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Research article

Schröder's method and the infinity point

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Abstract: This paper presents an initial investigation into the dynamic properties of a well-known iterative method for solving nonlinear equations: Schröder's method. We characterize the degree of the rational map induced by applying the method to polynomial equations, along with other dynamical features such as the nature of extraneous fixed points and the presence of attracting cycles. Particular attention is given to the significant dynamical differences between Schröder's method and other iterative methods, notably Newton's method, with a focus on the behavior at infinity.

Keywords: complex dynamics; numerical methods; Newton's method; Schröder's method; infinity point
Mathematics Subject Classification: 37F10, 65S05

1. Introduction

Schröder's method is a well-known iterative procedure for solving nonlinear equations, originally introduced by E. Schröder in 1870 (see [1] or [2]). While the method is applicable to a broad class of nonlinear equations, our focus is restricted to its application to polynomial equations defined over the complex plane.

$$p(z) = 0, \quad z \in \mathbb{C}. \tag{1.1}$$

In fact, Schröder's method for solving (1.1) defines a one-step nonlinear recurrence relation $z_{n+1} = S_p(z_n)$, $n \geq 0$, given by the iteration map

$$S_p(z) = z - \left(\frac{1}{1 - L_p(z)}\right) \frac{p(z)}{p'(z)}, \tag{1.2}$$

where $L_p(z)$ is defined by the quotient

$$L_p(z) = \frac{p(z)p''(z)}{p'(z)^2}. \tag{1.3}$$

The quotient defined in (1.3), along with its generalizations to higher dimensions and Banach spaces, plays a central role in both the concise formulation of various iterative methods and the analysis of their convergence properties. One of the earliest works highlighting the significance of this quotient —referred to as the degree of logarithmic convexity— is [3].

Originally, Schröder's method was conceived as a modification of Newton's method designed to achieve quadratic convergence even in the presence of multiple roots. It is straightforward to show that the method can be derived by applying Newton's method to the rational function $p(z)/p'(z)$. In brief, we can say that the high computational cost of Schröder's method (in general, methods using second derivatives have a cubic order of convergence [4]) is balanced with its robustness for the calculus of multiple roots. Perhaps due to this computational complexity, the dynamical behavior of Schröder's method has not been extensively studied. Nevertheless, we believe that the dynamical study of various iterative methods is valuable in its own right, as it can reveal behaviors that differ significantly from those exhibited by the most extensively studied method: Newton's method. For instance, such differences have been documented in the case of Halley's method [5] and Chebyshev's method [6]. The dynamical behavior of iterative methods, even in the simplest cases, provides valuable insights into the stability and global convergence of the method under consideration [7]. Schröder's method is a current topic of research. To confirm this statement, but without the aim of being very exhaustive, we show a couple of recent references: [8, 9]

We now turn our attention to the dynamical analysis of Schröder's method. In [1], Schröder himself made an initial exploration of the method's dynamics, specifically for the case $p(z) = z^2 - 1$. He observed that, in this instance, the iteration maps associated with Schröder's method, $S_p(z)$, and Newton's method, $N_p(z) = z - p(z)/p'(z)$, are inverses of each other, namely

$$S_p(z) = \frac{2}{z + \frac{1}{z}} = \frac{1}{N_p(z)}.$$

As a result, the dynamical behaviors of the two methods are closely related in this case. Schröder conducted his analysis using Cartesian and polar coordinates, but without employing complex numbers.

By using Cayley's theory and complex arithmetic, Schröder's method can be analyzed when applied to general quadratic polynomials.

$$p(z) = (z - a)(z - b), \quad a, b \in \mathbb{C}, \quad a \neq b,$$

is conjugated, via the Möbius transform $M(z) = (z - a)/(z - b)$ with the map $R(z) = -z^2$, that is, $R(z) = M \circ S_p \circ M^{-1}(z) = -z^2$. The Julia set of $R(z)$ is the unit circle $\{z \in \mathbb{C}; |z| = 1\}$ and the basin of attraction of $z = 0$ is $\{z \in \mathbb{C}; |z| < 1\}$. The basin of attraction of $z = \infty$ is $\{z \in \mathbb{C}; |z| > 1\}$. Consequently, for general quadratic polynomials $p(z) = (z - a)(z - b)$, the Julia set of $S_p(z)$ corresponds to the perpendicular bisector of the segment joining the roots a and b. The basin of attraction of the root $z = a$ is the half-plane containing $z = a$, and the basin of attraction of the root $z = b$ is the half-plane containing $z = b$. Up to this point, no significant differences have been observed between the dynamics of Newton's method and Schröder's method for quadratic polynomials.

However, differences begin to emerge when considering polynomials of higher degree. For example, in [6], the authors demonstrated that it is possible to analytically characterize the basins of attraction for Schröder's method applied to polynomials with two distinct roots $a, b \in \mathbb{C}$, each with integer multiplicities m and n, that is, $p(z) = (z-a)^m(z-b)^n$. Without loss of generality, we may assume $m \geq n$. If $J_{m,n,a,b}$ denotes the Julia set of Schröder's method applied to such polynomials, then:

- If $m = n$, $J_{m,n,a,b}$ is the line of points equidistant from a and b.
- If $m > n \geq 1$, then $J_{m,n,a,b}$ is a circle with center $c_{m,n,a,b}$ and radius $r_{m,n,a,b}$, where

$$c_{m,n,a,b} = \frac{bm^2 - an^2}{m^2 - n^2}, \qquad r_{m,n,a,b} = \frac{mn|a-b|}{m^2 - n^2}.$$

Let us note that, when $m > n \geq 1$, both $c_{m,n,a,b}$ and $r_{m,n,a,b}$ could be expressed in terms of the quotient $q = m/n > 1$, giving rise to

$$c_{q,a,b} = \frac{bq^2 - a}{q^2 - 1}, \qquad r_{q,a,b} = \frac{q|a-b|}{q^2 - 1}.$$

Consequently, the associated Julia set $J_{q,a,b}$ is the circle centered at $c_{q,a,b}$ with radius $r_{q,a,b}$. Thus, the basins of attraction of the roots a and b depend primarily on the roots themselves and on the quotient $q = m/n > 1$, rather than on the individual multiplicities m and n. Moreover, the evolution of the corresponding Julia set (which forms the boundary between the basins of attraction) can be visualized as a function of q. It transitions from the perpendicular bisector of the segment joining a and b (in the limiting case $q \to 1$) to circles of decreasing radius that ultimately collapse onto the root b as $q \to \infty$, since b has smaller multiplicity (see Figure 1 for an illustration of this behavior).

