AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 13082–13103.
DOI: 10.3934/math.2025587
Received: 23 January 2025
Revised: 19 May 2025
Accepted: 22 May 2025
Published: 06 June 2025

Research article

Bounds for the stop-loss distance of an independent random sum via Stein's method

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Abstract: Let $W = X_1 + X_2 + \cdots + X_N$ be a random sum and Z be the standard normal random variable. In this paper, we investigated uniform and non-uniform bounds of the stop-loss distance, which measures the difference between two random variables, W and Z, using the expression $|Eh_k(W) - Eh_k(Z)|$, where $h_k(x) = (x - k)^+$ is a call function. In particular, we focused on the case that $X_1, X_2, \ldots$ are independent random variables, and N is a non-negative, integer-valued random variable independent of the X_j's. Our methods were Stein's method and the concentration inequality approach. The value $Eh_k(W) = E(W - k)^+$ represents the excess over a threshold and is relevant to applications in collateralized debt obligations (CDOs) and the collective risk model.

Keywords: random sums; Stein's method; stop-loss distance; concentration inequality; normal approximation; non-uniform bounds
Mathematics Subject Classification: 60F05

1. Introduction

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables and let N be a non-negative, integer-valued random variable independent of the X_j's. The summation, $X_1 + X_2 + \cdots + X_N$, is called a random sum. Random sums appear frequently in modern probability theory and are widely applied across various fields, including finance for risk assessment and portfolio optimization, telecommunications for modeling call arrivals, operations research for inventory management, and particularly in insurance contexts (see Chapter 17 in [1–3] for additional examples).

The approximation of random sums $X_1 + X_2 + \cdots + X_N$ by the standard normal random variable Z was begun by Robbins [4] and Gnedenko & Korolev [5]. A widely used probability metric to compare such approximations is the Wasserstein distance. The Wasserstein distance between random variables W and Z, represented as $d_W(W, Z)$, is defined by

$$d_W(W, Z) = \sup_{h \in \mathcal{H}} |Eh(W) - Eh(Z)|,$$

where $\mathcal{H}$ is the class of all Lipschitz continuous functions h on $\mathbb{R}$ with a Lipschitz constant not greater than 1. Many authors have studied approximations of the random sums using this metric under various assumptions about the random variable N. In the absence of specific distributional assumptions about N, the results for Gaussian approximation are provided in works such as [6–8]. When N is treated as a known random variable, common distributional assumptions include Poisson (see [6, 9]), binomial [10, 11], negative binomial [9, 12], or mixed Poisson [9].

For $k > 0$, let $h_k(x) = (x - k)^+$ be a call function, where $x^+ = \max\{0, x\}$ and let W be any random variable. For an insurance context, if we consider W as the claim amount claimed by an insured and W as a retention, then $(W - k)^+$ represents excess money paid by a reinsurer (see [13] for more details). The main concern for a reinsurer is determining the magnitude of excess losses, i.e., $E(W - k)^+$. In this paper, we focus on a specialized metric known as the stop-loss distance. This distance is useful for modeling the excess over a set threshold, which is important for assessing potential losses. For any random variables W and Z, the stop-loss distance is defined for some $k \in \mathbb{R}^+$ as

$$d_{\mathrm{SL}}^{(k)}(W, Z) = |E(W - k)^+ - E(Z - k)^+| \tag{1.1}$$

and the uniform version as

$$d_{\mathrm{SL}}(W, Z) = \sup_{k \in \mathbb{R}^+} |E(W - k)^+ - E(Z - k)^+|. \tag{1.2}$$

The stop-loss distances can be classified into two distinct categories. For a fixed value of k in Eq (1.1), the resulting error bound typically depends on k, which is referred to as a non-uniform bound. In contrast, the error bound corresponding to Eq (1.2), which applies uniformly across all values of $k \in \mathbb{R}^+$, is called a uniform bound. In situations where the stop-loss threshold k is sufficiently large, non-uniform bounds provide a more precise approximation than uniform bounds.

We focus on the stop-loss distance because it is useful for estimating potential losses and can be applied in many different areas. In risk management and insurance, for example, it models excess losses in the collective risk model and default risks in financial products, such as collateralized debt obligations (CDOs). Its relevance also extends to finance and investment, where understanding the distribution of excess losses beyond a certain point is crucial for effective risk management. For instance, in an insurance context, if we consider $W := X_1 + X_2 + \cdots + X_N$ as the total claim amount with a retention k, where X_j denotes the claim amount from the j^{th} contract and N represents the number of claims, then $(W - k)^+$ represents excess money paid by a reinsurer. A key application of this setup is in modeling total claims from a portfolio of insurance contracts. There are two approaches to modeling W: the individual risk model and the collective risk model, which differ based on whether N is fixed or random. In the individual risk model, the total claims are represented as $X_1 + X_2 + \cdots + X_n$, where n is the fixed number of contracts in the portfolio. However, in many real-world insurance products, such as accident insurance, the number of claims is not fixed and can vary unpredictably. This uncertainty leads to the collective risk model, where N is treated as a random variable representing the number of claims. In practice, obtaining the precise value of $E(W-k)^+$ is extremely difficult. To address this, we estimate $E(W-k)^+$ using stop-loss distances with suitable limiting distributions Z. Beyond applications in the insurance context, this concept is also applied to risk management for CDOs, as discussed further in Section 4.

Throughout this paper, we let $X_1, X_2, \ldots$ be independent random variables with a zero mean and finite absolute third moment, and let N be a non-negative, integer-valued random variable with finite variance that is independent of the X_j's.

We will consistently use the following notation for any random variable X_i: $\sigma_i^2 = \text{Var}(X_i) = EX_i^2$ and $\gamma_i = E|X_i|^3$. For a positive integer n, let

$$s_n^2 = \sum_{i=1}^{n} \sigma_i^2 \quad \text{and} \quad \delta_n = \frac{1}{s_n^2 \sqrt{Es_N^2}} \sum_{i=1}^{n} \gamma_i.$$

For a non-negative integer-valued random variable N, we similarly define the random variables

$$s_N^2 = \sum_{i=1}^{N} \sigma_i^2 \quad \text{and} \quad \delta_N = \frac{1}{s_N^2 \sqrt{Es_N^2}} \sum_{i=1}^{N} \gamma_i.$$

Additionally, we denote

$$Y_i = \frac{X_i}{\sqrt{Es_N^2}}, \quad \text{and} \quad W_N = \sum_{i=1}^{N} Y_i.$$

From these, we observe that $EW_N = 0$ and $\text{Var}(W_N) = 1$, since $\text{Var}\left(\sum_{i=1}^{N} X_i\right) = Es_N^2$.

