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Research article

Exponential integral method for European option pricing

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Abstract: Since the official launch of domestic exchange-traded options in 2015, options have gradually become an important component of the financial market. With the diversification of option types and the expansion of market size, option pricing research has received widespread attention. In this paper, we propose a finite difference method based on exponential integrals for the pricing of European call options, based on the Black-Scholes differential equation. Through numerical analysis, this method discretizes only the price region and uses the exponential Euler method to solve the nonhomogeneous system of linear differential equations in the time direction. Numerical experiments have verified the effectiveness of this method, showing that it can stably solve option pricing problems, especially when the price is close to the exercise price, demonstrating superior numerical performance.

Keywords: option pricing; European options; exponential integral method; finite difference
Mathematics Subject Classification: 65L05, 65L07, 65L10, 65J15, 91G60

1. Introduction

Since the official launch of China's options market in 2015, options as an important financial derivative tool have gradually become a hot research topic in the domestic financial market. From the initial pilot between banks to the launch of the SSE 50ETF options and CSI 300ETF options, options have gradually become one of the major global ETF option types. The development of the options market provides investors with an important tool for risk aversion and also provides theoretical support for market managers to stabilize trading order.

Option pricing, as a core issue in option research, has always been one of the hot topics in both theoretical research and practical application. The Black-Scholes option pricing model is a classic option pricing theory [1, 2], which derives a closed solution for option pricing by assuming that the underlying asset price follows a geometric Brownian motion. However, the basic assumptions of the Black-Scholes model are somewhat limited in actual markets, especially the complexity of the market and the discontinuity of price fluctuations, which necessitate the use of numerical methods for solving in practical applications. Traditional numerical methods for option pricing can be broadly categorized into finite element methods, binomial tree models, Monte Carlo simulations, quadrature-based methods, Fourier transform approaches, and finite difference methods.

Finite element methods (FEM) has become a vital tool for pricing complex derivatives. The methodology was initially advanced by Holmes and Yang (2008) [3] through their front-fixing technique for American options, while Yan and Zhao (2008) [4] concurrently developed simplified FEM formulations for practical implementation. Andalaft-Chacur et al. (2011) [5] subsequently extended FEM applications to real options valuation. For complex pricing problems, Zhu and Chen (2013) [6] proposed inverse FEM for volatility calibration. Under stochastic volatility models, Kozpınar and Uzunca (2020) [7] applied discontinuous Galerkin FEM to Heston pricing. Multi-asset challenges were addressed by Zhang et al. (2015) [8], with Abdulle et al. (2023) [9] providing theoretical foundations through stabilized schemes. Future directions include machine learning integration and crypto-asset pricing.

The binomial option pricing model, introduced by Cox, Ross, and Rubinstein (1979) [10], provides a discrete-time framework that converges to the Black-Scholes formula. Subsequent refinements by Leisen and Reimer (1996, 1998) [11, 12] and Chang and Palmer (2007) [13] improved convergence properties for American options. The methodology has been extended to path-dependent derivatives (Reynaerts et al., 2006) [14] and GARCH volatility modeling (Gong and Xu, 2020) [15]. While tree methods remain computationally efficient for early-exercise pricing, they face challenges in high-dimensional problems.

Monte Carlo simulation is established as a fundamental technique in modern option pricing since Boyle's (1977) [16] seminal work. The methodology achieved a breakthrough with Longstaff and Schwartz's (2001) [17] least squares approach for American option valuation. Advancements demonstrate its expanding applications: Nouri and Abbasi (2017) [18] developed enhanced techniques for barrier options, Capuozzo et al. (2021) [19] introduced path integral implementations, while Xiang and Wang (2022) [20] applied quasi-Monte Carlo methods to sensitivity analysis. The frontier continues to advance with Xia and Grabchak's (2024) [21] extension to multi-asset pricing using tempered stable distributions. While particularly powerful for path-dependent and high-dimensional problems, ongoing refinements continue to address computational efficiency challenges.

Quadrature methods [22] extend traditional integration techniques and, through recent innovations, can be adapted to a wider range of models while maintaining high efficiency. Finally, Fourier transform methods [23], particularly the fast Fourier transform (FFT) techniques proposed by Kwok et al., have shown high efficiency in pricing options under Lévy processes, enabling fast and accurate valuation of exotic derivatives via convolution-based computations in the frequency domain. The rapid evolution of financial derivatives in the FinTech era has introduced unprecedented challenges for option pricing methodologies, particularly for digital assets like Bitcoin, which exhibit unique market microstructures. While the traditional Black-Scholes model, based on geometric Brownian motion assumptions (Merton, 1976) [24], has proven inadequate in capturing the complexities of modern financial markets, this limitation is especially pronounced in cryptocurrency markets. With the recent launch of Bitcoin derivatives by major exchanges such as Nasdaq and CME Group, developing pricing models that accurately reflect market dynamics has become critically important.

Finite difference methods (FDMs) have emerged as a cornerstone of numerical techniques in financial engineering due to their simplicity and effectiveness in handling complex boundary conditions

(Brennan and Schwartz, 1978) [25]. Research has achieved significant breakthroughs in this field:

In jump process modeling, Cont and Voltchkova (2005) [26] developed an explicit-implicit finite difference scheme for solving parabolic partial integro-differential equations arising from option pricing under Lévy processes. Their work on localization error analysis and rigorous stability proofs laid the foundation for subsequent studies. Kudryavtsev (2019) [27] further enhanced the accuracy of pricing barrier and American options under Lévy jump models by incorporating Wiener-Hopf factorization, maintaining computational efficiency while significantly improving precision.