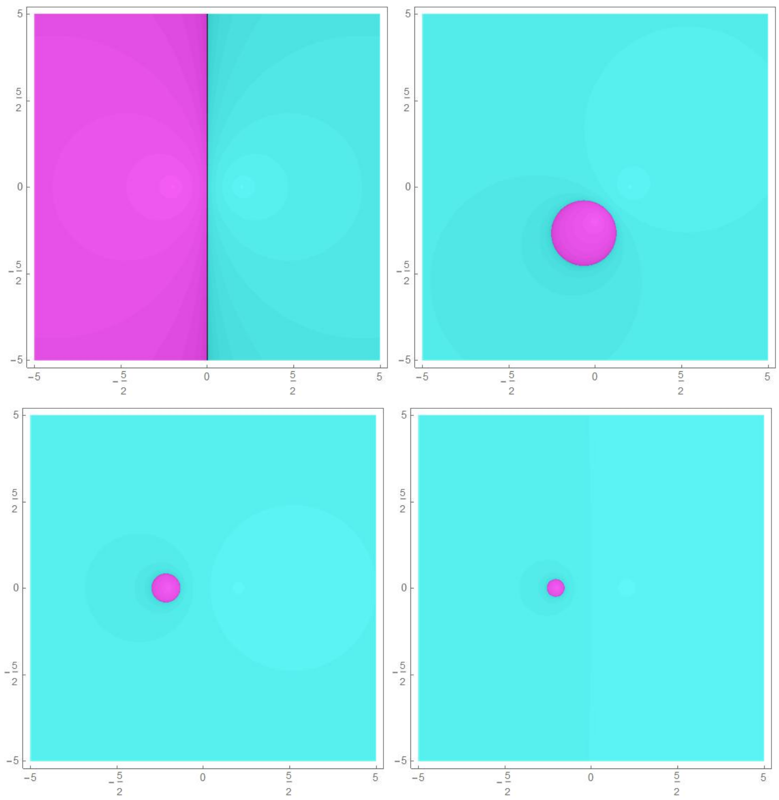

Figure 1. Evolution of the basins of attraction of Schröder's method applied to polynomials $p(z) = (z-1)^m(z+1)^n$ with $q = m/n$ increasing. We show the cases $q = 1, 2, 4, 8$ top left, top right, bottom left, and bottom right, respectively. The basin of the root $z = -1$ is colored in magenta, whereas the basin of the root $z = 1$ is colored in cyan.

In the remainder of the paper, we continue our investigation of notable properties of Schröder's method when applied to polynomials. In Section 2, we characterize the degree of the rational map arising from the application of Schröder's method. In Section 3, we demonstrate that Schröder's method admits extraneous fixed points, all of which are repelling, and we examine the potential existence of attracting cycles. Finally, in Section 4, we analyze the behavior of infinity as a fixed point of the iteration map defined in (1.2).

2. Degree of Schröder's method

In the study of the dynamical behavior of iterative methods for solving polynomial equations, one of the initial steps is to determine the degree of the associated rational map. While it is relatively straightforward to establish upper bounds for this degree, computing the exact value requires the introduction of the concept of a *special critical point*, as employed by Nayak and Pal [10] in their analysis of the rational map associated with Chebyshev's method (see also [11] for the broader Chebyshev–Halley family of methods).

Definition 1. *(Special critical point) Given a complex polynomial $p(z)$, a critical point $c \in \mathbb{C}$ is called special if $p(c) \neq 0$ but $p''(c) = 0$.*

For our convenience, let us write the iteration map for Schröder's method (1.2) in the equivalent way

$$S_p(z) = z - \frac{p(z)p'(z)}{p'(z)^2 - p(z)p''(z)}. \tag{2.1}$$

Theorem 1. *Let us consider $S_p(z)$, the iteration map of Schröder's method for solving polynomial equations, written as in* (2.1). *Then the degree of the rational map $S_p(z)$ is*

$$\deg(S_p(z)) = 2r + s - C - 2,$$

where

- *r is the number of distinct roots of $p(z)$.*
- *s is the number of special critical points of $p(z)$.*
- *$C = c_1 + \cdots + c_k$ is the sum of multiplicities of all the special critical points.*

Proof. First we note that $\deg(\text{Num}(S_p(z)) \leq \deg(\text{Den}(S_p(z)) \leq 2d - 2$. In fact, for a general monic polynomial written in the way $p(z) = z^d + a_{d-1}z^{d-1} + \cdots$ it is easy to prove, after a few algebraic manipulations, that

$$\text{Num}(S_p(z)) = zp'(z)^2 - zp(z)p''(z) - p(z)p'(z) = -a_{d-1}z^{2d-2} + P_{2d-3}(z),$$

$$\text{Den}(S_p(z)) = p'(z)^2 - p(z)p''(z) = -dz^{2d-2} + Q_{2d-3}(z),$$

where $P_{2d-3}(z)$ and $Q_{2d-3}(z)$ are polynomials of degree $2d - 3$.

Consequently, $\deg(S_p(z) \leq 2d - 2$. In addition, $\deg(S_p(z) < 2d - 2$ if $\text{Num}(S_p(z))$ and $\text{Den}(S_p(z))$ have common roots. For calculating the exact degree of $S_p(z)$, we must just analyze $\deg(\text{Den}(S_p(z))$.

The proof is based on the following factorizations of the polynomials $p(z)$, $p'(z)$ and $p''(z)$:

$$p(z) = \prod_{i=1}^{m}(z - \alpha_i)\prod_{j=1}^{n}(z - \beta_j)^{b_j}, \quad b_j \geq 2,$$

with $\deg(p) = m + B$, $B = \sum_{j=1}^{n} b_j$. Taking into account that a root of $p(z)$ with a certain multiplicity $k \geq 1$ is a root of $p'(z)$ with multiplicity $k - 1$ (and a similar criterion for $p''(z)$), we have

$$p'(z) = g(z) \prod_{j=1}^{n} (z - \beta_j)^{b_j - 1},$$

$$p''(z) = h(z) \prod_{j=1}^{n} (z - \beta_j)^{b_j - 2},$$

where $g(z)$ and $h(z)$ are polynomials that satisfy $g(\alpha_i) \neq 0$, $i = 1, \ldots, m$, $g(\beta_j) \neq 0$ and $h(\beta_j) \neq 0$ for $j = 1, \ldots, n$. In addition, $\deg(g) = \deg(p') - B + n = m + n - 1$ and the leading coefficient of $g(z)$ is d; $\deg(h) = \deg(p'') - B + 2n = m + 2n - 2$ and the leading coefficient of $h(z)$ is $d(d - 1)$.

