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Research article

Diverse wave solutions for the (2+1)-dimensional Zoomeron equation using the modified extended direct algebraic approach

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Abstract: This work used the modified extended direct algebraic expansion method to find exact soliton solutions for the (2+1)-dimensional nonlinear Zoomeron equation. The modified extended direct algebraic technique employs a wave transformation and, in order to determine solutions, it then performs an algebraic expansion, compares coefficients, and balances the equation. The results were an effective acquisition of a variety of solitons with unique wave characteristics including bright, kink, periodic, singular periodic, and dark solitons. A stability investigation has confirmed the structural integrity of these solutions under minor perturbations. In the form of 2D, contour, and 3D graphical representations, the stability and propagation of these solutions were further investigated. The findings illustrate how effectively this technique can solve higher-dimensional nonlinear equations and yield more soliton solutions. Beyond broadening our knowledge of nonlinear wave behavior, this research could be beneficial in nonlinear optics, fluid motion, and plasma systems.

Keywords: modified extended direct algebraic method; Zoomeron equation; soliton; stability
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

In different fields, such as quantum mechanics, solid state physics, plasma waves, fluid mechanics, field theory, optical fibers, and chemical physics, nonlinear partial differential equations (NLPDEs) are widely used to induce several different phenomena [1–3]. Exact analytical solutions of NLPDEs are essential in numerous applied scientific fields for precisely analyzing associated processes and comprehending the qualitative nature of multiple methods. Lately, scholars have proposed several kinds of methods to find exact solutions for NLPDEs. Some methods are the variational iteration method [4–6], tanh function technique [7], modified extended tanh function technique [8–10], sine-cosine technique [11, 12], exp-function technique [13], inverse scattering technique [14], Hirotas bilinear technique [15], the modified Sardar sub-equation method [16, 17], and homogeneous balance technique [18].

The (2+1)-dimensional Zoomeron equation (ZE) in mathematical physics has garnered significant attention due to its rich mathematical structure and diverse wave dynamics [19]. This equation, which generalizes certain integrable models, serves as an essential framework for investigating solitonic behavior in nonlinear media. Various analytical approaches have been designed to investigate its exact results, including solitons, periodic waves, and other nonlinear wave structures. The approximate similarity solutions to the Boiti-Leon-Mana-Pempinelli equation are derived by using the extended unified method [20]. The modified extended direct algebraic method is used to find special soliton solutions for the (3+1)-fractional modified Zakharov–Kuznetsov equation [21]. In [22], the exact solutions of the Klein Kramer equation were derived via the extended unified method. The sine-Gordon expansion technique and the generalized Riccati equation mapping technique provide further insights into nonlinear wave interactions [23]. These diverse methodologies significantly enhance the understanding of the (2+1)-dimensional ZE and its soliton behavior in nonlinear wave dynamics.

The first to suggest solitons with variable speeds were Calogero and Degasperis [24], who also observed a relationship between polarization effects and speed. The ability of the ZE to articulate localized wave formations and soliton-like solutions has garnered a lot of fascination among NLPDEs. A bilateral soliton was created as a result, with one appearing from one side and coming from a vast distance. The soliton boosts the boomerang to that side at the same speed in faraway times, while the other one rotates repeatedly around the fixed point as though it were pulling in the direction of a shift in the spatial field. These well-known boomeron and trappon soliton equations were used in [25] to construct the coupled boomeron equation and other specific integrable structures for coupled wave equations, respectively. One of these specific instances of the boomeron equation is the nonlinear ZE. It serves to explain phenomena pertaining to boomerons and trappons and shows how a monovariant field evolves. Consequently the ZE describes specific cases of solitons with unusual characteristics in a range of physical situations, specifically in fluid mechanics, laser physics, and nonlinear optics [26–28].

