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Research article

A family of trivariate Gould-Hopper-Bell-Apostol-Type polynomials: Construction and properties

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Abstract: In this study, we define and investigate a new hybrid family of special polynomials, named the trivariate Gould-Hopper-Bell-Apostol-type polynomials. By applying the monomiality principle, we construct their generating function, derive series representations, identify quasi-monomial operators, and formulate the corresponding differential equation. We also establish summation formulas and integral/differential representations. Furthermore, we investigate specific members of this family, including Gould-Hopper-Bell-Apostol-Bernoulli, Gould-Hopper-Bell-Apostol-Euler, and Gould-Hopper-Bell-Apostol-Genocchi polynomials, revealing analogous results for each. Finally, we utilize Mathematica for computational exploration, examining zero distributions and graphical representations.

Keywords: Apostol-type polynomials; Gould-Hopper polynomials; Bell polynomials; monomiality principle; generating function; differential equations; zero distributions
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1. Introduction

Special polynomials are highly important across mathematics, theoretical physics, and engineering due to their fundamental roles and applications, particularly in analyzing the differential equations common to physics and engineering problems. Furthermore, these special polynomials readily yield numerous useful identities and are foundational for defining new polynomial classes. Notably significant are the Gould-Hopper and Bell polynomials, prized for their wide-ranging use across mathematics [1–3].

The Apostol-type polynomials, particularly the Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, are significant in pure and applied mathematics, attracting considerable research. Studies by Luo et al. [4–6] and Ozarslan [7] explored fundamental properties and explicit series representations for these polynomials. Srivastava [8] focused on explicit representations via a generalized Hurwitz-Lerch; zeta function. Further, various extended forms of Apostol-type polynomials have been explored, for example, parametric extensions [9, 10], hybrid classes like truncated-exponential-Apostol-type polynomials [11]), unified formulas connecting to other families [12], and unified frameworks [13, 14].

The following notations and definitions will be employed consistently in this study: $\mathbb{R}$ refers to the set of real numbers, $\mathbb{C}$ refers to the set of complex numbers, $\mathbb{Z}$ refers to for the set of integers, $\mathbb{N}$ refers to the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ refers to the set of non-negative integers.

The Gould-Hopper polynomials $\mathcal{H}_\tau^{(r)}(\omega_1, \omega_2)$ [15] are defined as follows:

$$e^{\omega_1\mu+\omega_2\mu^r} = \sum_{\tau=0}^{\infty} \mathcal{H}_\tau^{(r)}(\omega_1, \omega_2)\frac{\mu^\tau}{\tau!}, \qquad r \in \mathbb{Z}^+ \tag{1.1}$$

and represented by the series

$$\mathcal{H}_\tau^{(r)}(\omega_1, \omega_2) = \tau! \sum_{\kappa=0}^{[\frac{\tau}{r}]} \frac{\omega_1^{\tau-r\kappa}\ \omega_2^{\kappa}}{(\tau - r\kappa)!\kappa!}. \tag{1.2}$$

The classical Bell polynomials $\mathcal{B}el_\tau(\omega)$ [16, 17] are defined by

$$e^{\omega(e^\mu-1)} = \sum_{\tau=0}^{\infty} \mathcal{B}el_\tau(\omega)\,\frac{\mu^\tau}{\tau!}. \tag{1.3}$$

The 2-variable Bell polynomials, denoted as $\mathcal{B}el_\tau(\omega_1, \omega_2)$, are defined as follows [18, 19]:

$$e^{\omega_1\mu}e^{\omega_2(e^\mu-1)} = \sum_{\tau=0}^{\infty} \mathcal{B}el_\tau(\omega_1, \omega_2)\,\frac{\mu^\tau}{\tau!}. \tag{1.4}$$

Recently, the Gould-Hopper-Bell polynomials (GHBelP) ${}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)$ were introduced in [20] by the generating function

$$e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)\frac{\mu^\tau}{\tau!} \tag{1.5}$$

and represented by the series

$${}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z) = \sum_{\kappa=0}^{\tau} \binom{\tau}{\kappa} \mathcal{H}_{\tau-\kappa}^{(r)}(\omega_1, \omega_2)\ \mathcal{B}el_\kappa(z). \tag{1.6}$$

Operational methods involving differential operators, derived from the monomiality principle, offer effective tools for studying classical polynomial classes and their diverse extensions. The concept of monomiality originates from the notion of poweroid introduced by Steffensen [21]. This concept was revisited and methodically applied by Dattoli [22]. In line with the monomiality principle [21, 22], a polynomial set $\rho_\tau(\omega)$ ($\tau \in \mathbb{N},\ \omega \in \mathbb{C}$) is termed *quasi-monomial* if it is possible to define "multiplicative" ($\hat{M}$) and "derivative" ($\hat{P}$) operators for which

$$\hat{M}\{\rho_\tau(\omega)\} = \rho_{\tau+1}(\omega), \tag{1.7}$$

$$\hat{P}\{\rho_\tau(\omega)\} = \tau\, \rho_{\tau-1}(\omega), \tag{1.8}$$

for all $\tau \in \mathbb{N}$. Moreover, these operators satisfy the relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1} \tag{1.9}$$

and therefore reveals the Weyl group structure. If the polynomial set $\{\rho_\tau(\omega)\}_{\tau\in\mathbb{N}}$ under consideration is quasi-monomial, its properties can be readily determined from the properties of the operators $\hat{M}$ and $\hat{P}$. Consequently, we have:

(i) Differential realizations of $\hat{M}$ and $\hat{P}$ imply that $\rho_\tau(\omega)$ fulfills the differential equation

$$\hat{M}\hat{P}\{\rho_\tau(\omega)\} = \tau\, \rho_\tau(\omega). \tag{1.10}$$

(ii) With the assumption that $\rho_0(\omega) = 1$, we have an explicit construction for the polynomials $\rho_\tau(\omega)$ as

$$\rho_\tau(\omega) = \hat{M}^\tau\{\rho_0(\omega)\} = \hat{M}^\tau\{1\}, \tag{1.11}$$

from which we derive the series definition of $\rho_\tau(\omega)$.

(iii) Based on identity (1.11), we can express the exponential generating function of $\rho_\tau(\omega)$ as follows:

$$\exp(\mu\hat{M})\{1\} = \sum_{\tau=0}^{\infty} \rho_\tau(\omega)\frac{\mu^\tau}{\tau!}, \qquad |\mu| < \infty. \tag{1.12}$$

The quasi-monomiality of the GHBelP ${}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)$ [20] is established via the following operators:

$$\hat{M}_{GH\mathcal{B}el} = \omega_1 + r\omega_2 D_{\omega_1}^{r-1} + ze^{D_{\omega_1}}, \qquad \left(D_{\omega_1} := \frac{\partial}{\partial \omega_1}\right) \tag{1.13}$$

and

$$\hat{P}_{GH\mathcal{B}el} := D_{\omega_1}. \tag{1.14}$$

Based on the monomiality principle, the GHBelP ${}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)$ fulfills the following identities:

$$\hat{M}_{GH\mathcal{B}el}\{{}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)\} = {}_{\mathcal{H}}\mathcal{B}el_{\tau+1}^{(r)}(\omega_1, \omega_2, z), \tag{1.15}$$

$$\hat{P}_{GH\mathcal{B}el}\{{}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)\} = \tau\, {}_{\mathcal{H}}\mathcal{B}el_{\tau-1}^{(r)}(\omega_1, \omega_2, z), \tag{1.16}$$

$$\hat{M}_{GH\mathcal{B}el}\hat{P}_{GH\mathcal{B}el}\{{}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)\} = \tau\, {}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z), \tag{1.17}$$

$$\exp(\hat{M}_{GH\mathcal{B}el}\,\mu)\{1\} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}el_\tau^{(r)}(\omega_1, \omega_2, z)\,\frac{\mu^\tau}{\tau!} \qquad (|\mu| < \infty). \tag{1.18}$$

The Apostol-Bernoulli $\mathfrak{B}_{\tau}^{(\sigma)}(\omega;\zeta)$ [5], Apostol-Euler $\mathcal{E}_{\tau}^{(\sigma)}(\omega;\zeta)$ [4], and Apostol-Genocchi $\mathcal{G}_{\tau}^{(\sigma)}(\omega;\zeta)$ [23] polynomials, all of order σ, are respectively defined by

$$\left(\frac{\mu}{\zeta e^{\mu}-1}\right)^{\sigma} e^{\omega\mu} = \sum_{\tau=0}^{\infty} \mathfrak{B}_{\tau}^{(\sigma)}(\omega;\zeta)\,\frac{\mu^{\tau}}{\tau!} \quad (|\mu+\log\ \zeta|<2\pi, 1^{\sigma}:=1), \tag{1.19}$$

$$\left(\frac{2}{\zeta e^{\mu}+1}\right)^{\sigma} e^{\omega\mu} = \sum_{\tau=0}^{\infty} \mathcal{E}_{\tau}^{(\sigma)}(\omega;\zeta)\,\frac{\mu^{\tau}}{\tau!} \quad (|\mu+\log\ \zeta|<\pi, 1^{\sigma}:=1), \tag{1.20}$$

$$\left(\frac{2\mu}{\zeta e^{\mu}+1}\right)^{\sigma} e^{\omega\mu} = \sum_{\tau=0}^{\infty} \mathcal{G}_{\tau}^{(\sigma)}(\omega;\zeta)\,\frac{\mu^{\tau}}{\tau!} \quad (|\mu+\log\ \zeta|<\pi, 1^{\sigma}:=1), \tag{1.21}$$

where σ and ζ are arbitrary real or complex parameters. When $\omega=0$ in Eqs (1.19)–(1.21), we get

$$\mathfrak{B}_{\tau}^{(\sigma)}(0;\zeta)=\mathfrak{B}_{\tau}^{(\sigma)}(\zeta),\ \mathcal{E}_{\tau}^{(\sigma)}(0;\zeta)=\mathcal{E}_{\tau}^{(\sigma)}(\zeta) \text{ and } \mathcal{G}_{\tau}^{(\sigma)}(0;\zeta)=\mathcal{G}_{\tau}^{(\sigma)}(\zeta),$$

which denote the Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi numbers of order σ, respectively.

The unified family of generalized Apostol-type polynomials $\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega;\delta,a,b)$ [7] are given by

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma} e^{\omega\mu} = \sum_{\tau=0}^{\infty} \mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega;\delta,a,b)\frac{\mu^{\tau}}{\tau!}\ \left(\left|\mu+b\log\left(\frac{\zeta}{a}\right)\right|<2\pi,\ \delta\in\mathbb{N},\ a,b\in\mathbb{R}^{+},\ \sigma,\zeta\in\mathbb{C}\right), \tag{1.22}$$

where $\mathsf{P}_{\tau,\zeta}^{(\sigma)}(0;\delta,a,b):=\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\delta,a,b)$ denotes the generalized Apostol-type numbers. Also, we note that

$$\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega;1,1,1)=\mathfrak{B}_{\tau}^{(\sigma)}(\omega;\zeta),\ \mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega;0,-1,1)=\mathcal{E}_{\tau}^{(\sigma)}(\omega;\zeta)$$

and

$$\mathsf{P}_{\tau,\frac{\zeta}{2}}^{(\sigma)}\left(\omega;1,-\frac{1}{2},1\right)=\mathcal{G}_{\tau}^{(\sigma)}(\omega;\zeta).$$

Special polynomials can be defined through several ways, such as generating functions, series representations, determinant representations, and differential and integral representations. The hybrid special polynomials can be defined mostly by means of the generating functions using several techniques. The choice of the most suitable technique is determined by specific properties inherent to the combined polynomials. Some of these techniques include the operational technique [24, 25] and series expansion technique [26, 27]. In recent years, there has been growing interest in a novel method concerning special functions, known as the determinant approach, which was introduced by Costabile et al. [26, 28, 29].

