Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 12888–12899.
DOI: 10.3934/math.2025579
Received: 17 March 2025
Revised: 20 May 2025
Accepted: 23 May 2025
Published: 04 June 2025

Research article

The orthogonal reflection method for the numerical solution of linear systems

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Abstract: This paper extends the convergence analysis of the reflection method from the case of 2 equations to the case of n equations. A novel approach called the orthogonal reflection method is also proposed. The orthogonal reflection method comprises two key steps. First, a Householder transformation is employed to derive an equivalent system of equations with an orthogonal coefficient matrix that maintains the same solution set as the original system. Second, the reflection method is applied to efficiently solve this transformed system. Compared with the reflection method, the orthogonal reflection method significantly enhances the convergence speed, especially when the angles are acute between the hyperplanes represented by the linear system. We also derive the convergence rate for it, demonstrating that the orthogonal reflection method is always convergent for an arbitrary point in $\mathbb{R}^n$. The necessity of orthogonalization is presented in the form of a theorem in $\mathbb{R}^2$. When the coefficient matrix has a large condition number, the orthogonal reflection method can still compute relatively accurate numerical solutions rapidly. By comparing with algorithms including Jacobi iteration, Gauss-Seidel iteration, the conjugate gradient method, GMRES, weighted RBAS, and the reflection method on coefficient matrices of 10×10 random matrices, 1000×1000 sparse matrices, and 1000×1000 randomly generated full-rank matrices, the efficiency and robustness of the orthogonal reflection method are demonstrated.

Keywords: orthogonal reflection method; reflection method; linear systems; numerical solution; Householder transformations
Mathematics Subject Classification: 65F10

1. Introduction

Solving systems of linear equations remains a problem worthy of attention. Different methods have been proposed, such as quantum algorithms [1], mixed-precision methods [2,3], and direct methods [4].

We consider the following linear system by iterative method:

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1.1}$$

which is perhaps the most important single problem in numerical analysis, in particular, arising from discretizations of differential equations. Denoting the set of $n \times n$ matrices with real entries as $\mathbb{R}^{n\times n}$, then the linear system (1.1) consists of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, a column vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$, and a column vector $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)^T \in \mathbb{R}^n$.

Common iteration methods include Jacobi iteration, Gauss-Seidel iteration, the successive over-relaxation method, and the conjugate gradient method [5]. In addition to the above, there are some other iterative methods. Two parameters, a drop-tolerance and a stability factor, are introduced in iterative refinement to solve sparse linear systems and it is effective in reducing both the computing time and the storage requirements [6]. The Huang algorithms, proposed in 1975 [7], offer a stable approach for the general solution of systems of linear equations. However, it occasionally falls short in precision. This imprecision stems from the challenge of pre-determining the rank of the coefficient matrix A and the number of linearly independent vectors in A. Then the numerical performance based upon the explicit QR factorization and the implicit LQ factorization associated with the Huang and the modified Huang algorithms in the ABS class is presented in [8]. The modified Huang algorithm seems more accurate. Later, in [9], Young provides a systematic development of a substantial portion of the theory of iterative methods for solving large linear systems, and the recent developments in Krylov subspace methods for linear systems are given in [10]. They both include the conjugate gradient (CG) mehtod, the minimal residual method (MINRES), and the generalized minimal residual method (GMRES). Recently, some iterative methods based on random projection were also provided in [11, 12]. It excels at solving sparse linear systems, but its convergence is affected by the blocking. A simpler and more intuitive method appears, that is, the reflection algorithm [13], which originates from the Cimmino's iterative method [14, 15]. It can be used in linear inequalities, semi-definite linear equations, and parallel computation, especially for image reconstruction; see [16–19]. Although this method is easier to understand, sometimes the rate of convergence is not satisfactory. Cimmino therefore upgraded his method by introducing probabilistic arguments. Consequently, the approximate solutions' accuracy will depend on "chance" [13]. In this paper, the reflection method is further studied, and a novel method called the orthogonal reflection method, which always converges for arbitrary initial values, is provided to overcome the time-consuming nature of the reflection method.

The rest of this paper is organized as follows. The reflection algorithm and its convergence analysis for the general case of n equations are elaborated in Section 2. Section 3 presents the orthogonal reflection algorithm, and some theorems related to its convergence rate are provided. In Section 4, many numerical experiment results are given to show the effectiveness of the orthogonal reflection method. Finally, the conclusions are presented in Section 5.