Let γ_j, $j = 1, \ldots, s$ be the special critical points of $p(z)$, with multiplicities c_j. Then the polynomials $g(z)$ and $h(z)$ can be factorized in the following way:

$$g(z) = \tilde{g}(z) \prod_{k=1}^{s} (z - \gamma_k)^{c_k},$$

$$h(z) = \tilde{h}(z) \prod_{k=1}^{s} (z - \gamma_k)^{c_k - 1},$$

where $\tilde{g}(z)$ and $\tilde{h}(z)$ are polynomials that satisfy $\tilde{g}(\alpha_i) \neq 0$, $i = 1, \ldots, m$; $\tilde{g}(\beta_j) \neq 0$ and $\tilde{h}(\beta_j) \neq 0$ for $j = 1, \ldots, n$; $\tilde{h}(\gamma_k) \neq 0$ for $k = 1, \ldots, s$. In addition, $\deg(\tilde{g}) = \deg(g) - C = m + n - C - 1$ and the leading coefficient of $\tilde{g}(z)$ is d; $\deg(\tilde{h}) = \deg(h) - C + s = m + 2n - 2 - C + s$ and the leading coefficient of $h(z)$ is $d(d - 1)$.

Therefore we can simplify the common roots in the following quotient

$$\frac{p(z)p'(z)}{p'(z)^2 - p(z)p''(z)} = \frac{\prod_{i=1}^{m}(z - \alpha_i) \prod_{j=1}^{n}(z - \beta_j) \prod_{k=1}^{s}(z - \gamma_k)\tilde{g}(z)}{\prod_{k=1}^{s}(z - \gamma_k)^{(c_k+1)}\tilde{g}(z)^2 - \prod_{i=1}^{m}(z - \alpha_i)\tilde{h}(z)}.$$

The degree of $S_p(z)$ coincides with the denominator of the previous quotient (note that there are not more common roots between the numerator and the denominator of $S_p(z)$). Note that the leading coefficient of the polynomial in the denominator of the previous quotient

$$\prod_{k=1}^{s} (z - \gamma_k)^{(c_k+1)} \tilde{g}(z)^2 - \prod_{i=1}^{m} (z - \alpha_i) \tilde{h}(z)$$

is $d^2 - d(d - 1) = d$, and its degree is $2m + 2n + s - C - 2 = 2r + s - C - 2$, where $r = m + n$ is the number of distinct roots of $p(z)$. So, we have proved the result. □

Corollary 1. *If $p(z)$ is a polynomial without special critical points, then the degree of the rational map $S_p(z)$ is $2r - 2$, where r is the number of distinct roots of $p(z)$.*

Taking into account the proof of Theorem 1, we can characterize the behavior of the infinity point for Schröder's method. As we can see, it is different from the behavior for the most famous iterative processes (Newton, Halley, and Chebyshev), for which infinity is a repelling fixed point.

Corollary 2. *The infinity point is not a fixed point for Schröder's method* (1.2)*.*

Proof. Following the proof of Theorem 1, and with the same notations, we have

$$S_p(\infty) = -\frac{a_{d-1}}{d},$$

so ∞ is not a fixed point for $S_p(z)$. □

3. Extraneous fixed points and cycles for Schröder's method

The existence of extraneous fixed points associated with an iterative method for solving a polynomial equation such as (1.1) is a classical subject in the dynamical analysis of these methods. Recall that extraneous fixed points are fixed points of the iteration map that do not correspond to roots of the equation. It is well known that Newton's method has no extraneous fixed points [4]. Similarly, Halley's method possesses only repelling extraneous fixed points (see [12]). In contrast, Chebyshev's method admits attracting extraneous fixed points, as demonstrated in [6] and [13], among others.

Theorem 2. *Let us consider $S_p(z)$, the iteration map of Schröder's method for solving polynomial equations, written as in* (2.1)*. Then the extraneous fixed points of $S_p(z)$ are solutions of $p'(z) = 0$, with $p(z) \neq 0$. All of them are repelling.*

Proof. Taking into account (2.1), the fixed points of $S_p(z)$ are the roots of $p(z)$ and the solutions of $p'(z) = 0$. Then the extraneous fixed points of Schröder's method are points $\omega \in \mathbb{C}$ such that $p'(\omega) = 0$ and $p(\omega) \neq 0$. Let $m \in \mathbb{N}$ the multiplicity of ω as a root of $p'(z)$, that is,

$$p'(z) = (z-\omega)^m g(z),$$

with $g(\omega) \neq 0$. By substituting in (2.1), we obtain

$$S_p(z) = z - (z-\omega)H(z), \quad H(z) = \frac{p(z)g(z)}{(z-\omega)^{m+1}g(z)^2 - p(z)(mg(z) + (z-\omega)g'(z))}.$$

Now, taking into account that the denominator in $H(\omega)$ is not zero, we deduce after a few algebraic manipulations

$$S_p'(\omega) = 1 - H(\omega) = 1 + \frac{1}{m} > 1.$$

Consequently, the multiplier of ω as a fixed point is bigger that one, and ω is a repelling fixed point. □

As discussed in the introduction, the behavior of Schröder's method for quadratic polynomials —and more generally, for polynomials with two distinct roots— is completely understood (see [14]). However, when the method is applied to polynomials of higher degree, its dynamical behavior becomes significantly more intricate. In such cases, there may exist open sets of initial guesses that fail to converge to any root of the given polynomial. Having established that Schröder's method does not possess attracting extraneous fixed points, we now focus on demonstrating the existence of attracting cycles. To this end, we employ the technique introduced by Roberts and Horgan–Kobelsky in [5]. In particular, we investigate the case of cubic polynomials.

As a preliminary step, we present a useful result known as the Scaling Theorem, which allows us —through appropriate changes of variables— to reduce the study of the dynamics of various iterative methods, possibly involving several parameters, to simpler representative cases (see [15] for applications to other iterative methods).