The generalized special polynomials enhance the applicability of classical special polynomials by integrating their advantages and offering increased adaptability. This evolution renders them more adaptable and potent in addressing intricate contemporary challenges across diverse fields. These polynomials prove especially effective in tackling multifaceted, cross-disciplinary issues and driving progress in both theoretical frameworks and practical applications within mathematics. By generalizing classical polynomials, researchers can unlock new tools for approximation, interpolation, and solving differential equations, while also gaining deeper insights into the relationships between different polynomial families.

In recent studies, various researchers have utilized operational techniques in combination with the monomiality principle to study classical special polynomials and to develop generalized classes. Notable contributions in this area are presented in several works [11, 24, 30–32], with further advancements and applications reported in several studies [33–35], as well as in extended formulations explored in recent literature [36–38]. Further, several researchers presented certain results for the hybrid form of special polynomials associated with the Apostol-type polynomials [39, 40].

In this work, in Section 2, by combining the Gould-Hopper-Bell polynomials and the unified Apostol-type polynomials, and in view of the monomiality principle, we provide a generalized class of hybrid special polynomials, referred to as the trivariate Gould-Hopper-Bell-Apostol-type polynomials. Next, the series representations, quasi-monomial operators, and differential equations are derived. In Section 3, we establish some summation formulae for the trivariate Gould-Hopper-Bell-Apostol-type polynomials. In Section 4, we investigate some related differential and integral identities. In Section 5, the Gould-Hopper-Bell-Apostol-Bernoulli, Gould-Hopper-Bell-Apostol-Euler, and Gould-Hopper-Bell-Apostol-Genocchi polynomials are introduced as specific cases, and their associated results are also discussed. Finally, the zero distributions and graphical representations are examined.

2. Family of trivariate Gould-Hopper-Bell-Apostol-type polynomials

In this section, we introduce a novel unified family of hybrid special polynomials, referred to as the trivariate Gould-Hopper-Bell-Apostol-type polynomials (TGHBelATP), through generating functions and series representations. Additionally, based on the principle of monomiality, the generating function is utilized to derive the related multiplicative and derivative operators, as well as the associated differential equation.

In the generating function (1.22), replacing ω by the multiplicative operator $\hat{M}_{GH\mathcal{B}el}$ (1.13) of the GHBelP ${}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(r)}(\omega_1, \omega_2, z)$, gives

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma}\ \exp(\hat{M}_{GH\mathcal{B}el}\,\mu)=\sum_{\tau=0}^{\infty}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\hat{M}_{GH\mathcal{B}el};\delta,a,b)\frac{\mu^{\tau}}{\tau!}. \tag{2.1}$$

Applying Eq (1.18) to the preceding equation, and denoting $\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\hat{M}_{GH\mathcal{B}el};\delta,a,b)$ by ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ (the trivariate Gould-Hopper-Bell-Apostol-type polynomials), yields:

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma}\ \left(\sum_{\tau=0}^{\infty}{}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(r)}(\omega_1,\omega_2,z)\right)=\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}. \tag{2.2}$$

Now, utilizing Eq (1.5) in the above equation, we arrive at the following definition.

Definition 1. *The trivariate Gould-Hopper-Bell-Apostol-type polynomials* ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ *of order* σ *are defined by the generating function:*

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma} e^{\omega_1\mu+\omega_2\mu^{r}+z(e^{\mu}-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}, \tag{2.3}$$

$$\left(\left|\mu+b\log\left(\frac{\zeta}{a}\right)\right|<2\pi,\ \delta\in\mathbb{N},\ a,b\in\mathbb{R}^{+},\ \sigma,\zeta\in\mathbb{C}\right).$$

Remark 1. *Setting* $z=0$ *in generating relation (2.3), we get the unified Gould-Hopper-Apostol-type polynomials* ${}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)$ *of order* σ *which are defined by:*

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma} e^{\omega_1\mu+\omega_2\mu^{r}} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}, \tag{2.4}$$

which is a special case of the polynomials defined by the generating function (2.1) in [41, P. 291].

Remark 2. *Setting* $\omega_2=0$ *in generating relation (2.3), we get the new 2-variable unified Bell-Apostol-type polynomials* ${}_{\mathcal{B}el}\mathsf{P}^{(\sigma)}_{\tau,\zeta}(\omega_1,z;\delta,a,b)$ *of order* σ *given by the generating function*

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma} e^{\omega_1\mu+z(e^{\mu}-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{B}el}\mathsf{P}^{(\sigma)}_{\tau,\zeta}(\omega_1,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}. \tag{2.5}$$

Remark 3. *Setting* $r=2$*,* $\sigma=1$ *in the generating relation (2.3), we get new special polynomials, called trivariate Hermite Kampé de Fériet-Bell-Apostol-type polynomials* ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$*, given by the generating function*

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right) e^{\omega_1\mu+\omega_2\mu^{2}+z(e^{\mu}-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}. \tag{2.6}$$

Remark 4. *Setting* $\sigma=1$ *in the generating relation (2.3), we get the trivariate Gould-Hopper-Bell-Apostol-type polynomials* ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$*, which are defined by the generating function*

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right) e^{\omega_1\mu+\omega_2\mu^{r}+z(e^{\mu}-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}. \tag{2.7}$$

Taking $\omega_1=\omega_2=0$ and $z=1$ in (2.3), we get unified Bell-Apostol-type numbers of order σ, which are defined by

$$\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma} e^{(e^{\mu}-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{B}el}\mathsf{P}^{(\sigma)}_{\tau,\zeta}(\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}. \tag{2.8}$$

Next, in view of generating function (2.3), we establish certain series representations of the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ.

Theorem 1. *The trivariate Gould-Hopper-Bell-Apostol-type polynomials* ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ *of order* σ *satisfy the following series representations:*

$${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa} {}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2;\delta,a,b)\,\mathcal{B}el_{\kappa}(z); \tag{2.9}$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathsf{P}^{(\sigma)}_{\tau-\kappa,\zeta}(0;\delta,a,b)\;{}_{\mathcal{H}}\mathcal{B}el^{(r)}_{\kappa}(\omega_1,\omega_2,z); \tag{2.10}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)=\tau!\sum_{\kappa=0}^{[\frac{\tau}{r}]}\frac{\omega_2^{\kappa}\;{}_{\mathcal{B}el}\mathsf{P}^{(\sigma)}_{\tau-r\kappa,\zeta}(\omega_1,z;\delta,a,b)}{\kappa!(\tau-r\kappa)!}. \tag{2.11}
$$

Proof. In view of generating relations (1.3), (2.3), and (2.4), we have

$$
\begin{aligned}
\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}&=\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma}e^{\omega_1\mu+\omega_2\mu^{r}+z(e^{\mu}-1)}\\
&=\left[\left(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^{b}e^{\mu}-a^{b}}\right)^{\sigma}e^{\omega_1\mu+\omega_2\mu^{r}}\right]\left[e^{z(e^{\mu}-1)}\right]\\
&=\left[\sum_{\tau=0}^{\infty}{}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}\right]\left[\sum_{\kappa=0}^{\infty}\mathcal{B}el_{\kappa}(z)\frac{\mu^{\kappa}}{\kappa!}\right]\\
&=\sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\infty}{}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)\,\mathcal{B}el_{\kappa}(z)\frac{\mu^{\tau+\kappa}}{\kappa!\,\tau!},
\end{aligned} \tag{2.12}
$$

which, upon substituting $\tau\longrightarrow\tau-\kappa$ and applying the Cauchy product rule, yields:

$$
\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}=\sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2;\delta,a,b)\,\mathcal{B}el_{\kappa}(z)\frac{\mu^{\tau}}{\tau!}, \tag{2.13}
$$

from which, by comparing the coefficients of powers of μ, we derive Eq (2.9). Similarly, the assertions in Eqs (2.10) and (2.11) can be proved. □

Remark 5. *Setting $r=2$, $\sigma=1$ in series representations (2.9)–(2.11), we find that the trivariate Hermite Kampé de Fériet-Bell-Apostol-type polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ satisfy the following series representations:*

$$
{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}}\mathsf{P}^{(2)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2;\delta,a,b)\;\mathcal{B}el_{\kappa}(z); \tag{2.14}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathsf{P}_{\tau-\kappa,\zeta}(0;\delta,a,b)\;{}_{\mathcal{H}}\mathcal{B}el^{(2)}_{\kappa}(\omega_1,\omega_2,z); \tag{2.15}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(2)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)=\tau!\sum_{\kappa=0}^{[\frac{\tau}{2}]}\frac{\omega_2^{\kappa}\;{}_{\mathcal{B}el}\mathsf{P}_{\tau-2\kappa,\zeta}(\omega_1,z;\delta,a,b)}{\kappa!(\tau-2\kappa)!}. \tag{2.16}
$$

Theorem 2. *For the unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$, the associated multiplicative and derivative operators demonstrating their quasi-monomial nature are:*

$$
\hat{M}_{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}}=\omega_1+r\omega_2D_{\omega_1}^{r-1}+ze^{D_{\omega_1}}+\frac{\sigma\delta(\zeta^{b}e^{D_{\omega_1}}-a^{b})-\sigma D_{\omega_1}\zeta^{b}e^{D_{\omega_1}}}{D_{\omega_1}(\zeta^{b}e^{D_{\omega_1}}-a^{b})} \tag{2.17}
$$

and

$$\hat{P}_{_{\mathcal{HB}el}\mathsf{P}} = D_{\omega_1}, \tag{2.18}$$

respectively.