2. Solving linear systems in the general case by the reflection method

2.1. Notation and preliminaries

We quickly present some notations and facts used throughout the paper. Let $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$, The linear system (1.1) can also be written in the following form:

$$\begin{cases} \langle \boldsymbol{a}_1, \boldsymbol{x} \rangle = b_1, \\ \langle \boldsymbol{a}_2, \boldsymbol{x} \rangle = b_2, \\ \quad \cdots \\ \langle \boldsymbol{a}_n, \boldsymbol{x} \rangle = b_n. \end{cases} \tag{2.1}$$

We emphasize that n can be viewed as the number of equations in the linear system (2.1). Geometrically, each equation represents a hyperplane, denoted as $H_i : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i, i = 1, 2, \cdots, n$, where $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in})^T$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$. $\nu_i = \frac{a_i}{\|a\|}$ is the unit normal of H_i. Since A is nonsingular, the system (2.1) has a unique solution; let us call it $S = (s_1, s_2, \cdots, s_n)^T$. That is, all the hyperplanes intersect at the point S. Let $P^{(0)} = \boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) \notin H_i, i = 1, 2, \cdots, n$ be an initial point and $P^{(i)} = \boldsymbol{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})$ be the point after the i-th iteration, $i = 1, 2, \cdots$.

2.2. Cimmino's method for the linear system of n equations

The reflection method is closely related to the Cimmino's method [13–15], and Cimmino's method for $n = 2$ is given in [13]. Now we generalize it to the system (2.1). Retaining the notation in Section 2.1, some content from [13] is restated in the form of the following multiple lemmas.

Lemma 1. *Consider the i-th hyperplane $H_i : \langle \mathbf{a}_i, \mathbf{x} \rangle = b_i$ in the linear system (2.1). Let $R_i = (r_1, r_2, \cdots, r_n)^T \in \mathbb{R}^n$ be the orthogonal projection of $P^{(0)}$ onto H_i, and let $Q_i \in \mathbb{R}^n$ be the symmetric point of $P^{(0)}$ with respect to R_i; see Figure 1 for related content in $\mathbb{R}^3$. Then*

$$R_i = \boldsymbol{x}^{(0)} + \frac{b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_i\|^2} \boldsymbol{a}_i, \qquad Q_i = \boldsymbol{x}^{(0)} + 2 \frac{b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_i\|^2} \boldsymbol{a}_i.$$

The proof of Lemma 1 is omitted here, since it is similar to [13, Lemma 2.1].

Lemma 2. *[13, page 1141] Retaining the previous notation, let $P^{(0)} \neq S$ and consider the point Q_i as the symmetric point of $P^{(0)}$ with respect to the hyperplane H_i, for each $i = 1, 2, \cdots, n$, then the centroid of $Q_1, Q_2, \cdots$, and Q_n (each bearing mass 1), denoted as $P^{(1)}$, satisfies*

$$dist(P^{(1)}, S) < dist(P^{(0)}, S),$$

where $dist(P^{(0)}, S)$ is the Euclidean distance between $P^{(0)}$ and S.

See Figure 2 for Lemma 2, taking the linear system $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$ as an example.

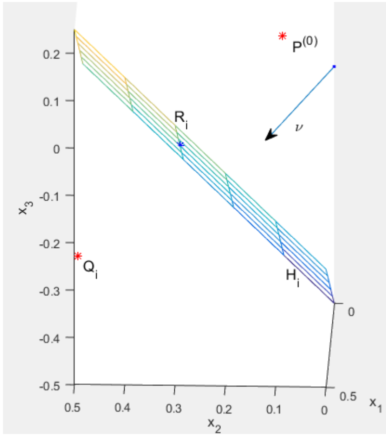

Figure 1. Illustration of Lemma 1.

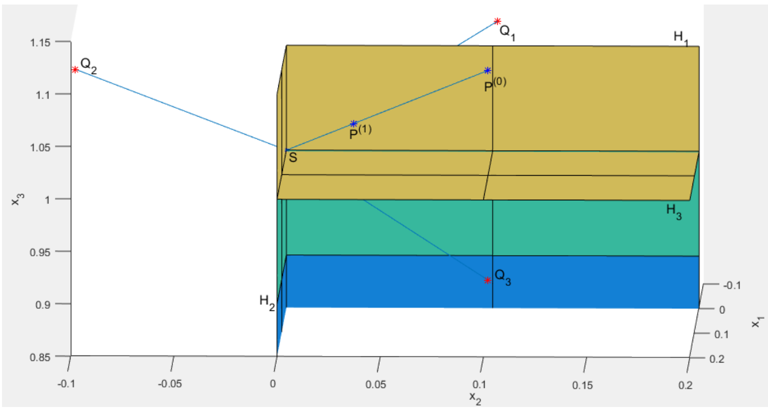

Figure 2. Illustration of $dist(P^{(1)}, S) < dist(P^{(0)}, S)$.