For American option pricing, several advancements have been made: Khaliq et al. (2008) [28] introduced adaptive θ-methods, Tangman et al. (2008) [29] designed high-order fast algorithms, and Hu et al. (2009) [30] optimized the convergence rate of explicit schemes. Notably, Gu and Kang (2014) [31] proposed a seven-point finite difference GMRES method, providing an efficient numerical framework for European option pricing.

In fractional models, Song and Wang (2013) [32] pioneered the application of finite difference methods to solve the fractional Black-Scholes equation. Roul and Goura (2021) [33] later developed a compact scheme that further improved computational accuracy. Additionally, Ma et al. (2017) [34] introduced a hybrid Laplace transform-finite difference method, offering new solutions for pricing American options under complex models.

The finite difference method discretizes continuous differential equations into algebraic equation groups, making it relatively simple to solve and suitable for handling complex boundary conditions. However, the discretization of time and space in the finite difference method may lead to error accumulation, especially when dealing with high volatility markets or when the underlying asset price is close to the exercise price, where numerical instability is likely to occur.

In general, an integrator is called as semi-analytical for solving PDEs, if it avoids the discretization of one derivation. However, most numerical simulators for PDEs depend on the complete discretization of each derication derivative. This paper thus marks an introductory foray towards the development of semi-analytical integrators based on an entirely equivalent system to the underlying system. We propose a finite difference method based on exponential integrals. This method discretizes only in the price direction and remains continuous in the time direction, solving non-homogeneous linear differential equation groups through exponential integrals, avoiding error accumulation from time discretization. We not only provide a more accurate numerical method for option pricing but also verify its effectiveness in practical market applications. The major research contents of this paper include: First, introducing the Black-Scholes differential equation for European options and its traditional finite difference solving method; second, proposing a finite difference solving strategy based on exponential integrals and deriving the specific steps of this method in detail; finally, conducting a comparative analysis of this method with traditional methods through numerical simulation to verify its accuracy and computational efficiency.

2. Theoretical basis

2.1. Definition and classification of options

An option refers to the right to buy or sell a certain amount of goods or financial instruments at a mutually agreed price within a future period. Option trading refers to the buying and selling activities of such buying and selling rights. In option trading, after paying a certain fee to the option seller, the option buyer has the right to buy or sell a certain amount of a specific commodity or futures contract from the option seller at a specific price in the future, without an obligation to do so (the buyer of spot rights has the right to choose to buy or sell, and the option seller must unconditionally obey the buyer's choice and fulfill the commitment at the time of trading). In practice, this means that when the price is favorable for the long position, the buyer has the right to choose to exercise the option. When the price is unfavorable for the long position, the buyer has the right to choose not to exercise the option. Options are divided into call options and put options, and according to the exercise time, they can be divided into European options and American options. European options refer to options that can be exercised only on the expiration date, while American options can be exercised on any trading day before or on the expiration date. Therefore, compared to European options, American options offer relatively greater rights to buyers, i.e., under the same conditions, American options are priced higher. Here, we mainly study European call options, which can be exercised only at maturity.

2.2. Black-Scholes model

Research on option pricing needs to satisfy some basic assumptions, which ensure the operability of option pricing research and its closer alignment with actual financial markets.

(1) Stock prices follow a generalized Wiener process, and stock prices follow a lognormal distribution.

(2) All transactions in the financial market can be smoothly completed, and there is no possibility of risk-free arbitrage in the economic activity.

(3) There are no transaction costs and tax costs in transactions.

(4) No dividend payments occur during the trading period.

(5) Investors can purchase any number of underlying assets and can use all proceeds.

(6) The traded option is a European option, i.e., the option is not exercised before maturity.

(7) The risk-free interest rate r in transactions is constant and known.

We consider S as the stock (underlying asset) price, K as the strike price, T as the expiration time, $f = f(S,t)$ as the price (option premium) of the European call option. σ as the annual volatility of the stock price, which is a constant. In this paper, we assume $\sigma^2 \geq \alpha > 0, \beta^* \geq r \geq \beta > 0$. Usually, we may suppose $0.05 \geq r \geq 0.01$. The change in stock prices can be represented by the following Black-Scholes European option pricing differential equation [33]:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{2.1}$$

To determine the value of a call option within the validity period T, solve the following definite problem in the region $\{0 \leq S \leq \infty, 0 \leq t \leq T\}$:

$$\begin{cases} \dfrac{\partial f}{\partial t} + rs\dfrac{\partial f}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 f}{\partial S^2} = rf, \\ f(S,T) = max\{S-K, 0\}. \end{cases} \tag{2.2}$$

For European call options, their boundary condition exists as

$$\begin{cases} f(0,t) = 0, \\ f(S,t) = S - Ke^{-r(T-t)}, \quad S \to \infty. \end{cases} \tag{2.3}$$

3. Common finite difference formats for Black-Scholes differential equations

3.1. Explicit finite difference method

First, we divide the time into intervals of equal length, each part has a length of $\Delta t = T/N$. Moreover, we set a sufficiently large S_{max}, such that S_{max} far exceeds K. We perform equal-length segmentation on the option's underlying asset price S, with each part having a length of $\Delta S = S_{max}/M$. Ultimately, a grid of size $(M+1)(N+1)$ is obtained, including boundary conditions. Each grid point represents the intersection of the option price $f_{i,j}$, while (i, j) represents the underlying asset price $j\Delta S$ at time $i\Delta t$, $i = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, M$.