Proof. Differentiating relation (2.3) partially with respect to μ, gives

$$\begin{aligned}\Big(\omega_1 + r\omega_2\mu^{r-1} + ze^{\mu} + \frac{\sigma\delta(\zeta^b e^{\mu} - a^b) - \sigma\mu\zeta^b e^{\mu}}{\mu(\zeta^b e^{\mu} - a^b)}\Big)\Big(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^b e^{\mu} - a^b}\Big)^{\sigma}\ e^{\omega_1\mu + \omega_2\mu^r + z(e^{\mu}-1)}\\ = \sum_{\tau=0}^{\infty} \tau\ {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau-1}}{\tau!},\end{aligned} \tag{2.19}$$

which, upon replacing τ by $\tau + 1$ in the right-hand side and using relation (2.3) the left-hand side, becomes

$$\begin{aligned}\Big(\omega_1 + r\omega_2\mu^{r-1} + ze^{\mu} + \frac{\sigma\delta(\zeta^b e^{\mu} - a^b) - \sigma\mu\zeta^b e^{\mu}}{\mu(\zeta^b e^{\mu} - a^b)}\Big)\sum_{\tau=0}^{\infty} {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau}}{\tau!}\\ = \sum_{\tau=0}^{\infty} {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau+1,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau}}{\tau!}.\end{aligned} \tag{2.20}$$

Matching the coefficients of corresponding powers of μ in Eq (2.20), we obtain

$$\begin{aligned}\Big(\omega_1 + r\omega_2 D_{\omega_1}^{r-1} + ze^{D_{\omega_1}} + \frac{\sigma\delta(\zeta^b e^{D_{\omega_1}} - a^b) - \sigma D_{\omega_1}\zeta^b e^{D_{\omega_1}}}{D_{\omega_1}(\zeta^b e^{D_{\omega_1}} - a^b)}\Big){}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\\ = {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau+1,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\ .\end{aligned} \tag{2.21}$$

Using Eq (1.7) (for ${}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)$) in (2.21), we obtain assertion (2.17).

Further, differentiating the left-hand side of (2.3) with respect to ω_1, we get

$$D_{\omega_1}\Big[\Big(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^b e^{\mu} - a^b}\Big)^{\sigma}\ e^{\omega_1\mu + \omega_2\mu^r + z(e^{\mu}-1)}\Big] = \mu\ \Big[\Big(\frac{2^{1-\delta}\,\mu^{\delta}}{\zeta^b e^{\mu} - a^b}\Big)^{\sigma}\ e^{\omega_1\mu + \omega_2\mu^r + z(e^{\mu}-1)}\Big]. \tag{2.22}$$

Using relation (2.3) in expression (2.22), gives

$$\begin{aligned}D_{\omega_1}\Big[\sum_{\tau=0}^{\infty} {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau}}{\tau!}\Big] = \mu\ \Big[\sum_{\tau=0}^{\infty} {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau}}{\tau!}\Big]\\ = \sum_{\tau=0}^{\infty} {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\,\frac{\mu^{\tau+1}}{\tau!},\end{aligned} \tag{2.23}$$

which, upon replacing τ by $\tau - 1$ in the right-hand side and then comparing the coefficients of corresponding powers of μ in the resulting equation, we obtain

$$D_{\omega_1}\Big\{{}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)\Big\} = \ \tau\ \ {}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau-1,\zeta}(\omega_1, \omega_2, z; \delta, a, b). \tag{2.24}$$

Using Eq (1.8) (for ${}_{\mathcal{HB}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1, \omega_2, z; \delta, a, b)$) in (2.24), we obtain assertion (2.18). □

Remark 6. *For $z = 0$, Theorem (2) gives the associated multiplicative and derivative operators demonstrating the quasi-monomial nature of the unified Gould-Hopper-Apostol-type polynomials ${}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)$ of order σ:*

$$\hat{M}_{\mathcal{H}\mathsf{P}} = \omega_1 + r\omega_2 D^{r-1}_{\omega_1} + \frac{\sigma\delta(\zeta^b e^{D_{\omega_1}} - a^b) - \sigma D_{\omega_1}\zeta^b e^{D_{\omega_1}}}{D_{\omega_1}(\zeta^b e^{D_{\omega_1}} - a^b)} \tag{2.25}$$

and

$$\hat{P}_{\mathcal{H}\mathsf{P}} = D_{\omega_1}, \tag{2.26}$$

respectively.

Theorem 3. *The unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ satisfy the following differential equation:*

$$\left(\omega_1 D_{\omega_1} + r\omega_2 D^{r}_{\omega_1} + z e^{D_{\omega_1}} D_{\omega_1} + \frac{\sigma\delta(\zeta^b e^{D_{\omega_1}} - a^b) - \sigma D_{\omega_1}\zeta^b e^{D_{\omega_1}}}{(\zeta^b e^{D_{\omega_1}} - a^b)} - \tau\right){}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b) = 0. \tag{2.27}$$

Proof. In view of Eq (1.10) (for ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$), utilizing operators (2.17) and (2.18), we get the asserted result (2.27). □

Remark 7. *For $z = 0$, Theorem (3) gives the following differential equation that is satisfied by the unified Gould-Hopper-Apostol-type polynomials ${}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b)$ of order σ:*

$$\left(\omega_1 D_{\omega_1} + r\omega_2 D^{r}_{\omega_1} + \frac{\sigma\delta(\zeta^b e^{D_{\omega_1}} - a^b) - \sigma D_{\omega_1}\zeta^b e^{D_{\omega_1}}}{(\zeta^b e^{D_{\omega_1}} - a^b)} - \tau\right){}_{\mathcal{H}}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2;\delta,a,b) = 0. \tag{2.28}$$

3. Summation formulae

In this section, we investigate certain remarkable summation identities for the trivariate Gould-Hopper-Bell-Apostol-type polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ.

Theorem 4. *For $\tau \in \mathbb{N}_0, \delta \in \mathbb{N}$ and $\sigma, \zeta \in \mathbb{C}$, we have*

$$\begin{aligned}
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\\
&\quad = \frac{1}{2}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}_{\kappa}\Big({}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1+1,\omega_2,z;\delta,a,b) + {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\Big).
\end{aligned} \tag{3.1}$$

Proof. From generating relation (2.3), it follows that:

$$\begin{aligned}
&(e^{\mu}+1)\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}\\
&\quad = \sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1+1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!} + \sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!},
\end{aligned} \tag{3.2}$$

which can be written as

$$\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!} \tag{3.3}$$
$$=\frac{1}{2}\Big(\sum_{\tau=0}^{\infty}\mathcal{E}_\tau\,\frac{\mu^\tau}{\tau!}\Big)\Big(\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1+1,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}+\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}\Big),$$

where, $\mathcal{E}_\tau$ represents the Euler numbers defined by [42]:

$$\frac{2}{e^\mu+1}=\sum_{\tau=0}^{\infty}\mathcal{E}_\tau\,\frac{\mu^\tau}{\tau!}. \tag{3.4}$$

By applying the Cauchy product rule to (3.3) and equating the corresponding powers of μ in the resulting equation, we obtain (3.1). □

Remark 8. *Setting $z=0$ in (3.1), we get the following summation formula:*

$${}_{\mathcal{H}}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2;\delta,a,b)=\frac{1}{2}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}_\kappa\Big({}_{\mathcal{H}}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1+1,\omega_2;\delta,a,b)+{}_{\mathcal{H}}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2;\delta,a,b)\Big). \tag{3.5}$$

Theorem 5. *The unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ satisfy the following summation formula:*

$${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\upsilon,\omega_2,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\,{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\sigma,r)}(\omega_1-\theta,\omega_2,z;\delta,a,b). \tag{3.6}$$

Proof. Replacing ω_1 by υ in (2.3), we have

$$\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\upsilon,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}=\Big(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu-a^b}\Big)^\sigma\,e^{\upsilon\mu+\omega_2\mu^r+z(e^\mu-1)}$$
$$=\Big(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu-a^b}\Big)^\sigma e^{(\omega_1-\theta)\mu}\,e^{-(\omega_1-\upsilon-\theta)\mu}\,e^{\omega_2\mu^r+z(e^\mu-1)}$$
$$=e^{(\upsilon-\omega_1+\theta)\mu}\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1-\theta,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}$$
$$=\Big(\sum_{\tau=0}^{\infty}\frac{(\upsilon-\omega_1+\theta)^\tau\,\mu^\tau}{\tau!}\Big)\Big(\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1-\theta,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}\Big)$$
$$=\sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\,{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\sigma,r)}(\omega_1-\theta,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{\tau!}. \tag{3.7}$$

From (3.7), we get asserted result (3.6). □

Remark 9. *For $\omega_2=0$ in (3.6), the unified polynomials ${}_{\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)$ of order σ satisfy the following summation formula:*

$${}_{\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\upsilon,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\,{}_{\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\sigma)}(\omega_1-\theta,z;\delta,a,b). \tag{3.8}$$

Theorem 6. *For $\tau \in \mathbb{N}_0, \delta \in \mathbb{N}$, and $\sigma, \beta, \zeta \in \mathbb{C}$, we have*

$$\begin{aligned}
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma+\beta,r)}(\omega_1 + x, \omega_2 + y, z + u; \delta, a, b) \\
&\quad = \sum_{\kappa=0}^{\tau} \binom{\tau}{\kappa} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1, \omega_2, z; \delta, a, b)\, {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\beta,r)}(x, y, u; \delta, a, b).
\end{aligned} \tag{3.9}$$

Proof. In (2.3), replacing ω_1, ω_2, z, and σ by $\omega_1 + x, \omega_2 + y, z + u$, and $\sigma + \beta$, respectively, we have

$$\begin{aligned}
&\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma+\beta,r)}(\omega_1 + x, \omega_2 + y, z + u; \delta, a, b)\, \frac{\mu^\tau}{\tau!} \\
&\quad = \left(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu - a^b}\right)^{\sigma+\beta} e^{(\omega_1+x)\mu+(\omega_2+y)\mu^r+(z+u)(e^\mu-1)} \\
&\quad = \left(\left(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu - a^b}\right)^{\sigma} e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)}\right)\left(\left(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu - a^b}\right)^{\beta} e^{x\mu+y\mu^r+u(e^\mu-1)}\right) \\
&\quad = \left(\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1, \omega_2, z; \delta, a, b)\, \frac{\mu^\tau}{\tau!}\right)\left(\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\beta,r)}(x, y, u; \delta, a, b)\, \frac{\mu^\tau}{\tau!}\right) \\
&\quad = \sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\tau} \binom{\tau}{\kappa} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1, \omega_2, z; \delta, a, b)\, {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\beta,r)}(x, y, u; \delta, a, b)\, \frac{\mu^\tau}{\tau!}.
\end{aligned} \tag{3.10}$$

From (3.10), we get the asserted result (3.9). □

Remark 10. *For $r = 2$ and $\sigma = 1$ in (3.9), the unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(2)}(\omega_1, \omega_2, z; \delta, a, b)$ satisfy the following summation formula:*

$$\begin{aligned}
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(1+\beta,2)}(\omega_1 + x, \omega_2 + y, z + u; \delta, a, b) \\
&\quad = \sum_{\kappa=0}^{\tau} \binom{\tau}{\kappa} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(2)}(\omega_1, \omega_2, z; \delta, a, b)\, {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\kappa,\zeta}^{(\beta,2)}(x, y, u; \delta, a, b).
\end{aligned} \tag{3.11}$$

The generalized Stirling numbers of the second kind are given as follows [13, 43]:

$$\sum_{\tau=0}^{\infty} S(\tau, \sigma, a, b, \zeta)\, \frac{\mu^\tau}{\tau!} = \frac{(\zeta^b e^\mu - a^b)^\sigma}{\sigma!}. \tag{3.12}$$