As previously discussed, numerous authors have explored approximations for the random sum $X_1 + X_2 + \cdots + X_N$ with a particular emphasis on Gaussian approximations. In this paper, we aim to find both uniform and non-uniform bounds for the stop-loss distance under the assumption mentioned earlier. A recent result on the Wasserstein distance was presented by Döbler [7] in 2015, which is stated as follows:

Theorem 1.1. *[7] Let $X_1, X_2, \ldots$ be identically distributed. Then*

$$d_W(W_N, Z) \le \frac{2\sqrt{Var(N)}}{EN} + \frac{3\gamma_1}{\sigma_1^3 \sqrt{EN}}.$$

We observe that $d_{\text{SL}}(W_N, Z) \le d_W(W_N, Z)$, since any call function is a Lipschitz continuous function. Using this inequality, Theorem 1.1 directly provides a uniform bound for $d_{\text{SL}}(W_N, Z)$ in the case where the X_j's are independent and identically distributed (i.i.d.). However, we cannot apply this to obtain a non-uniform bound. In this paper, we provide uniform and non-uniform bounds for the stop-loss distance without requiring the X_j's to be identically distributed. A key tool in our approach, as well as in Döbler's work [7], is Stein's method. Additionally, Stein's method has been applied in non-i.i.d. settings in works such as [14, 15]. Using Stein's method along with the concentration inequality approach outlined in Section 3, we derive the main results presented in the following theorems.

Theorem 1.2. *(Uniform bound)*

$$d_{SL}(W_N, Z) \le \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{Var(s_N^2)}}{Es_N^2}\right) + \frac{7.24E[\delta_N s_N]}{\sqrt{Es_N^2}} + \frac{3E[\delta_N s_N^2]}{Es_N^2}.$$

Theorem 1.3. *(Non-uniform bound) For $k \ge 3$, we have*

$$d_{SL}^{(k)}(W_N, Z) \le \frac{1}{1+k}\left[\frac{1.38\sqrt{Var(s_N^2)}}{Es_N^2} + \frac{25.047E[\delta_N s_N]}{\sqrt{Es_N^2}} + \frac{1.725E[\delta_N s_N^2]}{Es_N^2} + \frac{1.68E[\delta_N s_N^4]}{(Es_N^2)^2}\right].$$

To gain insight into the behavior of the bounds, which we expect to converge to zero, we often examine a specific case where the X_i's are assumed to be identically distributed. Under this assumption, we can directly derive the following corollary.

Corollary 1.1. *Let $X_1, X_2, \ldots$ be identically distributed. Then*

(i)

$$d_{SL}(W_N, Z) \le \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{Var(N)}}{EN}\right) + \frac{10.24\gamma_1}{\sigma_1^3\sqrt{EN}}, \quad \text{and}$$

(ii) for $k \ge 3$, we have

$$d_{SL}^{(k)}(W_N, Z) \le \frac{1}{1+k}\left[\frac{1.38\sqrt{Var(N)}}{EN} + \frac{26.772\gamma_1}{\sigma_1^3\sqrt{EN}} + \frac{1.68\gamma_1 EN^2}{\sigma_1^3(EN)^{\frac{5}{2}}}\right].$$

To ensure the convergence of each bound, it is essential to verify that all terms involving the random variable N such as $\frac{\sqrt{\text{Var}(N)}}{EN}$, $\frac{1}{\sqrt{EN}}$, and $\frac{EN^2}{(EN)^{\frac{5}{2}}}$, approach zero. These terms represent different aspects of the variability and scale of N, and their convergence is crucial for controlling the bounds effectively. As stated in the above corollary, in the trivial case where $N = n$, it is evident that the bound converges to zero as $n \to \infty$. To explore other possible and more complex forms of N, we provide the following remark, which illustrates the behavior of the bound in such cases.

Remark 1.1. We present the result from Corollary 1.1 for the case where N follows specific, well-known distributions:

(i) For $N \sim Bin(n, p), n \in \mathbb{N}, p \in (0, 1)$, we have

$$d_{\text{SL}}(W_N, Z) \le \frac{1}{\sqrt{np}}\left[\sqrt{\frac{2(1-p)}{\pi}} + \frac{10.24\gamma_1}{\sigma_1^3}\right] \quad \text{and}$$

$$d_{\text{SL}}^{(k)}(W_N, Z) \le \frac{1}{(1+k)\sqrt{np}}\left[1.38\sqrt{1-p} + \frac{26.772\gamma_1}{\sigma_1^3} + \frac{1.68\gamma_1((1-p)+np)}{np\sigma_1^3}\right], \quad \text{for } k \ge 3.$$

In this case, it is straightforward to observe that the bounds converge to zero as $n \to \infty$.

(ii) For $N \sim Poi(\lambda)$, $\lambda > 0$,

$$d_{\text{SL}}(W_N, Z) \le \frac{1}{\sqrt{\lambda}}\left[\sqrt{\frac{2}{\pi}} + \frac{10.24\gamma_1}{\sigma_1^3}\right] \quad \text{and}$$
$$d_{\text{SL}}^{(k)}(W_N, Z) \le \frac{1}{(1+k)\sqrt{\lambda}}\left[1.38 + \frac{26.772\gamma_1}{\sigma_1^3} + \frac{1.68\gamma_1(\lambda+1)}{\lambda\sigma_1^3}\right], \quad \text{for } k \ge 3.$$

As $\lambda \to \infty$, the bounds still converge to zero.

(iii) For $N \sim NB(r, p)$, $r > 0, p \in (0, 1)$,

$$d_{\text{SL}}(W_N, Z) \le \frac{1}{\sqrt{r(1-p)}}\left[\sqrt{\frac{2}{\pi}} + \frac{10.24\gamma_1\sqrt{p}}{\sigma_1^3}\right] \quad \text{and}$$
$$d_{\text{SL}}^{(k)}(W_N, Z) \le \frac{1}{(1+k)\sqrt{r(1-p)}}\left[1.38 + \frac{26.772\gamma_1\sqrt{p}}{\sigma_1^3} + \frac{1.68\gamma_1\sqrt{p}(1+r(1-p))}{r(1-p)\sigma_1^3}\right], \quad \text{for } k \ge 3.$$

In the final case of the remark, it is clear that both bounds approach zero, depending on the size of r.

We observe that to apply the theorem most effectively, it is important to carefully select an appropriate form of N along with its parameters, as these choices significantly influence the efficiency and tightness of the resulting bounds.

In the following remark, we present specific examples and situations where Döbler's approach yields better bounds and, conversely, where our results show improvements over Döbler's.

Remark 1.2. Let $X_1, X_2, \ldots$ be independent and identically distributed random walks with probability 0.5, i.e., $P(X_j = \pm 1) = 0.5$, and let N be a non-negative, integer-valued random variable independent of the X_j's.

(i) For $N \sim Bin(n, 0.5)$, we apply Theorem 1.1 to obtain that $d_{\text{SL}}(W_N, Z) \le \frac{6.25}{\sqrt{n}}$ while Corollary 1.1 provides $d_{\text{SL}}(W_N, Z) \le \frac{15.28}{\sqrt{n}}$.

(ii) For $N \sim NB(r, 0.001)$, using Theorem 1.1 and Corollary 1.1, we have, for $r \ge 1$, $d_{\text{SL}}(W_N, Z) \le \frac{3.2}{\sqrt{r}}$ and $d_{\text{SL}}(W_N, Z) \le \frac{1.13}{\sqrt{r}}$, respectively.

In this work, we discuss Stein's method, which is an important tool for our study, in Section 2. Subsequently, the detailed proof of the main result is presented in Section 3. Finally, we demonstrate the applications of our results in specific areas, such as collective risk models and CDOs.