Theorem 3. *(Scaling Theorem for Schröder method) Let f an analytic function defined in the Riemann sphere and let $T(z) = \alpha z + \beta$ be an affine transformation. If $g(z) = f(T(z))$, we have that $T \circ S_g \circ T^{-1} = S_f$. Hence, S_f and S_g are topologically conjugated, where S_f and S_g are, respectively, Schröder's iteration maps applied to f and g.*

Proof. Firstly, we calculate the composition $S_g(T^{-1}(z))$:

$$S_g(T^{-1}(z)) = T^{-1}(z) - \frac{g(T^{-1}(z))}{g'(T^{-1}(z))}\frac{1}{1 - L_g(T^{-1}(z))}. \tag{3.1}$$

Starting from the equality $g(T^{-1}(z)) = f(z)$, we have that

$$f'(z) = (g \circ T^{-1})'(z) = \frac{1}{\alpha}g'(T^{-1}(z))$$

and

$$f''(z) = (g \circ T^{-1})''(z) = (\frac{1}{\alpha}g'(T^{-1}(z)))' = \frac{1}{\alpha^2}g''(T^{-1}(z)).$$

Consequently,

$$L_g(T^{-1}(z)) = \frac{g(T^{-1}(z))g''(T^{-1}(z))}{g'(T^{-1}(z))^2} = L_f(z).$$

Therefore:

$$S_g(T^{-1}(z)) = T^{-1}(z) - \frac{1}{\alpha}\frac{f(z))}{f'(z))}\frac{1}{1 - L_f(z))},$$

$$T(S_g(T^{-1}(z))) = \alpha(T^{-1}(z) - \frac{1}{\alpha}\frac{f(z))}{f'(z))}\frac{1}{1 - L_f(z))}) + \beta.$$

As $T^{-1}(z) = (z - \beta)/\alpha$, we deduce the stated result $T(S_g(T^{-1}(z))) = S_f(z)$. □

The Scaling Theorem enables us to reduce the study of the dynamical behavior of Schröder's method, when applied to a general cubic polynomial, to a family of polynomials depending on a single complex parameter. One such family —among several, all of which are topologically conjugate— suitable for analyzing the dynamics of Schröder's iteration map for cubic polynomials is given by

$$p_\lambda(z) = (z^2 - 1)(z - \lambda), \quad \lambda \in \mathbb{C}. \tag{3.2}$$

Note that we can identify each polynomial in (3.2) with the corresponding complex number $\lambda \in \mathbb{C}$. So, for our convenience, we can simplify the notation employed until this moment, and we can denote by S_λ the iteration map obtained by applying Schröder's method (1.2) to a polynomial in (3.2) , that is

$$S_\lambda(z) = S_{p_\lambda}(z) = z - \left(\frac{1}{1 - L_{p_\lambda}(z)}\right)\frac{p_\lambda(z)}{p_\lambda{}'(z)}, \tag{3.3}$$

The parameter space is a powerful graphical tool that enhances our understanding of the behavior of the iteration map defined in (3.3). It has been employed by several authors to investigate other iterative processes (see, for instance, [6, 13, 16]). This space is constructed by tracking the orbits of the free critical points of the corresponding iteration map. Free critical points are those critical points of a root-finding method that do not coincide with any of the roots of the polynomial equation being solved. The strategy of analyzing the orbits of critical points is grounded in the following classical theorem (see [17]):

Theorem 4. *(Fatou-Julia) Every attracting cycle of a rational map attracts at least one critical point.*

In our case, for Schröder's iteration map (1.2), formally we have that its derivative is

$$S'_p(z) = -\frac{L_p(z) - 2L_p(z)^2 + L_{p'}(z)L_p(z)^2}{(1 - L_p(z))^2},$$

where

$$L_{p'}(z) = \frac{p'(z)p'''(z)}{p''(z)^2}.$$

In particular, for polynomials in the family (3.2) and the iteration map (3.3), to obtain the free critical points, we have to solve $S'_p(z) = 0$. After a few algebraic manipulations and discarding the three roots of $p_\lambda(z)$, we obtain the cubic equation

$$(\lambda^2 + 3)z^3 - 12\lambda z^2 + (9\lambda^2 + 3)z - 2\lambda^3 - 2\lambda = 0.$$

In this case, the study of the parameter plane associated with Schröder's method applied to the family of polynomials given by (3.2) requires the numerical solution of the preceding equation. This yields three free critical points, denoted ρ_i, $i = 1, 2, 3$. This represents a key distinction from other methods —such as Newton's, Halley's, or Chebyshev's— where the free critical points can be computed explicitly via linear or quadratic equations. To visualize the parameter planes corresponding to Schröder's method (3.3), a color palette with $3^3 = 27$ colors would be needed to represent all possible behaviors. However, due to the complexity of such a representation, we simplify the visualization by using only two colors: Black for values of λ where the orbit of at least one free critical point does not converge to a root, and white for values of λ where the orbits of all free critical points converge to roots. In the latter case, this guarantees that no attractors other than the roots are present. Some details of this parameter plane are shown in Figure 2.