Using forward and backward differences for the first-order derivatives of f, respectively, gives

$$\frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta S}, \tag{3.1}$$

and

$$\frac{\partial f}{\partial S} \approx \frac{f_{i,j} - f_{i,j-1}}{\Delta S}. \tag{3.2}$$

We can obtain

$$\frac{\partial f}{\partial S} \approx \frac{1}{2}(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} + \frac{f_{i,j} - f_{i,j-1}}{\Delta S}) = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}. \tag{3.3}$$

Use backward difference for the first-order derivative of time:

$$\frac{\partial f}{\partial t} \approx \frac{f_{i,j} - f_{i-1,j}}{\Delta t}. \tag{3.4}$$

Construct the following central difference for the second-order derivative of f:

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{\frac{f_{i,j+1}-f_{i,j}}{\Delta S} - \frac{f_{i,j}-f_{i,j-1}}{\Delta S}}{\Delta S} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2}. \tag{3.5}$$

Substituting (3.3)–(3.5) into Black-Scholes European option pricing differential Eq (2.1), we get

$$\frac{f_{i,j} - f_{i-1,j}}{\Delta t} + rS_j \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 S_j^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} = rf_{i,j}. \tag{3.6}$$

Since $S_j = j\Delta S$, we get

$$f_{i,j} - f_{i-1,j} + \frac{1}{2} rj\Delta t(f_{i,j+1} - f_{i,j-1}) + \frac{1}{2}\sigma^2 j^2 \Delta t(f_{i,j+1} + f_{i,j-1} - 2f_{i,j}) = r\Delta t f_{i,j}. \tag{3.7}$$

Therefore,

$$b_j f_{i,j} + a_j f_{i,j-1} + c_j f_{i,j+1} = f_{i-1,j}, \tag{3.8}$$

where

$$a_j = -\frac{1}{2} r j \Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t, b_j = 1 - r\Delta t - \sigma^2 j^2 \Delta t, c_j = \frac{1}{2} r j \Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t.$$

The matrix form of (3.8) is as follows

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{M-2} & b_{M-2} & c_{M-2} \\ & & & & a_{M-1} & b_{M-1} \end{bmatrix} \cdot \begin{bmatrix} f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{bmatrix} + \begin{bmatrix} a_1 f_{i+1,0} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{M-1} f_{i+1,M} \end{bmatrix} = \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix}.$$

For European call options, its terminal condition is

$$f(T,S) = max(S-K,0), \forall S > 0. \tag{3.9}$$

Then, obtain its boundary condition as

$$\begin{aligned} f_{N,j} &= max(j\Delta S - K, 0); j = 0,1,\cdots M, \\ f_{i,M} &= S_{max} - K, f_{i,0} = 0, i = 0,1,\cdots N. \end{aligned} \tag{3.10}$$

We denote this difference method by EDM.

3.2. Implicit finite difference method

The regional division of the implicit finite difference method is consistent with the explicit method, modifying only (3.4) using forward difference for the first-order derivative of time

$$\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i,j}}{\Delta t}. \tag{3.11}$$

Substituting (3.3), (3.5), (3.11) into Black-Scholes European option pricing differential Eq (2.1), we get

$$f_{i+1,j} - f_{i,j} + \frac{1}{2} r j \Delta t (f_{i,j+1} - f_{i,j-1}) + \frac{1}{2}\sigma^2 j^2 \Delta t (f_{i,j+1} + f_{i,j-1} - 2f_{i,j}) = r\Delta t f_{i,j}. \tag{3.12}$$

After calculation, we get

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}, \tag{3.13}$$

where $a_j = \frac{1}{2} r j \Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$, $b_j = r\Delta t + 1 + \sigma^2 j^2 \Delta t$, $c_j = -\frac{1}{2} r j \Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$.

Rewrite (3.13) into the following matrix form

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{M-2} & b_{M-2} & c_{M-2} \\ & & & & a_{M-1} & b_{M-1} \end{bmatrix} \cdot \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix} + \begin{bmatrix} a_1 f_{i,0} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{M-1} f_{i,M} \end{bmatrix} = \begin{bmatrix} f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{bmatrix}.$$

Its European call option boundary conditions are the same as the explicit conditions ((3.9) and (3.10)). We denote this difference method by IDM.

3.3. Crank-Nicolson Difference Method

The Crank-Nicolson difference method is obtained by averaging the finite difference method with the extrapolated finite difference method, representing a more precise numerical solution method.

Discretize each term in Eq (2.1) as follows:

$$\begin{aligned} &\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i,j}}{\Delta t}, \\ &\frac{\partial f}{\partial S} \approx \frac{1}{2}\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2}\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S}, \\ &\frac{\partial^2 f}{\partial S^2} \approx \frac{1}{2}\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} + \frac{1}{2}\frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{\Delta S^2}, \\ &rf = r\frac{f_{i,j} + f_{i+1,j}}{2}. \end{aligned}$$

Let coefficients a_j, b_j, and c_j be consistent with display format (3.8), so we get

$$f_{i+1,j} + \frac{1}{2}[(b_j - 1)f_{i+1,j} + a_j f_{i+1,j-1} + c_j f_{i+1,j+1}] = f_{i,j} - \frac{1}{2}[(b_j - 1)f_{i,j} + a_j f_{i,j-1} + c_j f_{i,j+1}]. \tag{3.14}$$