Theorem 7. *The unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1, \omega_2, z; \delta, a, b)$ satisfy the following summation formula:*

$${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\delta\varrho,\zeta}^{(\sigma-\varrho,r)}(\omega_1, \omega_2, z; \delta, a, b) = \frac{(\tau-\delta\varrho)!\,\varrho!}{2^{\varrho(1-\delta)}\tau!} \sum_{\kappa=0}^{\tau} \binom{\tau}{\kappa} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1, \omega_2, z; \delta, a, b)\, S(\kappa, \varrho, a, b, \zeta). \tag{3.13}$$

Proof. In view of (2.3) and (3.12), we can write

$$\sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma-\varrho,r)}(\omega_1, \omega_2, z; \delta, a, b)\, \frac{\mu^\tau}{\tau!} = \left(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu - a^b}\right)^{\sigma-\varrho} e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)}$$

$$= \frac{\varrho!}{(2^{1-\delta}\,\mu^{\delta})^{\varrho}}\left(\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}\right)\left(\sum_{\tau=0}^{\infty}S(\tau,\varrho,a,b,\zeta)\,\frac{\mu^{\tau}}{\tau!}\right)$$

$$= \frac{\varrho!}{(2^{1-\delta}\,\mu^{\delta})^{\varrho}}\sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,S(\kappa,\varrho,a,b,\zeta)\,\frac{\mu^{\tau}}{\tau!}. \tag{3.14}$$

Upon simplifying the aforementioned relation and equating the coefficients of $\frac{\mu^{\tau}}{\tau!}$ on both sides of the resulting equation, we obtain the stated result (3.13). □

Remark 11. *For $\omega_2 = 0$ in (3.13), the unified polynomials ${}_{\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)$ of order σ satisfy the following summation formula:*

$${}_{\mathcal{B}el}\mathsf{P}_{\tau-\varrho\,\delta,\zeta}^{(\sigma-\varrho)}(\omega_1,z;\delta,a,b) = \frac{(\tau-\varrho\,\delta)!\,\varrho!}{2^{\varrho(1-\delta)}\tau!}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)\,S(\kappa,\varrho,a,b,\zeta). \tag{3.15}$$

Theorem 8. *The unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ satisfy the following summation formula:*

$$\begin{aligned}&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,r)}(\omega_1+1,\omega_2,z;\delta,a,b)\\ &\quad= \frac{1}{\zeta^{b}}\left\{\frac{2^{1-\delta}(\tau+\delta)!}{\tau!}{}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(r)}(\omega_1,\omega_2,z) + a^{b}\,{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}.\end{aligned} \tag{3.16}$$

Proof. In view of (1.5) and (2.3) for $\sigma = 1$, we can write

$$\begin{aligned}&\sum_{\tau=0}^{\infty}{}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(r)}(\omega_1,\omega_2,z)\frac{\mu^{\tau}}{\tau!} = e^{\omega_1\mu+\omega_2\mu^{r}+z(e^{\mu}-1)}\\ &= \left(\frac{\zeta^{b}e^{\mu}-a^{b}}{2^{1-\delta}\,\mu^{\delta}}\right)\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(1,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}\\ &= \frac{1}{(2^{1-\delta}\,\mu^{\delta})}\left\{\zeta^{b}\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(1,r)}(\omega_1+1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!} - a^{b}\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(1,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau}}{\tau!}\right\}\\ &= \frac{1}{2^{1-\delta}}\sum_{\tau=0}^{\infty}\frac{\tau!}{(\tau+\delta)!}\left\{\zeta^{b}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,r)}(\omega_1+1,\omega_2,z;\delta,a,b) - a^{b}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}\frac{\mu^{\tau}}{\tau!},\end{aligned} \tag{3.17}$$

which, upon comparing the coefficients of $\frac{\mu^{\tau}}{\tau!}$ on both sides, yields the asserted result (3.16). □

Remark 12. *For $r = 2$ in (3.16), the unified polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(1,2)}(\omega_1,\omega_2,z;\delta,a,b)$ satisfy the following summation formula:*

$$\begin{aligned}&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,2)}(\omega_1+1,\omega_2,z;\delta,a,b)\\ &\quad= \frac{1}{\zeta^{b}}\left\{\frac{2^{1-\delta}(\tau+\delta)!}{\tau!}{}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(2)}(\omega_1,\omega_2,z) + a^{b}\,{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+\delta,\zeta}^{(1,2)}(\omega_1,\omega_2,z;\delta,a,b)\right\}.\end{aligned} \tag{3.18}$$

4. Differential and integral formulae

In this section, we establish some differential and integral formulae associated with the trivariate Gould-Hopper-Bell-Apostol-type polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ.

Theorem 9. *For $\nu,\tau\in\mathbb{N}_0,\delta\in\mathbb{N}$, and $\sigma,\zeta\in\mathbb{C}$, we have*

$$\frac{\partial^\nu}{\partial\omega_1^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}=\begin{cases}\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\nu,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)}{(\tau-\nu)!}, & \tau\geq\nu;\\ 0, & 0\leq\tau<\nu.\end{cases}\tag{4.1}$$

Proof. Differentiating the generating relation (2.3) ν times with respect to ω_1, we obtain

$$\begin{aligned}\sum_{\tau=0}^{\infty}\frac{\partial^\nu}{\partial\omega_1^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}\frac{\mu^\tau}{\tau!}&=\mu^\nu\left\{\left(\frac{2^{1-\delta}\,\mu^\delta}{\zeta^b e^\mu-a^b}\right)^\sigma e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)}\right\}\\&=\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^{\tau+\nu}}{\tau!}\\&=\sum_{\tau=\nu}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\nu,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\frac{\mu^\tau}{(\tau-\nu)!}.\end{aligned}\tag{4.2}$$

By simplifying Eq (4.2) and subsequently comparing the coefficients of $\frac{\mu^\tau}{\tau!}$ on both sides of the resultant equation, we arrive at the asserted result, as given by (4.1). □

Remark 13. *Setting $\omega_2=0$ in (4.1), we have*

$$\frac{\partial^\nu}{\partial\omega_1^\nu}\left\{{}_{\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)\right\}=\begin{cases}\frac{\tau!\ {}_{\mathcal{B}el}\mathsf{P}_{\tau-\nu,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)}{(\tau-\nu)!}, & \tau\geq\nu;\\ 0, & 0\leq\tau<\nu.\end{cases}\tag{4.3}$$

Similarly, upon differentiating relation (2.3) ν times with respect to ω_2, we can get the following result.

Theorem 10. *For $\nu,\tau\in\mathbb{N}_0,\delta\in\mathbb{N}$, and $\sigma,\zeta\in\mathbb{C}$, we have*

$$\frac{\partial^\nu}{\partial\omega_2^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}=\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-r\nu,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)}{(\tau-r\nu)!}.\tag{4.4}$$

Remark 14. *Setting $r=2$ and $\sigma=1$ in (4.4), we have*

$$\frac{\partial^\nu}{\partial\omega_2^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(2)}(\omega_1,\omega_2,z;\delta,a,b)\right\}=\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-2\nu,\zeta}^{(2)}(\omega_1,\omega_2,z;\delta,a,b)}{(\tau-2\nu)!}.\tag{4.5}$$

Theorem 11. *For $\tau\in\mathbb{N}_0,\delta\in\mathbb{N}$, and $\sigma,\zeta\in\mathbb{C}$, we have*

$$\begin{aligned}&\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}\\&\quad=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)-{}_{\mathcal{H}}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2;\delta,a,b)\,\mathcal{B}el_\kappa(z)\right\}.\end{aligned}\tag{4.6}$$

Proof. We start with generating relation (2.3). Differentiating it with respect to z, followed by simplification using equation (2.9), results in

$$\sum_{\tau=0}^{\infty}\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}\ \frac{\mu^\tau}{\tau!} \tag{4.7}$$
$$= (e^\mu-1)\left\{\left(\frac{2^{1-\delta}\ \mu^\delta}{\zeta^b e^\mu-a^b}\right)^\sigma\ e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)}\right\}$$
$$= \left\{\sum_{\tau=0}^{\infty}\frac{\mu^\tau}{\tau!}\right\}\left\{\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\ \frac{\mu^\tau}{\tau!}\right\}-\left\{\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\ \frac{\mu^\tau}{\tau!}\right\}$$
$$= \sum_{\tau=0}^{\infty}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)-{}_{\mathcal{H}}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2;\delta,a,b)\ \mathcal{B}el_\kappa(z)\right\}\ \frac{\mu^\tau}{\tau!}.$$

From (4.7), we get the asserted result (4.6). □

Remark 15. *Setting $\omega_2=0$ in (4.6), we have*

$$\frac{\partial}{\partial z}{}_{\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega_1,z;\delta,a,b)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,z;\delta,a,b)-\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma)}(\omega_1;\delta,a,b)\ \mathcal{B}el_\kappa(z)\right\}. \tag{4.8}$$

Similarly, we can derive the following outcome.

Corollary 1. *For $\tau\in\mathbb{N}_0,\delta\in\mathbb{N}$, and $\sigma,\zeta\in\mathbb{C}$, we have*

$$\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\right\}={}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1+1,\omega_2,z;\delta,a,b)-{}_{\mathcal{G}}\mathcal{B}el_\tau(\omega_1,\omega_2,z). \tag{4.9}$$

Theorem 12. *The following formula holds true:*

$$\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\ \mathrm{d}\omega_1$$
$$=\frac{1}{\tau+1}\Big[{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+1,\zeta}^{(\sigma,r)}(u+\gamma,\omega_2,z;\delta,a,b)-{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+1,\zeta}^{(\sigma,r)}(u,\omega_2,z;\delta,a,b)\Big]. \tag{4.10}$$

Proof. By integrating both sides of Eq (2.3) with respect to ω_1, we obtain

$$\sum_{\tau=0}^{\infty}\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\ \mathrm{d}\omega_1\ \frac{\mu^\tau}{\tau!}$$
$$=\frac{1}{\mu}\left[\left(\frac{2^{1-\delta}\ \mu^\delta}{\zeta^b e^\mu-a^b}\right)^\sigma\ e^{(u+\gamma)\mu+\omega_2\mu^r+z(e^\mu-1)}-\left(\frac{2^{1-\delta}\ \mu^\delta}{\zeta^b e^\mu-a^b}\right)^\sigma\ e^{u\mu+\omega_2\mu^r+z(e^\mu-1)}\right]$$
$$=\frac{1}{\mu}\Big[\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(u+\gamma,\omega_2,z;\delta,a,b)\frac{\mu^\tau}{\tau!}-\sum_{\tau=0}^{\infty}{}_{\mathcal{H}}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(u,\omega_2;\delta,a,b)\ \mathcal{B}el_\kappa(z)\ \frac{\mu^\tau}{\tau!}\Big]. \tag{4.11}$$

From (4.11), we get asserted result (4.10). □

Similarly, the following results can be proved.