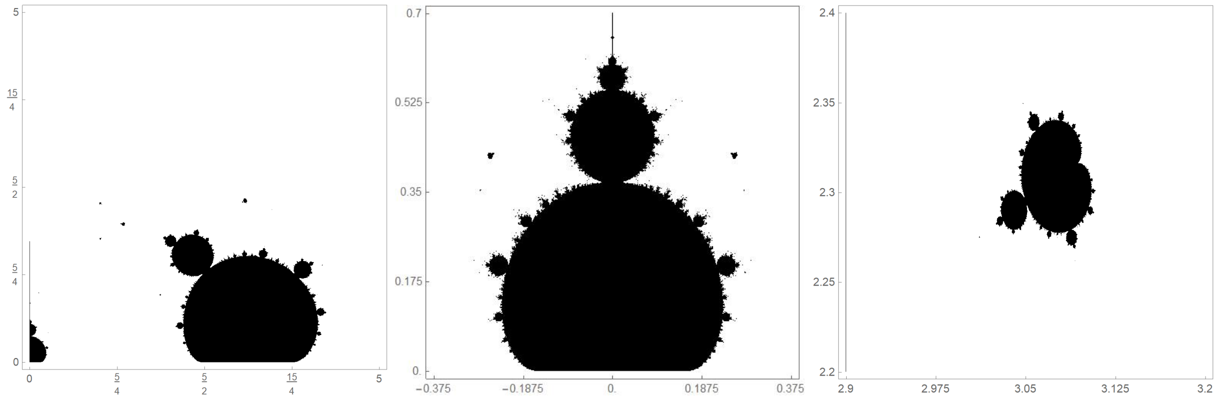

Figure 2. Three images of parameter plane of Schröder's method applied to the $p_\lambda(z)$ family. In black the values of λ for which at least the orbit of one of the three free critical points does not converge to any root, and in white the opposite case. On the left, we have drawn the parameter plane only in the first quadrant, taking into account the double symmetry with respect to the coordinate axes. Note that this parameter plane is not connected. There are multiple Mandelbrot-like sets scattered throughout the rest of the plane. For instance, in the right figure, we show one of them, corresponding to the values around the point $3.1 + 2.3i$. In the midddle, a magnification around the imaginary axes.

4. The behavior of the infinity point in Schröder's method

If we consider now the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the infinity point can be regarded as another point in the complex plane. Then it is possible to study its orbits by an iteration map. As it is well known (see [17] for details), the behavior of infinity related to an iteration map $R_1(z)$ can be studied via conjugation with the Möbius map $1/z$. In this way the behavior of ∞ for $R_1(z)$ is the same as the behavior of the origin for the map $R_2(z) = 1/R_1(1/z)$. In the case of Newton's method and other iterative processes, such as Halley's or Chebyshev's methods, the infinity point is a repelling fixed point.

A distinguishing feature of Schröder's method is that the infinity point is not a fixed point, as seen in Corollary 2. In this section, we explore in more detail the behavior of the infinity point for Schröder's method. We start with the cubic polynomials defined in (3.2).

4.1. The infinity point in Schröder's method (cubic polynomials)

Note that the behavior of ∞ for Schröder's map $S_\lambda(z)$ defined in (3.3) and applied to polynomials (3.2) is the same as the behavior of 0 for

$$T_\lambda(z) = \frac{1}{S_\lambda(1/z)} = \frac{(2\lambda^2+1)z^4 - 4\lambda z^3 + 2\lambda^2 z^2 - 4\lambda z + 3}{\lambda z^4 + 4\lambda^2 z^3 - 10\lambda z^2 + 4z + \lambda}. \tag{4.1}$$

As 0 is not a fixed point of $T_\lambda(z)$, ∞ is not a fixed point of $S_\lambda(z)$. Consequently, it makes sense to study the orbits of infinity by Schröder's map $S_\lambda(z)$ defined in (3.3) or, equivalently, the orbits of $z = 0$ by the iteration map $T_\lambda(z)$ defined in (4.1).

Definition 2. *We define the* parameters plane of ∞ for Schröder's method applied to cubic polynomials *as the parameter plane related to the orbits of* $z = 0$ *by* $T_\lambda(z)$ *defined in* (4.1). *It is obtained by identifying each polynomial in* (3.2) *with its corresponding parameter* λ *and by coloring each* $\lambda \in \mathbb{C}$ *depending on the convergence of the orbits of* $z = 0$ *by* $T_\lambda(z)$. *As* $T_\lambda(z)$ *has three attractive fixed points, we establish the following color code:*

- λ *is colored in pink if the orbit of* $z = 0$ *by* $T_\lambda(z)$ *converges to the fixed point* $z = 1$.
- λ *is colored in purple if the orbit of* $z = 0$ *by* $T_\lambda(z)$ *converges to the fixed point* $z = -1$.
- λ *is colored in green if the orbit of* $z = 0$ *by* $T_\lambda(z)$ *converges to the fixed point* $z = 1/\lambda$.

The resulting parameter plane is shown in Figure 3.

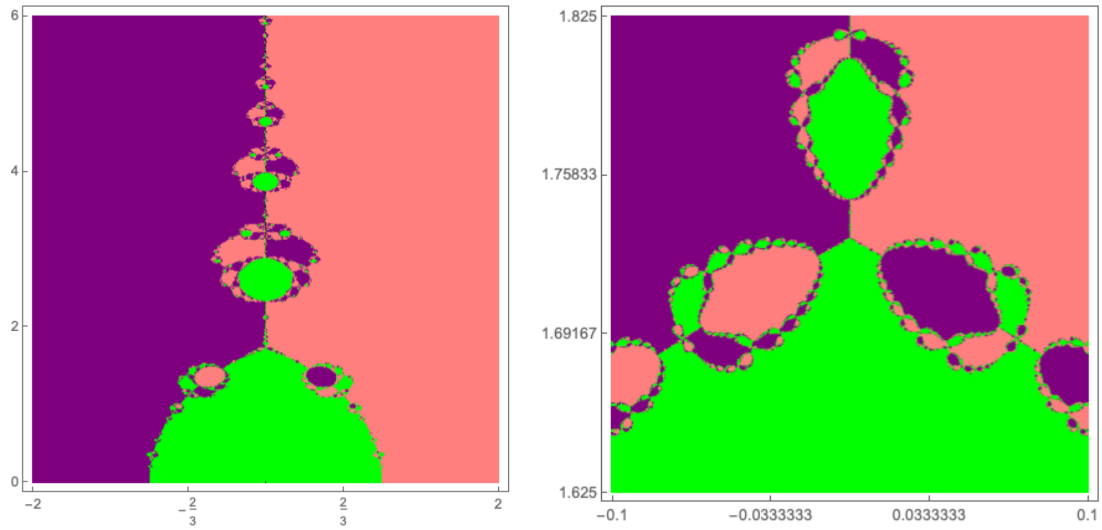

Figure 3. On the left, parameters plane of ∞ for Schröder's method in the rectangle $[-2, 2] \times [0, 6i]$. On the right, a detail around the point $\sqrt{3}i$.

Remark 1. *The map* $T_\lambda(z)$ *defined in* (4.1) *has two repelling fixed points,* $-\lambda \pm \sqrt{3 + \lambda^2}$, *both of them with multiplier equal to 2.*

We can prove that in Figure 3 there is a symmetry with respect to the real axis. In fact, we prove it in the following theorem.

