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Research article

Geometric perspectives on the variability of spiralike functions with respect to a boundary point in relation to Janowski functions

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Abstract: Investigating the variability domain in the geometric function theory yields profound insights into the behavior of geometric functions, thereby facilitating the examination of extremal problems and the derivation of bounds and inequalities. While the previous literature has examined similar classes, our approach offers significant advantages through a more generalized framework. Our study considers normalized analytic functions with specific positivity conditions which involve complex parameters. This investigation extends a previous work by analyzing a broader set of non-vanishing analytic functions. Unlike earlier studies that focused on specific parameter values, our approach allows for wider applications across multiple subclasses through the incorporation of additional parameters. We aim to determine the variability domain for the logarithm of these functions at fixed points within the unit disk as the functions range over a particular class defined by the specific parameter constraints. This generalized approach unifies several known results and provides a comprehensive framework to solve previously intractable boundary problems in the geometric function theory.

Keywords: Janowski functions; spiralike functions; boundary point; starlike functions; variability region
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1. Introduction

Let $H(E)$ be the space of analytic functions (AF) on the open unit disk

$$E = \{\tau : \tau \in \mathbb{C} \quad \text{and} \quad |\tau| < 1\},$$

equipped with the compact-open topology, where convergence is defined as a uniform convergence on compact subsets of E. Functions in $H(E)$ that map E univalently onto starlike domains with respect to the origin are defined as belonging to the subclass S^*. It is commonly known that $f \in S^*$ if and only if (iff) $f \in H(E)$ satisfies the normalization criteria

$$f'(0) - 1 = f(0) = 0;$$

if τ is in E, then

$$\Re\left(\frac{\tau f'(\tau)}{f(\tau)}\right) > 0.$$

Functions of the class S^* are referred to as starlike functions. For an in-depth examination of this class, we recommend consulting references [1, 2]. The study of starlike functions with respect to the interior points of the unit disk has a rich history. In contrast, starlike functions with respect to boundary points were largely unexamined until Robertson's pioneering work [3]. Building on Robertson's findings and subsequent characterizations, recent progress has been made in this area, as seen in [4, 5].

Consider the family P of holomorphic functions p, which map E onto the right half-plane, and have the series representation

$$p(\tau) = 1 + \sum_{n=1}^{\infty} c_n \tau^n,$$

and

$$p(0) = 1 \text{ and } \Re\{p(\tau)\} > 0.$$

We define B_0 as the set of analytic functions w in E, that is, $|w(\tau)| < 1$, and satisfy $w(0) = 0$. Consequently, for each $f \in H(E)$, there exists $w \in B_0$ such that

$$w(\tau) = \frac{p(\tau) - 1}{p(\tau) + 1} \qquad (\tau \in E).$$

A function f is subordinate to a function y, denoted by $f \prec y$, if there exists a Schwarz function w (i.e., $w \in B_0$) such that $f(\tau) = y(w(\tau))$ for all τ in E. This subordination implies that $f(0) = y(0)$, and the image of E under f is contained in the image of E under y, especially when y is univalent in E.

Define $P[\mathcal{U}, \mathcal{V}]$ as the set of AFs p that satisfy the condition $p(0) = 1$ and have the following representation:

$$p(\tau) = \frac{1 + \mathcal{U} w(\tau)}{1 + \mathcal{V} w(\tau)}, \tag{1.1}$$

where $w(\tau)$ is an Schwarz function. From (1.1), we have the following:

$$w(\tau) = \frac{p(\tau) - 1}{\mathcal{U} - \mathcal{V} p(\tau)}. \tag{1.2}$$

For a fixed point τ_0 in the unit disk E, the set of values of $\log\left(\frac{f(\tau_0)}{\tau_0}\right)$ obtained by considering all injective or univalent functions f forms a closed disk. This fact was first established by Grunsky [6]. Subsequently, the study of the regions of variability for specific subclasses of univalent functions has become an active area of research. Specifically, Yanagihara [7] characterized these regions for functions with bounded derivatives, which are defined by the conditions $|f'(\tau)| \leq 1$ and $\Re(f'(\tau)) > 0$. Furthermore, Ponnusamy [8] explored the regions of variability for the Kaplan family of functions, denoted by K. Furthermore, Yanagihara [9] determined the range of values for a subfamily of convex functions. Ponnusamy [10] explored similar problems for subfamilies of starlike functions (S^*) and the Kaplan family (K). Building on this work, Ponnusamy [11, 12] analyzed these aspects for spirallike functions and for spirallike functions with respect to a boundary point. Vasudevarao investigated the related results for functions with positive real parts, as detailed in [13]. Many authors have introduced new subfamilies of analytic and univalent functions and analyzed the regions of variability for these subfamilies. Notably, Chen and Aiwu [14] studied this for a linear combination of starlike and convex functions, while Ponnusamy et al. [15] investigated regions of variability for exponentially convex functions. Additionally, Raza et al. [16] and Haq [17] studied the regions of variability for Janowski functions. Recently, Raza et al [18] found the regions of variability $V_\kappa(\tau_0, \mathcal{U}, \mathcal{V})$ for $\log f'(\tau_0) = 0$ when f ranges over the class $\mathcal{V}_\alpha[\kappa, \mathcal{U}, \mathcal{V}]$. Furthermore, Bukhari et al. [19] investigated the regions of variability for Bazilevic functions.

This overview summarizes the key contributions of existing research articles published on the topic of the regions of variability. Although some research has been conducted on the topic of regions of variability for certain subclasses of AFs, it remains a relatively understudied area, with only a limited amount of work having been done thus far. Drawing inspiration from these studies, we aim to contribute to this area of research by investigating the regions of variability for a specific class of starlike functions related to Janowski functions. Our approach employs the Herglotz representation for Janowski functions, thus providing a novel perspective on this topic.

Let $\Omega_\rho(\mathcal{U}, \mathcal{V})$ denote the class of functions $f \in H(E)$, which are non-vanishing in E and satisfy $f(0) = 1$; for $\rho \in \mathbb{C}$, $\mathcal{U} \in \mathbb{C}$, $\mathcal{V} \in [-1, 0)$ such that

$$\Re\left\{p_f(\tau)\right\} > 0,$$

where

$$p_f(\tau) = \frac{2\pi}{\rho}\frac{\tau f'(\tau)}{f(\tau)} + \frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau}. \tag{1.3}$$

We note that, $P_f(0) = 1$. Notably, when $\mu = \pi$, the class $\Omega_\rho(\mathcal{U}, \mathcal{V})$ is equivalent to the class of Janowski starlike functions with respect to a boundary point. Specifically, for $\mathcal{U} = 1$ and $\mathcal{V} = -1$, the class $\Omega_\rho(\mathcal{U}, \mathcal{V})$ reduces to the fundamental class Ω_ρ, which has been previously studied in [20].