After arrangement, we get

$$\begin{aligned} &(1 - \beta_j)f_{i+1,j} - \alpha_j f_{i+1,j-1} - \gamma_j f_{i+1,j+1} = (1 + \beta_j)f_{i,j} + \alpha_j f_{i,j-1} + \gamma_j f_{i,j+1}, \\ &\alpha_j = -\frac{1}{4}\Delta t(\sigma^2 j^2 - rj), \beta_j = \frac{1}{2}\Delta t(r + \sigma^2 j^2), \gamma_j = -\frac{1}{4}\Delta t(rj + \sigma^2 j^2). \end{aligned} \tag{3.15}$$

Then, rewrite Eq (3.15) into matrix form as

$$\begin{bmatrix} 1+\beta_1 & \gamma_1 & & & & \\ \alpha_2 & 1+\beta_2 & \gamma_2 & & & \\ & \alpha_3 & 1+\beta_3 & \gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{M-2} & 1+\beta_{M-2} & \gamma_{M-2} \\ & & & & \alpha_{M-1} & 1+\beta_{M-1} \end{bmatrix} \cdot \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix} + \begin{bmatrix} \alpha_1 f_{i,0} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \gamma_{M-1} f_{i,M} \end{bmatrix} =$$

$$\begin{bmatrix} 1-\beta_1 & -\gamma_1 & & & & \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & & & \\ & -\alpha_3 & 1-\beta_3 & -\gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_{M-2} & 1-\beta_{M-2} & -\gamma_{M-2} \\ & & & & -\alpha_{M-1} & 1-\beta_{M-1} \end{bmatrix} \cdot \begin{bmatrix} f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ \vdots \\ f_{i+1,M-2} \\ f_{i+1,M-1} \end{bmatrix} + \begin{bmatrix} -\alpha_1 f_{i+1,0} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -\gamma_{M-1} f_{i+1,M} \end{bmatrix}.$$

Its European call option boundary conditions are also the same as the explicit conditions (3.9) and (3.10).

The three aforementioned finite difference schemes-explicit, implicit, and Crank-Nicolson each exhibit certain limitations when dealing with high volatility or prices close to the strike price [35] . The explicit scheme, due to its conditional stability imposed by the CFL condition, necessitates the use of small time steps to maintain numerical stability. This requirement not only increases the computational burden but also heightens the risk of numerical oscillations, particularly when the underlying asset prices experience rapid and large fluctuations associated with high volatility. Additionally, as the asset price nears the strike price, the payoff function's nonlinearity and discontinuity become more pronounced, further exacerbating the scheme's instability and reducing its accuracy. The implicit scheme, while unconditionally stable and thus free from the CFL condition's constraints, is not without its drawbacks. It requires solving a system of linear equations at each time step, which can be computationally intensive, especially for large-scale or high-dimensional problems. This computational demand can significantly slow down the solution process. Moreover, similar to the explicit scheme, the implicit method may also struggle with non-smooth initial conditions, potentially leading to numerical oscillations and reduced accuracy near the strike price. The Crank-Nicolson scheme, which combines the best features of both explicit and implicit schemes by employing central differencing in the time direction, offers enhanced accuracy and unconditional stability. However, it is not immune to certain issues. Despite its improved stability and precision, the Crank-Nicolson method requires solving a system of linear equations at each time step, similar to the implicit scheme. This requirement maintains a relatively high computational cost, which can be a significant drawback for complex problems. Furthermore, under specific circumstances, such as when dealing with highly nonlinear or degenerate cases like jump-diffusion processes, the Crank-Nicolson scheme may encounter numerical oscillations and stability problems. These issues can arise due to the complex nature of the underlying stochastic processes and the challenges in accurately approximating the solution in the presence of discontinuities or rapid changes in the payoff function.

3.4. Exponential integral method

3.4.1. Partial discretization of price

The above three methods all discretize the price S and time t directions of the differential equation simultaneously. We discretize only the price region S based on the exponential integral finite difference method and use a high-precision method to solve the differential equation group in the time region t, thus theoretically producing smaller errors and obtaining better numerical solutions.

This method discretizes in the price S direction, obtaining a solution for the differential equation group related only to time t, then performs high-precision solving on the semi-discrete differential equation group, substituting the corresponding time t to obtain the corresponding option pricing. The specific operation steps are as follows:

First, divide the underlying asset price S into equal intervals, each part having a length of $\Delta S = S_{max}/M$, divide the underlying asset price S_{max} into M equal intervals. The meaning of the underlying asset price is $f_j(t)$ at the time t.

From the above, we obtain the corresponding first-order and second-order derivatives related to price S as

$$\frac{\partial f}{\partial S} \approx \frac{f_{j+1}(t) - f_{j-1}(t)}{2\Delta S}, \tag{3.16}$$

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{f_{j+1}(t) - 2f_j(t) + f_{j-1}(t)}{\Delta S^2}. \tag{3.17}$$

Substitute (3.16) and (3.17) into the option pricing differential equation to get

$$\frac{\partial f_j}{\partial t} + rS_j \frac{f_{j+1}(t) - f_{j-1}(t)}{2\Delta S} + \frac{1}{2}\sigma^2 S_j^2 \frac{f_{j+1}(t) - 2f_j(t) + f_{j-1}(t)}{\Delta S^2} = rf_j(t). \tag{3.18}$$