2. Stein's method for normal approximation

In this section, we introduce the foundational tools for our work, beginning with the ingenious approach developed by Charles Stein in 1972, commonly referred to as Stein's method [16]. Let Φ be the distribution function of the standard normal Z, and C_{bd} be the set of continuous and piecewise continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$. Stein's method begins with the Stein equation for normal approximation,

$$xf(x) - f'(x) = h(x) - Eh(Z) \tag{2.1}$$

for a given function h and $f \in C_{bd}$. The solution of Eq (2.1) is

$$f(x) = e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} [h(t) - Eh(Z)]\, \mathrm{d}t,$$

see [17, p. 15].

In this work, we apply the Stein equation (2.1) with $h(x) = (x - k)^+$ and $k > 0$. Then we have

$$xf(x) - f'(x) = (x - k)^+ - E(Z - k)^+. \tag{2.2}$$

The solution of Eq (2.2) is

$$f_k(x) = \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} E(Z - k)^+ \Phi(x), & \text{if } x \le k, \\ 1 - \sqrt{2\pi} e^{\frac{x^2}{2}} (k + E(Z - k)^+) \Phi(-x), & \text{if } x > k, \end{cases}$$

and the expression for the first derivative, denoted as f_k', is as follows:

$$f_k'(x) = \begin{cases} E(Z - k)^+ \Big(1 + \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x)\Big), & \text{if } x < k, \\ \Big(k + E(Z - k)^+\Big)\Big(1 - \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(-x)\Big), & \text{if } x > k. \end{cases} \tag{2.3}$$

It is important to note that f_k' is not defined at the point $x = k$ due to the discontinuity of f_k at this point. However, by using the solution at $x = k$ in conjunction with Stein's equation (2.2), we can refine the expression for $f_k'(k)$ by

$$f_k'(k) = E(Z - k)^+ \Big(1 + \sqrt{2\pi} k e^{\frac{k^2}{2}} \Phi(k)\Big).$$

From this fact and (2.3), we have

$$f_k'(x) = \begin{cases} E(Z - k)^+ \Big(1 + \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x)\Big), & \text{if } x \le k, \\ \Big(k + E(Z - k)^+\Big)\Big(1 - \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(-x)\Big), & \text{if } x > k. \end{cases}$$

A crucial component of Stein's method is identifying the properties of the solution f_k. From Lemma 2.4 of Chen et al. [18] and some observations shown by Jongpreechaharn and Neammanee [19, p. 210–211], we have that

$$0 \le f_k'(x) \le \sqrt{\frac{2}{\pi}}, \quad \text{for } x \in \mathbb{R}. \tag{2.4}$$

For the non-uniform bound for f_k', we use some results from [19] to derive the bounds for f_k' in the following proposition.

Proposition 2.1. *For $k \ge 3$ and $x \in \mathbb{R}$, $0 \le f_k'(x) \le \frac{1.38}{1+k}$.*

Proof. Using the fact that $0 \le f_k'(x) \le \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}k^2} + \frac{1}{k}$ for all $x \in \mathbb{R}$ and $k \ge 1$ [19, p. 210–211], and

$$\frac{1}{k} \le \frac{4}{3(1 + k)} \quad \text{and} \quad e^{\frac{k^2}{2}} \ge 1 + \frac{k^2}{2} \ge 1 + k, \quad \text{for } k \ge 3,$$

we have

$$0 \leq f_k'(x) \leq \frac{1}{\sqrt{2\pi}(1+k)k^2} + \frac{4}{3(1+k)} \leq \frac{1.38}{1+k}.$$

□

Additional, the second derivative, denoted as f_k'', is expressed as follows:

$$f_k''(x) = \begin{cases} E(Z-k)^+\Big[x + \Big(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)\Big)\Big(x^2+1\Big)\Big], & \text{if } x < k, \\ \Big(k + E(Z-k)^+\Big)\Big[x - \Big(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(-x)\Big)\Big(x^2+1\Big)\Big], & \text{if } x > k. \end{cases}$$

We notice that f_k'' does not exist at $x = k$. It is important to highlight that this prevents the direct application of the mean value theorem on any interval $[a, b]$ where $k \in (a, b)$, since the theorem requires differentiability over the entire open interval (a, b). The uniform bound for f_k'' was observed by [20, p. 3501], indicating that

$$|f_k''(x)| \leq 2, \quad \text{for } x \in \mathbb{R} \setminus \{k\}. \tag{2.5}$$

Beyond the uniform bound for f_k'', Jongpreechaharn and Neammanee [20, p. 3502] utilized the result

$$E(Z-k)^+ \leq \frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} \min\left\{1, \frac{1}{k^2}\right\} \tag{2.6}$$

in [21, p. 115] to show that

$$|f_k''(x)| \leq \frac{1.43}{e^{\frac{3k^2}{8}}} \quad \text{for } x \leq \frac{k}{2}. \tag{2.7}$$

We employ some ideas from the process of finding (2.7) to obtain the following results.

Proposition 2.2. *For* $k \geq 3$*,*

(i) $|f_k''(x)| \leq \dfrac{0.23}{1+k}$ *for* $x \leq k-1$*,*
(ii) $|f_k''(x)| \leq 1.12$ *for* $x \in \mathbb{R} \setminus \{k\}$*.*

Proof. (i) For $x < 0$, by (2.6) and the fact that $|f_k''(x)| \leq 2E(Z-k)^+$ [20, p. 3502], we obtain

$$|f_k''(x)| \leq \frac{2e^{-\frac{k^2}{2}}}{k^2\sqrt{2\pi}} \leq \frac{8}{9(1+k)e^{\frac{9}{2}}\sqrt{2\pi}} \leq \frac{0.004}{1+k},$$

where we use $\dfrac{1}{k^2} \leq \dfrac{4}{9(1+k)}$ in the second inequality. Otherwise, for $0 \leq x \leq k-1$, we first observe that

$$0 \leq f_k''(x) = E(Z-k)^+\left[x + \sqrt{2\pi}\Phi(x)e^{\frac{x^2}{2}}(x^2+1)\right]$$

$$\le E(Z-k)^+\left[(k-1)+\sqrt{2\pi}e^{\frac{(k-1)^2}{2}}((k-1)^2+1)\right]$$
$$= A + B,$$

where

$$A = (k-1)E(Z-k)^+ \quad \text{and} \quad B = E(Z-k)^+\left[\sqrt{2\pi}e^{\frac{(k-1)^2}{2}}((k-1)^2+1)\right].$$

Using (2.6) along with the fact that $\dfrac{k-1}{k^2} \le \dfrac{1}{1+k}$, we obtain that

$$(k-1)E(Z-k)^+ \le \frac{(k-1)e^{-\frac{k^2}{2}}}{k^2\sqrt{2\pi}} \le \frac{1}{(1+k)e^{\frac{9}{2}}\sqrt{2\pi}} \le \frac{0.0045}{1+k}.$$

To bound B, we divide into two cases. If $3 \le k \le 4$, we can use (2.6) along with the facts that $\dfrac{1}{e^k} \le \dfrac{1}{5(1+k)}$ and $\dfrac{(k-1)^2}{k^2} \le \dfrac{9}{16}$ to show that

$$E(Z-k)^+\left[\sqrt{2\pi}(k-1)^2e^{\frac{(k-1)^2}{2}}\right] \le \frac{e^{-\frac{k^2}{2}}}{k^2\sqrt{2\pi}}\left[\sqrt{2\pi}(k-1)^2e^{\frac{(k-1)^2}{2}}\right] = \frac{(k-1)^2\sqrt{e}}{k^2e^k} < \frac{0.1855}{1+k}$$

and

$$E(Z-k)^+\left[\sqrt{2\pi}e^{\frac{(k-1)^2}{2}}\right] \le \frac{e^{-\frac{k^2}{2}}}{k^2\sqrt{2\pi}}\left[\sqrt{2\pi}e^{\frac{(k-1)^2}{2}}\right] = \frac{\sqrt{e}}{k^2e^k} \le \frac{0.04}{1+k}.$$

Thus, for $3 \le k \le 4$, $|f_k''(x)| \le \dfrac{0.23}{1+k}$. For $k \ge 4$, we follow the same argument, using the facts that $\dfrac{1}{e^k} \le \dfrac{0.1}{1+k}$ and $\dfrac{(k-1)^2}{k^2} \le 1$, to obtain that $|f_k''(x)| \le \dfrac{0.18}{1+k}$.