Lemma 3. *[13, page 1142] Retaining the previous notation, if $P^{(1)} \neq S$, we can determine $P^{(2)}$ in an analogous manner, followed successively by $P^{(3)}, \cdots, P^{(k)}, \cdots$, and $dist(P^{(0)}, S) > dist(P^{(1)}, S) > dist(P^{(2)}, S) > \cdots > dist(P^{(k)}, S) > \cdots$ holds. The sequence $P^{(k)}, k = 0, 1, \cdots$ converges to S as $k \to \infty$.*

2.3. The reflection method for solving linear system of n equations

For the reflection method, the centroid of the symmetric points serves as the initial point of the next iteration. This method is extended to solve the system (2.1) in this section.

Given the initial value $P^{(0)} = P^{(0)}(\boldsymbol{x}^{(0)})$, Q_i is the symmetric point of $P^{(0)}$ with respect to the hyperplane $H_i, i = 1, 2, \cdots, n$. By Lemma 1, we deduce that

$$\begin{aligned} Q_1 &= \boldsymbol{x}^{(0)} + 2\frac{b_1 - \langle \boldsymbol{a}_1, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_1\|^2}\boldsymbol{a}_1, \\ Q_2 &= \boldsymbol{x}^{(0)} + 2\frac{b_2 - \langle \boldsymbol{a}_2, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_2\|^2}\boldsymbol{a}_2, \\ &\cdots \\ Q_n &= \boldsymbol{x}^{(0)} + 2\frac{b_n - \langle \boldsymbol{a}_n, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_n\|^2}\boldsymbol{a}_n. \end{aligned} \tag{2.2}$$

Let $P^{(1)} = P^{(1)}(\boldsymbol{x}^{(1)})$ be the centroid of $Q_1, Q_2, \cdots, Q_n$; then

$$P^{(1)} \equiv \boldsymbol{x}^{(1)} = \frac{1}{n}(Q_1 + Q_2 + \cdots + Q_n). \tag{2.3}$$

Replace the Q_i in Eq (2.3) with the Q_i in Eq (2.2), $i = 1, 2, \cdots, n$, we have

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \frac{2}{n}\sum_{h=1}^{n}\frac{b_h - \langle \boldsymbol{a}_h, \boldsymbol{x}^{(0)}\rangle}{\|\boldsymbol{a}_h\|^2}\boldsymbol{a}_h. \tag{2.4}$$

Let $P^{(1)} = P^{(1)}(\boldsymbol{x}^{(1)})$ be the starting point of the next iteration, and so on. Then at step $v + 1$,

$$\boldsymbol{x}^{(v+1)} = \boldsymbol{x}^{(v)} + \frac{2}{n}\sum_{h=1}^{n}\frac{b_h - \langle \boldsymbol{a}_h, \boldsymbol{x}^{(v)}\rangle}{\|\boldsymbol{a}_h\|^2}\boldsymbol{a}_h, (v = 0, 1, 2, \cdots). \tag{2.5}$$

In matrix form, Eq (2.5) reads

$$\boldsymbol{x}^{(v+1)} = \boldsymbol{x}^{(v)} + \frac{2}{n}A^T D(\boldsymbol{b} - A\boldsymbol{x}^{(v)}), \tag{2.6}$$

where D is the diagonal matrix

$$D = \begin{pmatrix} \|\boldsymbol{a}_1\|^{-2} & 0 & \cdots & 0 \\ 0 & \|\boldsymbol{a}_2\|^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\boldsymbol{a}_n\|^{-2} \end{pmatrix}.$$

To make it easier to solve the system (2.1), we change the Eq (2.6) to

$$\boldsymbol{x}^{(v+1)} = (I - \frac{2}{n}A^T DA)\boldsymbol{x}^{(v)} + \frac{2}{n}A^T D\boldsymbol{b}, \tag{2.7}$$

where I is an identity matrix of n by n. According to Eq (2.7), we can find the numerical solution of the system (2.1).

2.4. Convergence analysis of the reflection method

Let $S = S(\xi)$, whose coordinates is ξ=$(\xi_1, \xi_2, \cdots, \xi_n)^T$, solve the system (2.1). Fix a point $P^{(0)} = P^{(0)}(\boldsymbol{x}^{(0)}) \neq S$. Since $\langle \boldsymbol{a}_i, \xi\rangle = b_i$, $i = 1, 2, \cdots, n$, we define the numbers $\eta_i = \langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)}\rangle - b_i = \langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} - \xi\rangle$ and according to the Eq (2.4), we obtain $\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} - \frac{2}{n}\sum\limits_{i=1}^{n}\frac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|^2}\langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} - \xi\rangle$. Now observe that