Proposition 1. *Let* $T^n_\lambda(z)$ *be the n-th composition of the map* $T_\lambda(z)$ *defined in* (4.1). *Then*

$$T^n_{\overline{\lambda}}(0) = \overline{T^n_\lambda(0)},$$

and, consequently,

$$\lim_{n\to\infty} T^n_{\overline{\lambda}}(0) = \lim_{n\to\infty} \overline{T^n_\lambda(0)}.$$

Proof. Let us prove that the above equality runs for all $n \in \mathbb{N}$ by induction. When $n = 1$,

$$\overline{T_\lambda(0)} = \overline{\left(\frac{3}{\lambda}\right)} = \frac{3}{\overline{\lambda}} = T_{\overline{\lambda}}(0).$$

Now let us assume that the equality in the theorem holds for $n = 1, 2 \ldots, k-1$. Let us see if it is also true for $n = k$.

$$\overline{T^k_{\overline{\lambda}}(0)} = \overline{T_{\overline{\lambda}}\left(T^{k-1}_{\overline{\lambda}}(0)\right)} = \overline{T_{\overline{\lambda}}\left(\overline{\overline{T^{k-1}_{\lambda}(0)}}\right)} \tag{4.2}$$

and since T is a rational function on λ and z, for all $(\lambda, z) \in \mathbb{C}^2$, $\overline{T_{\overline{\lambda}}(\overline{z})} = T_\lambda(z)$. Then, by taking into account (4.2), we conclude

$$\overline{T^k_{\overline{\lambda}}(0)} = T_\lambda\left(T^{k-1}_{\lambda}(0)\right) = T^k_\lambda(0)$$

and the proof is completed. □

A direct consequence of the behavior of the point at infinity in Schröder's method is the emergence of a dominant root when plotting the basins of attraction for the polynomials defined in (3.2). The orbit of infinity converges to this dominant root. Specifically, for values of λ in the pink region of Figure 3, the dominant root is $z = 1$. For λ in the purple zone, it is $z = -1$, and for λ in the green region, the dominant root is $z = \lambda$. Figure 4 provides a visual representation of the presence of these dominant roots. The basins of attraction for the three roots are colored as follows: cyan for the root $z = 1$, magenta for $z = -1$, and yellow for $z = \lambda$. In each case, black regions are also visible, corresponding to the presence of attracting cycles. For comparison, Figure 5 shows the basins of attraction of Newton's method applied to the same polynomials.

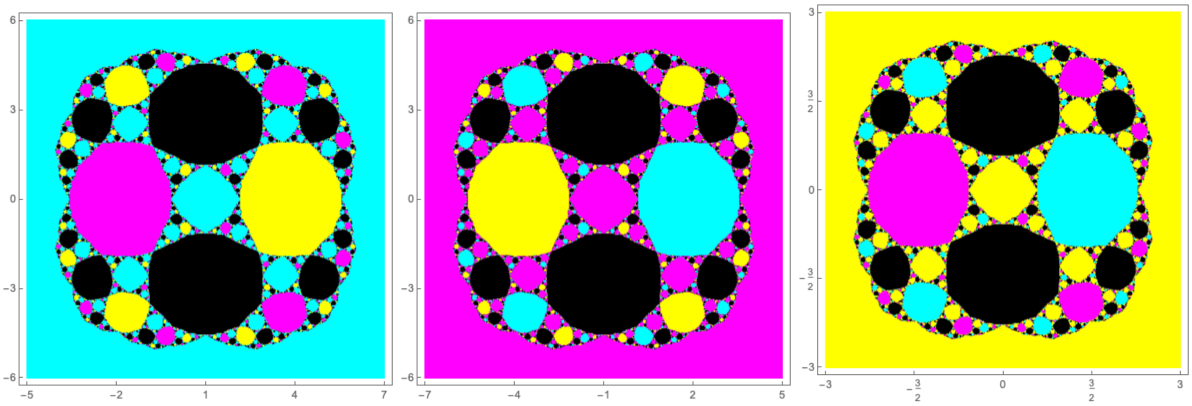

Figure 4. Basins of attraction of Schröder's method applied to polynomials (3.2) with $\lambda = 3$ (left), $\lambda = -3$ (middle), and $\lambda = 0$ (right).

The emergence of dominant roots might appear to contradict a result by Hubbard et al. [18], which asserts that each root is connected to the point at infinity. However, this is not the case. Hubbard's result specifically pertains to Newton's method applied to polynomials. In particular, it is shown that the immediate basin of a root —that is, the connected component of the basin of attraction containing the root itself— has a certain number of accesses to infinity. For each root ζ, an access to infinity is defined as an unbounded connected component W of the basin of attraction of ζ such that every point $\omega \in W$ can be joined to its image $N_p(\omega)$ by a curve entirely contained in W, where N_p denotes the Newton's map applied to the polynomial $p(z)$:

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

In Figure 5 we can appreciate the existence of these accesses to infinity for Newton's method applied to some particular polynomials.

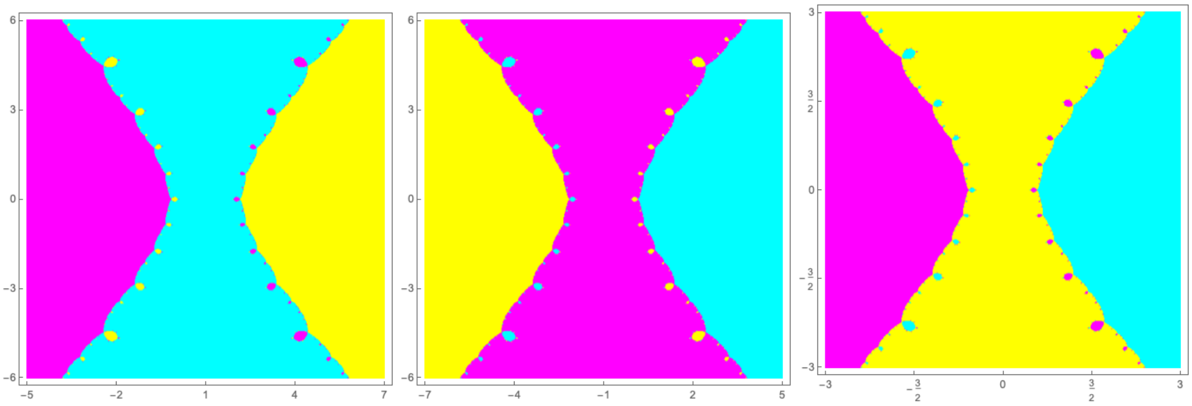

Figure 5. Basins of attraction of Newton's method applied to polynomials (3.2) with $\lambda = 3$ (left), $\lambda = -3$ (middle), and $\lambda = 0$ (right).

