Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 12935–12955.
DOI: 10.3934/math.2025582
Received: 25 January 2025
Revised: 01 May 2025
Accepted: 20 May 2025
Published: 5 June 2025

Research article

Recent advancements in M-cone metric space over Banach algebra endowed with binary relation

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Abstract: The objectives of this paper are three steps: first, to generalize the idea of complete M-cone metric spaces over Banach algebra; second, to present a new topological structure utilizing the concept introduced by Fernandez et al. (Fixed point results in M-cone metric space over Banach algebra with an application, *Filomat*, **36** (2022), 5547–5562.); and third, to explore the idea of Banach algebra type relational theoretic contractions and cyclic Banach algebra type contractions in M-cone metric spaces, establishing several fixed point results for these contractions. To illustrate the discussed concepts and results, several examples are provided. As an application, we discuss a solution of the nonlinear integral equation based on the main results.

Keywords: fixed point; M-cone metric spaces over Banach algebra; relation-theoretic contractions; nonlinear integral equation
Mathematics Subject Classification: 36A07, 48H11, 58H25

1. Introduction

Fixed point theory is a fundamental tool for addressing a wide range of problems across different fields of study. Numerous scholars have explored the potential to transform and revolutionize the concepts of metrics and metric spaces. The Banach contraction fixed point theorem has been extended in numerous ways over the years; we refer the reader to [1, 2] and the references therein for further details. In 2003, Kirk et al. [3] introduced the concept of a cyclic contraction, which extends the Banach contraction principle in a generalized metric space. They also established several fixed point results using this type of operator. The concept of a partial metric, which broadened and extended the idea of a metric space, was introduced by Matthews [4] in 1994. Indeed, the most significant difference from a metric is that self-distances of some points in partial metric spaces may not be zero. In the recent past, Huang and Zhang [5] proposed the concept of cone metric spaces as a broader generalization of metric spaces. Liu et al. [6] polished the concept of cone metric spaces and extended the fixed point results of Huang et al. [5] by developing cone metric spaces with Banach algebras. In this case, by substituting the real Banach space E with a Banach algebra β. However, they continued with results that focused entirely on normal cones. The Banach contraction Principle theorem in the setting of normal cone space with Banach algebra is stated as follows: Let (Ω, d) be a cone metric with Banach algebra, and let ξ be a normal cone with a normal constant g. Suppose that the self-mapping satisfies the generalized Lipschitz condition $d(\Upsilon(\varsigma), \Upsilon(s)) \leq gd(\varsigma, s)$, for all $\varsigma, s \in \Omega$, where $g \in \rho$ with $\rho(g) < 1$. Then Υ has a unique fixed point in Ω. And for any $\varsigma \in \Omega$, the iterative sequence $\{\Upsilon(s_n)\}$ converges to the fixed point. In 2015, Imdad et al. [7] introduced the notion of relation-contraction mapping and proved several fixed point results. After that, many researchers have generalized the idea of relation-theoretic contraction in various directions, including metric spaces, cone metric spaces, and C*-algebra valued metric spaces. See references [8–12].

In 2014, Asadi et al. [13] announced the idea of an M-metric space and established the Banach contraction principle within this context. Next, many fixed point results in M-metric spaces have been given by several mathematicians, see [14–16]. Very recently, Fernandez et al. [17] introduced the notion of M-cone metric spaces over Banach algebra, which generalizes both M-metric spaces and cone metric spaces over Banach algebra, and established some fixed point results. In our paper, we generalize the concept introduced by Fernandez et al. [17] with new topological properties and prove remarkable fixed point results in complete M-cone metric spaces over Banach algebra endowed with a binary relation.

2. Preliminaries

Throughout this paper, we use the symbol $\mathbb{N}$ to indicate the set of positive integers, and $\mathbb{N}_0$ to indicate the set of nonnegative integers. Similarly, we denote the set of real numbers and the set of positive real numbers as $\mathbb{R}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, respectively.

First, we review some established definitions and results that are utilized in this work.

Definition 2.1. [18] A real-Banach algebra β is a real Banach space that includes a multiplication operation defined as follows:

for all $\varsigma, s, l \in \beta$ and $\sigma \in \mathbb{R}$:
(i) $(\varsigma s) l = \varsigma (sl)$;
(ii) $\varsigma (s + l) = \varsigma s + \varsigma l$ and $(\varsigma + s) l = \varsigma l + sl$;
(iii) $\sigma (\varsigma s) = (\sigma \varsigma) s = \varsigma (\sigma s)$;
(iv) $\|\varsigma s\| \leq \|\varsigma\| \|s\|$.

In this paper, we assume that every Banach algebra has a unit (i.e., a multiplicative identity) e, which satisfies

$$e\varsigma = \varsigma e = \varsigma \text{ for all } s \text{ in } \beta.$$

An element $\varsigma \in A$ is considered invertible if there exists an element $s \in A$ such that $\varsigma s = s\varsigma = e$. The inverse of ς is denoted by ς^{-1}. Further details can be found in [18].

Definition 2.2. [18] A subset ξ of β of a Banach algebra is referred to as a cone if it satisfies specific properties (to be defined).

(i) ξ is non-empty, closed and $\{\vartheta, e\} \subset \beta$;
(ii) $\sigma\xi + \varsigma\xi \subset \beta$;
(iii) $\xi^2 = \xi\xi \subset \xi$;
(iv) $\xi \cap (-\xi) = \{\vartheta\}$.

Let ϑ and e represent the zero and unit elements of the Banach algebra β, respectively. For a given cone $\xi \subseteq \beta$, a partial ordering is defined as follows:

$$\varsigma \preceq s \text{ if and only if } s - \varsigma \text{ in } \xi.$$

It is well-known that $s \prec \varsigma$ indicates $\varsigma \preceq s$ and $\varsigma \neq s$. While $\varsigma \ll s$ implies $s - \varsigma$ in $int(\xi)$, where $int(\xi)$ denotes the interior of ξ. The cone ξ is said to be solid if $int(\xi) \neq \emptyset$. The cone ξ is said to be normal if there exists a constant $h > 1$ such that for all ς, s in ξ,the inequality holds.

$$\vartheta \preceq \varsigma \preceq s \text{ implies that } \|\varsigma\| \leq h\,\|s\|.$$

In this work, we consistently assume that β is a Banach algebra with a unit element ϑ. Moreover, ξ denotes a solid cone in β, and $\preceq$ represents the partial ordering associated with ξ. Here, we provide the basic flow chart of cone metric (CM), Banach alebra (BA), and cone metric over Banach algebra (CMB). See Figure 1.

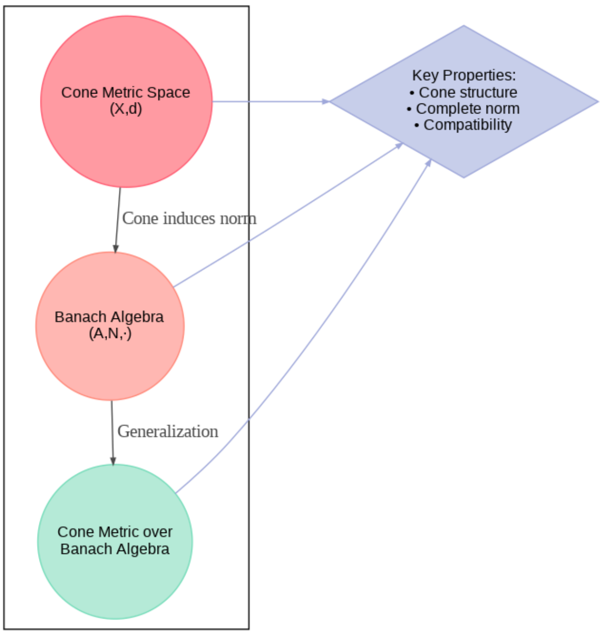

Figure 1. Flow chart of CM, BA, and CMB.

Some propositions of cone metric spaces with Banach algebra and relatedv concepts from [18, 19] are given as follows:

Proposition 2.1. *From the reference [18], let β be a Banach algebra with a unit element e, and let $\varsigma \in \beta$. If the spectral radius $\rho(g)$ of g satisfies $\rho(g) < 1$, that is,*

$$\rho(g) = \lim_{n\to\infty} \|g^n\|^{\frac{1}{n}} < 1.$$

Then, $(e - g)$ is invertible. In fact, its inverse is given by

$$(e - g)^{-1} = \sum_{i=0}^{\infty} g^i.$$

Proposition 2.2. *[20] If E is a real Banach space equipped with a cone ξ, then the following conditions hold:*

(i) $p \preceq ap$ with p in ξ and $0 \leq a < 1$, then $a = \vartheta$;

(ii) If $h \in \xi$, and $\vartheta \preceq h \ll \sigma$ for every $\vartheta \ll \sigma$, then $h = \vartheta$, and $\|g_n\| \to 0$ as $n \to \infty$, then for any $\vartheta \ll \sigma$ there exists n_0 in $\mathbb{N}$ such that, $g_n \ll \sigma$ for all $n < n_0$.

Remark 2.1. From the reference [18], it follows that the spectral radius $\rho(g)$ of g satisfies

$$\rho(g) \leq \|g\|, \text{ for all } g \text{ in } \xi,$$

where ξ is a Banach algebra with a unit element e.

Remark 2.2. If the condition $\rho(g) < 1$ in Proposition 2.1 is replaced with $g \leq 1$, the conclusion still holds (see [19]).

Remark 2.3. From the reference [19]. If in remark 2.2 $\rho(g) < 1$, then $\|g^n\|$ converges to 0 as n converges to $+\infty$.

Remark 2.4. If $p, q, c \in \xi$ and $p \ll q$, then $pc \ll qc$.