(*ii*) We first observe that, for $k-1 < x < k$, we make use of (2.6) to conclude that

$$\begin{aligned}
0 \le f_k''(x) &= E(Z-k)^+\left[x+\left(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)\right)(x^2+1)\right] \\
&\le \frac{e^{-\frac{k^2}{2}}}{k^2\sqrt{2\pi}}\left[k+\left(\sqrt{2\pi}e^{\frac{k^2}{2}}\Phi(k)\right)\left(k^2+1\right)\right] \\
&\le \frac{e^{-\frac{k^2}{2}}}{k\sqrt{2\pi}}+1+\frac{1}{k^2} \\
&< 1.12 \quad \text{for } k \ge 3. \qquad (2.8)
\end{aligned}$$

For the final case where $x > k$, we note that

$$f_k''(x) = \left(k+E(Z-k)^+\right)\left[x-\left(\sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(-x)\right)\left(x^2+1\right)\right].$$

To bound this term, we employ the Gaussian tail bound (see [18, p. 16, 38] and [22, p. 252]):

$$\frac{xe^{-\frac{x^2}{2}}}{(x^2+1)\sqrt{2\pi}} \leq \Phi(-x) \leq \min\left\{\frac{1}{2}, \frac{1}{x\sqrt{2\pi}}\right\} e^{\frac{-x^2}{2}} \quad \text{for } x > 0,$$

which gives us

$$x \leq \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(-x)(x^2+1) \leq x + \frac{1}{x}.$$

Thus, for $x > k$, we have $f_k''(x) \leq 0$ which leads to

$$\begin{aligned}
|f_k''(x)| &= \left(k + E(Z-k)^+\right)\left[\left(\sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(-x)\right)\left(x^2+1\right) - x\right] \\
&\leq \left(k + E(Z-k)^+\right)\frac{1}{x} \\
&\leq 1 + \frac{e^{-\frac{k^2}{2}}}{k^3\sqrt{2\pi}} \\
&\leq 1.01 \quad \text{for } k \geq 3, \qquad (2.9)
\end{aligned}$$

where we use (2.6) prior to the last inequality. For $x < k-1$, (*ii*) follows directly from (*i*). Using this fact, along with (2.8) and (2.9), we conclude that (*ii*) holds. □

3. Proof of our main results

We apply Stein's equation (2.2) to establish the following:

$$EW_N f_k(W_N) - Ef_k'(W_N) = E(W_N - k)^+ - E(Z-k)^+. \tag{3.1}$$

Thus, we can bound $|EW_N f_k(W_N) - Ef'(W_N)|$ instead of $|E(W_N - k)^+ - E(Z-k)^+|$. To handle the term $|EW_N f_k(W_N) - Ef'(W_N)|$, we use the concentration inequality approach as a complementary tool alongside Stein's method. This technique was first applied by Chen [23] and has been frequently employed to find bounds in this field (see [18, 20, 22, 24] for more examples). Furthermore, the concentration inequality approach has been applied in recent studies, including the work of [25] and Auld and Neammanee [26]. We recall that, for $i = 1, 2, \ldots, n$, $Y_i = \frac{X_i}{\sqrt{Es_N^2}}$ and, in context outside of the random sum framework, we denote

$$W_n = \sum_{i=1}^{n} Y_i, \quad \text{and} \quad W_n^{(i)} = W_n - Y_i.$$

From these notations, we can observe that

$$EW_n^2 = \sum_{i=1}^{n} EY_i^2 = \frac{s_n^2}{Es_N^2}, \quad \text{and} \quad \sum_{i=1}^{n} E|Y_i|^3 = \frac{\delta_n s_n^2}{Es_N^2}. \tag{3.2}$$

For $i = 1, 2, \ldots$, let $K_i(t) = E(Y_i I(0 \leq t \leq Y_i) - Y_i I(Y_i \leq t < 0))$. The K-function acts as a bridge that links the concentration inequality approach with Stein's method. It allows us to control and understand how well a test function approximates a distribution, especially in terms of its tail behavior and deviations from expected values. To illustrate its tail behavior more clearly, we provide examples in Figures 1 and 2.

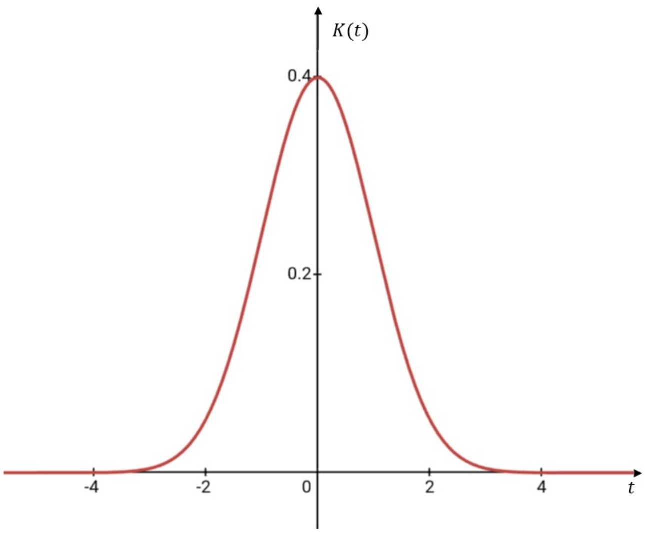

Figure 1. The case when Y_i has the standard normal distribution.

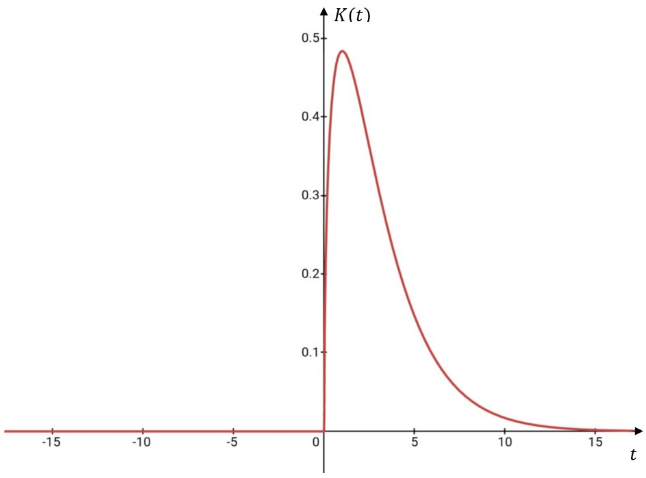

Figure 2. The case when $Y_i - 1$ follows the chi-square distribution with parameter 1.