Theorem 13. *The following formulas hold true:*

$$\int_u^{u+\gamma} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\mathrm{d}\omega_2$$
$$= \frac{\tau!}{(\tau+r)!}\Big[{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+r,\zeta}^{(\sigma,r)}(\omega_1,u+\gamma,z;\delta,a,b) - {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+r,\zeta}^{(\sigma,r)}(\omega_1,u,z;\delta,a,b)\Big], \tag{4.12}$$

$$\int_u^{u+\gamma} {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\mathrm{d}z$$
$$= \frac{1}{\tau+1}\sum_{k=0}^{\tau}\binom{\tau+1}{k}\mathfrak{B}_k\left[{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+1-k,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,u+\gamma;\delta,a,b) - {}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau+1-k,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,u;\delta,a,b)\right]. \tag{4.13}$$

In the next section, we turn to the consideration of several special cases of the trivariate Gould-Hopper-Bell-Apostol-type polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ.

5. Applications

In this section, some applications related to the established polynomials (TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ) are presented. Certain examples are investigated. Further, the zero distributions of the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ are examined.

5.1. Examples

Here, we introduce certain special members belonging to the unified family

$${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b),$$

with analogous results presented for each.

Example 1. *Gould-Hopper-Bell-Apostol-Bernoulli polynomials*

Since

$$\mathsf{P}_{\tau,\zeta}^{(\sigma)}(\omega;1,1,1) = \mathfrak{B}_\tau^{(\sigma)}(\omega;\zeta),$$

therefore, taking $\delta = a = b = 1$ in generating function (2.3), gives

$$\left(\frac{\mu}{\zeta e^\mu - 1}\right)^\sigma e^{\omega_1\mu+\omega_2\mu^r+z(e^\mu-1)} = \sum_{\tau=0}^{\infty} {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\,\frac{\mu^\tau}{\tau!}, \tag{5.1}$$

where ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)$ are referred to as the Gould-Hopper-Bell-Apostol-Bernoulli polynomials (GHBelBP) of order σ.

The series representations of the GHBelBP ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)$ of order σ are given as:

$${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma)}(z)\,\mathcal{H}_\kappa^{(r)}(\omega_1,\omega_2); \tag{5.2}$$

$${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2)\,\mathcal{B}el_\kappa(z); \tag{5.3}$$

$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma)}(0)\ {}_{\mathcal{H}}\mathcal{B}el_{\kappa}^{(r)}(\omega_1,\omega_2,z); \tag{5.4} $$

$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = \tau!\sum_{\kappa=0}^{[\frac{\tau}{r}]}\frac{\omega_2^{\kappa}\ {}_{\mathcal{B}el}\mathfrak{B}_{\tau-r\kappa,\zeta}^{(\sigma)}(\omega_1,z)}{\kappa!(\tau-r\kappa)!}. \tag{5.5} $$

Certain corresponding results related to the GHBelBP ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)$ of order σ are presented in Table 1.

Table 1. Findings for the Gould-Hopper-Bell-Apostol-Bernoulli polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)$.

Multiplicative and derivative operators	$\hat{M}_{\mathcal{H}\mathcal{B}el\mathfrak{B}} = \omega_1 + r\omega_2 D_{\omega_1}^{r-1} + ze^{D_{\omega_1}} + \frac{\sigma(\zeta e^{D_{\omega_1}}-1)-\sigma D_{\omega_1}\zeta e^{D_{\omega_1}}}{D_{\omega_1}(\zeta e^{D_{\omega_1}}-1)},\ \hat{P}_{\mathcal{H}\mathcal{B}el\mathfrak{B}} = D_{\omega_1}$
Differential equation	$\left(\omega_1 D_{\omega_1} + r\omega_2 D_{\omega_1}^{r} + ze^{D_{\omega_1}}D_{\omega_1} + \frac{\sigma(\zeta e^{D_{\omega_1}}-1)-\sigma D_{\omega_1}\zeta e^{D_{\omega_1}}}{(\zeta e^{D_{\omega_1}}-1)} - \tau\right){}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = 0$
Summation Formulae	${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) = \frac{1}{2}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}_{\kappa}\left({}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1+1,\omega_2,z) + {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\right)$ ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\upsilon,\omega_2,z) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\kappa,\zeta}^{(\sigma,r)}(\omega_1-\theta,\omega_2,z)$ ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma+\beta,r)}(\omega_1+x,\omega_2+y,z+u) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\kappa,\zeta}^{(\beta,r)}(x,y,u)$ ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\varrho,\zeta}^{(\sigma-\varrho,r)}(\omega_1,\omega_2,z) = \frac{(\tau-\varrho)!\ \varrho!}{\tau!}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\,S(\kappa,\varrho,\zeta)$ ${}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1,\zeta}^{(1,r)}(\omega_1+1,\omega_2,z) = \frac{1}{\zeta}\left\{(\tau+1){}_{\mathcal{H}}\mathcal{B}el_{\tau}^{(r)}(\omega_1,\omega_2,z) + {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1,\zeta}^{(1,r)}(\omega_1,\omega_2,z)\right\}$
Differential and Integral Formulae	$\frac{\partial^{\nu}}{\partial\omega_1^{\nu}}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\right\} = \begin{cases}\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\nu,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)}{(\tau-\nu)!}, & \tau\geq\nu;\\ 0, & 0\leq\tau<\nu.\end{cases}$ $\frac{\partial^{\nu}}{\partial\omega_2^{\nu}}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\right\} = \frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-r\nu,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)}{(\tau-r\nu)!}$ $\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\right\} = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z) - {}_{\mathcal{H}}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1,\omega_2)\ \mathcal{B}el_{\kappa}(z)\right\}$ $\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\right\} = {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau-\kappa,\zeta}^{(\sigma,r)}(\omega_1+1,\omega_2,z) - {}_{\mathcal{G}}\mathcal{B}el_{\tau}(\omega_1,\omega_2,z)$ $\int_{u}^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z)\,\mathrm{d}\omega_1 = \frac{1}{\tau+1}\left[{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1,\zeta}^{(\sigma,r)}(u+\gamma,\omega_2,z) - {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1,\zeta}^{(\sigma,r)}(u,\omega_2,z)\right]$ $\int_{u}^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)\,\mathrm{d}z$ $= \frac{1}{\tau+1}\sum_{k=0}^{\tau}\binom{\tau+1}{k}\mathfrak{B}_k\left[{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1-k,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,u+\gamma;\delta,a,b) - {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau+1-k,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,u;\delta,a,b)\right]$

The first few members of the GHBelBP

$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z), $$

for $\sigma = 1$ and $r = 3$ are given as:

$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{0,\zeta}^{(1,3)}(\omega_1,\omega_2,z) = 0, $$
$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{1,\zeta}^{(1,3)}(\omega_1,\omega_2,z) = \frac{1}{\zeta-1}, $$
$$ {}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}_{2,\zeta}^{(1,3)}(\omega_1,\omega_2,z) = -\frac{2\zeta}{(\zeta-1)^2} + \frac{2\zeta\omega_1}{(\zeta-1)^2} - \frac{2\omega_1}{(\zeta-1)^2} + \frac{2\zeta z}{(\zeta-1)^2} - \frac{2z}{(\zeta-1)^2}, $$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}^{(1,3)}_{3,\zeta}(\omega_1,\omega_2,z)=\frac{3\zeta^2}{(\zeta-1)^3}+\frac{3\zeta}{(\zeta-1)^3}+\frac{3\omega_1^2}{\zeta-1}-\frac{6\zeta\omega_1}{(\zeta-1)^2}+\frac{6\omega_1 z}{\zeta-1}+\frac{3z^2}{\zeta-1}
+\frac{3z}{\zeta-1}-\frac{6\zeta z}{(\zeta-1)^2},
$$

$$
\begin{aligned}
{}_{\mathcal{H}\mathcal{B}el}\mathfrak{B}^{(1,3)}_{4,\zeta}(\omega_1,\omega_2,z)=&-\frac{4\zeta^3}{(\zeta-1)^4}-\frac{16\zeta^2}{(\zeta-1)^4}-\frac{4\zeta}{(\zeta-1)^4}+\frac{4\omega_1^3}{\zeta-1}-\frac{12\zeta\omega_1^2}{(\zeta-1)^2}+\frac{12\omega_1^2 z}{\zeta-1}\\
&+\frac{12\zeta^2\omega_1}{(\zeta-1)^3}+\frac{12\zeta\omega_1}{(\zeta-1)^3}+\frac{12\omega_1 z^2}{\zeta-1}+\frac{12\omega_1 z}{\zeta-1}-\frac{24\zeta\omega_1 z}{(\zeta-1)^2}+\frac{24\omega_2}{\zeta-1}+\frac{4z^3}{\zeta-1}\\
&+\frac{12z^2}{\zeta-1}-\frac{12\zeta z^2}{(\zeta-1)^2}+\frac{12\zeta^2 z}{(\zeta-1)^3}+\frac{12\zeta z}{(\zeta-1)^3}+\frac{4z}{\zeta-1}-\frac{12\zeta z}{(\zeta-1)^2}.
\end{aligned}
$$

Example 2. *Gould-Hopper-Bell-Apostol-Euler polynomials*

Since

$$
\mathsf{P}^{(\sigma)}_{\tau,\zeta}(\omega;0,-1,1)=\mathcal{E}^{(\sigma)}_{\tau}(\omega;\zeta),
$$

therefore, taking $\delta=0, a=-1$ *and* $b=1$ *in generating function (2.3), gives*

$$
\left(\frac{2}{\zeta e^{\mu}+1}\right)^{\sigma} e^{\omega_1\mu+\omega_2\mu^r+z(e^{\mu}-1)}=\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\,\frac{\mu^{\tau}}{\tau!}, \tag{5.6}
$$

where

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)
$$

are referred to as the Gould-Hopper-Bell-Apostol-Euler polynomials (GHBelEP) of order σ.

The series representations of the GHBelEP

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)
$$

of order σ *are given as:*

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{B}el}\mathcal{E}^{(\sigma)}_{\tau-\kappa,\zeta}(z)\;\mathcal{H}^{(r)}_{\kappa}(\omega_1,\omega_2); \tag{5.7}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2)\;\mathcal{B}el_{\kappa}(z); \tag{5.8}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}^{(\sigma)}_{\tau-\kappa,\zeta}(0)\;{}_{\mathcal{H}}\mathcal{B}el^{(r)}_{\kappa}(\omega_1,\omega_2,z); \tag{5.9}
$$

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\tau!\sum_{\kappa=0}^{[\frac{\tau}{r}]}\frac{\omega_2^{\kappa}\;{}_{\mathcal{B}el}\mathcal{E}^{(\sigma)}_{\tau-r\kappa,\zeta}(\omega_1,z)}{\kappa!(\tau-r\kappa)!}. \tag{5.10}
$$

Certain corresponding results related to the GHBelEP ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$ *of order* σ *are presented in Table 2.*

Table 2. Findings for the Gould-Hopper-Bell-Apostol-Euler polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$.