For function f which belongs to the class $\Omega_\rho(\mathcal{U}, \mathcal{V})$, we denote the single-valued branch of the logarithm of f by $\log f$, such that $\log f(0) = 0$. The Herglotz representation for Janowski functions states that for any function f in the class $\Omega_\rho(\mathcal{U}, \mathcal{V})$, there exists a unique positive unit measure v on the interval $(-\pi, \pi]$ such that

$$\frac{2\pi}{\rho}\frac{\tau f'(\tau)}{f(\tau)} + \frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau} = \int_{-\pi}^{\pi} \frac{1+\mathcal{U}\tau e^{-it}}{1+\mathcal{V}\tau e^{-it}} dv(t). \tag{1.4}$$

Using (1.4), we obtain the expression for $\log f(\tau)$ as follows:

$$\frac{\tau f'(\tau)}{f(\tau)} = \frac{\rho}{2\pi}\left(\int_{-\pi}^{\pi} \frac{1+\mathcal{U}\tau e^{-it}}{(1+\mathcal{V}\tau e^{-it})}dv(t) - \frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau}\right),$$

$$\frac{f'(\tau)}{f(\tau)} = \frac{\rho}{2\pi}\left(\int_{-\pi}^{\pi} \frac{1+\mathcal{U}\tau e^{-it}}{(1+\mathcal{V}\tau e^{-it})\,\tau} - \frac{1+\mathcal{U}\tau}{(1+\mathcal{V}\tau)\,\tau}\right)dv(t).$$

Utilizing partial fraction decomposition and integration, we arrive at

$$\log f(\tau) = \frac{\rho\,(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_{-\pi}^{\pi}\log\left(\frac{1+\mathcal{V}\tau e^{-it}}{1+\mathcal{V}\tau}\right)dv(t),$$

and

$$f(\tau) = \exp\left(\frac{\rho\,(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_{-\pi}^{\pi}\log\left(\frac{1+\mathcal{V}\tau e^{-it}}{1+\mathcal{V}\tau}\right)dv(t)\right).$$

Since

$$p_f(\tau) = \frac{2\pi}{\rho}\frac{\tau f'(\tau)}{f(\tau)} + \frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau},$$

then,

$$p_f'(0) = \frac{2\pi}{\rho}\,[f'(0)] + (\mathcal{U}-\mathcal{V}). \tag{1.5}$$

From (1.3), we have the following:

$$p_f''(0) = \frac{4\pi}{\rho}f'(0) - \frac{4\pi}{\rho}\,(f'(0))^2 - 2\mathcal{V}(\mathcal{U}-\mathcal{V}). \tag{1.6}$$

For each $f \in \Omega_\rho\,(\mathcal{U},\mathcal{V})$, there exists an $w_f \in P[\mathcal{U},\mathcal{V}]$ of the form

$$w_f(\tau) = \frac{p_f(\tau)-1}{\mathcal{U}-\mathcal{V}p_f(\tau)}, \tag{1.7}$$

and conversely. From (1.7), we have the following:

$$p_f'(\tau) = \frac{(\mathcal{U}-\mathcal{V})\,w_f'(\tau)}{\left(1+\mathcal{U}w_f(\tau)\right)^2}, \tag{1.8}$$

$$p_f'(0) = (\mathcal{U}-\mathcal{V})\,w_f'(0) = \frac{2\pi}{\rho}f'(0) + (\mathcal{U}-\mathcal{V}). \tag{1.9}$$

Applying the classical Schwarz lemma, which states that $\left|w_f'(0)\right| \leq 1$, (see [21]), we derive the following inequality:

$$\left|p_f'(0)\right| = \left|\frac{2\pi}{\rho}f'(0) + (\mathcal{U}-\mathcal{V})\right| \leq (\mathcal{U}-\mathcal{V})\,.$$

Using (1.8), we obtain the following:

$$\frac{w_f''(0)}{2}=\frac{p_f''(0)}{2(\mathcal{U}-\mathcal{V})}+Y\kappa^2.$$

Solving (1.5) and (1.6), we have the following:

$$p_f''(0)=\frac{4\pi}{\rho}f'(0)-\frac{\rho}{\pi}(\mathcal{U}-\mathcal{V})^2(\kappa-1)^2-2\mathcal{V}(\mathcal{U}-\mathcal{V}).$$

Therefore,

$$\frac{w_f''(0)}{2}=\frac{2\pi}{\rho(\mathcal{U}-\mathcal{V})}f'(0)-\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}(\kappa-1)^2-\mathcal{V}\left(1-\kappa^2\right).$$

Now, if we let

$$g(\tau)=\begin{cases}\dfrac{\frac{w_f(\tau)}{\tau}-\kappa}{1-\overline{\kappa}\frac{w_f(\tau)}{\tau}}, & |\kappa|<1,\\ 0, & |\kappa|=1,\end{cases}$$

then this implies that

$$g'(0)=\begin{cases}\frac{1}{1-|\kappa|^2}\left(\frac{w_f(\tau)}{\tau}\right)'\Big|_{\tau=0}=\frac{1}{1-|\kappa|^2}\frac{w_f''(0)}{2}, & |\kappa|<1,\\ 0, & |\kappa|=1.\end{cases}$$

According to the Schwarz lemma, for $|\kappa|<1$, it follows that

$$|g(\tau)|\le|\tau|,$$

and

$$\left|g'(0)\right|\le 1,$$

iff

$$\frac{1}{1-|\kappa|^2}\frac{\left|w_f''(0)\right|}{2}\le 1,$$
$$\frac{1}{1-|\kappa|^2}\left(\frac{2\pi}{\rho(\mathcal{U}-\mathcal{V})}f'(0)-\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}(\kappa-1)^2-\mathcal{V}\left(1-\kappa^2\right)\right)\le 1,$$

iff

$$f'(0)=\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}\left[m\left(1-|\kappa|^2\right)+\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}(\kappa-1)^2+\mathcal{V}\left(1-\kappa^2\right)\right]$$

for some $m\in\mathbb{C}$ with $|m|\le 1$. Consequently, for $\kappa\in\overline{E}=\{\tau\in\mathbb{C}:|\tau|\le 1\}$ and for a fixed $\tau_0\in E$ we introduce the following:

$$\Omega_\rho(\kappa,\mathcal{U},\mathcal{V})=\left\{f\in\Omega_\rho(\mathcal{U},\mathcal{V}):f'(0)=\frac{\rho(\mathcal{U}-\mathcal{V})(\kappa-1)}{2\pi}\right\},$$

$$V(\tau_0,\kappa,\mathcal{U},\mathcal{V})=\left\{\log f(\tau_0):f\in\Omega_\rho(\kappa,\mathcal{U},\mathcal{V})\right\}.$$

Combining (1.9) with the normalization condition defined for the class $\Omega_\rho(\kappa,\mathcal{U},\mathcal{V})$, it is evident that $w_f'(0)=\kappa$.