The ZE, a nonlinear integrable PDE first derived by Calogero and Degasperis, plays a pivotal role in modeling complex wave phenomena across multiple physical domains including nonlinear optics, plasma physics, and fluid dynamics. The equation's integrability guarantees exact solvability through methods like the inverse scattering transform, while its physical relevance extends to optical pulse propagation in fibers, ion-acoustic waves in plasmas, and rogue wave dynamics in hydrodynamics. Previous analytical approaches including Hirota's bilinear method and Lie symmetry analysis have revealed various solution classes, yet significant gaps remain in understanding (2+1)-dimensional solutions with singularities or periodic structures. Our work advances this frontier by employing the modified extended direct algebraic method (MEDAM) to systematically derive new exact solutions—including bright/dark solitons, kinks, and singular periodic waves-while establishing their stability criteria and connecting them to concrete applications like optical signal processing and plasma wave control. This comprehensive treatment not only enriches nonlinear wave theory but also provides practical tools for wave manipulation in physical systems, demonstrating the continued relevance of the ZE in contemporary applied mathematics and theoretical physics.

Now let us explore the modified extended direct algebraic method (MEDAM) for analyzing the (2+1)-dimensional nonlinear ZE:

$$\left(\frac{\varpi_{xy}}{\varpi}\right)_{tt} - \left(\frac{\varpi_{xy}}{\varpi}\right)_{xx} + 2\left(\varpi^2\right)_{xt} = 0. \tag{1.1}$$

The amplitude of the associated wave node is symbolized by $\varpi(x, y, t)$. One such advancement that provides a methodical framework for generating exact solutions to nonlinear PDEs is the MEDAM. By adding more algebraic structures, this technique improves the conventional direct algebraic approach and thereby renders it possible to extract complex wave solutions, such as multi-wave, periodic waves, and soliton interactions. When applied to the (2+1)-dimensional ZE, this technique has produced explicit solutions that are consistent with well-known physical phenomena. The beneficial effect of MEDAM in nonlinear wave analysis was further demonstrated by El-Deeb and Samir [29], who used it to generate several solitonic wave solutions for an extended (2+1)-dimensional perturbed nonlinear Schrödinger equation. This approach has been found effective in solving a range of issues, including those using hyperbolic, rational functions and trigonometric. The goal is to build numerous types of wave structures and examine their consequences within the deeper context of nonlinear wave theory by utilizing the benefits of the MEDAM. In this study, the recommended model is executed using the MEDAM.

This paper introduces the MEDAM that yields exact soliton solutions of the nonlinear ZE. This expands the range of known solutions by obtaining bright, singular, periodic, singular periodic, dark, and kink solitons, each of which presents unique nonlinear wave characteristics. The ZE dynamic behavior and possible applications in modeling physical phenomena are simplified by these solutions. This research provides deeper insights into soliton dynamics and stability by incorporating complete 2D, 3D, and contour graphical illustrations, in contrast to earlier efforts that mostly focus on analytical methodologies. By applying the MEDAM to higher-dimensional nonlinear systems, we enhance the understanding of fluid dynamics, wave propagation, and optical fiber communications. By demonstrating that the MEDAM is efficient at solving nonlinear equations, the results enrich the theory of nonlinear waves and could potentially find application in complicated wave interactions, nonlinear optics, and plasma physics.

The MEDAM is a powerful analytical technique for obtaining exact solutions of NLPDEs. Its significance lies in its ability to generate a wide range of soliton and wave solutions including bright, dark, kink, periodic, and singular solitons through a systematic algebraic framework. Unlike traditional methods, the MEDAM enhances solution versatility by incorporating a generalized auxiliary function, enabling the derivation of complex wave structures in higher-dimensional systems. Its adaptability, simplicity, and symbolic computation compatibility make it an effective tool for analyzing nonlinear phenomena in fields such as optical fibers, plasma physics, and fluid dynamics. The MEDAM not only advances theoretical insights but also provides valuable analytical benchmarks for validating numerical models and guiding experimental research.

The article is organized as mentioned below. In Section 2, the suggested integration technique is described. Some results are extracted by applying the MEDAM in Section 3. Section 4 provides graphical illustration that specifies the characteristics of the derived solutions and the parametric affects are also discussed in this section. Physical interpretation and applications are provided in Section 5. The stability analysis of the proposed model is examined in Section 6. Finally, Section 7 provides our findings.