Multiplicative and derivative operators	$\hat{M}_{\mathcal{H}\mathcal{B}el\mathcal{E}} = \omega_1 + r\omega_2 D^{r-1}_{\omega_1} + ze^{D_{\omega_1}} - \frac{\sigma\zeta e^{D_{\omega_1}}}{\zeta e^{D_{\omega_1}}+1},\ \hat{P}_{\mathcal{H}\mathcal{B}el\mathcal{E}} = D_{\omega_1}$
Differential equation	$\left(\omega_1 D_{\omega_1} + r\omega_2 D^{r}_{\omega_1} + ze^{D_{\omega_1}}D_{\omega_1} - \frac{\sigma D_{\omega_1}\zeta e^{D_{\omega_1}}}{(\zeta e^{D_{\omega_1}}+1)} - \tau\right){}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z) = 0$
Summation Formulae	${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z) = \frac{1}{2}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}_\kappa\left({}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1+1,\omega_2,z) + {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\right)$ ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\upsilon,\omega_2,z) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\kappa,\zeta}(\omega_1-\theta,\omega_2,z)$ ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma+\beta,r)}_{\tau,\zeta}(\omega_1+x,\omega_2+y,z+u) = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\beta,r)}_{\kappa,\zeta}(x,y,u)$ ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma-\varrho,r)}_{\tau,\zeta}(\omega_1,\omega_2,z) = \frac{\varrho!}{2^\varrho}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\ S(\kappa,\varrho,\zeta)$ ${}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,r)}_{\tau,\zeta}(\omega_1+1,\omega_2,z) = \frac{1}{\zeta}\left\{2\ {}_{\mathcal{H}}\mathcal{B}el^{(r)}_{\tau}(\omega_1,\omega_2,z) - {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,r)}_{\tau+\delta,\zeta}(\omega_1,\omega_2,z)\right\}$
Differential and	$\frac{\partial^\nu}{\partial\omega_1^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \begin{cases}\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\nu,\zeta}(\omega_1,\omega_2,z)}{(\tau-\nu)!}, & \tau\geq\nu;\\ 0, & 0\leq\tau<\nu.\end{cases}$ $\frac{\partial^\nu}{\partial\omega_2^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-r\nu,\zeta}(\omega_1,\omega_2,z)}{(\tau-r\nu)!}$ $\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z) - {}_{\mathcal{H}}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2)\ \mathcal{B}el_\kappa(z)\right\}$ $\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1+1,\omega_2,z) - {}_{\mathcal{G}}\mathcal{B}el_\tau(\omega_1,\omega_2,z)$
Integral Formulae	$\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\,\mathrm{d}\omega_1 = \frac{1}{\tau+1}\left[{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau+1,\zeta}(u+\gamma,\omega_2,z) - {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau+1,\zeta}(u,\omega_2,z)\right]$ $\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\mathrm{d}z$ $= \frac{1}{\tau+1}\sum_{k=0}^{\tau}\binom{\tau+1}{k}\mathfrak{B}_k\left[{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau+1-k,\zeta}(\omega_1,\omega_2,u+\gamma;\delta,a,b) - {}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau+1-k,\zeta}(\omega_1,\omega_2,u;\delta,a,b)\right]$

The first few members of the GHBelEP

$$
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z),
$$

for $\sigma = 2$ and $r = 3$ are given as:

$$
\begin{aligned}
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,3)}_{0,\zeta}(\omega_1,\omega_2,z) &= \frac{4}{(\zeta+1)^2},\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,3)}_{1,\zeta}(\omega_1,\omega_2,z) &= -\frac{8\zeta}{(\zeta+1)^3} + \frac{4\zeta\omega_1}{(\zeta+1)^3} + \frac{4\omega_1}{(\zeta+1)^3} + \frac{4z}{(\zeta+1)^3} + \frac{4\zeta z}{(\zeta+1)^3},\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,3)}_{2,\zeta}(\omega_1,\omega_2,z) &= \frac{16\zeta^2}{(\zeta+1)^4} - \frac{8\zeta}{(\zeta+1)^4} + \frac{4\omega_1^2}{(\zeta+1)^2} - \frac{16\zeta\omega_1}{(\zeta+1)^3} + \frac{8\omega_1 z}{(\zeta+1)^2} + \frac{4z^2}{(\zeta+1)^2}\\
&\quad + \frac{4z}{(\zeta+1)^2} - \frac{16\zeta z}{(\zeta+1)^3},\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,3)}_{3,\zeta}(\omega_1,\omega_2,z) &= -\frac{32\zeta^3}{(\zeta+1)^5} + \frac{56\zeta^2}{(\zeta+1)^5} - \frac{8\zeta}{(\zeta+1)^5} + \frac{4\omega_1^3}{(\zeta+1)^2} - \frac{24\zeta\omega_1^2}{(\zeta+1)^3} + \frac{12\omega_1^2 z}{(\zeta+1)^2}\\
&\quad + \frac{48\zeta^2\omega_1}{(\zeta+1)^4} - \frac{24\zeta\omega_1}{(\zeta+1)^4} + \frac{12\omega_1 z^2}{(\zeta+1)^2} + \frac{12\omega_1 z}{(\zeta+1)^2} - \frac{48\zeta\omega_1 z}{(\zeta+1)^3} + \frac{24\omega_2}{(\zeta+1)^2} + \frac{4z^3}{(\zeta+1)^2}
\end{aligned}
$$

$$+\frac{12z^2}{(\zeta+1)^2}-\frac{24\zeta z^2}{(\zeta+1)^3}+\frac{48\zeta^2 z}{(\zeta+1)^4}+\frac{4z}{(\zeta+1)^2}-\frac{24\zeta z}{(\zeta+1)^3}-\frac{24\zeta z}{(\zeta+1)^4},$$

$$\begin{aligned}{}_{\mathcal{H}\mathcal{B}el}\mathcal{E}^{(1,3)}_{4,\zeta}(\omega_1,\omega_2,z) &= \frac{64\zeta^4}{(\zeta+1)^6}-\frac{264\zeta^3}{(\zeta+1)^6}+\frac{144\zeta^2}{(\zeta+1)^6}-\frac{8\zeta}{(\zeta+1)^6}+\frac{4\omega_1^4}{(\zeta+1)^2}-\frac{32\zeta\omega_1^3}{(\zeta+1)^3}\\
&+\frac{16\omega_1^3 z}{(\zeta+1)^2}+\frac{96\zeta^2\omega_1^2}{(\zeta+1)^4}-\frac{48\zeta\omega_1^2}{(\zeta+1)^4}+\frac{24\omega_1^2 z^2}{(\zeta+1)^2}+\frac{24\omega_1^2 z}{(\zeta+1)^2}-\frac{96\zeta\omega_1^2 z}{(\zeta+1)^3}-\frac{128\zeta^3\omega_1}{(\zeta+1)^5}\\
&+\frac{224\zeta^2\omega_1}{(\zeta+1)^5}-\frac{32\zeta\omega_1}{(\zeta+1)^5}+\frac{96\omega_1\omega_2}{(\zeta+1)^2}+\frac{16\omega_1 z^3}{(\zeta+1)^2}+\frac{48\omega_1 z^2}{(\zeta+1)^2}-\frac{96\zeta\omega_1 z^2}{(\zeta+1)^3}+\frac{192\zeta^2\omega_1 z}{(\zeta+1)^4}\\
&+\frac{16\omega_1 z}{(\zeta+1)^2}-\frac{96\zeta\omega_1 z}{(\zeta+1)^3}-\frac{96\zeta\omega_1 z}{(\zeta+1)^4}-\frac{192\zeta\omega_2}{(\zeta+1)^3}+\frac{96\omega_2 z}{(\zeta+1)^2}+\frac{4z^4}{(\zeta+1)^2}+\frac{24z^3}{(\zeta+1)^2}\\
&-\frac{32\zeta z^3}{(\zeta+1)^3}+\frac{96\zeta^2 z^2}{(\zeta+1)^4}+\frac{28z^2}{(\zeta+1)^2}-\frac{96\zeta z^2}{(\zeta+1)^3}-\frac{48\zeta z^2}{(\zeta+1)^4}-\frac{128\zeta^3 z}{(\zeta+1)^5}+\frac{96\zeta^2 z}{(\zeta+1)^4}\\
&+\frac{224\zeta^2 z}{(\zeta+1)^5}+\frac{4z}{(\zeta+1)^2}-\frac{32\zeta z}{(\zeta+1)^3}-\frac{48\zeta z}{(\zeta+1)^4}-\frac{32\zeta z}{(\zeta+1)^5}.\end{aligned}$$

Example 3. *Gould-Hopper-Bell-Apostol-Genocchi polynomials*

Since

$$\mathsf{P}^{(\sigma)}_{\tau,\frac{\zeta}{2}}\Big(\omega;1,-\frac{1}{2},1\Big)=\mathcal{G}^{(\sigma)}_{\tau}(\omega;\zeta),$$

therefore, taking $\delta=1, a=-\frac{1}{2}, b=1$ and $\zeta\longrightarrow\frac{\zeta}{2}$ in generating function (2.3), gives

$$\left(\frac{2\mu}{\zeta e^{\mu}+1}\right)^{\sigma} e^{\omega_1\mu+\omega_2\mu^r+z(e^{\mu}-1)}=\sum_{\tau=0}^{\infty}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\,\frac{\mu^{\tau}}{\tau!},\tag{5.11}$$

where ${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$ are referred to as the Gould-Hopper-Bell-Apostol-Genocchi polynomials (GHBelGP) of order σ.

The series representations of the GHBelGP ${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$ of order σ are given as:

$${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{B}el}\mathcal{G}^{(\sigma)}_{\tau-\kappa,\zeta}(z)\;\mathcal{H}^{(r)}_{\kappa}(\omega_1,\omega_2);\tag{5.12}$$

$${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2)\;\mathcal{B}el_{\kappa}(z);\tag{5.13}$$

$${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{G}^{(\sigma)}_{\tau-\kappa,\zeta}(0)\;{}_{\mathcal{H}}\mathcal{B}el^{(r)}_{\kappa}(\omega_1,\omega_2,z);\tag{5.14}$$

$${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)=\tau!\sum_{\kappa=0}^{[\frac{\tau}{r}]}\frac{\omega_2^{\kappa}\;{}_{\mathcal{B}el}\mathcal{G}^{(\sigma)}_{\tau-r\kappa,\zeta}(\omega_1,z)}{\kappa!(\tau-r\kappa)!}.\tag{5.15}$$

Certain corresponding results related to the GHBelGP ${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$ of order σ are presented in Table 3.

Table 3. Findings for the Gould-Hopper-Bell-Apostol-Genocchi polynomials ${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$.

