi(1)) \leq (D-E)(1-\delta)(S+\phi(0)) \tag{3.5}$$

holds.

Corollary 3.4. *Putting* $\rho = \delta = 0$ *in the above theorem, then* $M_\phi \in K(D,E)$ *if the inequality*

$$(1+|E|)(\phi''(1)+2\phi'(1))+(D-E)(\phi'(1)+\phi(1)) \leq (D-E)(S+\phi(0)) \tag{3.6}$$

holds.

Remark 3.2. (i) For $D = 1$ and $E = -1$, Theorem 3.2 leads to the outcome obtained by Porwal and Murugusundaramoorthy [33, Theorem 7, p. 227];

(ii) For $D = 1$ and $E = -1$, in Corollary 3.3 we get the outcome of Porwal [31, Theorem 11 with $\lambda = 0$].

Similarly, the next two main results give sufficient conditions for $\Phi_\phi(\mu;\cdot)$ defined by (1.7) to belongs to $S(D,E;\delta,\rho)$ and $K(D,E;\delta,\rho)$, respectively.

Theorem 3.3. *If the inequality*

$$\begin{aligned}\frac{1}{S}\Big[&(1+|E|)(1+\rho)\mu\phi''(1)+[(1+|E|)(1+\rho)(1+\mu)+\mu(D-E)(1-\delta)]\phi'(1)\\&+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\leq(D-E)(1-\delta)\end{aligned}\tag{3.7}$$

holds, then $\Phi_\phi(\mu;\cdot)\in S(D,E;\delta,\rho)$.

Proof. From assumption (3.7), we have

$$\begin{aligned}&\sum_{\jmath=2}^{\infty}[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)]\,[1+\mu(\jmath-1)]\,\frac{t_{\jmath-1}}{S}\\
=&\frac{1}{S}\Bigg[(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}+(1+|E|)(1+\rho)\mu\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-1)t_{\jmath-1}\\
&+\mu(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}+(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1}\Bigg]\\
=&\frac{1}{S}\Bigg[[(1+|E|)(1+\rho)(1+\mu)+\mu(D-E)(1-\delta)]\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}\\
&+(1+|E|)(1+\rho)\mu\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-2)t_{\jmath-1}+(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1}\Bigg]\\
=&\frac{1}{S}\Bigg[[(1+|E|)(1+\rho)(1+\mu)+\mu(D-E)(1-\delta)]\sum_{\jmath=1}^{\infty}\jmath t_\jmath\\
&+(1+|E|)(1+\rho)\mu\sum_{\jmath=1}^{\infty}\jmath(\jmath-1)t_\jmath+(D-E)(1-\delta)\sum_{\jmath=1}^{\infty}t_\jmath\Bigg]\\
=&\frac{1}{S}\Big[(1+|E|)(1+\rho)\mu\phi''(1)+[(1+|E|)(1+\rho)(1+\mu)+\mu(D-E)(1-\delta)]\phi'(1)\\
&+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\leq(D-E)(1-\delta)\end{aligned}$$

and using Lemma 2.1, we deduce that $\Phi_\phi(\mu;\cdot)\in S(A,B;\delta,\rho)$. □

The following two special cases of Theorem 3.3 show that this theorem generalizes the mentioned recent results.

Corollary 3.5. *Letting $\rho = 0$ in Theorem 3.3, then $\Phi_\phi(\mu;\cdot) \in S(D,E;\delta)$ if the inequality*

$$\begin{aligned}\frac{1}{S}\Big[&(1+|E|)\mu\phi''(1) + [(1+|E|)(1+\mu) + \mu(D-E)(1-\delta)]\phi'(1)\\ &+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big] \leq (D-E)(1-\delta)\end{aligned} \tag{3.8}$$

is satisfied.

Corollary 3.6. *If we put $\rho = \delta = 0$ in Theorem 3.3, then $\Phi_\phi(\mu;\cdot) \in S(D,E)$ if the inequality*

$$\begin{aligned}\frac{1}{S}[&(1+|E|)\mu\phi''(1) + [(1+|E|)(1+\mu) + \mu(D-E)]\phi'(1)\\ &+(D-E)(\phi(1)-\phi(0))] \leq (D-E)\end{aligned} \tag{3.9}$$

is satisfied.