Definition 2.3. [6] Consider a non-empty set Ω. Then, a mapping $\varrho : \Omega \times \Omega \to \beta$ is called a cone metric (or CM) on Ω if for all $\varsigma, s, l \in \Omega$, the following conditions hold:
(i)$\vartheta \preceq \varrho(\varsigma, s)$ and $\varrho(\varsigma, s) = \vartheta \Longleftrightarrow \varsigma = s$;
(ii) $\varrho(\varsigma, s) = \varrho(s, \varsigma)$;
(iv) $\varrho(\varsigma, s) \preceq \varrho(\varsigma, l) + \varrho(l, s)$;

Then, the pair (Ω, ϱ) is known as cone metric space over Banach algebra β.

After that, Fernandez et al. [11] gave the idea of partial cone metric over Banach algebra and opened the new window of research for researchers.

Definition 2.4. [11] Consider a non-empty set Ω. Then a mapping $P : \Omega \times \Omega \to \beta$ is called partial cone metric (or PCM) on Ω if for all $\varsigma, s, l \in \Omega$, the following conditions hold:
(i) $P(\varsigma, s) = P(\varsigma, \varsigma) = P(s, s) = \vartheta \Longleftrightarrow \varsigma = s$;
(ii) $P(\varsigma, \varsigma) \preceq P(\varsigma, s)$;
(iii) $P(\varsigma, s) = P(s, \varsigma)$;
(iv) $P(\varsigma, s) \preceq P(\varsigma, l) + P(l, s) - P(l, l)$;

Then, the pair (Ω, P) is known as PCM-space over Banach algebra β. Very recently, Fernandez 'et al. introduce the new concept of cone M-metric over Banach algebra in the following ways, see [17]

The following notations are required to define an M-cone metric space (CMM-space) over Banach algebra. The following notations are utilized in our presentation.

$$\begin{aligned} \mathbb{C}_{\varsigma,s} &= \min\{\mathbb{C}(\varsigma,\varsigma), \mathbb{C}(s,s)\}, \\ M_{\varsigma,s} &= \max\{\mathbb{C}(\varsigma,\varsigma), \mathbb{C}(s,s)\}. \end{aligned}$$

Definition 2.5. [17] Consider a non-empty set Ω. Then a mapping $\mathbb{C} : \Omega \times \Omega \to \beta$ cone metric (or CM) on Ω if for all $\varsigma, s, l \in \Omega$, the following conditions hold:
(i) $\mathbb{C}(\varsigma,\varsigma) = \mathbb{C}(s,s) = \mathbb{C}(\varsigma,s) \Longleftrightarrow \varsigma = s$;
(ii) $\mathbb{C}_{\varsigma,s} \preceq \mathbb{C}(\varsigma,s)$;
(iii) $\mathbb{C}(\varsigma,s) = \mathbb{C}(s,\varsigma)$;
(iv) $\mathbb{C}(\varsigma,s) - \mathbb{C}_{\varsigma,s} \preceq (\mathbb{C}(\varsigma,l) - \mathbb{C}_{\varsigma,l}) + (\mathbb{C}(l,s) - \mathbb{C}_{l,s})$.

Then the couple $(\Omega, \mathbb{C})$ is known as CMM-space over Banach algebra.

It is clear that every PCM-space over a Banach algebra is an CMM-space over a Banach algebra. However, the converse may not be true as seen in the following examples:

Definition 2.6. Suppose that $\beta = C_R^{'}[0,1]$ is the space of real-valued continuous functions on $[0,1]$ equipped with the norm

$$\|\varsigma\| = \|\varsigma\|_\infty + \left\|\varsigma^{'}\right\|_\infty,$$

where $\left\|\varsigma^{'}\right\|_\infty$ denotes the supremum norm. Under the usual multiplication. β is a real-unit Banach algebra with the unit element $e = 1$. Consider a cone $\xi = \{\varsigma \in \beta : \varsigma \geq 0\}$ in β. Moreover, ξ is a non-normal cone. Assume that $\Omega = [0, +\infty)$, and define $\mathbb{C} : \Omega \times \Omega \to \beta$

$$\mathbb{C}(\varsigma,s)(t) = \left(\frac{\varsigma + s}{2}\right)e^t \text{ for all } \varsigma, s \text{ in } \Omega.$$

Thus, $\mathbb{C}$ is an CMM on Ω, but it does not form a PCM-space over the Banach algebra according to the reference [17].

Remark 2.5. For all $\varsigma, s \in \Omega$
(i) $\vartheta \preceq M_{\varsigma,s} + \mathbb{C}_{\varsigma,s} = \mathbb{C}(\varsigma,\varsigma) + \mathbb{C}(s,s)$,
(ii) $\vartheta \preceq M_{\varsigma,s} - \mathbb{C}_{\varsigma,s} = |\mathbb{C}(\varsigma,\varsigma) - \mathbb{C}(s,s)|$,
(iii) $M_{\varsigma,s} - \mathbb{C}_{\varsigma,s} \preceq (M_{\varsigma,l} - \mathbb{C}_{\varsigma,l}) + (M_{l,s} - \mathbb{C}_{l,s})$.

Definition 2.7. [17] Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β. The collection of all m-open balls on Ω given by

$$B = \{B_m(\varsigma,\sigma) : \varsigma \in \Omega, \ \sigma \gg \vartheta\},$$

forms a basis for the topology on Ω.

Definition 2.8. [17] Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β, and let $\{\varsigma_n\}$ be a sequence in Ω. If for every $\sigma \in int\xi$, there exists a positive integer n_0 such that

$$\mathbb{C}(\varsigma_n,\varsigma) \ll \sigma + \mathbb{C}_{\varsigma_n,\varsigma} \text{ for all } n > n_0,$$

then $\{\varsigma_n\}$ is said to converge, and its limit is ς.

Definition 2.9. [17] Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β. A sequence $\{\varsigma_n\} \in \Omega$ is called a ϑ-Cauchy sequence if, for every $\sigma \gg \vartheta$ there exists $n_0 \in \mathbb{N}$ such that

$$\mathbb{C}(\varsigma_m, \varsigma_n) - m_{\varsigma_m,\varsigma_n} \ll \sigma \text{ and } M_{\varsigma_m,\varsigma_n} - \mathbb{C}_{\varsigma_m,\varsigma_n} \ll \sigma \text{ for all } m > n > n_0.$$

Definition 2.10. [17] Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β. The $(\Omega, \mathbb{C})$ is said to be ϑ-complete if every ϑ-Cauchy sequence $\{\varsigma_n\} \in \Omega$ converges to a point $\varsigma \in \Omega$, i.e.,

$$\lim_{n\to\infty} M_{\varsigma_n,\varsigma} = \lim_{n\to\infty} \mathbb{C}_{\varsigma_n,\varsigma} = \mathbb{C}(\varsigma, \varsigma) = \vartheta.$$

Theoretical relations have been extensively discussed in various research articles. For instance, refer to [7, 21]. Subsequently, we present basic concepts of relations.

Definition 2.11. [21] Consider Ω, be a nonempty set, and let $\acute{R}$ be a subset of $\Omega \times \Omega$ referred to as a binary relation on Ω. For every pair $\varsigma, s \in \Omega$ one of the following conditions applies:
(i) $(\varsigma, s) \in \acute{R}$ indicates that ς is $\acute{R}$-related to s or ς relates to s under $\acute{R}$. Sometimes, this is written as $\varsigma\acute{R}s$ instead of $(\varsigma, s) \in \acute{R}$.
(ii) $(\varsigma, s) \notin \acute{R}$ which means that ς is not $\acute{R}$-related to s or ς is not related to s under $\acute{R}$.

By definition, $\Omega \times \Omega$ and ϕ, as subsets of $\Omega \times \Omega$, are binary relations on Ω. These are universal the relation (or full relation) and the empty relation, respectively. Throughout this paper, $\acute{R}$ represents a nonempty binary relation. For simplicity, we will use the term binary relation to mean nonempty binary relation.

Definition 2.12. [7] Let $\acute{R}$ be a binary relation on a non-empty set Ω, and let $\varsigma, s \in \Omega$. We say that ς and s are $\acute{R}$-compareable if either $(\varsigma, s) \in \acute{R}$ or $(s, \varsigma) \in \acute{R}$. This is denoted by $[\varsigma, s] \in \acute{R}$.

Definition 2.13. [21] Let $\Omega \neq \phi$ and $\acute{R}$ be a binary relation on Ω.
(i) The inverse, transpose, or dual relation of $\acute{R}$ denoted by $\acute{R}^{-1}$ is defined as

$$\acute{R}^{-1} = \left\{(\varsigma, s) \in \Omega \times \Omega : (s, \varsigma) \in \acute{R}\right\}.$$

(ii) The reflexive closure of $\acute{R}$ is denoted by $\acute{R}^{\neq}$, is defined as $\acute{R}^{\neq} = \acute{R} \cup \nabla_{\varsigma}$.
(iii) The symmetric closure of $\acute{R}$, denoted by , $\acute{R}^S = \acute{R} \cup \acute{R}^{-1}$ is defined as, where $\acute{R}^{-1}$ represents the inverse of $\acute{R}$. In other words, $\acute{R}^S$ is the smallest symmetric relation on Ω that includes $\acute{R}$.

Proposition 2.3. *[7] For a binary relation $\acute{R}$ defined on a nonempty set Ω, $(s, \varsigma) \in \acute{R}^S \Leftrightarrow [\varsigma, s] \in \acute{R}$.*

Definition 2.14. [7] Let Ω be a nonempty set, and let $\acute{R}$ be a binary relation on Ω. Then:
(i) A sequence $\{\varsigma_n\} \in \Omega$ is said to be $\acute{R}$-preserving if $(\varsigma_n, \varsigma_{n+1}) \in \acute{R}\ \forall\ n \in \mathbb{N}$.
(ii) A self-map $\Upsilon : \Omega \to \Omega$ is referred to as Υ-closed if $(\Upsilon(\varsigma), \Upsilon(s)) \in \acute{R}$ whenever $(\varsigma, s) \in \acute{R}$.