$$\begin{aligned} \|\boldsymbol{x}^{(1)} - \xi\|^2 &= \|\boldsymbol{x}^{(0)} - \tfrac{2}{n}\sum_{i=1}^{n}\tfrac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|^2}\langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} - \xi\rangle - \xi\|^2 \\ &= \|\boldsymbol{x}^{(0)} - \xi\|^2 - \tfrac{4}{n}\sum_{i=1}^{n}\tfrac{\eta_i^2}{\|\boldsymbol{a}_i\|^2} + \tfrac{4}{n^2}\|\sum_{i=1}^{n}\tfrac{1}{\|\boldsymbol{a}_i\|^2}\eta_i\boldsymbol{a}_i\|^2 \\ &= \|\boldsymbol{x}^{(0)} - \xi\|^2 - \tfrac{4}{n^2}(\|\sum_{i=1}^{n}p_i\eta_i\boldsymbol{a}_i\|^2 - n\sum_{i=1}^{n}p_i\eta_i^2), \end{aligned}$$

where $p_i = \frac{1}{\|\boldsymbol{a}_i\|^2}$. Since $\|\sum\limits_{i=1}^{n}p_i\eta_i\boldsymbol{a}_i\|^2 \leq (\sum\limits_{i=1}^{n}p_i\eta_i^2)(\sum\limits_{i=1}^{n}p_i\|\boldsymbol{a}_i\|^2) = n(\sum\limits_{i=1}^{n}p_i\eta_i^2)$, we obtain $\|\boldsymbol{x}^{(1)} - \xi\| \leq \|\boldsymbol{x}^{(0)} - \xi\|$. That is, after v iterations, $\|\boldsymbol{x}^{(v+1)} - \xi\| \leq \|\boldsymbol{x}^{(v)} - \xi\|$ $(v = 0, 1, \cdots)$. It shows that if $x^{(v)} = \xi$, $x^{(v+1)} = x^{(v)}$ for any $v \in \mathcal{N}$.

3. The orthogonal reflection method

3.1. The orthogonal reflection method for solving a linear system of n equations

In this section, the orthogonal reflection method is presented to solve the linear system (2.1). Using householder transformations, matrix A can be decomposed into the product of an orthogonal matrix Q and an upper triangular matrix R. Householder transformations offer greater numerical stability compared to other orthogonalization methods, such as Gram-Schmidt orthogonalization. The operation steps of the orthogonal reflection method are as follows.

Step 1. Derive a linear system (3.1) with the same solution as (2.1) by householder transformations. By combining $A = QR$ and $A\boldsymbol{x} = \boldsymbol{b}$, we obtain $QR\boldsymbol{x} = \boldsymbol{b}$. So

$$Q\boldsymbol{x} = QR^{-1}Q^T\boldsymbol{b}, \tag{3.1}$$

where the row vectors of Q are mutually orthogonal.

Step 2. Solve the system (3.1) by the reflection method using the iterative formula (2.7) in Section 2.

3.2. Convergence analysis of the orthogonal reflection method

Given that the reflection method is employed in Step 2, the orthogonal reflection method always converges for any initial value. The convergence rate is computed and presented in the following theorems.

Theorem 1. *For the case of n = 2 in the system (2.1):* $\begin{cases} \langle \mathbf{a}_1, \mathbf{x} \rangle = b_1, \\ \langle \mathbf{a}_2, \mathbf{x} \rangle = b_2. \end{cases}$ *, the orthogonal reflection method only needs one iteration to achieve the exact solution.*

Proof. Let the two lines represented by $\langle \boldsymbol{a}_1, \boldsymbol{x} \rangle = b_1$ and $\langle \boldsymbol{a}_2, \boldsymbol{x} \rangle = b_2$ be r_1, r_2, see Figure 3(left). After householder transformations, they become r_1', r_2' respectively; see Figure 3(right). Let Q_1' and Q_2' be the symmetric points of $P^{(0)}$ with respect to r_1' and r_2'. $Q_1'P^{(0)}Q_2'$ is on a circumference with S as its center and they make a right triangle. So $P^{(1)} = \frac{Q_1'+Q_2'}{2} = S$. It needs one iteration to achieve the exact solution.

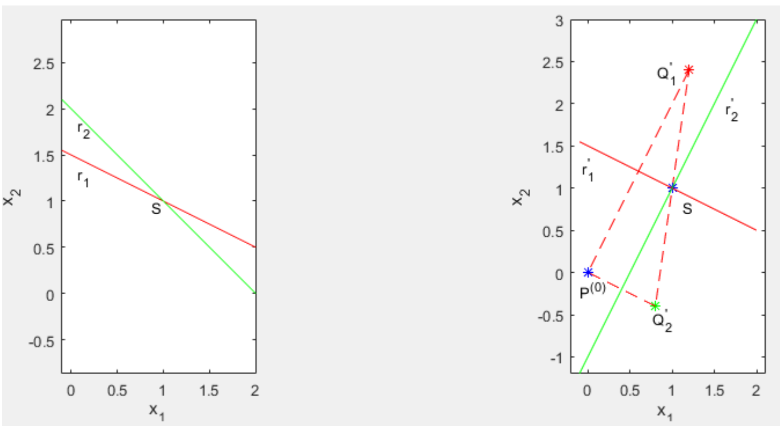

Figure 3. Illustration of Theorem 1.