Remark 3.3. (i) For $D = 1$, $E = -1$, and $\mu = 0$, Theorem 3.3 reduces to the result obtained by Porwal and Murugusundaramoorthy [33, Theorem 8, p. 227];

(ii) For $D = 1$, $E = -1$, and $\mu = 0$, Corollary 3.5 reduces to the theorem obtained by Porwal [31, Theorem 10 with $\lambda = 0$].

Theorem 3.4. *If the inequality*

$$\begin{aligned}\frac{1}{S}\Big[&\mu(1+|E|)(1+\rho)\phi'''(1) + \{(1+|E|)(1+\rho)(1+4\mu) + \mu(D-E)(1-\delta)\}\phi''(1)\\ &+\{2(1+|E|)(1+\rho)(1+\mu) + (1+2\mu)(D-E)(1-\delta)\}\phi'(1)\\ &+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big] \leq (D-E)(1-\delta),\end{aligned} \tag{3.10}$$

holds, then $\Phi_\phi(\mu;\cdot) \in K(D,E;\delta,\rho)$.

Proof. A simple computation combined with assumption (3.10) yields

$$\begin{aligned}&\sum_{\jmath=2}^{\infty} \jmath[(1+|E|)(1+\rho)(\jmath-1) + (D-E)(1-\delta)]\,[1+\mu(\jmath-1)]\,\frac{t_{\jmath-1}}{S}\\
&= \frac{1}{S}\Bigg[(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}\jmath(\jmath-1)t_{\jmath-1} + (D-E)(1-\delta)\sum_{\jmath=2}^{\infty}\jmath t_{\jmath-1}\\
&\quad+\mu(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}\jmath(\jmath-1)(\jmath-1)t_{\jmath-1} + \mu(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}\jmath(\jmath-1)t_{\jmath-1}\Bigg]\\
&= \frac{1}{S}\Bigg[\mu(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-2)(\jmath-3)t_{\jmath-1} + \{(1+|E|)(1+\rho)(1+4\mu)\\
&\quad+\mu(D-E)(1-\delta)\}\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-2)t_{\jmath-1} + \{2(1+|E|)(1+\rho)(1+\mu)\\
&\quad+(D-E)(1-\delta)(1+2\mu)\}\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1} + (D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1}\Bigg]\end{aligned}$$

$$= \frac{1}{S}\Bigg[\mu(1+|E|)(1+\rho)\sum_{\jmath=1}^{\infty}\jmath(\jmath-1)(\jmath-2)t_{\jmath}+\{(1+|E|)(1+\rho)(1+4\mu)$$
$$+\mu(D-E)(1-\delta)\}\sum_{\jmath=1}^{\infty}\jmath(\jmath-1)t_{\jmath}+\{2(1+|E|)(1+\rho)(1+\mu)$$
$$+(D-E)(1-\delta)(1+2\mu)\}\sum_{\jmath=1}^{\infty}\jmath t_{\jmath}+(D-E)(1-\delta)\sum_{\jmath=1}^{\infty}t_{\jmath}\Bigg]$$
$$= \frac{1}{S}\Big[\mu(1+|E|)(1+\rho)\phi'''(1)+\{(1+|E|)(1+\rho)(1+4\mu)+\mu(D-E)(1-\delta)\}\phi''(1)$$
$$+\{2(1+|E|)(1+\rho)(1+\mu)+(D-E)(1-\delta)(1+2\mu)\}\phi'(1)$$
$$+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta)$$

and from Lemma 2.1, we obtain the required result. □

In Theorem 3.4, letting the values $\rho = 0$ and $\rho = \delta = 0$, we obtain the next two results, respectively.

Corollary 3.7. *We have* $\Phi_\phi(\mu;\cdot) \in K(D,E;\delta)$ *if the inequality*

$$\begin{aligned}\frac{1}{S}\Big[\mu(1+|E|)\phi'''(1)+\{(1+|E|)(1+4\mu)+\mu(D-E)(1-\delta)\}\phi''(1)\\ +\{2(1+|E|)(1+\mu)+(1+2\mu)(D-E)(1-\delta)\}\phi'(1)\\ +(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta)\end{aligned} \tag{3.11}$$

holds.