Then, substitute $S_j = j\Delta S$ to get

$$\frac{\partial f_j}{\partial t} + \frac{1}{2} rj(f_{j+1}(t) - f_{j-1}(t)) + \frac{1}{2}\sigma^2 j^2 (f_{j+1}(t) - 2f_j(t) + f_{j-1}(t)) = rf_j(t). \tag{3.19}$$

That is

$$\begin{aligned} &f_j'(t) = a_j f_{j-1}(t) + b_j f_j(t) + c_j f_{j+1}(t), \\ &a_j = \frac{1}{2}(rj - \sigma^2 j^2), b_j = r + \sigma^2 j^2, c_j = -\frac{1}{2}(rj + \sigma^2 j^2). \end{aligned} \tag{3.20}$$

Equation (3.20) can be seen as a high-dimensional non-homogeneous linear differential equation group related only to time, namely

$$\frac{dF}{dt} = AF(t) + B(t), \tag{3.21}$$

where

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_M'(t) \end{bmatrix}, F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_M(t) \end{bmatrix}$$

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{M-1} & b_{M-1} & c_{M-1} \\ & & & & a_M & b_M \end{bmatrix}, B = \begin{bmatrix} a_1 f_0(t) \\ 0 \\ \vdots \\ \vdots \\ c_M f_{M+1}(t) \end{bmatrix}.$$

An algorithm for (3.21) is an exponential integrator if it involves the computation of matrix exponentials (or related matrix functions) and integrates the linear system exactly. In general, exponential integrators permit larger step sizes and achieve higher accuracy than non-exponential ones when (3.21) is a stiff differential equation such as highly oscillatory ODEs and semi-discrete time-dependent PDEs. Therefore, numerous exponential algorithms have been proposed (see, e.g., [36–38]).

In the boundary conditions of European call options, $f_0(t)$ and $f_{M+1}(t)$ are all known values, so $B(t)$ can be regarded as a constant vector. That is, rewrite the above Eq (3.21) as follows.

$$\frac{\mathrm{d}F}{\mathrm{d}t} = AF(t) + B. \tag{3.22}$$

Its European call option boundary conditions are $F(T) = \begin{bmatrix} max(S_0 - K, 0) \\ max(S_1 - K, 0) \\ \vdots \\ \vdots \\ max(S_M - K, 0) \end{bmatrix}$.

For matrix A, we have the following property.

Theorem 3.1. *A is an M-matrix.*

Proof. Obviously, by the expressions in (3.20), $b_j > 0$, $c_j < 0$, and $a_j < 0$, for each $j = 1, 2, \ldots, M$, since $r < \sigma^2 j$.

We also have that

$$b_1 + c_1 > 0,$$
$$a_j + b_j + c_j = r > 0, j = 2, \ldots, M - 1,$$
$$a_M + b_M > 0,$$

which means for a given vector $e = (1, 1, \cdots, 1)^T, Ae > 0$. Thus, A is an M-matrix [39]. □

3.4.2. Solving high-dimensional non-homogeneous system of linear differential equations

The differential equation group (3.22) is a nonhomogeneous linear differential equation group, so consider solving it directly. The first consideration is the method of variation of constants for solving nonhomogeneous linear differential equation groups.

According to the method of variation of constants, the solution for a nonhomogeneous differential equation with constant coefficient of the form $\frac{\mathrm{d}F}{\mathrm{d}t} = AF(t) + B(t)$ is

$$F(t) = exp[(t - t_0)A]\eta + \int_{t_0}^{t} exp[(t - s)A]B(s)ds, \tag{3.23}$$

with $\eta = F(t_0)$ the initial value condition.

Let $h = t - t_0$, then the above equation can be transformed into

$$F(t) = e^{Ah}F(t_0) + \int_0^h e^{(h-s)A}B(s + t_0)ds. \tag{3.24}$$

Since $B(t)$ can be regarded as a constant vector, the above equation becomes the following.

$$F(t) = e^{Ah}F(t_0) + \int_0^h e^{(h-s)A} ds \cdot B. \tag{3.25}$$

Therefore, we have

$$F(t+h) = e^{Ah}F(t) + \int_0^h e^{(h-s)A} ds \cdot B = e^{Ah}F(t) + (e^{Ah} - I)A^{-1}B. \tag{3.26}$$

From the boundary conditions (3.9) and (3.10) of the call option difference method, $F(T)$ is known, and since it is necessary to use known values $F(t+h)$ to backtrack $F(t)$, then

$$e^{Ah}F(t) = F(t+h) - (e^{Ah} - I)A^{-1}B. \tag{3.27}$$

Here, h is the time interval Δt, and through the above equation, we can gradually deduce all time points' option pricing values through forward iteration. We denote this exponential integral method based on semi-discrete differential equations by EIM.

Theorem 3.2. *Suppose $f(S,t)$ is the exact solution for system (2.2) and (2.3), $F(t)$ is the numerical solution of the semi-discrete exponential integral method EIM, then*

$$\|e(t)\| = O(\Delta S^2),$$

which means the method is consistent.