Proposition 2.4. *[7] Let Υ, Ω, and $\acute{R}$ be as defined in Definition 10. If $\acute{R}$ is Υ-closed, then $\acute{R}^S$ is also Υ-closed.*

Definition 2.15. [22, 23] Let Ω be a nonempty set, and let $\acute{R}$ be a binary relation on Ω;
(i) $W \subseteq \Omega$ is said to be $\acute{R}$-directed if, for every $\varsigma, s \in \Omega$, there exists $l \in \Omega$ such that $(\varsigma, l) \in \acute{R}$ and $(s, l) \in \acute{R}$,
(ii) For $\Upsilon \in \Omega$, a path of length i (where i is a natural number) in $\acute{R}$ from ς to s is a finite sequence $\{l_z\}_{z=0}^{i} \subseteq \Omega$ satisfying the following conditions:
(a) $l_0 = \varsigma$ and $l_0 = \varsigma$,
(b) $(l_z, l_{z+1}) \in \acute{R}$.

In this paper, we adopt the following notation as introduced in [7].
(i) $F(\Upsilon)$:the set of all fixed points of the mapping $\Upsilon : \Omega \to \Omega$.
(ii) $\Omega\left(\Upsilon, \acute{R}\right) = \{\varsigma \in \Omega : (\varsigma, \Upsilon(\varsigma))\} \in \acute{R}$ the set of points in Ω satisfying the relation $\acute{R}$ with Υ.
(iii) $\Psi\left(\varsigma, s, \acute{R}\right)$: the collection of all paths in $\acute{R} : \varsigma \to s$.

3. New topology on M-cone metric space over Banach algebra

In this section, we define the new base topology using the concept of Fenandez et al.'s [17] papsr and verify all topological properties.

Every CMM-over Banach algebra on Ω generates a T_0 topology τ_m (say) on Ω which has a base of collection of m-open balls

$$\{\mathbf{B}_m(\varsigma, \sigma) : \varsigma \in \Omega,\ \sigma \gg \vartheta\},$$

where

$$\mathbf{B}_m(\varsigma, \sigma) = \{s \in \Omega : \mathbb{C}(\varsigma, s) - m_{\varsigma,s} \ll \sigma\} \text{ for all } \varsigma \in \Omega,\ \sigma \gg \vartheta.$$

We now prove the equivalence between our new topology and the topology presented in reference [17] by stating a theorem that establishes this consequence.

Theorem 3.1. *(Equivalence of Topologies in Cone M-Metric over Banach algebra) . Let $\beta = C_R^{'}[0,1]$ be the Banach algebra of real valued continuous functions on $[0,1]$ equipped with the norm $\|\varsigma\| = \|\varsigma\|_\infty + \left\|\varsigma^{'}\right\|_\infty$, and a cone $\xi = \{\varsigma \in \beta : \varsigma \geq 0\} \in \beta$ which is closed and convex, but non-normal. Moreover, assume that $\Omega = [0, +\infty)$, and define $\mathbb{C} : \Omega \times \Omega \to \beta$*

$$\mathbb{C}(\varsigma, s)(t) = \left(\frac{\varsigma + s}{2}\right) e^t \textit{ for all } \varsigma, s \in \Omega,\ t \in [0,1].$$

Such that: Topology $(\mathbf{T}_{m_1})$

$$\mathbf{B}_m(\varsigma, \sigma) = \{s \in \Omega : \mathbb{C}(\varsigma, s) - \mathbb{C}_{\varsigma,s} \ll \sigma\} \textit{ for all } \varsigma \in \Omega,\ \sigma \gg \vartheta.$$

Topology (T_{m_2}) in reference [17]

$$B_m(\varsigma, \sigma) = \{s \in \Omega : \mathbb{C}(\varsigma, s) + \mathbb{C}(\varsigma, \varsigma) - \mathbb{C}_{\varsigma,s} - \mathbb{C}(s, s) \ll \sigma\}.$$

Then the topologies $(\mathbf{T}_{m_1})$ and (T_{m_2}) genrated $\mathbf{B}_m$ and B_m are equivalent.

Proof. To show $\mathbf{T}_{m_1}$and T_{m_2} are equavilent. We want to prove tha every $\mathbf{T}_{m_1}$-neighborhood cantains a T_{m_2}-neighborhood and every T_{m_2}-neighborhood contains a $\mathbf{T}_{m_1}$-neighborhood. First, we verify that $\mathbb{C}(\varsigma, \varsigma)$ and $\mathbb{C}(s, s)$ are, such that

$$\mathbb{C}(\varsigma, \varsigma)(t) = \varsigma e^t, \mathbb{C}(s, s) = s e^t.$$

As, $\min\{\mathbb{C}(\varsigma, \varsigma), \mathbb{C}(s, s)\}(t) = \min\{\varsigma, s\} e^t$.

For topology $(\mathbf{T}_{m_1})$,

$$\mathbb{C}(\varsigma, s) - \min\{\mathbb{C}(\varsigma, \varsigma), \mathbb{C}(s, s)\} = \left(\frac{\varsigma + s}{2} - \min\{\varsigma, s\}\right) e^t.$$

Let $\Xi_1 = \frac{\varsigma+s}{2} - \min\{\varsigma, s\}$. If $\varsigma \leq s$ then $\frac{s-\varsigma}{2}$. If $s \leq \varsigma$ then $\frac{\varsigma-s}{2}$. From both cases, we have $\Xi_1 = \frac{|s-\varsigma|}{2}$,

$$\mathbf{T}_{m_1}\text{-neighborhood: } \left(\frac{|s-\varsigma|}{2}\right)e^t \ll \sigma$$

For topology (T_{m_2})

$$\begin{aligned} & \mathbb{C}(\varsigma, s) + \mathbb{C}(\varsigma, \varsigma) - \min\{\mathbb{C}(\varsigma, \varsigma), \mathbb{C}(s, s)\} - \mathbb{C}(s, s) \\ = & \left(\frac{\varsigma + s}{2} + \varsigma - \min\{\varsigma, s\} - s\right)e^t. \end{aligned}$$

Simplifying the scalar part

$$\frac{3\varsigma - s}{2} - \min\{\varsigma, s\}.$$

If $\varsigma \leq s$ then $\min\{\varsigma, s\} = \varsigma$, thus, we have $\frac{3\varsigma-s}{2} - \varsigma = \frac{\varsigma-s}{2}$. If $s \leq \varsigma$ then $\min\{\varsigma, s\} = s$; thus, we obtain, $\frac{3\varsigma-s}{2} - s = \frac{3\varsigma-3s}{2}$. From both cases, the T_{m_2}-condition reduces to

$$\begin{cases} \left(\frac{\varsigma-s}{2}\right)e^t \ll \sigma & \text{if } \varsigma \leq s, \\ \left(\frac{3(\varsigma-s)}{2}\right)e^t \ll \sigma & \text{if } s \leq \varsigma. \end{cases}$$

We now aim to prove that $\mathbf{T}_{m_1} \subseteq T_{m_2}$.

For any element $s \in \mathbf{B}_m(\varsigma, \sigma)$, we have $\frac{|s-\varsigma|}{2}e^t \ll \sigma$; if $\varsigma \leq s$, the $\mathbf{T}_{m_1}$-condition is identical to T_{m_2}. If $s \leq \varsigma$, $\frac{(\varsigma-s)}{2}e^t \ll \sigma$, implies that $\frac{3(\varsigma-s)}{2}e^t \ll \frac{3}{2}\sigma$. Since, $\sigma \gg 0$, and $\frac{2}{2}\sigma \gg \sigma$. Therefore, $\mathbf{T}_{m_1}$-neighborhood implies T_{m_2}-neighborhood. i.e. $s \in B_m(\varsigma, \sigma)$.

Next, we show that, $T_{m_2} \subseteq \mathbf{T}_{m_1}$.

For $s \in B_m(\varsigma, \sigma)$: If $\varsigma \leq s$, the T_{m_2}-condition is identical to $\mathbf{T}_{m_1}$, if $s \leq \varsigma$, $\frac{3(\varsigma-s)}{2}e^t \ll \sigma$ implies that $\frac{(\varsigma-s)}{2}e^t \ll \frac{1}{3}\sigma \ll \sigma$. Hence, $s \in \mathbf{B}_m(\varsigma, \sigma)$. Thus, proof is done. □

Now, we describe the convergence, Cauchyness, and Completeness in CMM-space over Banach algebra according to our new topology:

Definition 3.1. Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β, and let $\{\varsigma_n\}$ be a sequence in Ω. If for every $\sigma \in int\xi$, there exists a $n_0 \in \mathbb{N}$ such that

$$\mathbb{C}(\varsigma_n, \varsigma) \ll \sigma + \mathbb{C}_{\varsigma_n,\varsigma} \text{ for all } n > n_0,$$

then $\{\varsigma_n\}$ is said to converge, and its limit is ς.