Corollary 3.8. *We have* $\Phi_\phi(\mu;\cdot) \in K(D,E)$ *if the inequality*

$$\frac{1}{S}\Big[\mu(1+|E|)\phi''(1)+\{(1+|E|)(1+4\mu)+\mu(D-E)\}\phi''(1)+\{2(1+|E|)(1+\mu)$$
$$+(1+2\mu)(D-E)\}\phi'(1)+(D-E)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta)$$

holds.

Remark 3.4. (i) For the special case $D = 1$, $E = -1$, and $\mu = 0$, Theorem 3.4 reduces to the result of Porwal and Murugusundaramoorthy [33, Theorem 7, p. 227];

(ii) Taking $D = 1$, $E = -1$, and $\mu = 0$ in Corollary 3.7 we obtain the result of Porwal [31, Theorem 11 with $\lambda = 0$].

The following two theorems give us sufficient conditions such that the images of the classes $\mathcal{H}(\lambda,\eta)$ by $\mathcal{P}_\phi$ and $\mathcal{Q}_\phi$ belong to $K(D,E;\delta,\rho)$, respectively.

Theorem 3.5. *If* $\omega \in \mathcal{H}(\lambda,\eta)$ *and the inequality*

$$\frac{2(1-\eta)\cos\lambda}{S}\Big[(1+|E|)(1+\rho)\phi'(1)+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta)$$

holds, then $\mathcal{P}_\phi(\omega) \in K(D,E;\delta,\rho)$.

Proof. Under the above assumptions, to prove $\mathcal{P}_\phi(\omega)(\in K(D,E;\delta,\rho)$, we will use Lemmas 2.2 and 2.3. Thus, it is easy to show that

$$\begin{aligned}
&\sum_{\jmath=2}^{\infty}\jmath[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)]\frac{2(1-\eta)\cos\lambda}{\jmath}\frac{t_{\jmath-1}}{S}\\
=&\frac{2(1-\eta)\cos\lambda}{S}\left[\sum_{\jmath=2}^{\infty}(1+|E|)(1+\rho)(\jmath-1)t_{\jmath-1}+(D-E)(1-\delta)t_{\jmath-1}\right]\\
=&\frac{2(1-\eta)\cos\lambda}{S}\left[(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}+(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1}\right]\\
=&\frac{2(1-\eta)\cos\lambda}{S}\left[(1+|E|)(1+\rho)\sum_{\jmath=1}^{\infty}\jmath t_{\jmath}+(D-E)(1-\delta)\sum_{\jmath=1}^{\infty}t_{\jmath}\right]\\
=&\frac{2(1-\eta)\cos\lambda}{S}\Big[(1+|E|)(1+\rho)\phi'(1)+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\leq(D-E)(1-\delta),
\end{aligned}$$

which completes the proof of Theorem 3.5. □

We emphasize the following special cases of this theorem, obtained for $\rho = 0$ and $\rho = \delta = 0$, respectively.

Corollary 3.9. *If $\omega \in \mathcal{H}(\lambda,\eta)$ and the inequality*

$$\frac{2(1-\eta)\cos\lambda}{S}\Big[(1+|E|)\phi'(1)+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\leq(D-E)(1-\delta)$$

holds, then $\mathcal{P}_\phi(\omega) \in K(D,E;\delta)$.

Corollary 3.10. *If $\omega \in \mathcal{H}(\lambda,\eta)$ and the inequality*

$$\frac{2(1-\eta)\cos\lambda}{S}\Big[(1+|E|)\phi'(1)+(D-E)(\phi(1)-\phi(0))\Big]\leq(D-E)$$

is satisfied, then $\mathcal{P}_\phi(\omega) \in K(D,E)$.

Theorem 3.6. *If $\omega \in \mathcal{H}(\lambda,\eta)$ and the inequality*

$$\begin{aligned}
&\frac{2(1-\eta)\cos\lambda}{S}\Big[\mu(1+|E|)(1+\rho)\phi''(1)+[(1+|E|)(1+\rho)(\mu+1)+\mu(D-E)(1-\delta)]\phi'(1)\\
&\quad+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\leq(D-E)(1-\delta)
\end{aligned}$$

holds, then $\mathcal{Q}_\phi(\omega) \in K(D,E;\delta,\rho)$.