Multiplicative and derivative operators	$\hat{M}_{\mathcal{H}\mathcal{B}el\mathcal{G}} = \omega_1 + r\omega_2 D^{r-1}_{\omega_1} + ze^{D_{\omega_1}} + \frac{\sigma(\zeta e^{D_{\omega_1}}+1)-\sigma D_{\omega_1}\zeta e^{D_{\omega_1}}}{D_{\omega_1}(\zeta e^{D_{\omega_1}}+1)},\ \hat{P}_{\mathcal{H}\mathcal{B}el\mathcal{G}} = D_{\omega_1}$
Differential equation	$\left(\omega_1 D_{\omega_1} + r\omega_2 D^{r}_{\omega_1} + ze^{D_{\omega_1}}D_{\omega_1} + \frac{\sigma(\zeta e^{D_{\omega_1}}+1)-\sigma D_{\omega_1}\zeta e^{D_{\omega_1}}}{(\zeta e^{D_{\omega_1}}+1)} - \tau\right){}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z) = 0$
Summation Formulae	${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z) = \frac{1}{2}\sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\mathcal{E}_\kappa\left({}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1+1,\omega_2,z) + {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\right)$
	${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\upsilon,\omega_2,z) = \sum\limits_{\kappa=0}^{\tau}\binom{\tau}{\kappa}(\upsilon-\omega_1+\theta)^{\tau-\kappa}\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\kappa,\zeta}(\omega_1-\theta,\omega_2,z)$
	${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma+\beta,r)}_{\tau,\zeta}(\omega_1+x,\omega_2+y,z+u) = \sum\limits_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\beta,r)}_{\kappa,\zeta}(x,y,u)$
	${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma-\varrho,r)}_{\tau-\varrho,\zeta}(\omega_1,\omega_2,z) = \frac{(\tau-\varrho)!\ \varrho!}{\tau!}\sum\limits_{\kappa=0}^{\tau}\binom{\tau}{\kappa}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z)\ S(\kappa,\varrho,\zeta)$
	${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,r)}_{\tau+1,\zeta}(\omega_1+1,\omega_2,z) = \frac{2}{\zeta}\left\{(\tau+1){}_{\mathcal{H}}\mathcal{B}el^{(r)}_{\tau}(\omega_1,\omega_2,z) - \frac{1}{2}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,r)}_{\tau+1,\zeta}(\omega_1,\omega_2,z)\right\}$
Differential and	$\frac{\partial^\nu}{\partial\omega_1^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \begin{cases}\frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\nu,\zeta}(\omega_1,\omega_2,z)}{(\tau-\nu)!}, & \tau\geq\nu;\\ 0, & 0\leq\tau<\nu.\end{cases}$
	$\frac{\partial^\nu}{\partial\omega_2^\nu}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \frac{\tau!\ {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-r\nu,\zeta}(\omega_1,\omega_2,z)}{(\tau-r\nu)!}$
	$\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = \sum_{\kappa=0}^{\tau}\binom{\tau}{\kappa}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2,z) - {}_{\mathcal{H}}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1,\omega_2)\ \mathcal{B}el_\kappa(z)\right\}$
	$\frac{\partial}{\partial z}\left\{{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\right\} = {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau-\kappa,\zeta}(\omega_1+1,\omega_2,z) - {}_{\mathcal{G}}\mathcal{B}el_\tau(\omega_1,\omega_2,z)$
Integral Formulae	$\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)\,\mathrm{d}\omega_1 = \frac{1}{\tau+1}\left[{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau+1,\zeta}(u+\gamma,\omega_2,z) - {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau+1,\zeta}(u,\omega_2,z)\right]$
	$\int_u^{u+\gamma}{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z;\delta,a,b)\,\mathrm{d}z$ $= \frac{1}{\tau+1}\sum_{k=0}^{\tau}\binom{\tau+1}{k}\mathfrak{B}_k\left[{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau+1-k,\zeta}(\omega_1,\omega_2,u+\gamma;\delta,a,b) - {}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau+1-k,\zeta}(\omega_1,\omega_2,u;\delta,a,b)\right]$

The first few members of the GHBelGP ${}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(\sigma,r)}_{\tau,\zeta}(\omega_1,\omega_2,z)$, for $\sigma=1$ and $r=3$ are given as:

$$
\begin{aligned}
{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,3)}_{0,\zeta}(\omega_1,\omega_2,z) &= 0,\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,3)}_{1,\zeta}(\omega_1,\omega_2,z) &= \frac{2}{\zeta+1},\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,3)}_{2,\zeta}(\omega_1,\omega_2,z) &= -\frac{4\zeta}{(\zeta+1)^2} + \frac{4\zeta\omega_1}{(\zeta+1)^2} + \frac{4\omega_1}{(\zeta+1)^2} + \frac{4z}{(\zeta+1)^2} + \frac{4\zeta z}{(\zeta+1)^2},\\
{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,3)}_{3,\zeta}(\omega_1,\omega_2,z) &= \frac{6\zeta^2}{(\zeta+1)^3} - \frac{6\zeta}{(\zeta+1)^3} + \frac{6\omega_1^2}{\zeta+1} - \frac{12\zeta\omega_1}{(\zeta+1)^2} + \frac{12\omega_1 z}{\zeta+1} + \frac{6z^2}{\zeta+1} + \frac{6z}{\zeta+1}\\
&\quad - \frac{12\zeta z}{(\zeta+1)^2},
\end{aligned}
$$

$$
\begin{aligned}
{}_{\mathcal{H}\mathcal{B}el}\mathcal{G}^{(1,3)}_{4,\zeta}(\omega_1,\omega_2,z) = &-\frac{8\zeta^3}{(\zeta+1)^4} + \frac{32\zeta^2}{(\zeta+1)^4} - \frac{8\zeta}{(\zeta+1)^4} + \frac{8\omega_1^3}{\zeta+1} - \frac{24\zeta\omega_1^2}{(\zeta+1)^2} + \frac{24\omega_1^2 z}{\zeta+1}\\
&+ \frac{24\zeta^2\omega_1}{(\zeta+1)^3} - \frac{24\zeta\omega_1}{(\zeta+1)^3} + \frac{24\omega_1 z^2}{\zeta+1} + \frac{24\omega_1 z}{\zeta+1} - \frac{48\zeta\omega_1 z}{(\zeta+1)^2} + \frac{48\omega_2}{\zeta+1} + \frac{8z^3}{\zeta+1} + \frac{24z^2}{\zeta+1}\\
&- \frac{24\zeta z^2}{(\zeta+1)^2} + \frac{24\zeta^2 z}{(\zeta+1)^3} + \frac{8z}{\zeta+1} - \frac{24\zeta z}{(\zeta+1)^2} - \frac{24\zeta z}{(\zeta+1)^3}.
\end{aligned}
$$

The established family in this study, the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ, can be considered as a generalization of the Gould-Hopper, Bell, unified Apostol-type, Gould-Hopper-Bell, Apostol Bernoulli, Apostol Euler, Apostol Genocchi, Gould-Hopper-Apostol-type, Apostol-type-Hermite, Bell-Apostol-type, Gould-Hopper-Bernoulli, Gould-Hopper-Euler, Gould-Hopper-Genocchi, Bell-Bernoulli, Bell-Euler, and Bell-Genocchi polynomials [39, 44, 45].

5.2. *Zeros and graphical representations*

In this subsection, we explore the distributions of zeros and present graphical illustrations of the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ of order σ for specific parameter values and indices.

In view of (2.3), we list the first six terms of the TGHBelATP

$${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$$

for $\delta=2,\sigma=1$, and $r=3$ as:

$$\begin{aligned}
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{0,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=0,\\
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{1,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=0,\\
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{2,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=\frac{1}{\zeta^b-a^b},\\
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{3,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=-\frac{3\zeta^b}{(a^b-\zeta^b)^2}-\frac{3\omega_1a^b}{(a^b-\zeta^b)^2}+\frac{3\omega_1\zeta^b}{(a^b-\zeta^b)^2}-\frac{3za^b}{(a^b-\zeta^b)^2}+\frac{3z\zeta^b}{(a^b-\zeta^b)^2},\\
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{4,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=\frac{6a^b\zeta^b}{(\zeta^b-a^b)^3}+\frac{6\zeta^{2b}}{(\zeta^b-a^b)^3}+\frac{6\omega_1^2}{\zeta^b-a^b}-\frac{12\omega_1\zeta^b}{(\zeta^b-a^b)^2}+\frac{12\omega_1z}{\zeta^b-a^b}\\
&\qquad+\frac{6z^2}{\zeta^b-a^b}-\frac{12z\zeta^b}{(\zeta^b-a^b)^2}+\frac{6z}{\zeta^b-a^b},\\
&{}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{5,\zeta}^{(1,3)}(\omega_1,\omega_2,z;2,a,b)=-\frac{40a^b\zeta^{2b}}{(a^b-\zeta^b)^4}-\frac{10\zeta^{3b}}{(a^b-\zeta^b)^4}+\frac{10\omega_1^3}{\zeta^b-a^b}-\frac{30\omega_1^2\zeta^b}{(\zeta^b-a^b)^2}+\frac{30\omega_1^2z}{\zeta^b-a^b}\\
&\qquad-\frac{30\omega_1a^b\zeta^b}{(a^b-\zeta^b)^3}-\frac{30\omega_1\zeta^{2b}}{(a^b-\zeta^b)^3}+\frac{30\omega_1z^2}{\zeta^b-a^b}+\frac{30\omega_1z}{\zeta^b-a^b}-\frac{60\omega_1z\zeta^b}{(\zeta^b-a^b)^2}\\
&\qquad+\frac{60\omega_2}{\zeta^b-a^b}+\frac{10z^3}{\zeta^b-a^b}+\frac{30z^2}{\zeta^b-a^b}-\frac{30z^2\zeta^b}{(\zeta^b-a^b)^2}-\frac{30za^b\zeta^b}{(a^b-\zeta^b)^3}\\
&\qquad+\frac{10z}{\zeta^b-a^b}-\frac{30z\zeta^b}{(\zeta^b-a^b)^2}-\frac{30z\zeta^{2b}}{(a^b-\zeta^b)^3}-\frac{10a^{2b}\zeta^b}{(a^b-\zeta^b)^4}.
\end{aligned}$$

To show the shapes of the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$ for $\tau=2;3;4;5;6;7,-100\leq\omega_1\leq100,\omega_2=\frac{1}{2},z=\frac{1}{3},\zeta=7,a=5,b=2,\sigma=1,r=3$, and $\delta=2$, Figure 1 is given.

Certain interesting zeros of the TGHBelATP ${}_{\mathcal{H}\mathcal{B}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)=0$, for $\sigma=1,r=3,\delta=2$, and $\tau=60$ are shown in Figure 2.