Chen [23, p. 100] showed that for all real t,

$$\int_{-\infty}^{\infty} K_i(t)dt = EY_i^2 \text{ and } \int_{-\infty}^{\infty} |t|K_i(t)dt = \frac{1}{2}E|Y_i|^3,$$

which implies the following:

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} K_i(t)dt = \sum_{i=1}^{n} EY_i^2 = \frac{s_n^2}{Es_N^2} \text{ and } \sum_{i=1}^{n} \int_{-\infty}^{\infty} |t|K_i(t)dt = \frac{1}{2}\sum_{i=1}^{n} E|Y_i|^3 = \frac{\delta_n s_n^2}{2Es_N^2}. \tag{3.3}$$

Using these results, together with Lyapunov's inequality, we derive that

$$\int_{-\infty}^{\infty} K_i(t) E\left(|Y_i| + |t|\right) dt = EY_i^2 E|Y_i| + \frac{1}{2} E|Y_i|^3 \leq \frac{3}{2} E|Y_i|^3.$$

This leads to the following bound:

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} K_i(t) E\left(|Y_i| + |t|\right) dt \leq \frac{3}{2} \sum_{i=1}^{n} E|Y_i|^3 = \frac{3\delta_n s_n^2}{2Es_N^2}. \tag{3.4}$$

In order to apply Stein's method, we encounter terms like $E[W_n f_k(W_n)]$ and $Ef_k'(W_n)$ that require careful handling. By the fact that Y_i is independent of $W_n^{(i)}$ and $EY_i = 0$, we have that

$$E[W_n f_k(W_n)] = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f_k'(W_n^{(i)} + t) K_i(t) dt, \tag{3.5}$$

and

$$Ef_k'(W_n) = \frac{Es_N^2}{s_n^2} \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f_k'(W_n) K_i(t) dt, \tag{3.6}$$

see [18, p. 20]. These results lead to the key expression:

$$\left|E[W_n f_k(W_n)] - Ef_k'(W_n)\right| = \left| \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left(f_k'(W_n^{(i)} + t) - \frac{Es_N^2}{s_n^2} f_k'(W_n) \right) K_i(t) dt \right|,$$

which plays a crucial role in our analysis. Beyond the importance of the K-function highlighted in its introduction, we will see that the K-function serves as a key tool in addressing the expression above, paving the way for the application of concentration inequalities. The use of these inequalities, however, will be demonstrated later in the proof of the main results. One famous concentration inequality theorem, which will be used in our work, is stated as follows:

Lemma 3.1. [18, p. 54] *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables with zero means, satisfying $\sum_{j=1}^{n} Var(\xi_j) = 1$. For all real $a < b$, and for every $1 \leq i \leq n$,*

$$P\left(a \leq \sum_{j=1, j\neq i}^{n} \xi_j \leq b \right) \leq \sqrt{2}(b-a) + 2(\sqrt{2}+1) \sum_{j=1}^{n} E|\xi_j|^3.$$

We are now ready to prove the main results.

3.1. Proof of Theorem 1.2

Proof of Theorem 1.2. From (3.3), (3.5), and (3.6), we have that

$$\left|E[W_n f_k(W_n)] - Ef_k'(W_n)\right| = \left| \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left(f_k'(W_n^{(i)} + t) - \frac{Es_N^2}{s_n^2} f_k'(W_n) \right) K_i(t) dt \right|$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t)dt \\
&\quad + \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| 1 - \frac{Es_N^2}{s_n^2} \right| |f_k'(W_n)| K_i(t)dt \\
&\leq \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t)dt \\
&\quad + \|f_k'\| \left| \frac{s_n^2 - Es_N^2}{s_n^2} \right| \sum_{i=1}^{n} E \int_{-\infty}^{\infty} K_i(t)dt \\
&= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t)dt + \|f_k'\| \left| \frac{s_n^2 - Es_N^2}{s_n^2} \right| \frac{s_n^2}{Es_N^2} \\
&\leq T + \|f_k'\| \left| \frac{s_n^2 - Es_N^2}{Es_N^2} \right|, \qquad (3.7)
\end{aligned}
$$

where $\|f_k'\| = \sup_{x\in\mathbb{R}} |f_k'(x)|$ and

$$
T = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t)dt.
$$

To bound T, we note that

$$
\begin{aligned}
T &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t) I(A_{i,t})dt + \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)}+t) - f_k'(W_n) \right| K_i(t) I(A_{i,t}^c)dt \\
&:= T_1 + T_2, \qquad (3.8)
\end{aligned}
$$

where

$$
A_{i,t} = \left\{ W_n^{(i)} + Y_i > k, W_n^{(i)} + t \leq k \right\} \cup \left\{ W_n^{(i)} + Y_i \leq k, W_n^{(i)} + t > k \right\}.
$$

To bound T_1, we observe that

$$
\begin{aligned}
I(A_{i,t}) &= I(k - Y_i < W_n^{(i)} \leq k - t) + I(k - t < W_n^{(i)} \leq k - Y_i) \\
&\leq I\left(k - |t| - |Y_i| < W_n^{(i)} \leq k + |t| + |Y_i|\right). \qquad (3.9)
\end{aligned}
$$

Using Lemma 3.1, (3.2), and the fact that $\mathrm{Var}\left(\frac{W_n \sqrt{Es_N^2}}{s_n}\right) = 1$, we have

$$
\begin{aligned}
&E\left[\mathbb{P}(k - |t| - |Y_i| < W_n^{(i)} \leq k + |t| + |Y_i| \mid Y_i)\right] \\
&\quad = E\left[\mathbb{P}\left(\frac{k - |t| - |Y_i|}{\sqrt{\mathrm{Var}(W_n)}} < \frac{W_n^{(i)}}{\sqrt{\mathrm{Var}(W_n)}} \leq \frac{k + |t| + |Y_i|}{\sqrt{\mathrm{Var}(W_n)}} \mid Y_i\right)\right] \\
&\quad = E\left[\mathbb{P}\left(\frac{(k - |t| - |Y_i|)\sqrt{Es_N^2}}{s_n} < \frac{W_n^{(i)}}{\sqrt{\mathrm{Var}(W_n)}} \leq \frac{(k + |t| + |Y_i|)\sqrt{Es_N^2}}{s_n} \mid Y_i\right)\right]
\end{aligned}
$$

$$\leq \frac{2\sqrt{2}(|t|+E|Y_i|)\sqrt{Es_N^2}}{s_n} + \frac{2(\sqrt{2}+1)(Es_N^2)^{\frac{3}{2}}}{s_n^3}\sum_{j=1}^{n} E\left|Y_j\right|^3$$
$$= \frac{2\sqrt{2}(|t|+E|Y_i|)\sqrt{Es_N^2}}{s_n} + \frac{2(\sqrt{2}+1)\delta_n\sqrt{Es_N^2}}{s_n Es_N^2}. \tag{3.10}$$