Proof. If $\omega \in \mathcal{H}(\lambda,\eta)$ and the inequality of the assumption holds, to prove $\mathcal{Q}_\phi(\omega) \in K(D,E;\delta,\rho)$, we will use Lemmas 2.2 and 2.3. A simple computation leads to

$$\begin{aligned}
&\sum_{\jmath=2}^{\infty}\jmath[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)][1+\mu(\jmath-1)]\frac{2(1-\eta)\cos\lambda}{\jmath}\frac{t_{\jmath-1}}{S}\\
=&\frac{2(1-\eta)\cos\lambda}{S}\sum_{\jmath=2}^{\infty}[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)][1+\mu(\jmath-1)]t_{\jmath-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(1-\eta)\cos\lambda}{S}\Bigg[(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}+(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1} \\
&\quad +\mu(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-1)t_{\jmath-1}+\mu(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}\Bigg] \\
&= \frac{2(1-\eta)\cos\lambda}{S}\Bigg[\mu(1+|E|)(1+\rho)\sum_{\jmath=2}^{\infty}(\jmath-1)(\jmath-2)t_{\jmath-1}+[(1+|E|)(1+\rho)(\mu+1) \\
&\quad +\mu(D-E)(1-\delta)]\sum_{\jmath=2}^{\infty}(\jmath-1)t_{\jmath-1}+(D-E)(1-\delta)\sum_{\jmath=2}^{\infty}t_{\jmath-1}\Bigg] \\
&= \frac{2(1-\eta)\cos\lambda}{S}\Bigg[\mu(1+|E|)(1+\rho)\sum_{\jmath-1}^{\infty}\jmath(\jmath-1)t_{\jmath}+[(1+|E|)(1+\rho)(\mu+1) \\
&\quad +\mu(D-E)(1-\delta)]\sum_{\jmath-1}^{\infty}\jmath t_{\jmath}+(D-E)(1-\delta)\sum_{\jmath-1}^{\infty}t_{\jmath}\Bigg] \\
&= \frac{2(1-\eta)\cos\lambda}{S}\Big[\mu(1+|E|)(1+\rho)\phi''(1)+[(1+|E|)(1+\rho)(\mu+1)+\mu(D-E)(1-\delta)]\phi'(1) \\
&\quad +(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta),
\end{aligned}
$$

which proves our result. □

Letting $\rho=0$ in the above theorem, we get the next result.

Corollary 3.11. *If $\omega\in\mathcal{H}(\lambda,\eta)$ and the inequality*

$$
\begin{aligned}
\frac{2(1-\eta)\cos\lambda}{S}\Big[\mu(1+|E|)\phi''(1)+[(1+|E|)(\mu+1)+\mu(D-E)(1-\delta)]\phi'(1) \\
+(D-E)(1-\delta)(\phi(1)-\phi(0))\Big]\le(D-E)(1-\delta)
\end{aligned}
$$

is satisfied, then $\mathcal{Q}_\phi(\omega)\in K(D,E;\delta)$.

For $\rho=\delta=0$, the above theorem leads to the next result.

Corollary 3.12. *If $\omega\in\mathcal{H}(\lambda,\eta)$ and the inequality*

$$
\begin{aligned}
\frac{2(1-\eta)\cos\lambda}{S}\Big[\mu(1+|E|)\phi''(1)+[(1+|E|)(\mu+1)+\mu(D-E)]\phi'(1) \\
+(D-E)(\phi(1)-\phi(0))\Big]\le(D-E)
\end{aligned}
$$

is satisfied, then $\mathcal{Q}_\phi(\omega)\in K(D,E)$.

Theorem 3.7. *Assume that inequality* (3.1) *is true. The next implication holds:*
(i) If $\omega\in\mathcal{S}$, then $\mathcal{R}_\phi(\omega)\in S(D,E;\delta,\rho)$;
(ii) If $\omega\in\mathcal{K}$, then $\mathcal{P}_\phi(\omega)\in S(D,E;\delta,\rho)$.