Proof. We note that in the construction process of the EIM method, the central difference is first applied to the derivatives related to price S using Eqs (3.16) and (3.17), resulting in an error of $O(\Delta S^2)$. After obtaining the semi-discrete system (3.22), the method of variation of constants is used in the time direction, yielding an exact expression for the semi-discrete system without introducing any temporal error. Therefore, the overall error of this scheme is $O(\Delta S^2)$. □

Theorem 3.3. *The semi-discrete exponential integral method EIM is unconditionally stable.*

Proof. Since A is an M-matrix, the all eigenvalues of A are non-negative. Therefore, $\|(e^{Ah})\| \geq 1$. Then, for the growth factor matrix $(e^{Ah})^{-1}$ from the $n-th$ step to $(n+1)-th$ step, we have $\|(e^{Ah})^{-1}\| \leq 1$. □

Lemma 3.1. *(Lax equivalence theorem [40]) Given a properly posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.*

Theorem 3.4. *The semi-discrete exponential integral method EIM is convergent.*

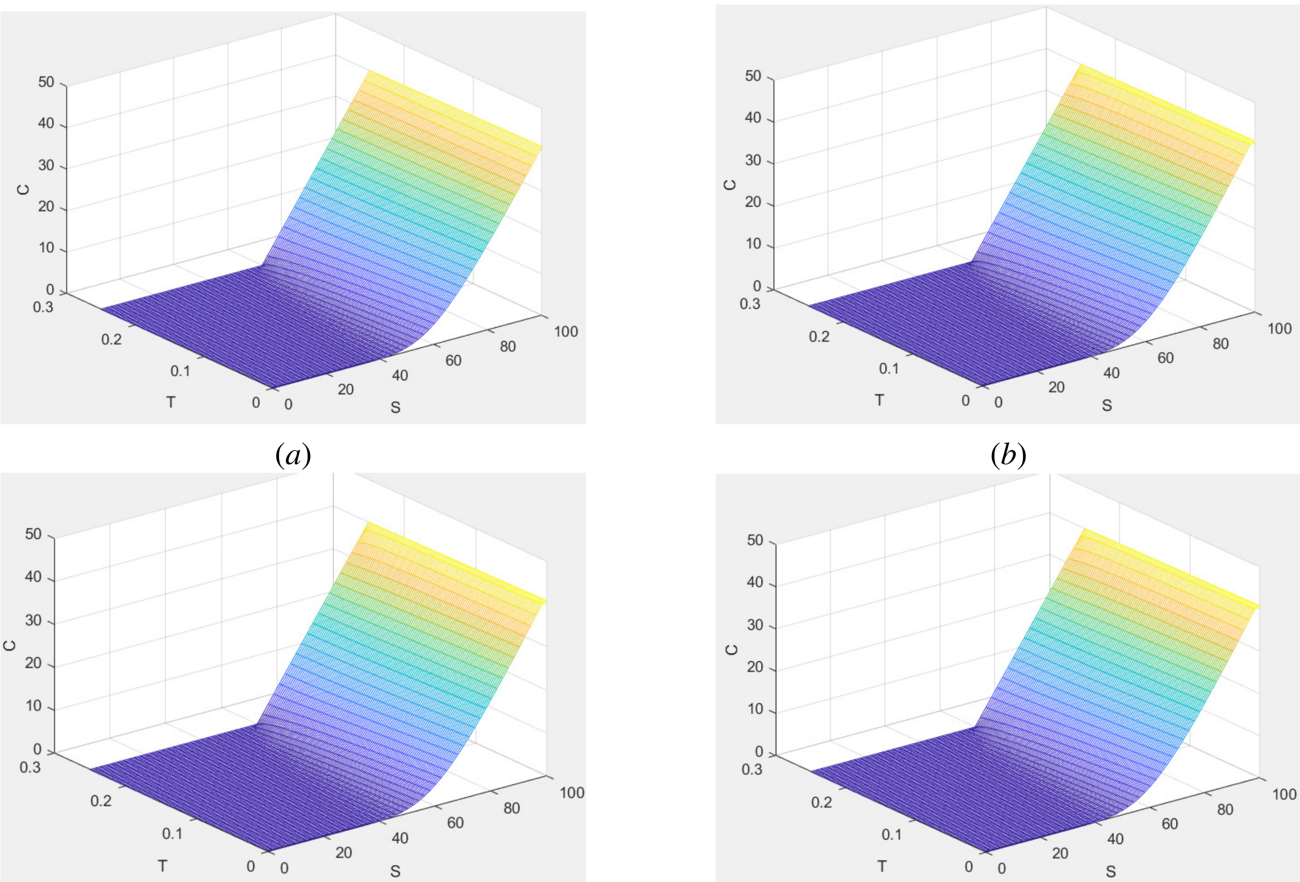
Proof. Since the semi-discrete exponential integral method (EIM) is consistent (Theorem 3.2) and unconditionally stable (Theorem 3.3), it is convergent according to the Lax Equivalence Theorem (Lemma 3.1). □

Compared with the EDM, IDM, and C-N methods, the EIM method, by employing exponential integration in the time direction, avoids the accumulation of discretization errors that come with time discretization. This enables it to better maintain the stability of the numerical solution when dealing with non-smooth initial conditions, thereby reducing the occurrence of spurious oscillations. Moreover, the EIM method significantly reduces computational costs by minimizing the number of time steps and circumventing time-discretization errors. The central differencing used in the spatial direction also helps to lighten the computational load. When it comes to handling nonlinear or degenerate problems, such as the jump-diffusion model, the EIM method demonstrates robust stability. It can effectively deal with nonlinear and discontinuous terms, thus preventing the instability of the numerical solution.