In the case of Schröder's method, the existence of the accesses to infinity is not guaranteed. However, this does not contradict Hubbard's result, as Schröder's method corresponds to applying Newton's method to the rational function $p(z)/p'(z)$ and not to a polynomial.

Nevertheless, there are instances in which Schröder's method does not exhibit dominant roots. In other words, the immediate basin of attraction for each root remains connected to the point at infinity. One such case occurs when $\lambda = \sqrt{3}i$, corresponding to the configuration where the three roots form an equilateral triangle. In this case, after some algebraic simplification, the iteration map $S_{\sqrt{3}i}(z)$ defined in (3.3) can be expressed in the form

$$S_{\sqrt{3}i}(z) = \frac{\sqrt{3}z^3 - 3iz^2 - 9\sqrt{3}z + 3i}{-3iz^3 - 3\sqrt{3}z^2 + 3iz - 5\sqrt{3}}.$$

The corresponding function $T_{\sqrt{3}i}(z)$ defined in (4.1) is

$$T_{\sqrt{3}i}(z) = -\frac{5\sqrt{3}z^3 - 3iz^2 + 3\sqrt{3}z + 3i}{3iz^3 - 9\sqrt{3}z^2 - 3iz + \sqrt{3}}.$$

In this case, 0 is a pre-image of the repelling fixed point $z = -\sqrt{3}i$, that is, $T_{\sqrt{3}i}(0) = -\sqrt{3}i$. This is why there is no dominant root for Schröder's method in this case, as we can see on the left side of Figure 6, where there are three accesses to infinity (one for each root).

We can search for more λ's for which their basins of attraction don't have a dominant root, finding λ such that

$$T_\lambda^n(0) = -\lambda \pm \sqrt{\lambda^2 + 3},$$

that is, the orbit of 0 by $T_\lambda(0)$ reaches one of the two repelling fixed points $-\lambda \pm \sqrt{\lambda^2 + 3}$. For instance, we can numerically find one of these values: $b = \sqrt{\tau} \approx 5.58299$, where τ is the only positive root of $15t^3 - 459t^2 - 243t - 729 = 0$. On the right side of Figure 6, we can see a case with two accesses to infinity.

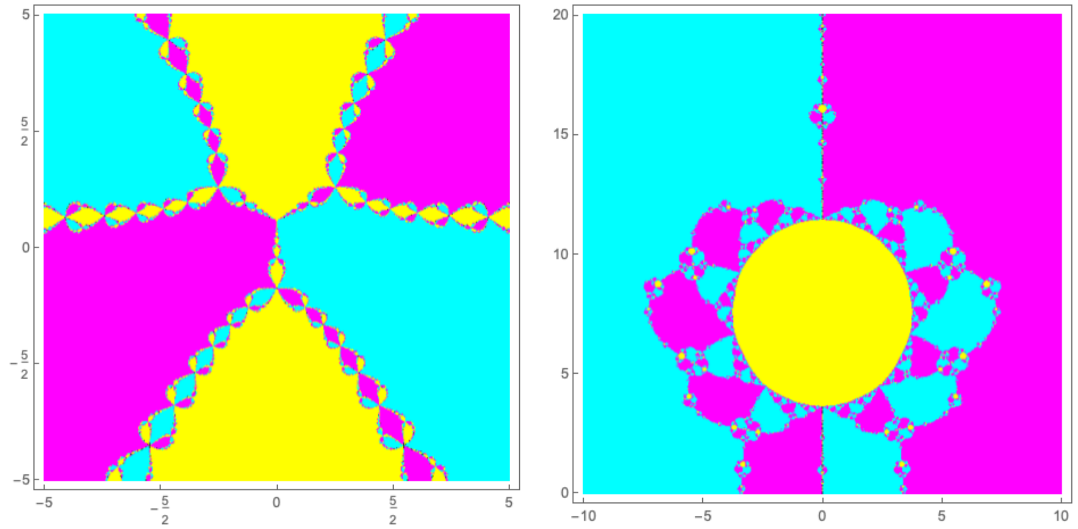

Figure 6. Basins of attraction of Schröder's method applied to polynomials (3.2) with $\lambda = \sqrt{3}i$ and with $\lambda = 5.58299i$.

4.2. The infinity point in Schröder's method (other polynomials)

We can extend the use of the parameters plane of ∞ for other polynomials. The following step could be to consider polynomials in the form

$$p_n(z) = (z^2 - 1)(z - \lambda)^n, \quad \lambda \in \mathbb{C}. \tag{4.3}$$

Let us define $S_{\lambda,n}(z)$ the Schröder's map applied to polynomials (4.3) and

$$T_{\lambda,n}(z) = \frac{1}{S_{\lambda,n}(1/z)} = \frac{(n + 2\lambda^2)z^4 - 4\lambda z^3 + 2(\lambda^2 - n + 1)z^2 - 4\lambda z + n + 2}{\lambda n z^4 + 4\lambda^2 z^3 - 2\lambda(n + 4)z^2 + 4z + \lambda n}. \tag{4.4}$$

By following the orbits of 0 by the iteration map $T_{\lambda,n}(z)$ defined in (4.4) and with the same color code given in Definition 2, we can plot the parameters plane of ∞ for Schröder's method applied to polynomials (4.3). For instance, in Figure 7 we can see the parameters plane of ∞ for $p_2(z) = (z^2 - 1)(z - \lambda)^2$ and $p_3(z) = (z^2 - 1)(z - \lambda)^3$.

We may conjecture that, when a multiple root is present, it tends to be dominant in the sense that it attracts the orbit of the point at infinity (as indicated by the green regions in Figure 7). Consequently, in the basins of attraction corresponding to polynomials with λ in the green area, the multiple root appears as the dominant root. In contrast, for polynomials associated with λ in the pink or purple areas, the dominant root is one of the simple roots ($z = 1$ or $z = -1$, respectively).

We verify this conjecture in specific cases. For example, Figure 8 displays the basins of attraction for Schröder's method applied to the polynomials $(z^2 - 1)z^2$ and $(z^2 - 1)(z - 4)^2$. In the first case, the basin of attraction of the multiple root $z = 0$, colored in yellow is the dominant one, whereas in the second case, the basin of attraction of the simple root $z = 1$, colored in cyan is the dominant one.