Proof. (i) From Lemma 2.1, we just have to demonstrate that

$$\sum_{J=2}^{\infty} [(1+|E|)(1+\rho)(J-1)+(D-E)(1-\delta)]\frac{|c_J|}{J}\frac{t_{J-1}}{S} \leq (D-E)(1-\delta).$$

Using the well-known fact $|c_J| \leq J$, $J \geq 2$, for all $\omega \in \mathcal{S}$, we have

$$\begin{aligned}
&\sum_{J=2}^{\infty} [(1+|E|)(1+\rho)(J-1)+(D-E)(1-\delta)]\frac{|c_J|}{J}\frac{t_{J-1}}{S} \\
&\leq \sum_{J=2}^{\infty} [(1+|E|)(1+\rho)(J-1)+(D-E)(1-\delta)]\frac{t_{J-1}}{S} \\
&= \frac{1}{S}\left[(1+|E|)(1+\rho)\sum_{J=2}^{\infty}(J-1)t_{J-1}+(D-E)(1-\delta)\sum_{J=2}^{\infty}t_{J-1}\right] \\
&= \frac{1}{S}\left[(1+|E|)(1+\rho)\sum_{k=1}^{\infty}kt_k+(D-E)(1-\delta)\sum_{J=1}^{\infty}t_k\right] \\
&= \frac{1}{S}[(1+|E|)(1+\rho)\phi'(1)+(D-E)(1-\delta)(\phi(1)-\phi(0))] \leq (D-E)(1-\delta),
\end{aligned}$$

which proves our result.

(ii) Using Lemma 2.1, we just have to demonstrate that

$$\sum_{J=2}^{\infty} [(1+|E|)(1+\rho)(J-1)+(D-E)(1-\delta)]|c_J|\frac{t_{J-1}}{S} \leq (D-E)(1-\delta).$$

The proof of the above inequality is similar to Theorem 3.1 using that $|c_J| \leq 1$ for all $\omega \in \mathcal{K}$, so we omit it. □

Taking $\rho = 0$ and $\rho = \delta = 0$ in this theorem, we get the next two special cases, respectively.

Corollary 3.13. *Suppose that inequality* (3.2) *holds. Then, the following implications hold:*
(i) If $\omega \in \mathcal{S}$, then $\mathcal{R}_\phi(\omega) \in S(D,E;\delta)$;
(ii) If $\omega \in \mathcal{K}$, then $\mathcal{P}_\phi(\omega) \in S(D,E;\delta)$.

Corollary 3.14. *Suppose that inequality* (3.3) *holds. Then, the following implications are true:*
(i) If $\omega \in \mathcal{S}$, then $\mathcal{R}_\phi(\omega) \in S(D,E)$;
(ii)If $\omega \in \mathcal{K}$, then $\mathcal{P}_\phi(\omega) \in S(D,E)$.

Remark 3.5. (i) If we take $t_J = \frac{m^J}{J!}$ with $m > 0$ in Theorem 3.7, we get the result of El-Ashwah and Kota [23, Theorem 2.4];

(ii) Considering $t_J = \frac{m^J}{J!}$ where $m > 0$ in Corollary 3.13, we get the result of El-Ashwah and Kota [23, Corallary 2.7];

(iii) For the special case $D = \gamma$, $E = -\gamma$, and $t_J = \frac{m^J}{J!}$ with $m > 0$ in Corollary 3.13, we get the result of El-Ashwah and Kota [23, Corallary 2.8].

Theorem 3.8. *If inequality* (3.4) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{P}_\phi(\omega) \in S(D, E; \delta, \rho)$*;*
(ii) $\omega \in \mathcal{S}$ *implies* $\mathcal{R}_\phi(\omega) \in K(D, E; \delta, \rho)$*.*

Proof. (i) From Lemma 2.1, we just have to demonstrate that

$$\sum_{\jmath=2}^{\infty} [(1 + |E|)(1 + \rho)(\jmath - 1) + (D - E)(1 - \delta)]|c_\jmath| \frac{t_{\jmath-1}}{S} \leq (D - E)(1 - \delta).$$