Applying this inequality along with (3.3), (3.4), (3.9), and (3.10), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} E\int_{-\infty}^{\infty} K_i(t) I(A_{i,t})dt \\
&\leq \sum_{i=1}^{n} E\int_{-\infty}^{\infty} K_i(t) I\left(k-|t|-|Y_i| < W_n^{(i)} \leq k+|t|+|Y_i|\right)dt \\
&= \sum_{i=1}^{n}\int_{-\infty}^{\infty} E\Big[E\Big(K_i(t) I\left(k-|t|-|Y_i| < W_n^{(i)} \leq k+|t|+|Y_i|\right) \mid Y_i\Big)\Big]dt \\
&= \sum_{i=1}^{n}\int_{-\infty}^{\infty} K_i(t) E\Big(\mathbb{P}\left(k-|t|-|Y_i| < W_n^{(i)} \leq k+|t|+|Y_i| \mid Y_i\right)\Big)dt \\
&\leq \sum_{i=1}^{n}\int_{-\infty}^{\infty} K_i(t) E\left[\frac{2\sqrt{2}(|t|+E|Y_i|)\sqrt{Es_N^2}}{s_n} + \frac{2(\sqrt{2}+1)\delta_n\sqrt{Es_N^2}}{s_n Es_N^2}\right]dt \\
&\leq \sum_{i=1}^{n}\left[\frac{3\sqrt{2}E|Y_i|^3\sqrt{Es_N^2}}{s_n} + \frac{2(\sqrt{2}+1)\delta_n EY_i^2\sqrt{Es_N^2}}{s_n Es_N^2}\right] \\
&\leq \frac{(5\sqrt{2}+2)\delta_n s_n}{\sqrt{Es_N^2}}.
\end{aligned}$$

Combining this fact and (2.4), we derive the bound for T_1 as follows:

$$T_1 \leq \sqrt{\frac{2}{\pi}}\left(\frac{(5\sqrt{2}+2)\delta_n s_n}{\sqrt{Es_N^2}}\right) \leq \frac{7.24\delta_n s_n}{\sqrt{Es_N^2}}. \tag{3.11}$$

To bound T_2, we note that

$$I\left(W_n^{(i)}+t \leq k, W_n^{(i)}+Y_i \leq k\right) + I\left(W_n^{(i)}+t > k, W_n^{(i)}+Y_i > k\right) = I(A_{i,t}^c) \leq 1.$$

Since f_k'' exists on $(-\infty, k)$ and (k, ∞), we can apply the mean value theorem on $A_{i,t}^c$, (2.5) and (3.4) to obtain that

$$T_2 = \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t) - f_k'(W_n)\right| K_i(t) I(A_{i,t}^c)dt$$

$$\leq 2\sum_{i=1}^{n} E\int_{-\infty}^{\infty}(|t|+|Y_i|)K_i(t)I(A_{i,t}^c)dt$$
$$\leq \frac{3\delta_n s_n^2}{Es_N^2}. \tag{3.12}$$

From (2.4), (3.1), (3.7), (3.8), (3.11), and (3.12), we have

$$\begin{aligned}|E(W_N-k)^+ - E(Z-k)^+| &\leq \sum_{n=1}^{\infty}\mathbb{P}(N=n)\left|E[W_n f_k(W_n)] - Ef_k'(W_n)\right| \\ &\leq \sum_{n=1}^{\infty}\mathbb{P}(N=n)\left[\frac{7.24\delta_n s_n}{\sqrt{Es_N^2}} + \frac{3\delta_n s_n^2}{Es_N^2} + \sqrt{\frac{2}{\pi}}\left|\frac{s_n^2 - Es_N^2}{Es_N^2}\right|\right] \\ &\leq \frac{7.24E[\delta_N s_N]}{\sqrt{Es_N^2}} + \frac{3E[\delta_N s_N^2]}{Es_N^2} + \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{\mathrm{Var}(s_N^2)}}{Es_N^2}\right).\end{aligned}$$

□

3.2. Proof of Theorem 1.3

Proof of Theorem 1.3. From the proof of Theorem 1.2, we start to bound T. Note that

$$\begin{aligned}T =& \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t) - f_k'(W_n)\right|K_i(t)I(B_{1,i,t})dt + \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t) - f_k'(W_n)\right|K_i(t)I(B_{2,i,t})dt \\ &+ \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t) - f_k'(W_n)\right|K_i(t)I(B_{3,i,t})dt \\ :=& S_1 + S_2 + S_3,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}B_{1,i,t} &= \left\{W_n^{(i)} + Y_i > k-1, W_n^{(i)} + t \leq k-1\right\} \cup \left\{W_n^{(i)} + Y_i \leq k-1, W_n^{(i)} + t > k-1\right\} \\ &\quad \cup \left\{k-1 < W_n \leq k, W_n^{(i)} + t > k\right\} \cup \left\{W_n > k, k-1 < W_n^{(i)} + t \leq k\right\}, \\ B_{2,i,t} &= \left\{W_n \leq k-1, W_n^{(i)} + t \leq k-1\right\}, \quad \text{and} \\ B_{3,i,t} &= \left\{k-1 < W_n \leq k, k-1 < W_n^{(i)} + t \leq k\right\} \cup \left\{W_n > k, W_n^{(i)} + t > k\right\}.\end{aligned}$$

To bound S_1, we note from (3.9) that

$$\begin{aligned}I(B_{1,i,t}) &= I(k-1-Y_i < W_n^{(i)} \leq k-1-t) + I(k-1-t < W_n^{(i)} \leq k-1-Y_i) \\ &\quad + I(k-1 < W_n \leq k, W_n^{(i)} + t > k) + I(W_n > k, k-1 < W_n^{(i)} + t \leq k) \\ &\leq I\left(k-1-|t|-|Y_i| < W_n^{(i)} \leq k-1+|t|+|Y_i|\right) \\ &\quad + I\left(k-|t|-|Y_i| < W_n^{(i)} \leq k+|t|+|Y_i|\right).\end{aligned}$$

By applying Lemma 3.1 in a similar fashion as (3.10), we derive the following inequality:

$$E\Big[\mathbb{P}(k-1-|t|-|Y_i|<W_n^{(i)}\le k-1+|t|+|Y_i|\mid Y_i)\Big]$$
$$=\frac{2\sqrt{2}(|t|+E|Y_i|)\sqrt{Es_N^2}}{s_n}+\frac{2(\sqrt{2}+1)\delta_n\sqrt{Es_N^2}}{s_nEs_N^2}.$$

Applying this inequality along with (3.3), (3.4), and (3.10), we obtain that

$$\begin{aligned}\sum_{i=1}^n E\int_{-\infty}^{\infty}K_i(t)I(B_{1,i,t})dt &= \sum_{i=1}^n\int_{-\infty}^{\infty}K_i(t)\Big[E\Big(\mathbb{P}\left(k-1-|t|-|Y_i|<W_n^{(i)}\le k-1+|t|+|Y_i|\right)\mid Y_i\Big)\\ &\quad+E\Big(\mathbb{P}\left(k-|t|-|Y_i|<W_n^{(i)}\le k+|t|+|Y_i|\right)\mid Y_i\Big)\Big]dt\\ &\le\sum_{i=1}^n\int_{-\infty}^{\infty}K_i(t)E\Big[\frac{4\sqrt{2}(|t|+|Y_i|)\sqrt{Es_N^2}}{s_n}+\frac{4(\sqrt{2}+1)\delta_n\sqrt{Es_N^2}}{s_nEs_N^2}\Big]dt\\ &=\frac{18.15\delta_ns_n}{\sqrt{Es_N^2}}.\end{aligned}$$