Remark 1.1. While we define the class $\Omega_\rho(U, V)$ with $U \in \mathbb{C}$, it is important to note that the class becomes empty when U and V are both real with $U < V$. This is because for τ approaching $-1/V$ from within the unit disk, the term $(1 + U\tau)/(1 + V\tau)$ would have a negative real part, making it impossible to satisfy the condition $\Re\{p_f(\tau)\} > 0$ for all $\tau \in E$. Therefore, when dealing with real parameters, the meaningful case is $U > V$.

Variability domains play a crucial role in the geometric function theory for determining bounds and solving extremal problems. While our work focuses on this area, we acknowledge related developments in broader mathematical fields that employ similar analytical techniques.

Recent work by Chalishajar et al. [22] explored the analyticity properties of weighted composition semigroups on spaces of holomorphic functions, thus demonstrating the importance of functional analysis techniques that we also utilize in our investigation. The study of decay properties in stochastic systems, as seen in Kasinathan et al. [23], offers methodological parallels to our approach for the boundary characterization of variability domains.

Furthermore, the controllability analysis of higher-order fractional systems by Chalishajar et al. [24] employed parameter-based techniques that, while applied in a different context, share mathematical foundations with our parametric approach to the function classes. Similarly, Sandrasekaran et al. [25] examined the qualitative behavior of stochastic systems with complex potentials, using analytical methods that inform our treatment of complex-valued functions. For more recent studies, we refer the readers to see [26–28].

The primary objective of this paper is to extend the existing research on variability domains by investigating and determining the region of variability, denoted as $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$, for the values of $\log f(\tau_0)$ as f varies over the class $\Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$. Unlike previous studies that focused on specific parameter values, our generalized approach incorporates the parameters $\mathcal{U}$ and $\mathcal{V}$ to create a comprehensive framework that unifies several known results and enables solutions to previously challenging extremal problems in geometric function theory. By examining this extended class of non-vanishing analytic functions, we provide new insights into the behavior of the geometric functions that facilitate the derivation of improved bounds and inequalities.

2. A set of lemmas

To present our main theorem, we first require some known lemmas:

Lemma 2.1. *[21]. Let f be an AF in E with*

$$f(\tau) = \tau^k + \dots .$$

If

$$\Re\left(1 + \frac{\tau f'(\tau)}{f'(\tau)}\right) > 0, \quad \tau \in E,$$

then $f \in (S^*)^k$.

Lemma 2.2. *[29]. For $\phi \in \mathbb{R}$ and $\kappa \in E$, consider the function defined by the following integral:*

$$G(\tau) = \int_0^\tau \frac{e^{i\phi}\zeta^2}{\left(1 + (\overline{\kappa}e^{i\phi} + \mathcal{V}\kappa)\,\zeta + \mathcal{V}e^{i\phi}\zeta^2\right)^2} d\zeta, \quad |\kappa| < 1.$$

The function has a zero of order 2 at the origin and no other zero in E. Furthermore, there exists a starlike normalized univalent function $G_0 \in S^*$ in $E : G(\tau) = \frac{1}{2}e^{i\phi}G_0^2(\tau)$.

3. Basic properties of the class $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$

We initiate our analysis by examining certain general characteristics of the set $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$, such as compactness and convexity, which will provide a foundation for our further investigations.

Proposition 3.1. *(i) $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is compact.*
(ii) $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is convex.
(iii) For $|\kappa| = 1$ or $\tau_0 = 0$, then

$$V(\tau_0, \kappa, \mathcal{U}, \mathcal{V}) = \frac{\rho(\mathcal{U} - \mathcal{V})}{2\pi\mathcal{V}} \log\left(\frac{1 + \mathcal{V}\kappa\tau_0}{1 + \mathcal{V}\tau_0}\right). \tag{2.3}$$

(iv) For $|\kappa| < 1$ and $\tau_0 \in E\backslash\{0\}$, $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ has an interior point of $\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi Y} \log\left(\frac{1+\mathcal{V}\kappa\tau_0}{1+\mathcal{V}\tau_0}\right)$.

Proof. (i) Since $\Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$ is a compact subset of $H(E)$, it follows that $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is also compact.

(ii) If f_1 and $f_2 \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$ and $0 \leq t \leq 1$, then

$$\log f_t(\tau) = (1 - t)\log f_1(\tau) + t \log f_1(\tau)$$

is obviously in $\Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$. Furthermore, as a consequence of the formulation of f_t, $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is convex.

(iii) If $\tau_0 = 0$, then trivially holds. Applying the classical Schwarz lemma [30] to the case $|\kappa| = 1$ and $\left|w_f'(0)\right| = 1$, we obtain $w_f(\tau) = \kappa\tau$, which implies

$$p_f(\tau) = \frac{1 + \mathcal{U}\kappa\tau}{1 + \mathcal{V}\kappa\tau}.$$

This implies that

$$f(\tau) = \left(\frac{1 + \mathcal{V}\kappa\tau}{1 + \mathcal{V}\tau}\right)^{\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}}.$$

Consequently,

$$V(\tau_0, \kappa, \mathcal{U}, \mathcal{V}) = \frac{\rho(\mathcal{U} - \mathcal{V})}{2\pi Y} \log\left(\frac{1 + \mathcal{V}\kappa\tau_0}{1 + \mathcal{V}\tau_0}\right).$$

(iv) For $|\kappa| < 1$ and $m \in \overline{E}$, we define

$$\delta(\tau, \kappa) = \frac{\tau + \kappa}{1 + \overline{\kappa}\tau},$$

and

$$H_{m,\kappa}(\tau) = \exp\left(\frac{\rho(\mathcal{U} - \mathcal{V})}{2\pi\mathcal{V}} \int_o^\tau \frac{\delta(m\zeta, \kappa) - 1}{(1 + \delta(m\zeta, \kappa)\mathcal{V}\zeta)(1 + \mathcal{V}\zeta)} d\zeta\right), \ \tau \in E. \tag{3.1}$$

First, we claim that $H_{m,\kappa}(\tau) \in \Omega_{\rho}(\kappa, \mathcal{U}, \mathcal{V})$. For this, we compute the following:

$$\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} = \frac{\rho}{2\pi\mathcal{V}}\left(\frac{(\mathcal{U}-\mathcal{V})\,\delta(m\tau,\kappa)-(\mathcal{U}-\mathcal{V})}{(1+\delta(m\tau,\kappa)\mathcal{V}\tau)(1+\mathcal{V}\tau)}\right);$$

therefore, we see easily that

$$\frac{2\pi}{\rho}\frac{\tau H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} + \frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau} = \frac{1+\mathcal{U}\delta(m\tau,\kappa)\,\tau}{1+\mathcal{V}\delta(m\tau,\kappa)\,\tau}.$$