In fact, when the angle between the two lines is acute, the effect of orthogonal reflection method is better than the reflection method; see Table 1 in Section 4.

Theorem 2. *If we use the orthogonal reflection method to solve the system (2.1) ($n > 2$), the iteration direction is always along $P^{(0)}S$, $\frac{\|P^{(k+1)}S\|}{\|P^{(k)}S\|} = \frac{n-2}{n}, k = 1, 2, \cdots$. We need $\ulcorner \frac{lg\frac{\varepsilon}{\|P^{(0)}S\|}}{lg(n-2)-lgn} \urcorner$ iterations to achieve the exact solution.*

Proof. Without loss of generality, we translate the coordinate origin to the exact solution S of the system (3.1) (S also solves the system (2.1)). The orthogonal hyperplanes of the system (3.1) are used as the coordinate planes. The reflection method is adopted in this new coordinate system. In the new coordinate system, let $P^{(0)}$ be $P^{(0)'} = (x_1^{(0)'}, x_2^{(0)'}, \cdots, x_n^{(0)'})^T$. The exact solution S is denoted as $S' = (0, 0, \cdots, 0)^T$ and the symmetric points of $P^{(0)'}$ with respect to all the orthogonal hyperplanes are $Q_i', i = 1, 2, \cdots, n$. That is $Q_1' = (-x_1^{(0)'}, x_2^{(0)'}, \cdots, x_n^{(0)'})^T, Q_2' = (x_1^{(0)'}, -x_2^{(0)'}, \cdots, x_n^{(0)'})^T, \cdots, Q_n' = (x_1^{(0)'}, x_2^{(0)'}, \cdots, -x_n^{(0)'})^T$. So $P^{(1)'} = \frac{Q_1'+Q_2'+\cdots+Q_n'}{n} = \frac{n-2}{n}(x_1^{(0)'}, x_2^{(0)'}, \cdots, x_n^{(0)'})^T = \frac{n-2}{n}P^{(0)'}$. Thus the iteration direction is along $P^{(0)'}S'$. That is the direction of $P^{(0)}S$. Let $\{P^{(k)'}\}$ be the sequence of iterations generated by the orthogonal reflection algorithm. $d_0' = \|P^{(0)'}S'\|, d_1' = \|P^{(1)'}S'\|, \cdots, d_k' = \|P^{(k)'}S'\|, \cdots$. Obviously, $\frac{d_1'}{d_0'} = \frac{n-2}{n}$. In the same way, $\frac{d_2'}{d_1'} = \frac{n-2}{n}, \cdots, \frac{d_{k-1}}{d_k} = \frac{n-2}{n}, \cdots$. Hence, the algorithm shows the linear convergence with constant $\frac{n-2}{n}$. Let the accuracy ε be reached after k iterations; then $\|P^{(0)}S\|(\frac{n-2}{n})^k = \varepsilon$, $k = \ulcorner \frac{lg\frac{\varepsilon}{\|P^{(0)}S\|}}{lg(n-2)-lgn} \urcorner$.

Remark 1. *Without considering householder transformations, the orthogonal reflection method shows the linear convergence with constant $\frac{n-2}{n}$. It is evident that the increasing magnitude of $\frac{n-2}{n}$ with larger n leads to degraded convergence speed.*

Since the direction of the orthogonal reflection method is along $P^{(0)}S$, obviously, its iterative path is shorter than the reflection method when angles are acute; see Figure 4 for $n = 3$. Of course it converges faster. Example 2 in Section 4 also arrives at the same conclusion; also see Table 2 in Section 4.

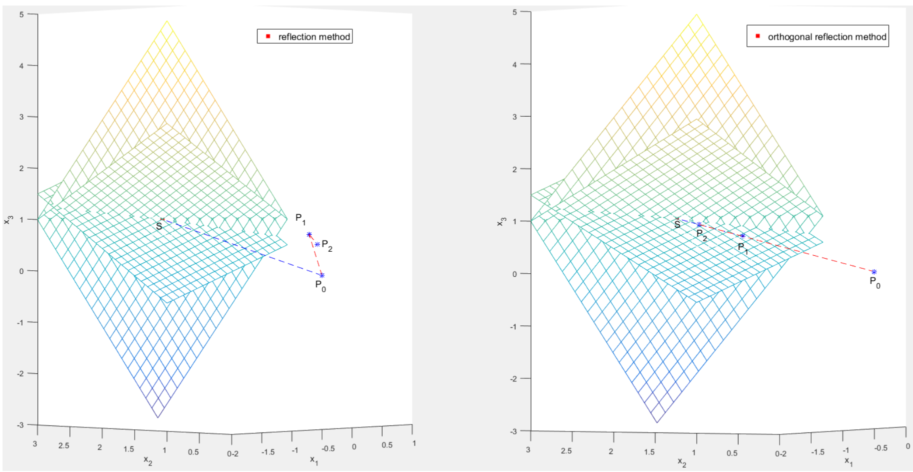

Figure 4. The iteration direction of reflection method (left) and the orthogonal reflection method (right).