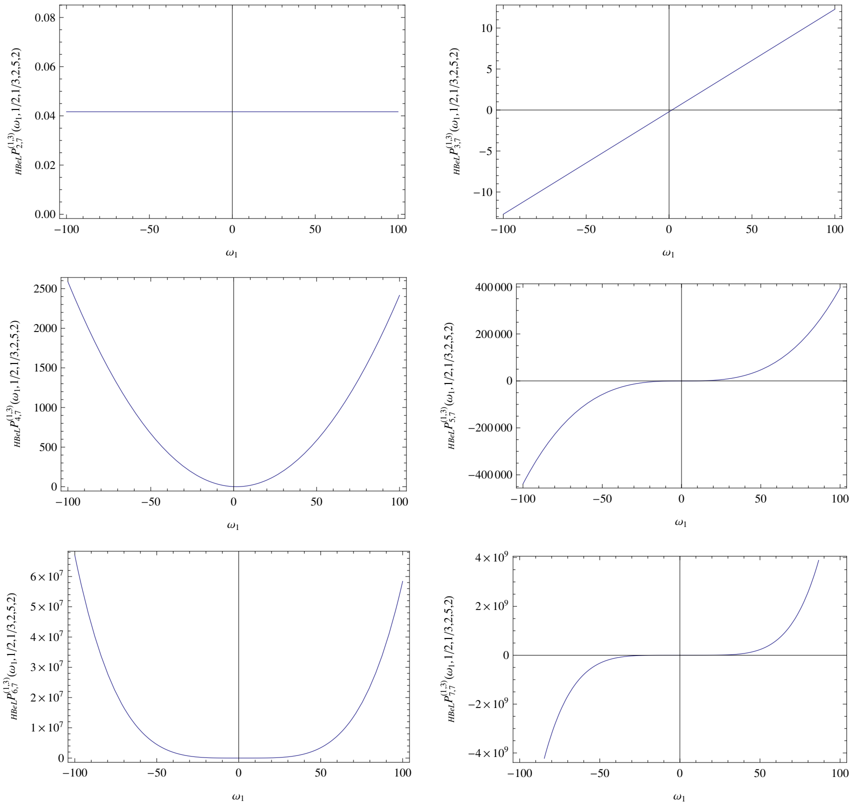
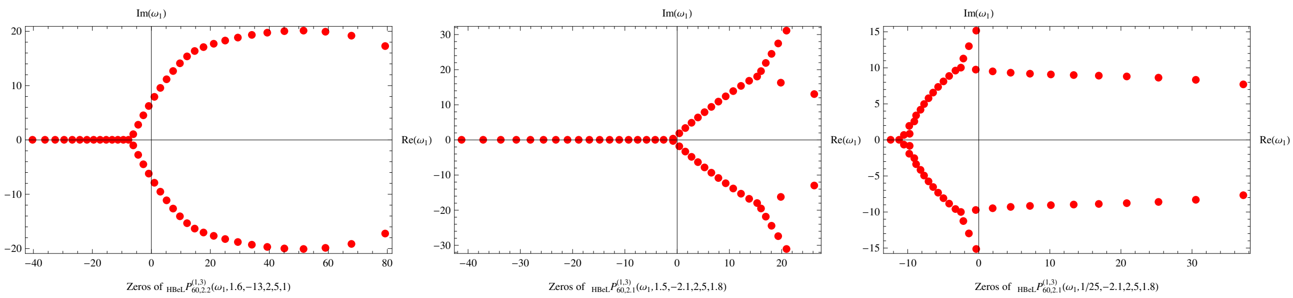


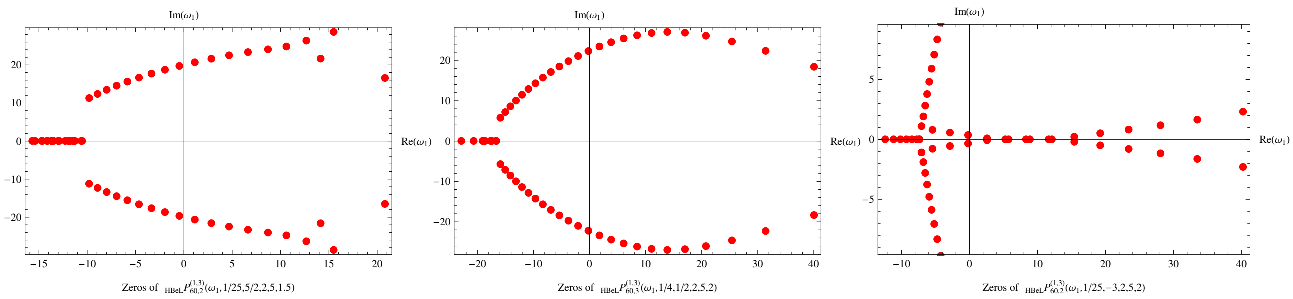
Figure 1. Curves of ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$.



(a) $\omega_2 = 1.6, z = -13, \zeta = 2.2, a = 5, b = 1$

(b) $\omega_2 = 1.5, z = -2.1, \zeta = 2.1, a = 5, b = 1.8$

(c) $\omega_2 = 1/25, z = -2.1, \zeta = 2.1, a = 5, b = 1.8$



(d) $\omega_2 = 1/25, z = 5/2, \zeta = 2, a = 5, b = 1.5$

(e) $\omega_2 = (1/4), z = 1/2, \zeta = 3, a = 5, b = 2$

(f) $\omega_2 = 1/25, z = -3, \zeta = 2, a = 5, b = 2$

Figure 2. Zeros of ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b) = 0$.

Remark 16. *We observed that zeros the TGHBelATP* ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)$*, for* $\sigma=1, r=3, \delta=2$*, and* $\tau=60$ *have the following properties:*

(1) When τ *is assigned a non-negative value* $m \geq 2$*, the TGHBelATP* ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z,a,b)$ *possesses* $m-2$ *zeros.*

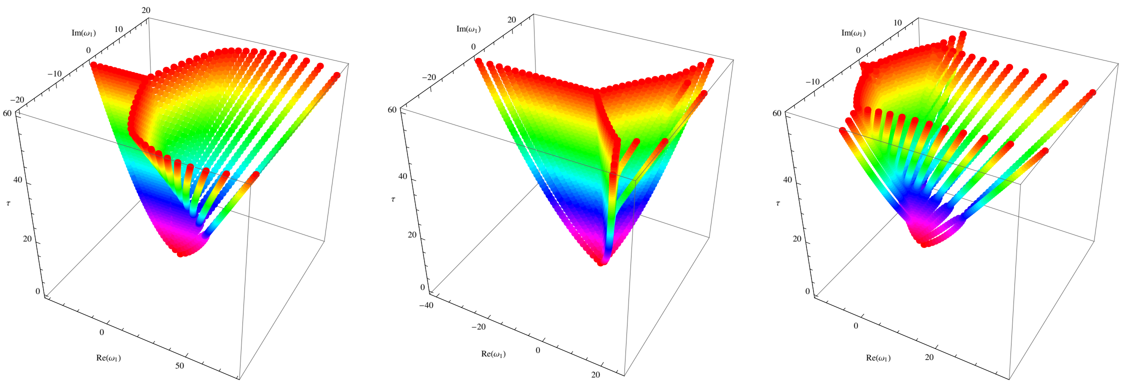
(2) Altering the variables, parameters, or indices generates distinct zero distributions and varied graphical configurations.

(3) The zeros (complex zeros) of ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z,a,b)=0$ *exhibit symmetry about the real axis.*

The stacking structures of approximation zeros of the TGHBelATP

$${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)=0,$$

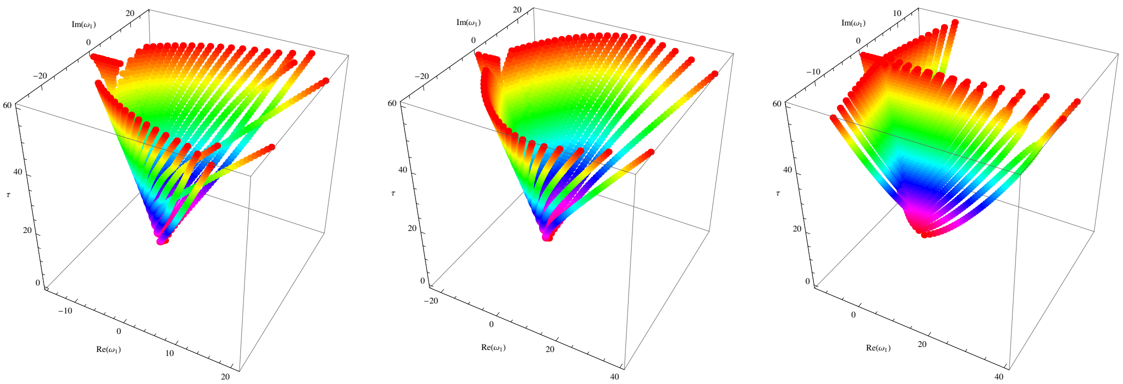
for $\sigma=1, r=3, \delta=2$, and $3\leq\tau\leq 60$, give 3D structures, which are presented in Figure 3.



(a) $\omega_2 = 1.6, z = -13, \zeta = 2.2, a = 5, b = 1$

(b) $\omega_2 = 1.5, z = -2.1, \zeta = 2.1, a = 5, b = 1.8$

(c) $\omega_2 = 1/25, z = -2.1, \zeta = 2.1, a = 5, b = 1.8$



(d) $\omega_2 = 1/25, z = 5/2, \zeta = 2, a = 5, b = 1.5$

(e) $\omega_2 = 1/4, z = 1/2, \zeta = 3, a = 5, b = 2$

(f) $\omega_2 = 1/25, z = -3, \zeta = 2, a = 5, b = 2$

Figure 3. Stacking structure zeros of ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)=0$. This figure shows the 3D plot of the zeros of the TGHBelATP ${}_{\mathcal{HB}el}\mathsf{P}_{\tau,\zeta}^{(\sigma,r)}(\omega_1,\omega_2,z;\delta,a,b)=0$, for $\sigma=1, r=3, \delta=2$ and $3\leq\tau\leq 60$.

6. Conclusions

The hybrid form of special polynomials has attracted significant attention from numerous researchers. In this work, we presented and explored a novel hybrid class of special polynomials, referred to as the trivariate Gould-Hopper-Bell-Apostol-type polynomials. By employing the monomiality principle, we constructed the associated generating function, series representations, quasi-monomial operators, and differential equation. Additionally, summation formulae, differential representations, and integral representations were derived, providing a comprehensive framework for the study of these polynomials.

Special examples of this unified family-such as the Gould-Hopper-Bell-Apostol-Bernoulli, Gould-Hopper-Bell-Apostol-Euler, and Gould-Hopper-Bell-Apostol-Genocchi polynomials were examined, revealing analogous results for each. These results underscore the adaptability and relevance of the unified family across diverse mathematical contexts. Furthermore, computational investigations using Mathematica were conducted to explore the zero distributions and graphical representations of the trivariate Gould-Hopper-Bell-Apostol-type polynomials. The visual and numerical analyses offer a more profound understanding of the behavior and characteristics of these polynomials.

In summary, this work not only establishes a new class of polynomials, but also lays the groundwork for further research into their theoretical and practical applications. The results presented here contribute to the broader field of special functions and polynomial theory, offering a unified approach to studying diverse polynomial families. Future research could explore the degenerate forms of the established special polynomials in this study, along with their associated applications.

Author contributions

Rabeb Sidaoui: Conceptualization, Methodology, Investigation; Abdulghani Muhyi: Conceptualization, Formal analysis, Writing-oiginal draft preparation; Khaled Aldwoah: Writing-review & editing, Supervision, Project administration; Ayman Alahmade: Methodology, Investigation, Data curation; Mohammed Rabih: Validation, Visualization; Amer Alsulami: Software, Data curation; Khidir Mohamed: Validation, Resources, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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