As $\delta(m\tau,\kappa)$ lies in the unit disk E, $H_{m,\kappa}(\tau) \in \Omega_{\rho}(\kappa, \mathcal{U}, \mathcal{V})$ and the claims follows. Additionally, we observe that

$$w_{H_{m,\kappa}(\tau)} = \tau\delta(m\tau,\kappa). \tag{3.2}$$

Next, we claim that the mapping $E \ni m \to \log H_{m,\kappa}(\tau_0)$ is a non-constant AF of m for each fixed $\tau_0 \in E\backslash\{0\}$ and $\kappa \in E$. To achieve this, we set

$$h(\tau) = \frac{2\pi\mathcal{V}}{\rho(\mathcal{U}-\mathcal{V})\left(1-|\kappa|^2\right)}\frac{\partial}{\partial m}\left(\log H_{m,\kappa}(\tau)\right)\Big|_{m=0}.$$

A computation gives the following:

$$h(\tau) = \int_0^{\tau}\frac{\zeta}{(1+Y\kappa\zeta)^2}d\zeta.$$

By taking the logarithmic derivative

$$\Re\left(\frac{\tau h''(\tau)}{h'(\tau)}\right) = \Re\left(\frac{1-\mathcal{V}\kappa\tau}{1+\mathcal{V}\kappa\tau}\right) > 0, \qquad (\tau \in E).$$

Applying Lemma 2.1 yields the existence of a function $h_0 \in S^*$ which satisfies $h = h_0^2$. The univalence of h_0 combined with the condition $h_0(0) = 0$ implies that $h(\tau_0) \neq 0$ for all $\tau_0 \in E\backslash\{0\}$. Consequently, the mapping $E \ni m \to \log H_{m,\kappa}(\tau_0)$ is a non-constant AF of “m”, and hence, it is an open mapping. Thus, $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ contains the open set

$$\{\log H_{m,\kappa}(\tau_0) : |m| < 1\}.$$

In particular,

$$\log H_{m,\kappa}(\tau_0) = \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\log\left(\frac{1+\mathcal{V}\kappa\tau_0}{1+\mathcal{V}\tau_0}\right)$$

is an interior point of

$$\{\log H_{m,\kappa}(\tau_0) : m \in E\} \subset V(\tau_0, \kappa, \mathcal{U}, \mathcal{V}).$$

□

Remark 3.2. Since $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is a compact convex subset of $\mathbb{C}$ and has a nonempty interior; the boundary $\partial V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is a Jordan curve and $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ is the union of $\partial V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ and its inner domain.

Remark 3.3. If the variability domain $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ fails to be compact, then several significant consequences would arise. First, the existence of extremal functions for optimization problems over this class could not be guaranteed, as continuous functions may not attain their extreme values on non-compact sets. Second, the boundary of the variability region would potentially contain points that cannot be realized by any function in the class $\Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$. Finally, the connection between geometric properties of the functions and the shape of the variability domain would become less direct, thus obscuring the geometric interpretation of our results.

4. Main results

Now, we prove that the $\log f(\tau)$ is contained in some closed disk for $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$, with center $\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} Q(\tau, \kappa, \mathcal{U}, \mathcal{V})$ and $\frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi} R(\tau, \kappa, \mathcal{U}, \mathcal{V})$.

Theorem 4.1. *If $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$, then*

$$\left|\frac{f'(\tau)}{f(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} Q(\tau, \kappa, \mathcal{U}, \mathcal{V})\right| \leq \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi} R(\tau, \kappa, \mathcal{U}, \mathcal{V}), \tag{4.1}$$

where $Q(\tau, \kappa, \mathcal{U}, \mathcal{V})$ and $R(\tau, \kappa, \mathcal{U}, \mathcal{V})$ are given by (4.8) and (4.9), respectively. For each $\tau \in E\backslash\{0\}$, the equality holds iff $f = H_{e^{i\phi},\kappa}$ for some $\phi \in \mathbb{R}$.

Proof. Since $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$, then by using the Schwarz lemma [30] for $w_f \in B_0$ with $w_f'(0) = \kappa$ such that

$$\left|\frac{\frac{w_f(\tau)}{\tau} - \kappa}{1 - \overline{\kappa}\frac{w_f(\tau)}{\tau}}\right| \leq |\tau|, \ \tau \in E. \tag{4.2}$$

From (1.7), we have the following:

$$\left|\frac{1+\kappa\mathcal{V}\tau}{-\overline{\kappa}-\mathcal{V}\tau}\right|\left|\frac{p_f(\tau) - \frac{1+\mathcal{U}\kappa\tau}{1+\mathcal{V}\kappa\tau}}{p_f(\tau) + \frac{\overline{\kappa}+\mathcal{U}\tau}{-\overline{\kappa}-\mathcal{V}\tau}}\right| \leq |\tau|, \ \tau \in E.$$

Using (1.3), we obtain the following:

$$\left|\frac{\frac{f'(\tau)}{f(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})(\kappa-1)}{2\pi(1+\mathcal{V}\tau)(1+\kappa\mathcal{V}\tau)}}{\frac{f'(\tau)}{f(\tau)} + \frac{\rho(\mathcal{U}-\mathcal{V})(1-\overline{\kappa})}{2\pi(1+\mathcal{V}\tau)(-\overline{\kappa}-\mathcal{V}\tau)}}\right| \leq |\tau|\left|\frac{-\overline{\kappa}-\mathcal{V}\tau}{1+\kappa\mathcal{V}\tau}\right|, \ \tau \in E.$$

Hence,

$$\left|\frac{\frac{f'(\tau)}{f(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} H(\tau, \kappa, \mathcal{U}, \mathcal{V})}{\frac{f'(\tau)}{f(\tau)} + \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} E(\tau, \kappa, \mathcal{U}, \mathcal{V})}\right| \leq |\tau|\,|\mathcal{U}(\tau, \kappa, \mathcal{U}, \mathcal{V})|, \ \tau \in E, \tag{4.3}$$

where

$$H(\tau, \kappa, \mathcal{U}, \mathcal{V}) = \frac{(\kappa-1)}{(1+\mathcal{V}\tau)(1+\kappa\mathcal{V}\tau)}, \tag{4.4}$$

$$E(\tau,\kappa,\mathcal{U},\mathcal{V}) = \frac{(1-\overline{\kappa})}{(1+\mathcal{V}\tau)(-\overline{\kappa}-\mathcal{V}\tau)}, \tag{4.5}$$

$$\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V}) = \frac{-\overline{\kappa}-\mathcal{V}\tau}{1+\kappa\mathcal{V}\tau}. \tag{4.6}$$

A brief calculation demonstrates that the inequalities (4.3) and (4.7) are equivalent:

$$\begin{aligned}&\left|\frac{f'(\tau)}{f(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}\frac{H(\tau,\kappa,\mathcal{U},\mathcal{V}) + |\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 E(\tau,\kappa,\mathcal{U},\mathcal{V})}{1-|\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2}\right| \\ &\leq \frac{|\rho(\mathcal{U}-\mathcal{V})||\tau||\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})||H(\tau,\kappa,\mathcal{U},\mathcal{V}) + E(\tau,\kappa,\mathcal{U},\mathcal{V})|}{2\pi\left(1-|\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2\right)}.\end{aligned} \tag{4.7}$$