Definition 3.2. Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β. A sequence $\{\varsigma_n\} \in \Omega$ is called a ϑ-Cauchy sequence if, for every $\sigma \gg \vartheta$, there exists a $n_0 \in \mathbb{N}$ such that

$$\mathbb{C}(\varsigma_m, \varsigma_n) - \mathbb{C}_{\varsigma_m,\varsigma_n} \ll \sigma \text{ and } M_{\varsigma_m,\varsigma_n} - \mathbb{C}_{\varsigma_m,\varsigma_n} \ll \sigma \text{ for all } m > n > n_0.$$

Definition 3.3. Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β. The $(\Omega, \mathbb{C})$ is said to be ϑ-complete if every ϑ-Cauchy sequence $\{\varsigma_n\} \in \Omega$ converges to a point $\varsigma \in \Omega$, i.e.,

$$\lim_{n\to\infty}(\mathbb{C}(\varsigma_n, \varsigma) - \mathbb{C}_{\varsigma_n,\varsigma}) = \vartheta \text{ and } \lim_{n\to\infty}(M_{\varsigma_n,\varsigma} - \mathbb{C}_{\varsigma_n,\varsigma}) = \vartheta.$$

Here, we will prove the equivalence of the two definitions (2.10 and 3.3) of ϑ-completeness for an CMM-space $(\Omega, \mathbb{C})$ over a Banach algebra:

Proposition 3.1. *Let $(\Omega, \mathbb{C})$ be a CMM-space over a Banach algebra β with cone ξ, and let $\vartheta \in \xi$. The following two statements of ϑ-completeness are equivalent.*
(i) A sequence $\{\varsigma_n\} \in \Omega$ converges to $\varsigma \in \Omega$ if:

$$\lim_{n\to\infty} M_{\varsigma_{n,\varsigma}} = \mathbb{C}(\varsigma, \varsigma) = \vartheta \textit{ and } \lim_{n\to\infty} \mathbb{C}_{\varsigma_{n,\varsigma}} = \mathbb{C}(\varsigma, \varsigma) = \vartheta.$$

(ii) A sequence $\{\varsigma_n\} \in \Omega$ converges to $\varsigma \in \Omega$ if:

$$\lim_{n\to\infty} \left(M_{\varsigma_{n,\varsigma}} - \mathbb{C}_{\varsigma_{n,\varsigma}}\right) = \lim_{n\to\infty} (\vartheta - \vartheta) = \vartheta.$$

Proof. (i) $\Rightarrow$ (ii)

From condition (i), we have

$$\lim_{n\to\infty} M_{\varsigma_{n,\varsigma}} = \mathbb{C}(\varsigma, \varsigma) = \vartheta \text{ and } \lim_{n\to\infty} \mathbb{C}_{\varsigma_{n,\varsigma}} = \mathbb{C}(\varsigma, \varsigma) = \vartheta.$$

Since $M_{\varsigma_{n,\varsigma}} \geq \mathbb{C}_{\varsigma_{n,\varsigma}}$, for all n, we can write

$$M_{\varsigma_{n,\varsigma}} - \mathbb{C}_{\varsigma_{n,\varsigma}} \geq \vartheta.$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} \left(M_{\varsigma_{n,\varsigma}} - \mathbb{C}_{\varsigma_{n,\varsigma}}\right) = \lim_{n\to\infty} (\vartheta - \vartheta) = \vartheta.$$

Thus, condition (ii) holds if the condition (i) is true.
condition (ii)$\Rightarrow$condition (i)

From condition (ii), we know that

$$\lim_{n\to\infty} \left(M_{\varsigma_{n,\varsigma}} - \mathbb{C}_{\varsigma_{n,\varsigma}}\right) = \vartheta.$$

Since,

$$\begin{aligned} M_{\varsigma_{n,\varsigma}} &= \max\{\mathbb{C}(\varsigma, \varsigma), \mathbb{C}(\varsigma, \varsigma)\}, \\ \mathbb{C}_{\varsigma_{n,\varsigma}} &= \min\{\mathbb{C}(\varsigma, \varsigma), \mathbb{C}(\varsigma, \varsigma)\}. \end{aligned}$$

Therefore, the condition $\lim_{n\to\infty} \left(M_{\varsigma_{n,\varsigma}} - \mathbb{C}_{\varsigma_{n,\varsigma}}\right) = \vartheta$ implies that:

$$\lim_{n\to\infty} M_{\varsigma_{n,\varsigma}} = \lim_{n\to\infty} \mathbb{C}_{\varsigma_{n,\varsigma}} = \vartheta.$$

This is equivalent to the condition (i), as it guarantees that ς_n converges to $\varsigma \in \Omega$ with $\mathbb{C}(\varsigma, \varsigma) = \vartheta$. □

4. Main results

In this section, we first prove some propositions and consider the existence of fixed points for mappings in complete CMM-space over a Banach algebra endowed with binary relation.

Definition 4.1. Let $(\Omega, \mathbb{C})$ be a complete CMM-space over a Banach algebra β,with a solid cone ξ and endowed with binary relation $\acute{R}$. A self-mapping Υ on Ω is referred to as relational theoretic contraction if there exists $g \in \xi$ so that $\rho(g) < 1$ and

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \leq g\mathbb{C}(\varsigma, s)$$

for all $\varsigma, s \in \Omega$ with $(\varsigma, s) \in \acute{R}$. The vector g is referred to as the vector of Υ.

Proposition 4.1. *Let $(\Omega, \mathbb{C})$ be a complete CMM-space over a Banach algebra β with a solid cone ξ and endowed with binary relation $\acute{R}$. Let $\Upsilon : \Omega \to \Omega$ be a Banach algebra type relational theoretic mapping on Ω and vector $g \in \xi$ and $\rho(g) < 1$. The following contractive conditions are equivalent:*

(i)

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \leq g\mathbb{C}(\varsigma, s) \text{ for all } \varsigma, s \in \Omega \text{ with } (\varsigma, s) \in \acute{R}.$$

(ii)

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \leq g\mathbb{C}(\varsigma, s) \text{ for all } \varsigma, s \in \Omega \text{ with } [\varsigma, s] \in \acute{R}.$$

Proof. We need to show that the two contractive conditions are equivalent, where $[\varsigma, s] \in \acute{R}$ refers to the symmetric closure of $\acute{R}$ i.e., $[\varsigma, s] \in \acute{R}$ if and only if $(\varsigma, s) \in \acute{R}$ or $(s, \varsigma) \in \acute{R}$.

(i)⇒(ii)

Assume condition (i)

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \leq g\mathbb{C}(\varsigma, s) \text{ for all } \varsigma, s \in \Omega \text{ with } (\varsigma, s) \in \acute{R}.$$

Now, assume $[\varsigma, s] \in \acute{R}$. By the definition of the symmetric closure $[\varsigma, s] \in \acute{R}$, we have either $[\varsigma, s] \in \acute{R}$ or $(\varsigma, s) \in \acute{R}$.

If $(\varsigma, s) \in \acute{R}$, then by (i)

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \leq g\mathbb{C}(\varsigma, s).$$

If $(s, \varsigma) \in \acute{R}$, note that in CMM-space over Banach algebra

$$\mathbb{C}(\varsigma, s) = \mathbb{C}(s, \varsigma).$$

From (i)

$$\mathbb{C}(\Upsilon(s), \Upsilon(\varsigma)) \leq g\mathbb{C}(\varsigma, s).$$

Using the symmetric property of CMM-space over Banach algebra $\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) = \mathbb{C}(\Upsilon(s), \Upsilon(\varsigma))$, which implies that

$$\mathbb{C}(\Upsilon(s), \Upsilon(\varsigma)) \leq g\mathbb{C}(\varsigma, s).$$

Thus,

$$\mathbb{C}(\Upsilon(s), \Upsilon(\varsigma)) \leq g\mathbb{C}(\varsigma, s),$$

holds for all $\varsigma, s \in \Omega$ with $[s, \varsigma] \in \acute{R}$ Therefore, condition (ii) is satisfied.

(ii)⇒(i)

Assume condition (ii)

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \preceq g\mathbb{C}(\varsigma, s) \text{ for all } \varsigma, s \in \Omega \text{ with } [\varsigma, s] \in \acute{R}.$$

Now, suppose $(\varsigma, s) \in \acute{R}$. By definition of $[\varsigma, s] \in \acute{R}$, if $(\varsigma, s) \in \acute{R}$ then $[\varsigma, s] \in \acute{R}$, since $\acute{R} \subseteq \left[\acute{R}\right]$, where $\left[\acute{R}\right]$ is the symmetric closure of $\acute{R}$. Hence, by (ii)

$$\mathbb{C}(\Upsilon(s), \Upsilon(\varsigma)) \preceq g\mathbb{C}(\varsigma, s).$$

This shows that condition (i) is satisfied. □

Definition 4.2. Let $(\Omega, \mathbb{C})$ be a CMM-space over Banach algebra β with ξ as the underlying solid cone and $\acute{R}$ as a binary relation on Ω, we say that $(\Omega, \mathbb{C})$ is $\acute{R}$-preserving if for each sequence $\{\varsigma_n\} \in \Omega$, then

$$(\varsigma_n, \varsigma_{n+1}) \in \acute{R} \text{ for all } n \in \mathbb{N}.$$

Definition 4.3. Let $(\Omega, \mathbb{C})$ be a CMM-space over Banach algebra β with ξ as the underlying solid cone and $\acute{R}$ as a binary relation on Ω, we say that $(\Omega, \mathbb{C})$ is $\acute{R}$-regular if for each sequence $\{\varsigma_n\} \in \Omega$, if for every $\sigma \in int\xi$,

$$\left.\begin{array}{c}(\varsigma_n, \varsigma_{n+1}) \in \acute{R} \text{ for all } n \in \mathbb{N} \\ \mathbb{C}(\varsigma_n, \varsigma) - \mathbb{C}_{\varsigma_n,\varsigma} \ll \sigma \text{ i.e } \varsigma_n \to \varsigma \in \acute{R}\end{array}\right\} \Rightarrow (\varsigma_n, \varsigma) \in \acute{R}.$$

Definition 4.4. Let $(\Omega, \mathbb{C})$ be a CMM-space over Banach algebra β with ξ as the underlying solid cone, and let $\acute{R}$ be a binary relation on. The relation $\acute{R}$ is said to be $\mathbb{C}$-self-closed if, for every $\acute{R}$-preserving sequence $\{\varsigma_n\}$ with ς_n converging to ς as converges to ∞ there exists a subsequence $\left\{\varsigma_{n_\chi}\right\}$ of $\{\varsigma_n\}$ such that $\left[\varsigma_{n_\chi}, \varsigma\right] \in \acute{R}$ for all $\chi \in \mathbb{N}$.