Combining this fact and Proposition 2.1, we derive the bound for S_1 as follows:

$$S_1\le\frac{18.15\delta_ns_n\|f_k'\|}{\sqrt{Es_N^2}}\le\frac{1}{1+k}\left(\frac{25.047\delta_ns_n}{\sqrt{Es_N^2}}\right).\tag{3.14}$$

To bound S_2, we use the mean value theorem, Proposition 2.2(i), and (3.4) to obtain that

$$\begin{aligned}S_2&=\sum_{i=1}^n E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t)-f_k'(W_n)\right|K_i(t)I\Big(W_n^{(i)}+t\le k-1,W_n^{(i)}+Y_i\le k-1\Big)dt\\ &\le\sum_{i=1}^n E\int_{-\infty}^{\infty}|f_k''(w)|(|t|+|Y_i|)K_i(t)dt,\quad\text{for some } w\le k-1\\ &\le\frac{0.23}{1+k}\left[\sum_{i=1}^n E\int_{-\infty}^{\infty}(|t|+|Y_i|)K_i(t)dt\right]\\ &\le\frac{0.23}{1+k}\left[\frac{3\delta_ns_n^2}{2Es_N^2}\right]\\ &=\frac{0.345\delta_ns_n^2}{(1+k)Es_N^2}.\end{aligned}\tag{3.15}$$

To bound S_3, we note that

$$S_3=\sum_{i=1}^n E\int_{-\infty}^{\infty}\left|f_k'(W_n^{(i)}+t)-f_k'(W_n)\right|K_i(t)I(B_{3,i,t})dt$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)} + t) - f_k'(W_n) \right| K_i(t) I(B_{3,i,t}) I(|Y_i| \le 1) dt \\
&\quad + \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \left| f_k'(W_n^{(i)} + t) - f_k'(W_n) \right| K_i(t) I(B_{3,i,t}) I(|Y_i| > 1) dt \\
&:= S_{3,1} + S_{3,2}.
\end{aligned}
$$

For $S_{3,1}$, we note that

$$
\begin{aligned}
I(k-1 < W_n \le k, k-1 < W_n^{(i)} + t \le k) I(|Y_i| \le 1) \\
&\le I\,(W_n > k-1)\, I(|Y_i| \le 1) \\
&\le I\left(W_n^{(i)} > k-2\right) I(|Y_i| \le 1).
\end{aligned}
$$

Similarly,

$$
I(W_n > k, W_n^{(i)} + t > k) I(|Y_i| \le 1) \le I\,(W_n > k)\, I(|Y_i| \le 1) \le I\left(W_n^{(i)} > k-2\right) I(|Y_i| \le 1).
$$

From these observations and since $\{k-1 < W_n \le k, k-1 < W_n^{(i)} + t \le k\}$ and $\{W_n > k, W_n^{(i)} + t > k\}$ are disjoint, we conclude that

$$
I(B_{3,i,t}) I(|Y_i| \le 1) \le I\left(W_n^{(i)} > k-2\right) I(|Y_i| \le 1).
$$

By applying the mean value theorem in conjunction with the above fact, Proposition 2.2(ii), and (3.2), we obtain

$$
\begin{aligned}
S_{3,1} &\le 1.12 \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|Y_i| + |t|) K_i(t) I(B_{3,i,t}) I(|Y_i| \le 1) dt \\
&\le 1.12 \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|Y_i| + |t|) K_i(t) I\left(W_n^{(i)} > k-2\right) I(|Y_i| \le 1) dt \\
&\le 1.12 \sum_{i=1}^{n} \left[\frac{3E|Y_i|^3}{2} P\left(W_n^{(i)} > k-2\right) \right] \\
&\le 1.68 \sum_{i=1}^{n} \left[E|Y_i|^3 \left(\frac{E(W_n^{(i)})^2}{(k-2)^2} \right) \right] \\
&\le 1.68 \sum_{i=1}^{n} \left[E|Y_i|^3 \left(\frac{s_n^2}{(1+k) E s_N^2} \right) \right] \\
&= \frac{1.68 \delta_n s_n^4}{(1+k)(E s_N^2)^2},
\end{aligned}
$$

where we use the fact that for $k \ge 3$, $\dfrac{1}{(k-2)^2} \le \dfrac{1}{1+k}$ in the fifth inequality. To bound $S_{3,2}$, we use Proposition 2.1, (3.2), and (3.3) to obtain

$$
S_{3,2} \le \frac{1.38}{1+k} \sum_{i=1}^{n} E \int_{-\infty}^{\infty} K_i(t) I(|Y_i| > 1) dt
$$

$$\leq \frac{1.38}{1+k}\sum_{i=1}^{n} EY_i^2 P(|Y_i|>1)$$
$$\leq \frac{1.38}{1+k}\sum_{i=1}^{n} E|Y_i|^3$$
$$= \frac{1.38\delta_n s_n^2}{(1+k)Es_N^2}.$$

Then

$$S_3 \leq \frac{1.68\delta_n s_n^4}{(1+k)(Es_N^2)^2} + \frac{1.38\delta_n s_n^2}{(1+k)Es_N^2}. \tag{3.16}$$

Combining (3.7) and (3.13)–(3.16), we have

$$\left|E[W_n f_k(W_n)] - Ef_k'(W_n)\right| \leq \frac{1}{1+k}\left[\frac{25.047\delta_n s_n}{\sqrt{Es_N^2}} + \frac{1.725\delta_n s_n^2}{Es_N^2} + \frac{1.68\delta_n s_n^4}{(1+k)(Es_N^2)^2} + 1.38\left|\frac{s_n^2 - Es_N^2}{Es_N^2}\right|\right].$$

Therefore, using this inequality along with (3.1), we conclude that

$$|E(W_N-k)^+ - E(Z-k)^+| \leq \frac{1}{1+k}\left[\frac{25.047E[\delta_N s_N]}{\sqrt{Es_N^2}} + \frac{1.725E[\delta_N s_N^2]}{Es_N^2} + \frac{1.68E[\delta_N s_N^4]}{(Es_N^2)^2} + \frac{1.38\sqrt{\mathrm{Var}}(s_N^2)}{Es_N^2}\right].$$

□

3.3. Proof of Corollary 1.1

Since the X_j's are identically distributed, we have $\delta_n = \frac{\gamma_1}{\sigma_1^3\sqrt{EN}}$, which implies that

$$Es_N^2 = \sigma_1^2 EN, \quad \frac{\delta_n s_n}{\sqrt{Es_N^2}} = \frac{\gamma_1\sqrt{n}}{\sigma_1^3 EN}, \quad \text{and} \quad \frac{\delta_n s_n^2}{Es_N^2} = \frac{\gamma_1 n}{\sigma_1^3 (EN)^{\frac{3}{2}}}.$$

Then

$$\frac{E[\delta_N s_N]}{\sqrt{Es_N^2}} = \frac{E[\delta_N s_N^2]}{Es_N^2} = \frac{\gamma_1}{\sigma_1^3\sqrt{EN}}.$$