Since Step 2 of the orthogonal reflection method is essentially a reflection process, it is necessary to analyze the relationship between the angle and convergence rate in the reflection method. This analysis serves to both demonstrate the necessity of orthogonalization and identify the key factors for improving the convergence speed of the orthogonal reflection method.

Theorem 3. *Let the angle between H_1 and H_2 be α; then $\frac{\|P^{(k+1)}-S\|}{\|P^{(k)}-S\|} = cos\alpha$, $k = 1, 2, \cdots\cdots$.*

Proof. The definitions of $P^{(0)}$, $P^{(1)}$, Q_1, Q_2, S, H_1, and H_2 remain consistent with the previous ones; see Figure 5. Due to symmetry, $\|S - P^{(0)}\| = \|S - Q_1\| = \|S - Q_2\|$. Line Q_1Q_2 intersects lines SH_1, $SP^{(0)}$, and SH_2 at distinct points M_1, M_2, and M_3 respectively. Let $\angle M_1SM_2 = \alpha$ be the angle, $\angle Q_1SM_1 = \beta$, $\angle P^{(1)}SM_2 = \delta$, and $\angle M_2SQ_2 = \gamma$. Since Q_1 and Q_2 are the reflection of points of $P^{(0)}$ about the lines H_1 and H_2, it follows that $\angle M_1SP^{(0)} = \beta$ and $\alpha - \beta = \gamma$. $P^{(1)}$ is the midpoint of points Q_1 and Q_2, we get $\Delta SP^{(1)}Q_1$ is a right-angled triangle with the right angle at vertex $P^{(1)}$ and $\alpha - \delta + \beta = \gamma + \delta$. So $\beta = \delta$. $\angle Q_1SP^{(1)} = \alpha - \delta + \beta = \alpha$. $\frac{\|P^{(1)}-S\|}{\|P^{(0)}-S\|} = cos\alpha$. Hence $\frac{\|P^{(k+1)}-S\|}{\|P^{(k)}-S\|} = cos\alpha$, for $k = 1, 2, \cdots$.

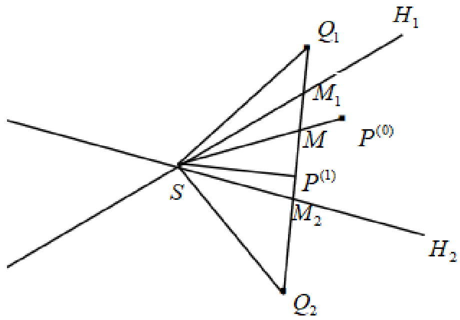

Figure 5. The relationship between the angle and convergence speed in $\mathbb{R}^2$.

Remark 2. *The algorithm shows the linear convergence with constant $cos\alpha$. As the angle α decreases, $cos\alpha$ increases, resulting in slower convergence rates and a larger condition number $\kappa(A)$, indicating progressively worse matrix ill-conditioning. When $\alpha = \frac{\pi}{2}$ (where $cos\alpha = 0$), the reflection method achieves optimal superlinear convergence with $\kappa(A) = 1$, representing perfect numerical stability. The relationship between the angle and the convergence rate of the reflection method in $\mathbb{R}^n$ can be analyzed with reference to Section 2.3. If the householder transformations are sufficiently accurate and fast, the orthogonal reflection method outperforms the reflection method. One might consider orthogonalizing only a portion of the hyperplanes, which can be feasible in certain scenarios. However, sometimes this may produce new small angles, so it is preferable to orthogonalize all coefficients of the linear system.*

4. Numerical results

In this section, we present numerical results to compare with the orthogonal reflection method. The running times presented in this study are all recorded in seconds. The tolerance is 1.0×10^{-7} for all examples.

Example 1. Solve a system of equations: $\begin{cases} x_1 + x_2 = 2 \\ x_1 + 1.0001x_2 = 2.0001 \end{cases}$. The initial value is $(0, 0)^T$. The exact solution is $(1, 1)^T$. The angle is small. The condition number of matrix A is 4.0002×10^4. The comparison of convergence rates for the two methods is shown in Table 1. After 1000 iterations, although the reflection method is theoretically guaranteed to converge eventually, the residual of it remains large. Notably, the orthogonal reflection method is better than the reflection method.