Theorem 4.1. *Suppose that $(\Omega, \mathbb{C})$ is a CMM-space over Banach algebra β with ξ as the underlying solid cone and $\acute{R}$ as a binary relation on Ω. Let $\Upsilon : \Omega \to \Omega$ be a Banach algebra type relational theoretic contraction with a vector g and $\rho(g) < 1$, satisfying the following assumptions:*
(i) The class $\Omega\left(\Upsilon, \acute{R}\right)$ is nonempty;
(ii) $\acute{R}$ is Υ-closed;
(iii) Either Υ is continuous or $\acute{R}$ is $\mathbb{C}$-sel-closed.
Under these assumptions, Υ has a fixed point $\varsigma \in \Omega$.

Proof. Since, $\Omega\left(\Upsilon, \acute{R}\right) \neq \emptyset$, let $\varsigma_0 \in \Omega\left(\Upsilon, \acute{R}\right)$ it follows that $(\varsigma_0, \Upsilon(\varsigma_0)) \in \acute{R}$. We define a sequence $\{\varsigma_n\}$ is defined by $\varsigma_n = \Upsilon(\varsigma_{n-1})$ for all $n \in \mathbb{N}$. We will establish that the sequence $\{\varsigma_n\}$ is $\acute{R}$-preserving. From the definition of relation, we have $(\varsigma_0, \varsigma_1) \in \acute{R}$. Using the Υ-closedness of $\acute{R}$, we get

$$(\Upsilon(\varsigma_0), \Upsilon(\varsigma_1)) = (\varsigma_0, \varsigma_1) \in \acute{R}.$$

Repeating this argument yields

$$(\varsigma_{n-1}, \varsigma_n) \in \acute{R} \; \forall n \in \mathbb{N}.$$

Therefore, the sequence $\{\varsigma_n\}$ is $\acute{R}$-preserving. Given that Υ is a relational theoretic contraction with a vector g, and the sequence $\{\varsigma_n\}$ is $\acute{R}$-preserving, it follows that this property holds for all $n \in \mathbb{N}$.

$$\mathbb{C}(\varsigma_n, \varsigma_{n+1}) = \mathbb{C}(\Upsilon(\varsigma_{n-1}), \Upsilon(\varsigma_n)) \preceq g\mathbb{C}(\varsigma_{n-1}, \varsigma_n).$$

By induction, this implies

$$\mathbb{C}(\varsigma_n, \varsigma_{n+1}) \preceq g^n \mathbb{C}(\varsigma_0, \varsigma_1) \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

Therefore, for $n < m$

$$\begin{aligned}
\mathbb{C}(\varsigma_n, \varsigma_m) - \mathbb{C}_{\varsigma_n,\varsigma_m} &\preceq \mathbb{C}(\varsigma_n, \varsigma_{n+1}) - \mathbb{C}_{\varsigma_n,\varsigma_{n+1}} + \mathbb{C}(\varsigma_{n+1}, \varsigma_{n+2}) - \mathbb{C}_{\varsigma_{n+1},\varsigma_{n+2}} + ... + \mathbb{C}(\varsigma_{m-1}, \varsigma_m) - \mathbb{C}_{\varsigma_{m-1},\varsigma_m} \\
&\preceq \mathbb{C}(\varsigma_n, \varsigma_{n+1}) + \mathbb{C}(\varsigma_{n+1}, \varsigma_{n+2}) + ... + \mathbb{C}(\varsigma_{m-1}, \varsigma_m) \\
&\preceq g^n \mathbb{C}(\varsigma_0, \varsigma_1) + g^{n+1} \mathbb{C}(\varsigma_0, \varsigma_1) + ... g^{m-1} \mathbb{C}(\varsigma_0, \varsigma_1) \\
&= \left(e + g + ... + g^{m-n-1}\right) g^n \mathbb{C}(\varsigma_0, \varsigma_1) \\
&\preceq \left(\sum_{i=0}^{\infty} g^i\right) g^n \mathbb{C}(\varsigma_0, \varsigma_1) \\
&= (e - g)^{-1} g^n \mathbb{C}(\varsigma_0, \varsigma_1).
\end{aligned}$$

Given that $\rho(g) < 1$ Remark 2.3 implies $\|g^n\|$ converges to 0 as n converges to ∞. Consequently, by Proposition 2.1, every $\sigma \in \beta$ with $\sigma \gg \vartheta$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathbb{C}(\varsigma_n, \varsigma_m) - \mathbb{C}_{\varsigma_n,\varsigma_m} \preceq (e - g)^{-1} g^n \mathbb{C}(\varsigma_0, \varsigma_1) \ll \sigma \text{ for all } n > n_0.$$

And

$$\mathbb{C}(\varsigma_n, \varsigma_m) - \mathbb{C}_{\varsigma_n,\varsigma_m} \ll \sigma \text{ for all } n > n_0.$$

This indicates that the sequence $\{\varsigma_n\}$ is ϑ-Cauchy. Due to the ϑ-completeness of Ω, there exists $\varsigma \in \Omega$ such that

$$\lim_{n\to\infty} \mathbb{C}(\varsigma_n, \varsigma) - \mathbb{C}_{\varsigma_n,\varsigma} = \vartheta.$$

As $\varsigma_n \to \infty$ as $n \to \infty$ for some n. Therefore, $\lim_{n\to\infty} \mathbb{C}(\varsigma_n, \varsigma) - \mathbb{C}_{\varsigma_n,\varsigma} \to \vartheta$ as $n \to \infty$. We demonstrate that ς^* is a fixed point of Υ. Based on (iii), we examine the following two cases:

C-1 Assume Υ is continuous. Then, $\varsigma_{n+1} = \Upsilon(\varsigma_n)$ converges to ς^* with respect to t_m topology. By the uniqueness of limits, we have ς^*, meaning ς^* is a fixed point of Υ. Assume $\acute{R}$ is $\mathbb{C}$-self-closed. Since, $\{\varsigma_n\}$ is $\acute{R}$-preserving and $\varsigma_n \to \varsigma$ as $n \to \infty$, the $\mathbb{C}$-self-closedness of $\acute{R}$ ensures the existence of a subsequence ς_{n_χ} of $\{\varsigma_n\}$ such that $\left[\varsigma_{n_\chi}, \varsigma\right] \in \acute{R}$ for all $\chi \in \mathbb{N}$. Given that Υ is a relational theoretic contraction with a g, we apply Proposition 3.1 to obtain

$$\begin{aligned}
\mathbb{C}(\varsigma^*, \Upsilon(\varsigma^*)) - \mathbb{C}_{\varsigma^*,\Upsilon(\varsigma^*)} &\preceq \mathbb{C}\left(\varsigma^*, \varsigma_{n_\chi+1}\right) - \mathbb{C}_{\varsigma^*,\varsigma_{n_\chi+1}} + \mathbb{C}\left(\varsigma_{n_\chi+1,} \Upsilon(\varsigma^*)\right) - \mathbb{C}_{\varsigma_{n_\chi+1,}\Upsilon(\varsigma^*)} \\
&\preceq \mathbb{C}\left(\varsigma^*, \varsigma_{n_\chi+1}\right) + \mathbb{C}\left(\varsigma_{n_\chi+1,} \Upsilon(\varsigma^*)\right) \\
&= \mathbb{C}\left(\varsigma^*, \varsigma_{n_\chi+1}\right) + \mathbb{C}\left(\Upsilon\left(\varsigma_{n_\chi}\right), \Upsilon(\varsigma^*)\right) \\
&\preceq \mathbb{C}\left(\varsigma^*, \varsigma_{n_\chi+1}\right) + g\mathbb{C}\left(\varsigma_{n_\chi}, \varsigma^*\right).
\end{aligned}$$

As $\varsigma_n \to \varsigma$ as $n \to \infty$, for each $\sigma \in \xi$ with $\vartheta \ll \sigma$ and for every m in $\mathbb{N}$, there exists $\chi(m)$ such that $\mathbb{C}\left(\varsigma^*, \varsigma_{n_\chi}\right) \ll \frac{\sigma}{m}\ \forall \chi > \chi(m)$. Consequently, by Remark 1.4, we have $\mathbb{C}\left(\varsigma_{n_\chi}, \varsigma^*\right) \ll \frac{g\sigma}{m}$. Thus, this inequality implies that

$$\mathbb{C}(\varsigma^*, \Upsilon(\varsigma^*)) - \mathbb{C}_{\varsigma^*,\Upsilon(\varsigma^*)} \preceq \frac{\sigma}{m} + \frac{g\sigma}{m} = \frac{\sigma}{m}(e + g),\ \forall \chi > \chi(m),\ m \in \mathbb{N}.$$

Since, $\mathbb{C}_{\varsigma^*,\Upsilon(\varsigma^*)} \to \vartheta$, this shows that

$$\frac{\sigma}{m}(e+g) - \mathbb{C}(\varsigma^*, \Upsilon(\varsigma^*)) \in \xi, \ \forall, \ m \in \mathbb{N}.$$

Since ξ is closed, taking the limit as $m \to \infty$ gives

$$\vartheta - \mathbb{C}(\varsigma^*, \Upsilon(\varsigma^*)) \in \xi.$$

By definition, this implies

$$\mathbb{C}(\varsigma^*, \Upsilon(\varsigma^*)) = \vartheta,$$

it implies that $\Upsilon(\varsigma^*) = \varsigma^*$ Hence, ς^* is a fixed point of Υ. □

Now, we prove a result of uniqueness.

Theorem 4.2. *If all the conditions of Theorem 4.1 are satisfied and the set* $\Psi\left(\varsigma, s, \acute{R}^S\right)$ *is nonempty with* $\varsigma, s \in \Omega,$ *for every* $\varsigma, s \in F(\Upsilon)$*, then* Υ *has a unique fixed point* $\varsigma^* \in \Omega$.