Using (4.4)–(4.6), we can easily see that

$$\begin{aligned}&1-|\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 \\ &= \frac{1-\mathcal{V}^2|\tau|^4 + 2\mathcal{V}\left(1-|\tau|^2\right)\Re(\kappa\tau) + |\kappa|^2|\tau|^2\left(\mathcal{V}^2-1\right)}{|1+\mathcal{V}\kappa\tau|^2}.\end{aligned}$$

Additionally,

$$H(\tau,\kappa,\mathcal{U},\mathcal{V}) + E(\tau,\kappa,\mathcal{U},\mathcal{V}) = \frac{1-|\kappa|^2}{(1+\mathcal{V}\kappa\tau)(\overline{\kappa}-\mathcal{V}\tau)},$$

and

$$\begin{aligned}&H(\tau,\kappa,\mathcal{U},\mathcal{V}) + |\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 E(\tau,\kappa,\mathcal{U},\mathcal{V}) \\ &= \frac{(\kappa-1)(1+\mathcal{V}\overline{\kappa\tau}) + |\tau|^2(-\mathcal{V}\overline{\tau}-\kappa)(1-\overline{\kappa})}{(1+\mathcal{V}\tau)|1+Y\kappa\tau|^2}.\end{aligned}$$

Thus, by a simple computation, we see that

$$\begin{aligned}Q(\tau,\kappa,\mathcal{U},\mathcal{V}) &= \frac{H(\tau,\kappa,\mathcal{U},\mathcal{V}) + |\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 E(\tau,\kappa,\mathcal{U},\mathcal{V})}{1-|\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2} \\ &= \frac{(\kappa-1)(1+\mathcal{V}\overline{\kappa\tau}) + |\tau|^2(-\mathcal{V}\overline{\tau}-\kappa)(1-\overline{\kappa})}{(1+\mathcal{V}\tau)\left(1-\mathcal{V}^2|\tau|^4 + 2\mathcal{V}\left(1-|\tau|^2\right)\Re(\kappa\tau) + |\kappa|^2|\tau|^2(\mathcal{V}^2-1)\right)},\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}R(\tau,\kappa,\mathcal{U},\mathcal{V}) &= \frac{|\tau||\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})||H(\tau,\kappa,\mathcal{U},\mathcal{V}) + E(\tau,\kappa,\mathcal{U},\mathcal{V})|}{\left(1-|\tau|^2|\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2\right)}, \\ &= \frac{|\tau|\left(1-|\kappa|^2\right)}{1-\mathcal{V}^2|\tau|^4 + 2\mathcal{V}\left(1-|\tau|^2\right)\Re(\kappa\tau) + |\kappa|^2|\tau|^2(\mathcal{V}^2-1)}.\end{aligned} \tag{4.9}$$

It is readily apparent that the equality in (4.1) occurs for some $\tau \in E$ iff $f = H_{e^{i\phi},\kappa}$ for $\phi \in \mathbb{R}$. Conversely, if the equality occurs for some $\tau \in E\backslash\{0\}$ in (4.1), then the equality must also hold in (4.2). By the Schwarz lemma, this implies the existence of a $\phi \in \mathbb{R}$ such that $w_f(\tau) = \tau\delta\left(e^{i\phi}\tau,\kappa\right)$ for some $\tau \in E$. This implies that $f = H_{e^{i\phi},\kappa}$. □

Corollary 4.2. *If* $f \in \Omega_\rho(0, \mathcal{U}, \mathcal{V})$, *then*

$$\left|\frac{f'(\tau)}{f(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}\left(\frac{-1-\mathcal{V}|\tau|^2\overline{\tau}}{(1+\mathcal{V}\tau)\left(1-\mathcal{V}^2|\tau|^4\right)}\right)\right| \leq \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi}\frac{|\tau|}{\left(1-\mathcal{V}^2|\tau|^4\right)},\ \tau \in E.$$

For each $\tau \in E\backslash\{0\}$, the equality holds iff $f = H_{e^{i\phi},0}$ for some $\phi \in \mathbb{R}$.

For $\mathcal{U} = 1$ and $\mathcal{V} = -1$, we have following known result.

Corollary 4.3. *[12]. If* $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$, *then*

$$\left|\frac{f'(\tau)}{f(\tau)} - \frac{\rho}{\pi}Q(\tau,\kappa)\right| \leq \frac{|\rho|}{\pi}R(\tau,\kappa),$$

where

$$Q(\tau,\kappa) = \frac{(\kappa-1)(1-2\overline{\kappa\tau}) + |\tau|^2(\overline{\tau}-\kappa)(1-\overline{\kappa})}{(1-\tau)\,1-|\tau|^4 + 2\left(1-|\tau|^2\right)\Re(\kappa\tau)},$$

and

$$R(\tau,\kappa) = \frac{|\tau|\left(1-|\kappa|^2\right)}{1-|\tau|^4 - 2\left(1-|\tau|^2\right)\Re(\kappa\tau)}.$$

For each $\tau \in E\backslash\{0\}$, the equality holds iff $f = H_{e^{i\phi},\kappa}$ for some $\phi \in \mathbb{R}$.

Corollary 4.4. *Let* $\gamma : \tau(t)$, $0 \leq t \leq 1$, *be a* $\mathcal{V}^1$*-curve in E with* $\tau(0) = 0$ *and* $\tau(1) = \tau_0$*; then, we have*

$$V(\tau_0, \kappa, \mathcal{U}, \mathcal{V}) \subset \left\{\begin{array}{c} w \in \mathbb{C} : \left|w - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}L(\gamma,\kappa,\mathcal{U},\mathcal{V})\right| \\ \leq \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi}M(\gamma,\kappa,\mathcal{U},\mathcal{V}) \end{array}\right\},$$

where

$$L(\gamma,\kappa,\mathcal{U},\mathcal{V}) = \int_0^1 Q(\tau(t),\kappa,\mathcal{U},\mathcal{V})\,\tau'(t)\,dt, \tag{4.10}$$

and

$$M(\gamma,\kappa,\mathcal{U},\mathcal{V}) = \int_0^1 R(\tau(t),\kappa,\mathcal{U},\mathcal{V})\left|\tau'(t)\right|dt. \tag{4.11}$$

Proof. For $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$, it follows from Proposition 4.1 that

$$\left|\log f(\tau_0) - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}L(\gamma,\kappa,\mathcal{U},\mathcal{V})\right| = \left|\int_0^1\left(\frac{f'(\tau(t))}{f(\tau(t))} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}Q(\tau(t),\kappa,\mathcal{U},\mathcal{V})\right)\tau'(t)\,dt\right|$$

$$\leq \int_0^1 \left| \frac{f'(\tau(t))}{f(\tau(t))} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} Q(\tau(t), \kappa, \mathcal{U}, \mathcal{V}) \right| \left|\tau'(t)\right| dt$$

$$\leq \frac{|\rho| \, |(\mathcal{U}-\mathcal{V})|}{2\pi} \int_0^1 R(\tau(t), \kappa, \mathcal{U}, \mathcal{V}) \left|\tau'(t)\right| dt$$

$$= \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi} M(\gamma, \kappa, \mathcal{U}, \mathcal{V}).$$

Since $\log f(\tau_0) \in V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ was arbitrary, the conclusion follows. □

In Theorem 4.5, we show that $\log H_{e^{i\phi},\kappa}(\tau_0)$ lies on the boundary of $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$.