Table 1. Numerical experiments for Example 1.

methods	iterations	time	residual
reflection method	1000	0.004s	2.8285
orthogonal reflection method	1	0.001s	$4.4409e-16$

Example 2. Solve a system of equations: $\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \\ 2x_1 + 2x_2 + x_3 = 3 \end{cases}$. The initial value is $(0, 0, 0)^T$. The exact solution is $(-1, 2, 1)^T$. We also find that both methods converge to exact solutions $(-1, 2, 1)^T$. The angle between the two planes $x_1 + x_2 + x_3 = 2$ and $2x_1 + 2x_2 + x_3 = 3$ is only 0.2756. The orthogonal reflection method performs better than the reflection method; see Table 2.

Table 2. Numerical experiments for Example 2.

methods	iterations	time	numerical solution
reflection method	9867	0.024s	$(-1, 2, 1)^T$
orthogonal reflection method	16	0.006s	$(-1, 2, 1)^T$

The relative residual errors $\frac{\|b-A*x\|}{\|b\|}$ change with the number of iterations and can be found in Figure 6. It is plotted on a logarithmic scale on the vertical axis.

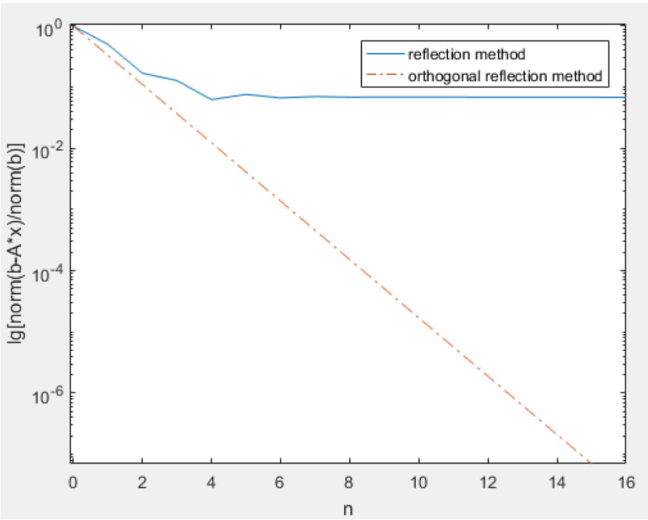

Figure 6. Convergence history for the system in Example 2.

Figure 6 shows that the orthogonal reflection method is superior to the reflection method. After the fourth iteration, the relative residual of the reflection method decreases very slowly, but the orthogonal reflection method reaches the solution approximately at the 15th iteration.

Example 3. We mainly compare the orthogonal reflection method with Jacobi iteration, Gauss-Seidel iteration, CG (conjugate gradients) [20], GMRES (generalized minimal residual) [9, 10] weighted RBAS [12], and the reflection method. Three kinds of linear systems are randomly generated by

Matlab. But it is the same linear system for all the methods in each case. The initial values are $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$, zero vector, and zero vector, respectively. The condition numbers of the coefficient matrices are 88.4448, 25.7619, and 1.4511×10^5, respectively. The sparse matrix is also 1000×1000. We mainly use "speye(1000)" in Matlab to control its sparsity. The time taken by different methods is shown in Table 3.

Table 3. Numerical experiments for Example 3.

methods	matrix(10-by-10)	sparse matrix	matrix(1000-by-1000)
Jacobi iteration	0.002s	not converge	not converge
Gauss-Seidel iteration	0.001s	not converge	0.022s
CG	not converge	not converge	not converge
GMRES	not converge	0.008s	1.645s
weighted RBAS	not converge	0.008s	1.730s
Reflection method	0.003s	6.265s	>1481s
Orthogonal reflection method	0.002s	1.812s	1.453s

Jacobi iteration, Gauss-Seidel iteration, and CG require A to meet certain conditions to achieve convergence. GMRES is sensitive to initial values. Weighted RBAS is affected by the blocking. So they sometimes do not converge. The reflection method and the orthogonal reflection method always converge for any initial value. Especially for matrix (1000-by-1000) where the coefficient matrix is ill-conditioned (condition numbers= 1.4511×10^5), the orthogonal reflection method still yields excellent results. This not only demonstrates that the orthogonal reflection method is superior to the reflection method but also shows that the orthogonal reflection method can effectively solve ill-conditioned systems of equations to a certain extent.

5. Conclusions

In this paper, we have derived the iterative formula of the reflection method to solve the system (2.1), and its convergence analysis is also presented. Besides, we introduce an enhanced approach called the orthogonal reflection method to refine the performance of the reflection method. The convergence rates for this improved method are provided in Theorems 1 and 2. The necessity of orthogonality is given in Theorem 3. Numerical experiments show the efficiency and robustness of the orthogonal reflection method, which outperforms the other methods in aspects of both the number of iteration steps and the computing time. Householder transformations are used in the orthogonal reflection to get the solution set of the original system. The performance of the orthogonal reflection method is correlated with the orthogonalization technique employed. Inadequate orthogonalization could lead to degraded outcomes. We will further investigate orthogonalization methods specifically adapted for the orthogonal reflection algorithms, and the application of the orthogonal reflection method in various fields will also be explored.