Proof. Using arguments similar to those in the proof of Theorem 4.1, we establish the existence of a fixed point $\varsigma^* \in \Omega$. Assume that the set $\Psi\left(\varsigma, s, R^S\right)$ is nonempty for every $\varsigma, s \in \Omega$. We will prove that ς^* is the unique fixed point of Υ. Contrary to this, assume that $s^* \in F(\Upsilon)$ and $\varsigma^* \neq s^*$

$$\varsigma^* = \Upsilon(\varsigma^*) \neq s^* = \Upsilon(s^*) \tag{4.2}$$

By assumption, there exists a path $\{l_z\}_{z=0}^{i} \subseteq \Omega$ of length r in R^S such that

$$l_0 = \varsigma^*, \ l_i = s^*, [l_z, l_{z+1}] \text{ in } \acute{R} \text{ for each } z = 0, 1, ...i-1. \tag{4.3}$$

Now, utilizing Proposition 2.3, we deduce that R^S is also Υ-closed, and thus

$$[\Upsilon^n(\varsigma_z), \Upsilon^n(\varsigma_{z+1})] \text{ in } \acute{R} \text{ for } z = 0, 1, ...i-1, \text{ for each } n \geq 0. \tag{4.4}$$

By applying Proposition 3.1 along with conditions (4.2), (2.3), and (2.4), we derive the following

$$\begin{aligned}
\mathbb{C}(\varsigma^*, s^*) &= \mathbb{C}_{\varsigma^*, s^*} = \mathbb{C}(\Upsilon^n(\varsigma^*), \Upsilon^n(s^*)) - \mathbb{C}_{\Upsilon^n(\varsigma^*), \Upsilon^n(s^*)} \\
&= \mathbb{C}(\Upsilon^n(l_0), \Upsilon^n(l_r)) - \mathbb{C}_{\Upsilon^n(l_0), \Upsilon^n(l_r)} \\
&\leq \sum_{i=0}^{r-1} \left(\mathbb{C}(\Upsilon^n(l_z), \Upsilon^n(l_{z+1})) - \mathbb{C}_{\Upsilon^n(l_z), \Upsilon^n(l_{z+1})} \right) \\
&\leq \sum_{i=0}^{r-1} \mathbb{C}(\Upsilon^n(l_z), \Upsilon^n(l_{z+1})) \\
&\leq g \sum_{i=0}^{r-1} \mathbb{C}\left(\Upsilon^{n-1}(l_z), \Upsilon^{n-1}(l_{i+1})\right) \\
&\leq g^2 \sum_{i=0}^{r-1} \mathbb{C}\left(\Upsilon^{n-2}(l_z), \Upsilon^{n-2}(l_{i+1})\right)
\end{aligned}$$

$$\leq \quad ..., \leq g^n \sum_{z=0}^{i-1} \mathbb{C}(l_z, l_{i+1}), \text{ for all } n \in \mathbb{N}.$$

Given that, $\rho(g) < 1$, it follows from Remark 2.2 that $\|g^n\| \to 0$ when $n \to \infty$, and therefore

$$\left\| g^n \sum_{z=0}^{i-1} m(l_z, l_{i+1}) \right\| \leq \|g^n\| \left\| \sum_{z=0}^{i-1} m(l_z, l_{i+1}) \right\| \to \vartheta \text{ as } n \to \infty.$$

Thus, by Lemma 2.1, it follows that for every $\sigma \in \beta$ with $\vartheta \ll \sigma$, there exists $n \in \mathbb{N}$ such that

$$\mathbb{C}(\varsigma^*, s^*) \preceq g^n \sum_{z=0}^{i-1} \mathbb{C}(l_z, l_{i+1}) \ll \sigma.$$

This implies that $m(\varsigma^*, s^*) = \vartheta$, which in turn means $\varsigma^* = s^*$. □

Example 4.1. Assume, $\beta = C_R^1[0,1] \times C_R^1[0,1]$, equipped with the norm:

$$\|\varsigma_1, \varsigma_2\| = \|\varsigma_1\|_\infty + \|\varsigma_2\|_\infty + \left\|\varsigma_1'\right\|_\infty + \left\|\varsigma_2'\right\|_\infty.$$

Define the multiplication on β as:

$$\varsigma s = (\varsigma_1 s_1, \varsigma_1 s_2 + \varsigma_2 s_1) \text{ for all } \varsigma = (\varsigma_1, \varsigma_2), \ s = (s_1, s_2) \in \beta.$$

Thus, β becomes a Banach algebra under the standard addition of functions and scalar multiplication on the Cartesian product $C_R^1[0,1] \times C_R^1[0,1]$ with the unit element $e = (0,1)$. Define

$$\xi = \{(\varsigma_1(t), \varsigma_2(t)) \in \beta : \varsigma_1(t), \varsigma_2(t) \geq 0, \ t \text{ in } [0,1] \ \}.$$

Therefore, ξ is not normal and a solid cone. Let $\Omega = R^+ \times R^+$, $R^+ = [0, \infty)$, and the CMM over Banach algebra is defined as

$$\mathbb{C}((\varsigma_1, \varsigma_2), (s_1, s_2)) = \left(\left| \frac{\varsigma_1 + s_1}{2} \right|, \left| \frac{\varsigma_2 + s_2}{2} \right| \right) e^t \text{ in } \xi.$$

Thus, $(\Omega, \mathbb{C})$ is a complete CMM over Banach algebra. Let ζ_0^Q denote the set of all sequences of nonnegative rational numbers that converge to zero. Clearly, $\zeta_0^{Q^+} \neq 0$. For a $\lambda \in Q^+$ define the mapping and the binary relation on $\acute{R}$,$\Upsilon : \Omega \to \Omega$ respectively

$$\Upsilon((\varsigma_1, \varsigma_2)) = \begin{cases} \frac{\varsigma_1}{4}, \frac{\varsigma_1}{4} + \lambda \frac{\varsigma_1}{2} & \text{if } \varsigma_1, \varsigma_2 \in Q^+, \\ \left(\varsigma_1 \left| \varsigma_1^2 - 1 \right|, \varsigma_1 \left| \varsigma_1^2 - 1 \right| \right) & \text{otherwise.} \end{cases}$$

And

$$\acute{R} = \{(\varsigma_n, s_n), (t_n, v_n), \ n \text{ in } \mathbb{N} : \ \{\varsigma_n\}, \ \{s_n\} \ \{t_n\}, \ \{v_n\} \text{ in } \mathbb{N}\}.$$

Therefore, Υ is a relational theoretic contraction with the vector $g(t) = (g_1, g_2)$, where g_1 in $\left[\frac{1}{2}, 1\right)$ and g_2 in $[g, \infty)$, with $\rho(g) = g_1$. In particular, if $((\varsigma_n, s_n), (t_n, v_n))$ in $\acute{R}$ it follows that $\varsigma_n, s_n, t_n, v_n$ in Q^+. Hence,

$$\mathbb{C}(\Upsilon(\varsigma_n, s_n), \Upsilon(t_n, v_n)) \ = \ \left(\frac{1}{2} \left| \frac{\varsigma_n + t_n}{2} \right|, \frac{1}{2} \left(\frac{s_n + v_n}{2} \right) + \lambda \left(\frac{\varsigma_n + t_n}{2} \right) \right) e^t$$

$$
\begin{aligned}
&\preceq \left(g_1\left|\frac{\varsigma_n+t_n}{2}\right|, g_1\left|\frac{s_n+v_n}{2}\right|+g_2\left(\left|\frac{\varsigma_n+t_n}{2}\right|\right)\right)e^t \\
&= (g_1,g_2)\left(\left|\frac{\varsigma_n+t_n}{2}\right|, \left|\frac{s_n+v_n}{2}\right|\right)e^t \\
&= \mathbb{C}\left((\varsigma_1,\varsigma_2),(s_1,s_2)\right).
\end{aligned}
$$

As, $\zeta_0^{Q^+}$ is non-empty, consider $\{\varsigma_n\} \in \zeta_0^{Q^+}$. Since $\lambda \in Q^+$, we obtain

$$((\varsigma_n,\varsigma_n), \Upsilon(\varsigma_n,\varsigma_n)) = \left((\varsigma_n,\varsigma_n), \left(\frac{\varsigma_n}{4}, \frac{\varsigma_n}{4}+\lambda\frac{\varsigma_n}{2}\right)\right) \in \acute{R}.$$

Thus, $\acute{R}$ is Υ-closed. Moreover, if $\{(\varsigma_n, s_n)\}$ is a sequence in Ω such that

$$((\varsigma_n,s_n),(\varsigma_{n+1},s_{n+1})) \in \acute{R}, \tag{4.5}$$

and (ς_n, s_n) converges to (ς, s) with respect to t_m-topology as n converges to ∞. So $\{\varsigma_n\}, \{s_n\} \in \zeta_0^{Q^+}$, leading to $(\varsigma, s) = (0,0)$, we are in the first case. Therefore, according to the definition of, $\acute{R}$ we conclude:

$$((\varsigma_n,s_n),(\varsigma,s)) \in \acute{R},\ \forall\, n \text{ in } \mathbb{N}. \tag{4.6}$$

Consequently, $\acute{R}$ is $\mathbb{C}$-self-closed. As a result, all the conditions of Theorem 4.3 are fulfilled, leading to the conclusion that the mapping Υ has a fixed point. Indeed, Υ has fixed points in Ω. Which are $F(\Upsilon) = \left\{(0,0), \left(\sqrt{2},0\right), \left(0,\sqrt{2}\right), \left(\sqrt{2},\sqrt{2}\right)\right\}$.

5. Cyclic operator in cone M-metric over Banach algebra with relation

In [3], Kirk et al. introduced the concept of a cyclic contraction, which extends the Banach contraction. This concept is utilized in the workframe of CMM over Banach algebras with binary relation.