Since $|c_\jmath| \leq \jmath$, $\jmath \geq 2$, for all $\omega \in \mathcal{S}$, using assumption (3.4), we have

$$\begin{aligned}
&\sum_{\jmath=2}^{\infty} [(1 + |E|)(1 + \rho)(\jmath - 1) + (D - E)(1 - \delta)]|c_\jmath| \frac{t_{\jmath-1}}{S} \\
&\leq \sum_{\jmath=2}^{\infty} \jmath[(1 + |E|)(1 + \rho)(\jmath - 1) + (D - E)(1 - \delta)] \frac{t_{\jmath-1}}{S} \\
&= \frac{1}{S} \left[(1 + |E|)(1 + \rho) \sum_{\jmath=2}^{\infty} (\jmath - 1)t_{\jmath-1} + (D - E)(1 - \delta) \sum_{\jmath=2}^{\infty} t_{\jmath-1} \right] \\
&= \frac{1}{S} \left[(1 + |E|)(1 + \rho) \sum_{\jmath=1}^{\infty} \jmath t_\jmath + (D - E)(1 - \delta) \sum_{\jmath=1}^{\infty} t_\jmath \right] \\
&= \frac{1}{S} \left[(1 + |E|)(1 + \rho)\phi'(1) + (D - E)(1 - \delta)(\phi(1) - \phi(0)) \right] \leq (D - E)(1 - \delta),
\end{aligned}$$

and the proof is complete.

(ii) From Lemma 2.2, we just have to demonstrate that

$$\sum_{\jmath=2}^{\infty} \jmath[(1 + |E|)(1 + \rho)(\jmath - 1) + (D - E)(1 - \delta)] \frac{|c_\jmath|}{\jmath} \frac{t_{\jmath-1}}{S} \leq (D - E)(1 - \delta).$$

The proof is similar to Theorem 3.2 using $|c_\jmath| \leq \jmath$, $\jmath \geq 2$, for all $\omega \in \mathcal{S}$, so we omit it. □

If we take in Theorem 3.8 the particular cases $\rho = 0$ and $\rho = \delta = 0$, we obtain the next two corollaries, respectively.

Corollary 3.15. *If inequality* (3.5) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{P}_\phi(\omega) \in S(D, E; \delta)$*;*
(ii) $\omega \in \mathcal{S}$ *implies* $\mathcal{R}_\phi(\omega) \in K(D, E; \delta)$*.*

Corollary 3.16. *If inequality* (3.6) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{P}_\phi(\omega) \in S(A, B)$*;*
(ii)$\omega \in \mathcal{S}$ *implies* $\mathcal{R}_\phi(\omega) \in K(A, B)$*.*

Remark 3.6. (i) If we take $t_\jmath = \frac{m^\jmath}{\jmath!}$ with $m > 0$ in Theorem 3.8(i), we get the result of El-Ashwah and Kota [23, Theorem 2.3];

(ii) Putting $t_\jmath = \frac{m^\jmath}{\jmath!}$ with $m > 0$ in Corollary 3.15(i), we get the result due to El-Ashwah and Kota [23, Corallary 2.5];

(iii) For $D = \gamma$, $E = -\gamma$, and $t_\jmath = \frac{m^\jmath}{\jmath!}$ with $m > 0$ Corollary 3.15(i) reduces to the result obtained by El-Ashwah and Kota [23, Corallary 2.6].

Theorem 3.9. *Assuming that inequality* (3.7) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in S(D, E; \delta, \rho)$*;*
(ii) $\omega \in \mathcal{K}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E; \delta, \rho)$.

Proof. (i) In view of Lemma 2.1, we must demonstrate that

$$\sum_{J=2}^{\infty} [(1 + |E|)(1 + \rho)(J - 1) + (D - E)(1 - \delta)][1 + \mu(J - 1)]\frac{|c_J|}{J}\frac{t_{J-1}}{S} \leq (D - E)(1 - \delta).$$

The proof is similar to Theorem 3.3, using $|c_J| \leq J$, $J \geq 2$ for all $\omega \in \mathcal{S}$, so we will omit it.