Author contributions

Wenyue Feng: Methodology, validation, writing-original draft; Hailong Zhu: Conceptualization, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artiffcial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National NSF of China (NO. 11671118), NSF of Anhui Province of China(NO. 2024AH050013) and University-level Project(NO. ACKYC24051)

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. M. Childs, R. Kothari, R. D. Somma, Quantum algorithm for systems of linear equations with exponentially improved dependence on precision, *SIAM J. Comput.*, **46** (2017), 1920–1950. https://doi.org/10.1137/16M1087072
2. E. Carson, N. J. Higham, Accelerating the solution of linear systems by iterative refinement in three precisions, *SIAM J. Sci. Comput.*, **40** (2018), 817–847. https://doi.org/10.1137/17m1140819
3. N. J. Higham, S. Pranesh, Exploiting lower precision arithmetic in solving symmetric positive definite linear systems and least squares problem, *SIAM J. Sci. Comput.*, **43** (2021), 258–277. https://doi.org/10.1137/19M1298263
4. T. A. Davis, S. Rajamanickam, W. M. S. Lakhdar, A survey of direct methods for sparse linear systems, *Acta Numer.*, **25** (2016), 383–566. https://doi.org/10.1017/S0962492916000076
5. H. A. V. D. Vorst, *Computer solution of large linear systems*, Elsevier, 1999. https://doi.org/10.1016/0377-0427(94)90302-6
6. Z. Zlatev, Use of iterative refinement in the solution of sparse linear systems, *SIAM J. Numer. Anal.*, **19** (1982), 381–399. https://doi.org/10.1137/0719024
7. H. Y. Huang, A direct method for the general solution of a system of linear equations, *J. Optimiz. Theory Appl.*, **16** (1975), 429–445. https://doi.org/10.1007/bf00933852
8. E. Spedicato, M. T. Vespucci, Variations on the Gram-Schmidt and the Huang algorithms for linear systems: A numerical study, *Appl. Math.*, **38** (1993), 81–100. https://doi.org/10.21136/AM.1993.104537
9. D. M. Young, *Iterative solution of large linear systems*, Elsevier, 2014.

10. V. Simoncini, D. B. Szyld, Recent developments in Krylov subspace methods for linear systems, *Numer. Linear Algebr.*, **14** (2007), 1–59. https://doi.org/10.1002/nla.499
11. L. Tondji, D. A. Lorenz, Faster randomized block sparse Kaczmarz by averaging, *Numer. Algorithms*, **93** (2023), 1417–1451. https://doi.org/10.1007/s11075-022-01473-x
12. V. Patel, M. Jahangoshahi, D. A. Maldonado, Randomized block adaptive linear system solvers, *SIAM J. Matrix Anal. A.*, **44** (2023), 1349–1369. https://doi.org/10.1137/22M1488715
13. M. Guida, C. Sbordone, The reflection method for the numerical solution of linear systems, *SIAM Rev.*, **65** (2023), 1137–1151. https://doi.org/10.1137/22M1470463
14. G. Cimmino, Approximate computation of the solutions of systems of linear equations, *Rend. Accad. Sci. Fis. Mat. Napoli*, **89** (2022), 65–72.
15. M. Benzi, Gianfranco Cimmino's contributions to numerical mathematics, *Atti del Seminario di Analisi Matematica, Dipartimento di Matematica dell'Universita di Bologna. Volume Speciale: Ciclo di Conferenze in Ricordo di Gianfranco Cimmino*, 2004, 87–109.
16. Y. Censor, T. Elfving, New methods for linear inequalities, *Linear Algebra Appl.*, **42** (1982), 199–211. https://doi.org/10.1016/0024-3795(82)90149-5
17. T. Elfving, A projection method for semidefinite linear systems and its applications, *Linear Algebra Appl.*, **391** (2004), 57–73. https://doi.org/10.1016/j.laa.2003.11.025
18. Y. Censor, M. D. Altschuler, W. D. Powlis, On the use of Cimmino's simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning, *Inverse Probl.*, **4** (1988), 607. https://doi.org/10.1088/0266-5611/4/3/006
19. S. Petra, C. Popa, C. Schnörr, *Extended and constrained Cimmino-type algorithms with applications in tomographic image reconstruction*, Universitatsbibliothek Heidelberg, 2008. https://doi.org/10.11588/heidok.00008798
20. M. R. Hestenes, E. Stiefel, Methods of conjugate Gradients for solving linear systems, *J. Res. Natl. Bureau Stand.*, **49** (1952), 409–436. https://doi.org/10.6028/jres.049.044