4. Numerical experiments

To verify the effectiveness of the above exponential integral finite difference method, numerical simulations were conducted using the above four difference methods for comparison. To ensure that the option pricing conditions of the four difference methods are consistent, we select the current option exercise price K as 60 RMB, risk-free interest rate as $r = 0.05$, and volatility as $\sigma = 0.4$. Thus, choose 3 months as time interval, that is $\frac{1}{4}$ years, take time step as $\Delta t = \frac{\frac{1}{4}}{N} = \frac{0.25}{100} = \frac{1}{400}$, and price step as $\Delta S = \frac{S_{max}}{M} = \frac{100}{50} = 2$.

Figure 1 gives the numerical solution graphs for the explicit finite difference method, implicit finite difference method, C-N difference method, and exponential integral method, respectively.



(c) (d)

Figure 1. Numerical solutions of four difference methods.

The C value on the z-axis represents the numerical solution of the option price. The specific image analysis is as follows. The basic shape and trend of the numerical solutions of the above four methods are the same, with the numerical solution gradually increasing as the price and expiration time T increase.

Under the same price S conditions, the numerical solutions change slightly under different expiration time T conditions, and the smaller the expiration time T, the smaller the value of the numerical solution.

The impact of price S is greater compared to expiration time T on the numerical solutions, at around a price S of 40 RMB, the numerical solution starts to increase gradually from 0 and reaches a maximum value, which is about 40 RMB at the price S of 100 RMB.

Since there is almost no difference in the numerical solutions of the four methods from the graph, we consider the error between the numerical solutions of the four methods and the analytical solution of the option pricing differential equation for analysis. The analytical solution selected from the option price differential equation is the option pricing formula, i.e.,

$$f(S,t) = SN(d_1) - Ke^{rT}N(d_2),$$

where $d_1 = \dfrac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$. Figure 2 shows the error diagram between the numerical solutions of the four methods and the analytical solution of the option pricing differential equation (the error E is analytical solution - numerical solution).

The above images (a), (b), (c), and (d) respectively show the error images between the numerical solutions of the explicit difference method, implicit difference method, C-N format difference method, and exponential integral method and the analytical solution. The color legend shown is illustrated in the error image (a).

Looking at the order of magnitude of errors from the images, the error of the explicit difference method reaches nearly 3, while the error is generally within 1; moreover, the errors of the implicit difference method and C-N format difference method reach -1.5, and the error based on the exponential integral difference method is highest at about -1, with most error values between -1 and -0.5. The specific maximum error values are shown in Table 1 (the sign only indicates the direction of error):

Table 1. The maximum error of four difference methods.

Method	EDM	IDM	C-N	EIM
Maximum error	2.913136	-1.510988	-1.515075	-1.237535

To more intuitively display the effect of numerical solutions of the four models, we use evaluation indicators MAE (Mean Absolute Error), MSE (Mean Square Error), and RMSE (Root Mean Square Error) as criteria. The smaller the evaluation value, the better the model's prediction effect. The calculation formulas for the three indicators are as follows:

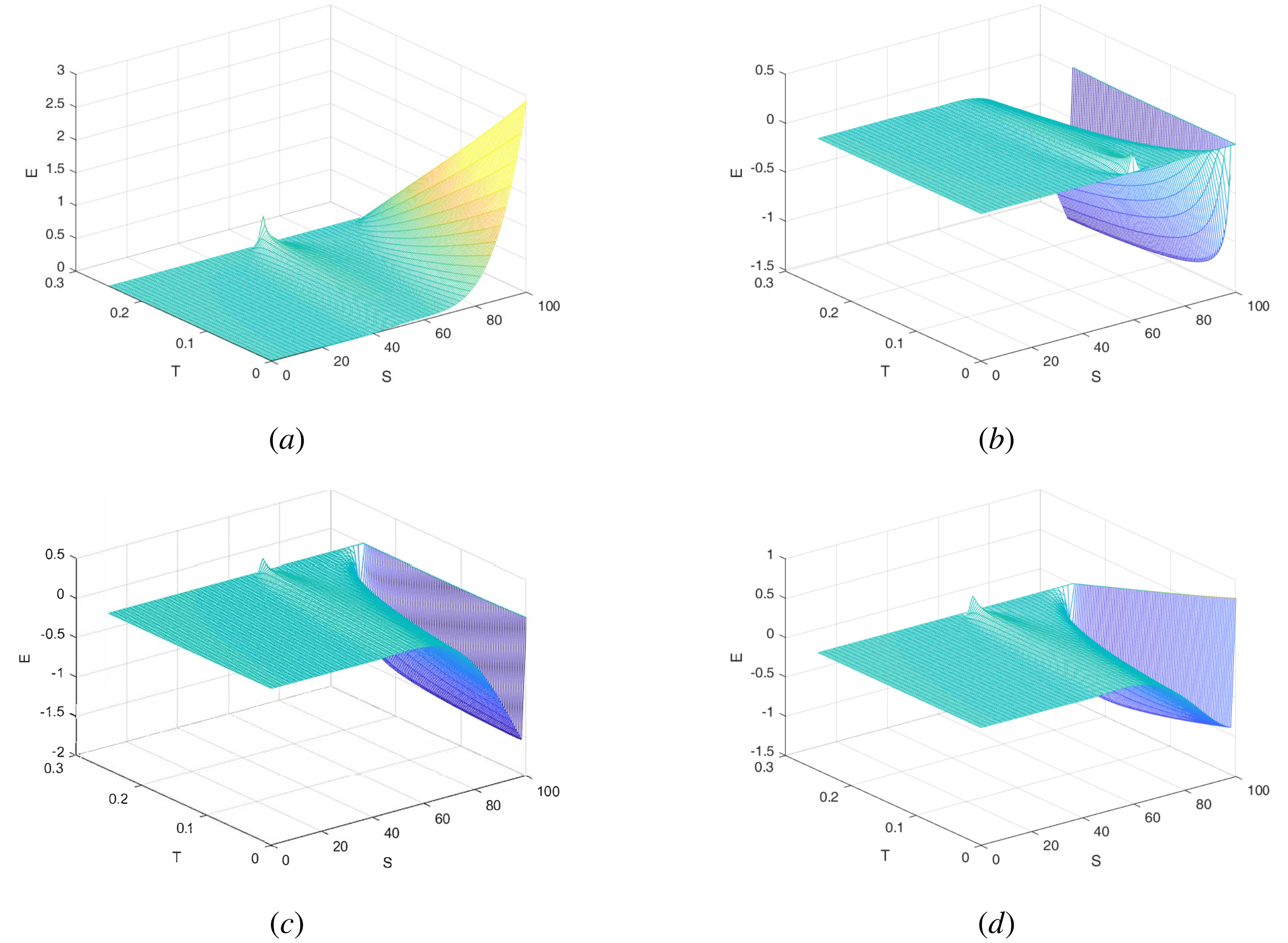


Figure 2. Error diagram of numerical solutions of four difference methods.

$$
\begin{aligned}
MAE &= \frac{1}{N}\sum_{i=1}^{N}|y_i - \hat{y}_i|, \\
MSE &= \frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2, \\
RMSE &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}
\end{aligned}
\tag{4.1}
$$

with N the length of the data set, y_i the true value at the current period, and $\hat{y}_i$ the predicted value by the model at the current period. Since ΔS and Δt divide S and T into 51 and 101 parts, respectively, the total amount of data is 5151 pieces. The corresponding results are presented in Table 2.

In summary, we conducted numerical simulations for pricing European call options, verifying the effectiveness of the exponential integral method. Numerical experiments show that the proposed exponential integral method in this paper is superior to traditional Crank-Nicolson methods in terms of error control and computational efficiency.

Table 2. Evaluation index values of four difference methods.

Method	MAE	MSE	RMSE
EDM	0.135348	0.154061	0.392507
IDM	0.141455	0.128015	0.357791
C-N	0.140689	0.128768	0.358842
EIM	0.122434	0.083638	0.289203

Numerical experiment results indicate that when the underlying asset price is close to the exercise price, the exponential integral method can better avoid numerical oscillations, especially under high volatility conditions, showing more stable numerical convergence.

In order to estimate the error and verify the convergence order of the proposed scheme EIM in spatial direction, we solve the problem with different spatial stepsizes on the interval $S \in [0, 100]$. The errors and convergence rates for the proposed scheme are shown in Table 3. Since the scheme EIM is almost exact in temporal direction, the convergence rates and errors are computed by

$$Order = log_2 \left| \frac{Error_{\Delta S}}{Error_{\Delta S/2}} \right|, \tag{4.2}$$

and the $Error_{\Delta S}$ is the maximum error values with stepsize ΔS after one time step calculating.

Table 3. Spatial errors and convergence rate of the scheme EIM with different spatial stepsizes.

ΔS =4	ΔS	$\Delta S/2$	$\Delta S/4$
Error	-1.9812	-0.7778	-0.1691
Order	*	1.35	2.20

5. Conclusions

We begin with the difference method and introduce the basic difference format of the finite difference method and its related formulas in the Black-Scholes differential equation. Then it discretizes in the price direction to obtain a semi-discrete differential equation related to the time direction, and solves it specifically through exponential integral methods. Finally, it conducts actual solving through numerical simulation for the model proposed in this paper. This method discretizes in the price direction and uses exponential integral methods to solve in the time direction, thereby effectively improving the precision and stability of numerical calculations. Compared with traditional difference methods, this method can handle option pricing problems with higher precision, especially showing significant advantages under conditions of large market fluctuations.

However, the method proposed in this paper also has certain limitations. First, although the EIM method performs well in handling high volatility and prices close to the strike price, our model is based on some assumptions, such as the underlying asset price following geometric Brownian motion. These assumptions may not hold entirely in real markets, especially when extreme volatility or nonlinear behavior occurs. Second, despite the EIM method's superiority in computational efficiency over traditional methods, computational complexity remains a challenge when dealing with large-scale or high-dimensional problems. Further optimization of computational efficiency is necessary, especially in real-time computing and high-frequency trading scenarios. Finally, the EIM method is mainly applied to European option pricing. Further research and extension are needed for pricing American options, path-dependent options, and other complex financial derivatives. Future research directions include: Considering more complex market models, such as jump-diffusion [41] or stochastic volatility models, to enhance the model's applicability and accuracy; exploring more efficient numerical algorithms, such as parallel computing or adaptive mesh techniques, to further reduce computational costs and improve computational efficiency; extending the EIM method to pricing American options, path-dependent options, and other complex financial derivatives to verify its effectiveness and stability in a broader range of scenarios; and conducting empirical studies with real market data to validate the EIM method's performance in practical applications and compare it with existing methods.

Author contributions

Xun Lu: methodology, formal analysis, investigation, data curation, writing original draft; Wei Shi: conceptualization, methodology, software, validation, supervision, funding acquisition; Changhao Yang: methodology, software, validation, investigation, data curation; Fan Yang: methodology, validation, writing review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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