Theorem 4.5. *Let $\tau_0 \in E\backslash\{0\}$. Then, for $\phi \in (-\pi, \pi]$, we have $\log H_{e^{i\phi},\kappa}(\tau_0) \in \partial V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$. Furthermore, if $\log f(\tau_0) = \log H_{e^{i\phi},\kappa}(\tau_0)$ for some $f \in \Omega_\rho(\kappa, \mathcal{U}, \mathcal{V})$ and $\phi \in (-\pi, \pi]$, then $f = H_{e^{i\phi},\kappa}$.*

Proof. From (3.1), we have the following:

$$\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} = \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{\delta(m\tau, \kappa) - 1}{(1 + \delta(m\tau, \kappa)\mathcal{V}\tau)(1 + \mathcal{V}\tau)} \right)$$
$$= \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{(\kappa - 1) + (1 - \overline{\kappa}) mz}{(1 + \mathcal{V}\tau)(1 + (\overline{\kappa}m + \mathcal{V}\kappa)\tau + \mathcal{V}m\tau^2)} \right). \tag{4.12}$$

Using (4.4) and (4.12), we have the following:

$$\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} H(\tau, \kappa, \mathcal{U}, \mathcal{V})$$
$$= \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{(\kappa - 1) + (1 - \overline{\kappa}) m\tau}{(1 + \mathcal{V}\tau)(1 + (\overline{\kappa}m + \mathcal{V}\kappa)\tau + \mathcal{V}m\tau^2)} - \frac{(\kappa - 1)}{(1 + \mathcal{V}\tau)(1 + \kappa\mathcal{V}\tau)} \right)$$
$$= \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{\left(1 - |\kappa|^2\right) m\tau}{(1 + \kappa\mathcal{V}\tau)(1 + (\overline{\kappa}m + \mathcal{V}\kappa)\tau + \mathcal{V}m\tau^2)} \right). \tag{4.13}$$

Again, using (4.12) and (4.5), we have the following:

$$\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} + \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} E(\tau, \kappa, \mathcal{U}, \mathcal{V})$$
$$= \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{(\kappa - 1) + (1 - \overline{\kappa}) m\tau}{(1 + \mathcal{V}\tau)(1 + (\overline{\kappa}m + \mathcal{V}\kappa)\tau + \mathcal{V}m\tau^2)} - \frac{(1 - \overline{\kappa})}{(1 + \mathcal{V}\tau)(-\overline{\kappa} - \mathcal{V}\tau)} \right)$$
$$= \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \left(\frac{1 - |\kappa|^2}{(-\overline{\kappa} - \mathcal{V}\tau)(1 + (\overline{\kappa}m + \mathcal{V}\kappa)\tau + \mathcal{V}m\tau^2)} \right). \tag{4.14}$$

Hence, we obtain that

$$\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau, \kappa, \mathcal{U}, \mathcal{V})$$

$$
\begin{aligned}
&= \left\{ \frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \times \right. \\
&\left. \left(\frac{H(\tau,\kappa,\mathcal{U},\mathcal{V}) + |\tau|^2 |\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 E(\tau,\kappa,\mathcal{U},\mathcal{V})}{1-|\tau|^2 |\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2} \right) \right\} \\
&= \frac{1}{1-|\tau|^2 |\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2} \left\{ \left(\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} H(\tau,\kappa,\mathcal{U},\mathcal{V}) \right) \right. \\
&\left. - |\tau|^2 |\mathcal{U}(\tau,\kappa,\mathcal{U},\mathcal{V})|^2 \left(\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} + \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} E(\tau,\kappa,\mathcal{U},\mathcal{V}) \right) \right\}. \qquad (4.15)
\end{aligned}
$$

Using (4.13), (4.14), and (4.15), we have the following:

$$
\begin{aligned}
&\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau,\kappa,\mathcal{U},\mathcal{V}) \\
&= \frac{\rho(\mathcal{U}-\mathcal{V})\left(1-|\kappa|^2\right)\tau\left[m\left(1+\mathcal{V}\overline{\kappa\tau}-\overline{\tau}\left(-\mathcal{V}\tau-\kappa\right)\right)\right]}{2\pi\mathcal{V}\left(1-\mathcal{V}^2|\tau|^4+2\mathcal{V}\left(1-|\tau|^2\right)\Re(\kappa\tau)+|\kappa|^2|\tau|^2\left(\mathcal{V}^2-1\right)\right)\left(1+(\overline{\kappa}m+\mathcal{V}\kappa)\tau+\mathcal{V}m\tau^2\right)} \\
&= R(\tau,\kappa,\mathcal{U},\mathcal{V}) \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \frac{az}{|\tau|} \left(\frac{\left|1+(\overline{\kappa}m+\mathcal{V}\kappa)\tau+\mathcal{V}m\tau^2\right|^2}{\left(1+(\overline{\kappa}m+\mathcal{V}\kappa)\tau+\mathcal{V}m\tau^2\right)^2} \right).
\end{aligned}
$$

Now, by substituting $m = e^{i\phi}$, we easily see that

$$
\begin{aligned}
&\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau,\kappa,\mathcal{U},\mathcal{V}) \\
&= R(\tau,\kappa,\mathcal{U},\mathcal{V}) \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \frac{e^{i\phi}\tau}{|\tau|} \left(\frac{\left|1+\left(\overline{\kappa}e^{i\phi}+\mathcal{V}\kappa\right)\tau+\mathcal{V}e^{i\phi}\tau^2\right|^2}{\left(1+(\overline{\kappa}e^{i\phi}+\mathcal{V}\kappa)\tau+\mathcal{V}e^{i\phi}\tau^2\right)^2} \right).
\end{aligned}
$$

Putting $G(\tau)$ as in Lemma 2.2, we obtain the following:

$$
\begin{aligned}
&\frac{H'_{m,\kappa}(\tau)}{H_{m,\kappa}(\tau)} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau,\kappa,\mathcal{U},\mathcal{V}) \\
&= \left| \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \right| R(\tau,\kappa,\mathcal{U},\mathcal{V}) \frac{G'(\tau)}{|G'(\tau)|}. \qquad (4.16)
\end{aligned}
$$

As in Lemma 2.2, we write $G = 2^{-1}e^{i\phi}G_0^2$, where G_0 is starlike in E with $G_0(0) = G_0'(0) - 1 = 0$. Consequently, for any $\tau_0 \in E\backslash\{0\}$, the line segment connecting 0 to $G_0(\tau_0)$ is entirely contained within $G_0(E)$. We define γ_0 as the curve given by the following:

$$
\gamma_0 : \tau(t) = G_0^{-1}(tG_0(\tau_0)), \quad t \in [0,1]. \qquad (4.17)
$$