Definition 5.1. Let $(\Omega, \mathbb{C})$ be a CMM-space over Banach algebra β with ξ as the underlying solid cone and $\acute{R}$ as a binary relation on Ω, $E \subset \Omega$ and $F \subset \Omega$, $\Omega = E \cup F$, and $\Upsilon : \Omega \to \Omega$ a mapping on Ω. Then we say that Υ cyclic Banach algebra type relational theoretic contraction with respect to the pair (E, F) if and only if satisfies the following

(i) there is a vector $g \in \xi$ and $\rho(g) < 1$,

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \preceq g\mathbb{C}(\varsigma, s) \text{ for all } \varsigma \in E, s \in F \text{ with } (\varsigma, s) \in \acute{R}. \tag{5.1}$$

Theorem 5.1. *Let $(\Omega, \mathbb{C})$ be a CMM-space over Banach algebra β with ξ as the underlying solid cone and $\acute{R}$ as a binary relation on Ω, $E \subset \Omega$ and $F \subset \Omega$, $\Omega = E \cup F$, and $\Upsilon : \Omega \to \Omega$ a mapping on Ω. Assume that the successive axioms hold:*

(i) $\Upsilon(E) \subset F$ and $\Upsilon(F) \subset E$,

(ii) there is a vector $g \in \xi$ and $\rho(g) < 1$ such that

$$\mathbb{C}(\Upsilon(\varsigma), \Upsilon(s)) \preceq g\mathbb{C}(\varsigma, s) \textit{ for all } \varsigma \in E, s \in F \textit{ with } (\varsigma, s) \in \acute{R}. \tag{5.2}$$

Then, Υ has a fixed point.

Proof. Suppose that $\Theta = E \cup F$. Thus Θ is a closed subspace, so $(\Omega, \mathbb{C})$ is a CMM-space over Banach algebra β. Define a binary relation $\acute{R}$ on the set Ω, specified by the condition $\acute{R} = E \times F$ it implies that (ς, s) in $\acute{R}$ iff (ς, s) in $E \times F$ for all $\varsigma, s \in \Omega$. Consider the symmetric relation

$$\Gamma = \acute{R} \cup \acute{R}^{-1}.$$

We state that $(\Omega, \mathbb{C}, \Gamma)$ is regular. Consider a sequence $\{\varsigma_n\} \in \Omega$ and let $\varsigma \in \Omega$ be a point such that

$$(\varsigma_n, \varsigma) \in \acute{R},$$

for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \mathbb{C}(\varsigma_n, \varsigma) = \lim_{n\to\infty} = \mathbb{C}_{\varsigma_n,\varsigma} = \mathbb{C}(\varsigma, \varsigma).$$

By applying the definition of Γ, we obtain

$$(\varsigma_n, \varsigma_{n+1}) \in E \times F \cup F \times E \text{ for all } n \in \mathbb{N}. \tag{5.3}$$

Directly, the product structure $\Gamma \times \Gamma$ with the CMM-space over Banach algebra given by

$$C((\varsigma_1, s_1), (\varsigma_{2,}\varsigma_2)) = \left(\frac{\mathbb{C}(\varsigma_1, s_1) + \mathbb{C}(\varsigma_2, s_2)}{2}\right) e^t \text{ for all } (\varsigma_1, s_1), (\varsigma_{2,}\varsigma_2) \text{ in } \Gamma \times \Gamma.$$

Considering the completeness of $(\Omega, \mathbb{C})$. It follows that $(\Gamma \times \Gamma, \mathbb{C})$. Furthermore, $E \times F$ and $F \times E$ are closed in $(\Gamma \times \Gamma, \mathbb{C})$, since E and F are closed in $(\Omega, \mathbb{C})$. Now, by considering the limit as $n \to \infty$ in (5.3). We reach at

$$(\varsigma, \varsigma) \in (E \times F \cup F \times E).$$

This leads to the conclusion that

$$\varsigma \in E \cap F.$$

Moreover, from equation (4.3), we obtain

$$\varsigma_n \in E \cup F.$$

Therefore, we have $\varsigma_n \Gamma \varsigma$ for all $n \in \mathbb{N}$. Thus, our assertion is confirmed. Furthermore, since Υ is a self-mapping and by condition (i), we obtain for all $\varsigma, s \in \Omega$

$$(\varsigma, s) \in E \times F \Rightarrow (\Upsilon(\varsigma) \times \Upsilon(s)) \in E \times F,$$

and

$$(s, \varsigma) \in F \times E \Rightarrow (\Upsilon(s) \times \Upsilon(\varsigma)) \in F \times E.$$

Hence, $\acute{R}$ is Υ-closed. As E is non-empty, there exists ς_0 in E so that $\Upsilon(\varsigma_0)$ in Ω, meaning $\varsigma_0 \acute{R} \Upsilon(\varsigma_0)$. Thus, all the assumptions of Theorem 4.3 hold. Hence, $F(\Upsilon) \neq \phi$, and $F(\Upsilon)$ is contained in $E \cap F$. Consequently, since $\varsigma \acute{R} s$ for all $\varsigma, s \in E \cap F$. Thus, $E \cap F$ is Γ-directed Hence, the all assumptions of Theorem 4.2 are fulfilled, implying that Υ has a unique fixed point. This concludes the proof. □

6. Application

In this section, we investigate the existence of a solution to a Volterra-type integral equation using Theorem 2.6. Let us consider the following Volterra-type integral equation:

$$\varsigma(t) = \int_0^\Delta H(t, \varsigma, \varsigma(t))\, \mathbb{C}\varsigma, \ t \in [0, \Delta], \tag{6.1}$$

where $H : [0, \Delta] \times [0, \Delta] \times [0, \Delta] \to [0, \Delta]$. Let us consider $C([0, \Delta], R)$ the class of continuous functions on $[0, \Delta]$, $\Delta > 0$. Assume $\beta = C[0, \Delta]$, equipped with the norm

$$\|\varsigma\| = \|\varsigma\|_\infty + \left\|\varsigma^{'}\right\|_\infty .$$

Using the standard multiplication, β forms a Banach algebra with the unit $e = 1$. The CMM, $\mathbb{C}$ is defined as follows:

$$\mathbb{C}(\varsigma, s)(t) = \sup_{t \in [a,b]} \left(\frac{\varsigma + s}{2}\right) e^t, \text{ for all } \varsigma, s \in C([0, \Delta], R). \tag{6.2}$$

Observe that $C([0, \Delta], R)$, $\mathbb{C}$ forms a ϑ-complete CMM-space over the Banach algebra $C([0, \Delta], R)$.

Theorem 6.1. *Consider Volterra-type integral Eq (6.1) Assume that $\varsigma, s \in$ in $C([0, \Delta], R)$*

$$\left|\frac{H(t, \varsigma, \varsigma(t)) + H(t, \varsigma, s(t))}{2}\right| \leq g \left|\frac{\varsigma(t) + s(t)}{2}\right|$$

for all $t, \varsigma \in [0, \Delta]$ where $\alpha \in [0, 1]$. If Eq (6.1) has a solution, then a unique solution to the integral Eq (6.1) exists.

Proof. We define an operator $\Upsilon : \Omega \to \Omega$ by

$$\Upsilon(\varsigma)(t) = \int_0^\Delta H(t, \varsigma, \varsigma(t))\, \mathbb{C}\varsigma \text{ for all } t, \varsigma \in [0, \Delta].$$

It is straightforward to verify that Υ is well-defined and $\preceq$ that on $\acute{R}$. Observe that ς is a fixed point of Υ if Eq (6.1) has a solution. Let

$$(\varsigma, s) \in \acute{R} = \{\varsigma \preceq s : \mathbb{C}(\varsigma, s) \succeq \vartheta, \text{ where } \mathbb{C} \text{ is a } CMM\text{-space over Banach algebra}\},$$

Since $\acute{R}$ is Υ-closed, then $\Upsilon(\varsigma) \preceq \Upsilon(s)$. To prove $\Upsilon : \Omega \to \Omega$ is a relational theoretic contraction with a contractive vector g. We have

$$\begin{aligned} \left|\frac{\Upsilon(\varsigma)(t) + \Upsilon(s)(t)}{2}\right| &= \left|\int_0^\Delta \left(\frac{H(t, \varsigma, \varsigma(t)) + H(t, \varsigma, s(t))}{2}\right) \mathbb{C}\varsigma\right| e^t \\ &\preceq \left(\int_0^\Delta \left|\frac{H(t, \varsigma, \varsigma(t)) + H(t, \varsigma, s(t))}{2}\right| \mathbb{C}\varsigma\right) e^t \\ &\preceq \left(g \int_0^\Delta \left|\frac{\varsigma(t) + s(t)}{2}\right| \mathbb{C}\varsigma\right) e^t \end{aligned}$$

$$\leq \left(g \int_0^{\Delta} \frac{|\varsigma(t)| + |s(t)|}{2} \mathbb{C}\varsigma \right) e^t$$
$$\leq \left(g \sup_{t\in[a,b]} \frac{|\varsigma(t)| + |s(t)|}{2} \int_0^{\Delta} \mathbb{C}\varsigma \right) e^t$$
$$\leq g\mathbb{C}(\varsigma, s).$$

Thus, Υ is a relational theoretic contraction with an g. Inequality (6.2) holds. Since ς_n is a $\acute{R}$-preserving sequence, $\{\varsigma_n\} \in C([0,\Delta], R)$, and $\varsigma_n \to \varsigma$ for some $\varsigma \in C([0,\Delta], R)$. We have $\varsigma_0(t) \preceq \varsigma_1(t) \preceq \varsigma_2(t) \preceq ... \preceq \varsigma_n(t) \preceq \varsigma_{n+1}(t) \preceq ...$, for all $t \in [0,\Delta]$, which leads to $\varsigma_n(t) \preceq \varsigma(t)$, for all $t \in [0,\Delta]$. Therefore, $\varsigma, s \in F(\Upsilon)$. Then $l = \max\{\varsigma, s\} \in C([0,\Delta], R)$, and thus $\varsigma \preceq l$, and $s \preceq l$. Therefore, all the axioms of Theorem 4.2 is satisfied, and the integral Eq (6.1) has a solution. □

7. Example of integral equation

In this section, we give the real-world example of the integral Eq (6.1) to show that all conditions of Theorem 6.1 are satisfied.