(ii) Again using Lemma 2.1, it is required to demonstrate that

$$\sum_{J=2}^{\infty} [(1 + |E|)(1 + \rho)(J - 1) + (D - E)(1 - \delta)]\,[1 + \mu(J - 1)]|c_J|\frac{t_{J-1}}{S} \leq (D - E)(1 - \delta).$$

Since this proof is similar to Theorem 3.3, using $|c_J| \leq 1$, $J \geq 2$ for all $\omega \in \mathcal{K}$, we omit it. □

Letting $\rho = 0$ and $\rho = \delta = 0$ in the previously discussed theorem, we get the next two corollaries respectively.

Corollary 3.17. *If inequality* (3.8) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in S(D, E; \delta)$*;*
(ii) $\omega \in \mathcal{K}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E; \delta)$.

Corollary 3.18. *If inequality* (3.9) *holds, then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in S(D, E)$*;*
(ii) $\omega \in \mathcal{K}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E)$.

Remark 3.7. (i) Taking $t_J = \frac{m^J}{J!}$ with $m > 0$ in Theorem 3.9, we get the result of El-Ashwah and Kota [23, Theorem 2.7];

(ii) For $t_J = \frac{m^J}{J!}$ with $m > 0$, Corollary 3.13 reduces to the result due to El-Ashwah and Kota [23, Corallary 2.13];

(iii) For the particular case $D = \gamma$, $E = -\gamma$, and $t_J = \frac{m^J}{J!}$ with $m > 0$, Corollary 3.13 leads to the result of El-Ashwah and Kota [23, Corallary 2.14].

Theorem 3.10. *Suppose that inequality* (3.10) *holds. Then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E; \delta, \rho)$*;*
(ii) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in K(D, E; \delta, \rho)$.

Proof. Using Lemma 2.1, we have

$$\sum_{J=2}^{\infty} [(1 + |E|)(1 + \rho)(J - 1) + (D - E)(1 - \delta)]\,[1 + \mu(J - 1)]|c_J|\frac{t_{J-1}}{S} \leq (D - E)(1 - \delta).$$

Since the proof is similar to that of Theorem 3.4 using the fact $|c_J| \leq J$, $J \geq 2$ for all $\omega \in \mathcal{S}$, we will omit it.

(ii) From Lemma 2.2, we have

$$\sum_{\jmath=2}^{\infty} \jmath[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)][1+\mu(\jmath-1)]\frac{|c_\jmath|}{\jmath}\frac{t_{\jmath-1}}{S} \leq (D-E)(1-\delta).$$

The proof is similar to Theorem 3.4 using the above mentioned inequality, so we will omit it. □

For the particular cases $\rho = 0$ and $\rho = \delta = 0$, Theorem 3.10 gives us the following results.

Corollary 3.19. *Assume that inequality* (3.11) *holds. Then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E; \delta)$*;*
(ii) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in K(D, E; \delta)$*.*

Corollary 3.20. *Assume that inequality* (3.11) *holds. Then:*
(i) $\omega \in \mathcal{S}$ *implies* $\mathcal{Q}_\phi(\omega) \in S(D, E)$*;*
(ii) $\omega \in \mathcal{S}$ *implies* $\mathcal{G}_\phi(\omega) \in K(D, E)$*.*

Remark 3.8. (1) Putting $t_\jmath = \frac{m^\jmath}{\jmath!}$ with $m > 0$ in Theorem 3.10(i), we get the results acquired by El-Ashwah and Kota [23, Theorem 2.6];
(2) For $t_\jmath = \dfrac{m^\jmath}{\jmath!}$ with $m > 0$, Corollary 3.19(i) leads to the result of El-Ashwah and Kota [23, Corollary 2.11];
(3) For the particular values $D = \gamma$, $E = -\gamma$, and $t_\jmath = \frac{m^\jmath}{\jmath!}$ with $m > 0$ in Corollary 3.19(i), we found the result of El-Ashwah and Kota [23, Corallary 2.12].

4. Connection with a function obtained by the Alexander–type integral operator

Let M_ϕ be defined by (1.6), and defining with the aid of an integral the function M_ϕ by

$$\Omega_\phi(\xi) = \int_0^\xi \frac{M_\phi(t)}{t}dt, \ \xi \in \mathbb{D},$$

we obtain a sufficient condition for $\Omega_\phi \in K(D, E; \delta, \rho)$.