Moreover,

$$
G(\tau(t)) = \frac{1}{2}e^{i\phi}(G_0(\tau(t)))^2 = \frac{1}{2}e^{i\phi}(tG_0(\tau_0))^2 = t^2 G(\tau_0).
$$

Differentiating the expression with respect to t results in the following:

$$G^{'}(\tau(t))\tau^{'}(t) = 2tG(\tau_0),\ t \in [0,1]. \tag{4.18}$$

Using (4.18) and (4.16), we have the following:

$$\begin{aligned}
&\log H_{e^{i\phi},\kappa}(\tau) - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} L(\gamma_0,\kappa,\mathcal{U},\mathcal{V}) \\
&= \int_0^1 \left(\frac{H^{'}_{e^{i\phi},\kappa}(\tau(t))}{H_{e^{i\phi},\kappa}(\tau(t))} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \right) \tau^{'}(t)\, dt \\
&= \left| \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \right| \int_0^1 R(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \frac{G^{'}(\tau(t))\tau^{'}(t)}{|G^{'}(\tau(t))\tau^{'}(t)|} \left|\tau^{'}(t)\right| dt \\
&= \frac{G^{'}(\tau_0)}{|G(\tau_0)|} \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi\mathcal{V}} \int_0^1 R(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \left|\tau^{'}(t)\right| dt \\
&= \frac{G^{'}(\tau_0)}{|G(\tau_0)|} \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi\mathcal{V}} M(\gamma_0,\kappa,\mathcal{U},\mathcal{V}),
\end{aligned}$$

where $L(\gamma_0,\kappa,\mathcal{U},\mathcal{V})$ and $M(\gamma_0,\kappa,\mathcal{U},\mathcal{V})$ are defined in (4.10) and (4.11), respectively. Thus, we have the following:

$$\log H_{e^{i\phi},\kappa}(\tau) \in \partial\overline{E}\left(\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} L(\gamma_0,\kappa,\mathcal{U},\mathcal{V}),\ \left| \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \right| M(\gamma_0,\kappa,\mathcal{U},\mathcal{V}) \right).$$

Additionally, from Corollary 4.4, we have the following:

$$\log H_{e^{i\phi},\kappa}(\tau) \in V(\tau_0,\kappa,\mathcal{U},\mathcal{V}) \subset \overline{E}\left(\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} L(\gamma_0,\kappa,\mathcal{U},\mathcal{V}),\ \left| \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \right| M(\gamma_0,\kappa,\mathcal{U},\mathcal{V}) \right).$$

Hence, we conclude that $\log H_{e^{i\phi},\kappa}(\tau) \in \partial V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$. To establish uniqueness, assume that there exists a function $f \in \Omega_\rho(\kappa,\mathcal{U},\mathcal{V})$ and an angle $\phi \in (-\pi,\pi]$ such that

$$\log f(\tau_0) = \log H_{e^{i\phi},\kappa}(\tau).$$

Consider the following:

$$h(t) = \frac{\overline{G'(\tau_0)}}{|G(\tau_0)|} \left(\frac{f'(\tau(t))}{f(\tau(t))} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} Q(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \right) \tau^{'}(t), \tag{4.19}$$

where $\gamma_0 : \tau(t),\ t \in [0,1]$, is given by (4.17). Consequently, $h(t)$ is a continuous function in $[0,1]$ and satisfies the following:

$$|h(t)| \leq \left| \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}} \right| R(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \left|\tau^{'}(t)\right|.$$

Moreover, (4.19) implies that

$$\int_0^1 \Re h(t) = \int_0^1 \Re\left(\frac{\overline{G'(\tau_0)}}{|G(\tau_0)|} \left(\frac{f'(\tau(t))}{f(\tau(t))} - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi} Q(\tau(t),\kappa,\mathcal{U},\mathcal{V}) \right) \tau^{'}(t) \right)$$

$$= \Re\left(\frac{\overline{G'(\tau_0)}}{|G(\tau_0)|}\left(\log(f(\tau_0)) - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}L(\gamma_0,\kappa,\mathcal{U},\mathcal{V})\right)\right)$$
$$= \Re\left(\frac{\overline{G'(\tau_0)}}{|G(\tau_0)|}\left(H_{e^{i\phi},\kappa}(\tau_0) - \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi}L(\gamma_0,\kappa,\mathcal{U},\mathcal{V})\right)\right)$$
$$= \frac{|\rho(\mathcal{U}-\mathcal{V})|}{2\pi}\int_0^1 R(\tau(t),\kappa,\mathcal{U},\mathcal{V})\left|\tau'(t)\right| dt, \text{ for all } t\in[0,1].$$

From (4.16) and (4.18), it follows that

$$\frac{f'}{f} = \frac{H'_{e^{i\phi},\kappa}(\tau_0)}{H_{e^{i\phi},\kappa}(\tau_0)} \text{ on } \gamma_0.$$

An application of the Identity Theorem for AF yields the following:

$$\frac{f'}{f} = \frac{H'_{e^{i\phi},\kappa}(\tau_0)}{H_{e^{i\phi},\kappa}(\tau_0)} \text{ in } E.$$

Thus, after normalization,

$$f = H_{e^{i\phi},\kappa}(\tau_0) \text{ in } E.$$

□

Theorem 4.6. *For $\kappa \in E$ and $\tau_0 \in E\backslash\{0\}$, the boundary $\partial V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$ of the Jordan curve given by the following:*

$$(-\pi,\pi] \ni \phi \to \log H_{e^{i\phi},\kappa}(\tau_0) = \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_o^{\tau_0}\frac{\delta\left(e^{i\phi}\zeta,\kappa\right)-1}{(1+\mathcal{V}\zeta\delta(e^{i\phi}\zeta,\kappa))(1+\mathcal{V}\zeta)}d\zeta.$$

If $\log f(\tau_0) = \log H_{e^{i\phi},\kappa}(\tau_0)$ for some $f\in\Omega_\rho(\kappa,\mathcal{U},\mathcal{V})$ and $\phi\in(-\pi,\pi]$, then $f(\tau) = H_{e^{i\phi},\kappa}(\tau)$.