In conclusion, the proof can be finalized by noting that

$$\frac{E[\delta_N s_N^4]}{(Es_N^2)^2} = \frac{\gamma_1 EN^2}{\sigma_1^3 (EN)^{\frac{5}{2}}}.$$

4. Applications

4.1. Collateralized debt obligations (CDOs)

A fascinating example of utilizing the call function has emerged in the pricing of CDO tranches. A CDO is a complex financial product that pools together various debt instruments, such as bonds and loans, and then divides the pool into different tranches with varying levels of risk and return. Each tranche is assigned a different payment priority and interest rate. Normally, the tranches primarily used in CDOs are typically known as senior, mezzanine, and junior. The senior tranche includes securities with high credit ratings and tends to be low risk, and therefore has lower returns. Investors have the option to invest in various tranches based on their preferences. In the conventional pricing model for a CDO with a total of n portfolios, each portfolio i is assumed to possess a recovery rate $R > 0$. This recovery rate signifies the proportion of bad debt that can be recovered. We can obtain the percentage loss at the time T by the total loss on the portfolio,

$$L(T) = \frac{(1-R)}{n}\sum_{i=1}^{n} I_{\{\tau_i \le T\}},$$

where τ_i is the default time of the ith portfolio and I_A is an indicator function of the set A.

For each CDO tranche, there exists a detachment point (a limit above which the tranche loss does not increase) and an attachment point (a limit below which the tranche bears none of the loss). Since the cash flow of a CDO tranche is driven by its loss, the pricing problem can be reduced to the problem of calculating the expectation of a call function, i.e., $E(L(T) - l)^+$ where l is the attachment or the detachment point of the tranche (see [27–30] for more details).

The approximation of the stop-loss model for CDOs has been widely studied, as evidenced by [20,27,31,32]. To apply Theorem 1.3 effectively in the Gaussian approximation, we need to centralize the summand variables. Specifically, for each $i = 1, 2, \dots, n$, define

$$\xi_i = \frac{1}{n}(1-R)I_{\{\tau_i \le T\}} \quad \text{and} \quad X_i = \xi_i - \mu_i,$$

where $\mu_i = E\xi_i = \dfrac{(1-R)p_i}{n}$, $p_i = P(\tau_i \le T)$. Then we let

$$Y_i = \frac{X_i}{s_n} \quad \text{and} \quad W_n = \sum_{i=1}^{n} Y_i = \frac{L(T) - \mu}{s_n},$$

where $\mu := \sum_{i=1}^{n} \mu_i$. Then we can apply Theorems 1.2 and 1.3 to obtain that

$$\left|E\,(W_n - k)^+ - E\,(Z - k)^+\right| \le 10.24\delta_n$$

and for $k \ge 3$,

$$\left|E\,(W_n - k)^+ - E\,(Z - k)^+\right| \le \frac{28.452\delta_n}{1+k},$$

where

$$s_n^2 = \sum_{i=1}^{n} \mathrm{Var}(X_i) = \left(\frac{1-R}{n}\right)^2 \sum_{i=1}^{n} p_i(1-p_i) \quad \text{and} \quad \delta_n = \left(\frac{1-R}{ns_n}\right)^3 \sum_{i=1}^{n} p_i(1-p_i)(1-2p_i+2p_i^2).$$

Based on the behavior of the bounds above, we observe that both bounds converge to zero at the rate $O(\delta_n) = O\left(\frac{1}{\sqrt{n}}\right)$.

4.2. Collective risk model

As previously discussed, the collective risk model represents total claims as $\xi_1 + \xi_2 + \cdots + \xi_N$, where ξ_i's are independent and identically distributed, each ξ_i represents the claim amount from the i^{th} claim, and N represents either the number of claims or the number of claim payments. Allow us to provide a more precise explanation of why we regard N as a random variable: When selling insurance that permits individuals to make multiple claims within a given period, such as accident insurance, we face uncertainty regarding the exact number of claims. To address this uncertainty, we use a random variable, N, to interpret it. Commonly used distributions for the number of claims include the binomial, Poisson, and negative binomial distributions.

Before applying our results, we need to denote that for $i = 1, 2 \ldots$,

$$X_i = \xi_i - \mu, \quad Y_i = \frac{X_i}{\sigma\sqrt{EN}}, \quad \text{and} \quad W_N = \sum_{i=1}^{N} Y_i,$$

where $\mu = E\xi_i$ and $\sigma^2 = \mathrm{Var}(\xi_i)$. To align our results more closely with realistic scenarios, we consider ξ as the claim amount in a health insurance contract. It is common to model the random index N, representing the number of claims, using a Poisson distribution, as it is well-suited for rare events (e.g., the likelihood of a single person frequently claiming health insurance is relatively low). Referring to Remark 1.1, we observe that if $N \sim Poi(\lambda)$, then

$$\left|E(W_N-k)^+ - E(Z-k)^+\right| \le \frac{1}{\sqrt{\lambda}}\left[\sqrt{\frac{2}{\pi}} + \frac{10.24\gamma_1}{\sigma_1^3}\right]$$

and for $k \ge 3$,

$$\left|E(W_N-k)^+ - E(Z-k)^+\right| \le \frac{1}{(1+k)\sqrt{\lambda}}\left[1.38 + \frac{28.46\gamma_1}{\sigma_1^3} + \frac{1.68\gamma_1}{\lambda\sigma_1^3}\right].$$

As noted in Remark 1.1, these bounds converge to zero at a rate of $O\left(\frac{1}{\lambda}\right)$ when $\lambda \to \infty$.

5. Conclusions

In this work, we studied the uniform and non-uniform bounds for the stop-loss distance between a random sum $W = X_1 + X_2 + \cdots + X_N$ and the standard normal random variable Z. The stop-loss distance, expressed as $|Eh_k(W) - Eh_k(Z)|$, with $h_k(x) = (x-k)^+$, serves as a crucial measure of the deviation between the two random variables.

Our approach combined Stein's method with concentration inequalities, leveraging their strengths to derive precise bounds. By focusing on the case where $X_1, X_2, \ldots, X_N$ are independent random variables and N is a non-negative integer-valued random variable independent of the X_j's, we provided detailed insights into the behavior of the stop-loss distance under various conditions.

In particular, for the i.i.d. case, our results demonstrated improvements over Döbler's results in specific scenarios, as illustrated with examples in the paper. Furthermore, our investigation into non-uniform bounds represents a novel contribution, offering a detailed characterization of the stop-loss distance for random sums compared to the normal distribution.

Lastly, we explored practical applications of our results, emphasizing their relevance to financial and insurance contexts, such as collateralized debt obligations (CDOs) and the collective risk model. These applications highlight the utility of our findings in understanding and managing risk in real-world scenarios.

Author contributions

Punyapat Kammoo: Visualization, writing – original draft, funding acquisition, formal analysis, validation, writing – review & editing; Kritsana Neammanee: Conceptualization, methodology, project administration, visualization, writing – original draft, Formal analysis, Validation , Writing – review & editing; Kittipong Laipaporn: Conceptualization, methodology, project administration, funding acquisition, formal analysis, validation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Development and Promotion of Science and Technology Talents Project (DPST).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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