Theorem 4.1. *If inequality* (3.1) *holds, then* $\Omega_\phi \in K(D, E; \delta, \rho)$*.*

Proof. Since

$$\Omega_\phi(\xi) = \xi + \sum_{\jmath=2}^{\infty} \frac{t_{\jmath-1}}{\jmath S}\xi^\jmath, \ \xi \in \mathbb{D},$$

by Lemma 2.2, we have $\sum_{\jmath=2}^{\infty} \jmath\left[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)\right]\frac{t_{\jmath-1}}{\jmath S} \leq (D-E)(1-\delta)$. Now,

$$\begin{aligned}
&\sum_{\jmath=2}^{\infty} \jmath\left[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)\right]\frac{t_{\jmath-1}}{\jmath S}\\
&= \frac{1}{S}\sum_{\jmath=2}^{\infty}\left[(1+|E|)(1+\rho)(\jmath-1)+(D-E)(1-\delta)\right]t_{\jmath-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{S}[(1+|E|)(1+\rho)\sum_{J=2}^{\infty}(J-1)t_{J-1} + (D-E)(1-\delta)\sum_{J=2}^{\infty}t_{J-1}] \\
&= \frac{1}{S}[(1+|E|)(1+\rho)\sum_{J-1}^{\infty}Jt_J + (D-E)(1-\delta)\sum_{J-1}^{\infty}t_J] \\
&= \frac{1}{S}[(1+|E|)(1+\rho)\phi'(1) + (D-E)(1-\delta)(\phi(1)-\phi(0))] \leq (D-E)(1-\delta).
\end{aligned}
$$

□

For $\rho = 0$ and $\rho = \delta = 0$, we get the following corollaries respectively.

Corollary 4.1. *If inequality* (3.2) *holds, then* $\Omega_\phi \in K(D, E; \delta)$.

Corollary 4.2. *If inequality* (3.3) *holds, then* $\Omega_\phi \in K(D, E)$.

Remark 4.1. Putting $D = 1$ and $E = -1$ in Theorem 4.1, we obtained the results acquired by Porwal and Murugusundaramoorthy [33, Theorem 15, p.228].

Remark 4.2.
- (i) Taking $t_J = \frac{m^J}{J!}$, the previous theorem's results reduces to the corresponding result for the Poisson distribution series;
- (ii) Taking $t_J = \frac{(a)_J}{(c)_J}\frac{m!}{J!}$, previous theorem's result will give the result for the special case involving the confluent hypergeometric distribution series;
- (iii) Taking $t_J = \frac{(a)_J(b)_J}{(c)_J}\frac{m!}{J!}$, the previous theorem's result will reduce to those for the hypergeometric distribution series;
- (iv) Taking $t_J = \frac{m^J}{\Gamma(\alpha J+\beta)}$, the previous theorem's result gives the corresponding result for to the Mittag–Leffler type Poisson distribution series.

5. Conclusions

The sufficient criteria for generalized distribution series on several subclasses of univalent functions are discussed in this article, more exactly that simple sufficient conditions such that a few analytic functions defined by generalized distribution series belong to the new defined general classes. The generalized distribution series integral operator was explored, and we also obtained several inclusion relations. Moreover, the image by Alexander of one of these functions was discussed in Section 4.

Recently, researchers have looked into the Hankel determinant [17] and bounds for generalized distribution probabilities [37]. We believe that future research may focus on additional characteristics of this series of distributions, such inequalities involving different integrals, Turan-type inequalities [38], neighborhood problems [39], convolution features, and partial sums [40], etc. We anticipate that this series of distributions will have a significant impact in a number of mathematical, scientific, and technological fields, and some generalizations of these results and notions could be found according to recent results in this direction. In the future, we will extend these results with the q-version of these distributions.

Author contributions

All authors have been working together for the actual study. Ekram E. Ali: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Resources, Supervision, Writing–original draft;

Rabha M. El-Ashwah: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing–original draft, Writing–review & editing; Wafaa Y. Kota: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation, Visualization, Writing–original draft; Abeer M. Albalahi: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Visualization, Writing–original draft; Teodor Bulboacă: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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