Proof. Initially, we must establish that the closed curve

$$(-\pi,\pi] \ni \phi \to \log H_{e^{i\phi},\kappa}(\tau_0) = \frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_o^{\tau_0}\frac{\delta\left(e^{i\phi}\zeta,\kappa\right)-1}{(1+\mathcal{V}\zeta\delta(e^{i\phi}\zeta,\kappa))(1+\mathcal{V}\zeta)}d\zeta$$

is simple. Suppose that

$$\log H_{e^{i\phi_1},\kappa}(\tau_0) = \log H_{e^{i\phi_2},\kappa}(\tau_0)$$

for some $\phi_1,\phi_2\in(-\pi,\pi]$ with $\phi_1\neq\phi_2$. Then, from Theorem 4.5, we have the following:

$$H_{e^{i\phi_1},\kappa} = H_{e^{i\phi_2},\kappa}.$$

From (3.2), this gives a contradiction that

$$\mathcal{U}\left(\frac{w_{H_{e^{i\phi_1},\kappa}}}{\tau},\kappa,\mathcal{U},\mathcal{V}\right) = \mathcal{U}\left(\frac{w_{H_{e^{i\phi_2},\kappa}}}{\tau},\kappa,\mathcal{U},\mathcal{V}\right)$$

$$\mathcal{U}\left(\delta\left(e^{i\phi_1}\tau,\kappa\right),\kappa,\mathcal{U},\mathcal{V}\right)=\mathcal{U}\left(\delta\left(e^{i\phi_2}\tau,\kappa\right),\kappa,\mathcal{U},\mathcal{V}\right)$$
$$\frac{-\overline{\kappa}-\mathcal{V}\delta\left(e^{i\phi_1}\tau,\kappa\right)}{1+\kappa\mathcal{V}\delta\left(e^{i\phi_1}\tau,\kappa\right)}=\frac{-\overline{\kappa}-\mathcal{V}\delta\left(e^{i\phi_2}\tau,\kappa\right)}{1+\kappa\mathcal{V}\delta\left(e^{i\phi_2}\tau,\kappa\right)}$$
$$\frac{-\overline{\kappa}-\mathcal{V}\delta\left(e^{i\phi_1}\tau,\kappa\right)}{1+\kappa\mathcal{V}\delta\left(e^{i\phi_1}\tau,\kappa\right)}=\frac{-\overline{\kappa}-\mathcal{V}\delta\left(e^{i\phi_2}\tau,\kappa\right)}{1+\kappa\mathcal{V}\delta\left(e^{i\phi_2}\tau,\kappa\right)}$$
$$\frac{-\overline{\kappa}-\mathcal{V}\left(\frac{e^{i\phi_1}\tau+\kappa}{1+\overline{\kappa}e^{i\phi_1}\tau}\right)}{1+\kappa\mathcal{V}\left(\frac{e^{i\phi_1}\tau+\kappa}{1+\overline{\kappa}e^{i\phi_1}\tau}\right)}=\frac{-\overline{\kappa}-\mathcal{V}\left(\frac{e^{i\phi_2}\tau+\kappa}{1+\overline{\kappa}e^{i\phi_2}\tau}\right)}{1+\kappa\mathcal{V}\left(\frac{e^{i\phi_1}\tau+\kappa}{1+\overline{\kappa}e^{i\phi_2}\tau}\right)}$$
$$\frac{-\overline{\kappa}\left(1+\overline{\kappa}e^{i\phi_1}\tau\right)-\mathcal{V}\left(e^{i\phi_1}\tau+\kappa\right)}{\left(1+\overline{\kappa}e^{i\phi_1}\tau\right)+\kappa\mathcal{V}\left(e^{i\phi_1}\tau+\kappa\right)}=\frac{-\overline{\kappa}\left(1+\overline{\kappa}e^{i\phi_2}\tau\right)-\mathcal{V}\left(e^{i\phi_2}\tau+\kappa\right)}{\left(1+\overline{\kappa}e^{i\phi_2}\tau\right)+\kappa\mathcal{V}\left(e^{i\phi_1}\tau+\kappa\right)}.$$

After some simplification, we obtain the following:

$$\tau e^{i\phi_1}=\tau e^{i\phi_2}.$$

The contradiction implies the curve's simplicity. Therefore, $V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$ forms a compact convex subset of $\mathbb{C}$ with a nonempty interior. Moreover, the curve is contained within the boundary $\partial V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$,

$$(-\pi,\pi]\ni\phi\to\log H_{e^{i\phi},\kappa}(\tau_0)=\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_o^{\tau_0}\frac{\delta\left(e^{i\phi}\zeta,\kappa\right)-1}{\left(1+\mathcal{V}\zeta\delta\left(e^{i\phi}\zeta,\kappa\right)\right)\left(1+\mathcal{V}\zeta\right)}d\zeta.$$

Since no simple closed curve can enclose another simple closed curve except for itself, we conclude that $\partial V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$ is determined by the following:

$$(-\pi,\pi]\ni\phi\to\log H_{e^{i\phi},\kappa}(\tau_0)=\frac{\rho(\mathcal{U}-\mathcal{V})}{2\pi\mathcal{V}}\int_o^{\tau_0}\frac{\delta\left(e^{i\phi}\zeta,\kappa\right)-1}{\left(1+\mathcal{V}\zeta\delta\left(e^{i\phi}\zeta,\kappa\right)\right)\left(1+\mathcal{V}\zeta\right)}d\zeta.$$

□

5. Conclusions

In this work, we considered the class $\Omega_\rho(\mathcal{U},\mathcal{V})$ of the non-vanishing AF f in the unit disk E, normalized such that $f(0)=1$ with $\rho\in\mathbb{C}$ satisfying $\Re(\rho)>0$, and fulfilling the condition $\Re\left(p_f(\tau)\right)>0$ in E, where

$$p_f(\tau)=\frac{2\pi}{\rho}\frac{\tau f'(\tau)}{f(\tau)}+\frac{1+\mathcal{U}\tau}{1+\mathcal{V}\tau}.$$

Our goal was to determine the region of variability $V(\tau_0,\kappa,\mathcal{U},\mathcal{V})$ for $\log(f(\tau_0)$ as f ranges over the following class:

$$\Omega_\rho(\kappa,\mathcal{U},\mathcal{V})=\left\{f\in\Omega_\rho:f'(0)=\frac{\rho(\mathcal{U}-\mathcal{V})(\kappa-1)}{2\pi\mathcal{V}},\ \mathcal{U}\in\mathbb{C},\ \mathcal{V}\in[-1,0)\right\}.$$

Through this investigation, we established explicit conditions and structural insights regarding the region of variability. The results provide a detailed characterization of $V(\tau_0, \kappa, \mathcal{U}, \mathcal{V})$ in terms of the parameters ρ, $\mathcal{U}$, and $\mathcal{V}$. This enhances the understanding of the behavior of the logarithmic mapping of functions within this class and their dependency on the defining parameters.

Our findings significantly contribute to the broader theory of AFs, particularly in understanding classes defined by non-vanishing conditions and real part constraints. Furthermore, the methodology employed can serve as a framework to examine the variability regions for other classes of functions with similar or more generalized constraints.

Future research could extend these results by exploring the impact of higher-order parameter variations or applying the established framework to other domains in the geometric function theory.

Author contributions

Bilal Khan: Methodology, conceptualization, writing-original draft preparation, reviewing, editing; Wafa F. Alfwzan: Methodology, conceptualization, writing-original draft, reviewing, editing; Khadijah M. Abualnaja: Methodology, conceptualization, writing-original draft preparation, reviewing, editing; Manuela Oliveira: Methodology, conceptualization, writing-original draft, reviewing, editing. All authors have read and approved the manuscript.

Use of Generative-AI tools declaration

The authors declare they has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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