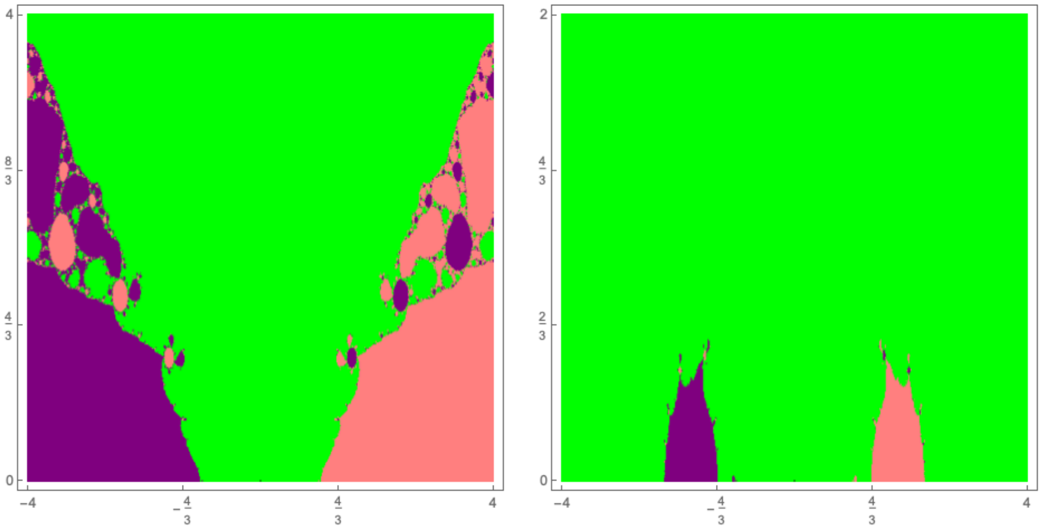

Figure 7. Parameters plane of ∞ for Schröder's method applied to polynomials $p_2(z) = (z^2 - 1)(z - \lambda)^2$ and $p_3(z) = (z^2 - 1)(z - \lambda)^3$.

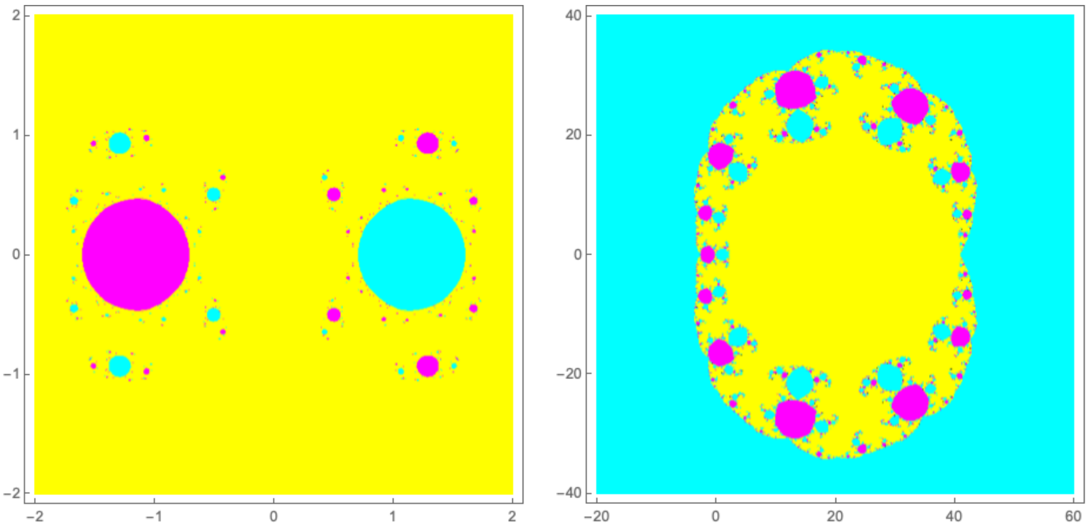

Figure 8. Basins of attraction of Schröder's method applied to the polynomials $(z^2 - 1)z^2$ and $(z^2 - 1)(z - 4)^2$.

5. Conclusions

This paper presents a comprehensive study on the dynamics of Schröder's iterative method for solving polynomial equations, providing a thorough comparison with other well-known iterative methods, particularly Newton's method. The differences between these approaches emerge even in the simplest scenarios, such as polynomials with two roots of different multiplicities, highlighting the distinct nature of Schröder's approach. In the case of Schröder's method, the dynamics are significantly simpler, as the Julia set consists of straight lines and circles.

Throughout the analysis, we have examined several properties of Schröder's iteration map, including the degree of the associated rational map, the repelling nature of extraneous fixed points, and the occurrence of attractive cycles.

One of the most significant findings is the unique behavior of the infinity point, which differs from Newton's method and other iterative techniques where it generally acts as a repelling fixed point. Instead, in Schröder's method, the infinity point exhibits a non-repelling character that influences the overall dynamics of the method. Additionally, we have conducted a detailed study of the infinity point behavior for cubic polynomials by introducing a novel graphical tool: the parameter plane of the infinity point. This tool has proven effective in visualizing and understanding the influence of the infinity point on the basins of attraction of the roots. A notable outcome of this analysis is the general appearance of a dominant root when plotting these basins, a phenomenon intrinsically linked to the particular behavior of the infinity point under Schröder's method.

Furthermore, our work suggests that the differences in dynamics are not limited to cubic polynomials. The distinctive behavior of the infinity point and its implications for the basins of attraction may extend to polynomials of higher degrees. As such, a natural extension of this work involves the study of the infinity point's behavior for other classes of polynomials and the potential application of our findings to broader families of iterative methods.

Future research could also explore the potential connections between the dynamics of Schröder's method and other iterative schemes, particularly in terms of their efficiency, convergence rates, and sensitivity to initial conditions. Additionally, further refinement of the parameter plane tool could enhance its applicability and provide deeper insights into the global dynamics of Schröder's iterative method.

Author contributions

Both authors have been working together in the mathematical development of the manuscript. J. M. Gutiérrez: Writing–review & editing, Methodology, Conceptualization, Supervision; V. Galilea: Writing–original draft, Software, Methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare they have no conflict of interest in the research related with this paper.

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