Example 7.1. Consider a Volterra-type integral equation

$$\varsigma(t) = 1 + \int_0^{\Delta} \frac{1}{2}\varsigma(t)\mathbb{C}\varsigma,\ t \in [0,\Delta],$$

where $\varsigma(t)$ is unknown function (i.e temperature, population), and the kernel is $H(t,\varsigma,\varsigma(t)) = \frac{1}{2}\varsigma(t)$.

Proof. First, we use the Picard iteration method. Table 1 appears to show an iterative approximation method for solving an integral equation with the following structure:

Table 1. This table illustrates an iterative process to approximate the exponential function $e^{\frac{t}{2}}$. Starting with an initial guess $\varsigma_0(t)$=1, each subsequent iteration refines the approximation by integrating the previous result.

Iteration	Computation	Result
$\varsigma_0(t)$	initial guess	1
$\varsigma_1(t)$	$1 + \int_0^t \frac{1}{2}.1\mathbb{C}\varsigma$	$1 + \frac{t}{2}$
$\varsigma_2(t)$	$1 + \int_0^t \frac{1}{2}\left(1 + \frac{\varsigma}{2}\right)\mathbb{C}\varsigma$	$1 + \frac{t}{2} + \frac{t^2}{2}$
...	...	...
$\varsigma_n(t)$	....	$\sum_{k=0}^{n} \frac{\left(\frac{t}{2}\right)^k}{k!}.$

As $n \to \infty$, this converges to

$$\varsigma(t) = e^{\frac{t}{2}}$$

Since, the given pace is cone $\mathbb{C}$-metric space over a Banach algebra, $C([0,\Delta], R)$ with sup-norm

$$\|\varsigma\| = \sup_{t\in[0,1]} |\varsigma(t)| + \sup_{t\in[0,1]} \left|\varsigma'(t)\right|.$$

To show that Υ is a contraction. Now, we define an operator $\Upsilon : \Omega \to \Omega$ by

$$\Upsilon(\varsigma)(t) = 1 + \int_0^t \frac{1}{2}\varsigma(t)\,\mathbb{C}\varsigma$$

form the contraction, we deduce that

$$\begin{aligned} \frac{1}{2}\|H(t,\varsigma,\varsigma(t)) + H(t,s,s(t))\| &= \frac{1}{2}\left\|\frac{1}{2}\varsigma(t) + \frac{1}{2}s(t)\right\| \\ &\leq \frac{1}{2}\|\varsigma(t) + s(t)\|. \end{aligned}$$

Thus, satisfied the contraction condition, now define a binary relation $\varsigma \preceq s$ if and only if $\varsigma(t) \leq s(t)$ for $t \in [0,t]$. If $\varsigma \preceq s$, then

$$\Upsilon(\varsigma)(t) = 1 + \int_0^t \frac{1}{2}\varsigma(t)\,\mathbb{C}\varsigma \leq 1 + \int_0^t \frac{1}{2}s(t)\,\mathbb{C}\varsigma = \Upsilon(s)(t).$$

So, give operator is order preserving starting from $\varsigma_0(t) = 1$, the iteration $\varsigma_{n+1} = \Upsilon(\varsigma_n)$ converges monotonically to the solution $\varsigma(t) = e^{\frac{t}{2}}$. See Figure 2. □

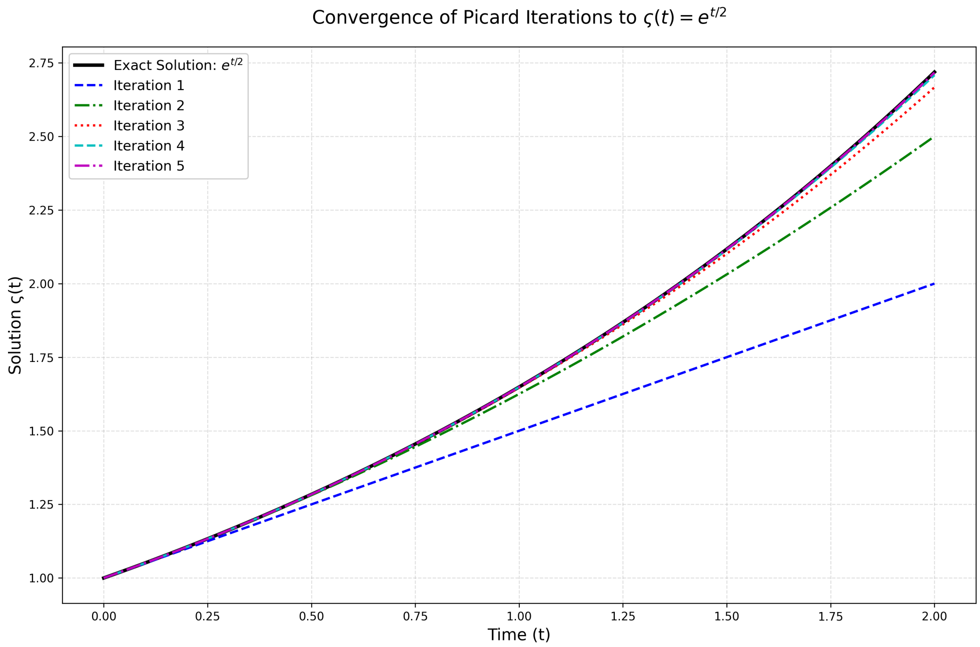

Figure 2. Convergence behavior of iteration.

8. Conclusions

In this paper, we have unified the concept of a cone $\mathbb{C}$-metric space over a Banach algebra endowed with a binary relation, and have established several significant fixed point results within the context of such spaces. Additionally, as an application, we have used these results to prove the existence of solutions for a class of nonlinear integral equations. Our proposed work will be extended in various directions. In essence, this research has contributed valuable insights into the field of nonlinear contraction principles and will open new avenues for future exploration in related domains such as generalized metric spaces, m_b-metric spaces, quasi m-metric spaces, extended m-metric spaces, and rectangular m-metric spaces.

Author contributions

M. Tariq: writing—original draft, S. Mansou: methodology, A. Assiry: conceptualization, J. U. Rehman: Formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number 25UQU4331214GSSR05.

Funding

This research was funded by Umm Al-Qura University, Saudi Arabia, under grant number 25UQU4331214GSSR05.

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales, *Fund. Math.*, **3** (1922), 133–181.
2. K. C. Border, *Fixed point theorems with applications to economics and game theory*, Cambridge University Press, 1985. https://doi.org/10.1017/CBO9780511625756
3. W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed points for mapping satisfying cyclic contractive conditions, *Fixed Point Theor.*, **4** (2003), 79–89.
4. S. G. Matthews, Partial metric topology, *Ann. NY Acad. Sci.*, **728** (1994) 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x
5. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087
6. H. Liu, S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, *Fixed Point Theory Appl.*, **2013** (2013), 320. https://doi.org/10.1186/1687-1812-2013-320

7. A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17** (2015), 693–702. https://doi.org/10.1007/s11784-015-0247-y
8. S. K. Malhotra, J. B. Sharma, S. Shukla, Relation-theoretic contraction principle in cone metric spaces with Banach algebra, *Ser. A Appl. Math. Inf. Mech.*, **8** (2016), 87–100. https://doi.org/10.5937/SPSUNP1601087M
9. M. Tariq, M. Arshad, M. Abbas, E. Ameer, S. Mansour, H. Aydi, A relation theoretic m-metric fixed point algorithm and related applications, *AIMS Math.*, **8** (2023), 19504–19525. https://doi.org/10.3934/math.2023995
10. A. Malhotra, D. Kumar, C. Park, C*-algebra valued R-metric space and fixed point theorems, *AIMS Math.*, **7** (2022), 6550–6564. https://doi.org/10.3934/math.2022365
11. J. Fernandez, Partial cone metric spaces over Banach algebra and generalized Lipschitz mappings with applications, 2016.
12. H. Huang, W. S. Du, J. Y. Chen, On BID-cone b-metric spaces over Banach algebras: New topological properties and fixed point theorems, *Mathematics*, **10** (2022), 1425. https://doi.org/10.3390/math10091425
13. M. Asadi, E. Karapinar, P. Salimi, New extension of p-metric spaces with fixed points results on M-metric spaces, *J. Inequal. Appl.*, **2014** (2014), 18. https://doi.org/10.1186/1029-242X-2014-18
14. M. Tariq, S. Mansour, M. Abbas, A. Assiry, A solution to the non-cooperative equilibrium problem for two and three players using the fixed-point technique, *Symmetry*, **17** (2025), 544. https://doi.org/10.3390/sym17040544
15. I. Ayoob, N. Z. Chuan, N. Mlaiki, Quasi M-metric spaces, *AIMS Math.*, **8** (2023), 10228–10248. https://doi.org/10.3934/math.2023518
16. M. Tariq, E. Ameer, A. Ali, H. A. Hammad, F. Jarad, Applying fixed point techniques for obtaining a positive definite solution to nonlinear matrix equation, *AIMS Math.*, **8** (2023), 3842–3859. https://doi.org/10.3934/math.2023191
17. J. Fernandeza, N. Malviyab, E. Gili, Fixed point results in M-cone metric space over Banach algebra with an application, *Filomat*, **36** (2022), 5547–5562. https://doi.org/10.2298/FIL2216547F
18. W. Rudin, *Functional analysis*, New York, 1991.
19. S. Xu, S. Radenovic, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl.*, **2014** (2014), 102. https://doi.org/10.1186/1687-1812-2014-102
20. S. Radenovic, B. E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, *Comput. Math. Appl.*, **57** (2009), 1701–1707. https://doi.org/10.1016/j.camwa.2009.03.058
21. S. Lipschutz, *Schaum's outline of theory and problems of set theory and related topics*, 1964.
22. B. Kolman, R. C. Busby, S. Ross, *Discrete mathematical structures*, 1995.
23. B. Samet, M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.*, **13** (2012), 82–97.

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