2. Description of the integration technique

This section presents the MEDA technique, taking NLPDE into consideration as follows:

$$P(H, H_x, H_y, H_t, H_{xx}, H_{yy}, H_{xt}, \ldots) = 0. \tag{2.1}$$

Using the MEDA approach [30–32], the following steps are taken to tackle Eq (2.1):
Step 1. Utilize the transformation of traveling waves described below.

$$H(x, y, t) = B(\eta), \eta = x + y - \mu t. \tag{2.2}$$

The ordinary differential equation (ODE) is then obtained by using Eq (2.1):

$$\aleph(B, B', B'', B''', \ldots) = 0. \tag{2.3}$$

Step 2. The following is the purported solution to the extracted ODE:

$$B(\eta) = \sum_{r=-l}^{l} b_r \sigma(\eta)^r, \tag{2.4}$$

where σ meets the subsequent requirement,

$$\sigma^{'}(\eta) = \sqrt{u_0 + u_1\sigma(\eta) + u_2\sigma^2(\eta) + u_3\sigma^3(\eta) + u_4\sigma^4(\eta) + u_6\sigma^6(\eta)}, \tag{2.5}$$

where $\sigma = \sigma(\eta)$ and $\sigma^{'} = \frac{d\sigma}{d\eta}$.
Step 3. The balancing rule is utilized to assess the integer l.
Step 4. The consequence of Eqs (2.5) and (2.4) can be transferred into Eq (2.3) and we accumulate all $\sigma(\eta)^r$. A set of nonlinear equations is then produced by setting all of the coefficients to zero. To determine the unknowns, this derived system of equations is handled by using the Mathematica program.
Step 5. Several cases can be obtained by assigning various possible choices for u_0, u_1, u_2, u_3, u_4, and u_6.
Case 1: If $u_0 = u_1 = u_3 = u_6 = 0$, then the following genres are obtained:
Genre 1. If $u_2 > 0$ and $u_4 < 0$, then $\sigma(\eta) = \sqrt{-\frac{u_2}{u_4}} \operatorname{sech}(\sqrt{u_2}\,\eta)$.
Genre 2. If $u_2 < 0$ and $u_4 > 0$, then $\sigma(\eta) = \sqrt{-\frac{u_2}{u_4}} \sec(\sqrt{-u_2}\,\eta)$.

Genre 3. If $u_2 > 0$ and $u_4 > 0$, then $\sigma(\eta) = \sqrt{-\frac{u_2}{u_4}}\csc\left(\sqrt{-u_2}\,\eta\right)$.

Case 2: If $u_0 = \frac{u_2^2}{4u_4}$ and $u_1 = u_3 = u_6 = 0$, then the following genres are obtained:

Genre 1. If $u_2 < 0$ and $u_4 > 0$, then $\sigma(\eta) = \sqrt{-\frac{u_2}{2u_4}}\tanh\left(\sqrt{-\frac{u_2}{2}}\,\eta\right)$.

Genre 2. If $u_2 > 0$ and $u_4 > 0$, then $\sigma(\eta) = \sqrt{\frac{u_2}{2u_4}}\tan\left(\sqrt{\frac{u_2}{2}}\,\eta\right)$.

Genre 3. If $u_2 > 0$ and $u_4 > 0$, then $\sigma(\eta) = \sqrt{\frac{u_2}{u_4}}\operatorname{csch}\left(\sqrt{2u_2}\,\eta\right)$.

Case 3: If $u_3 = u_4 = u_6 = 0$, then the following genres are obtained:

Genre 1. If $u_2 < 0$ and $u_0 = 0$, then $\sigma(\eta) = \frac{u_1\sinh(2\sqrt{u_2}\eta)}{2u_2} - \frac{u_1}{2u_2}$.

Genre 2. If $u_2 < 0$ and $u_0 = 0$, then $\sigma(\eta) = \frac{u_1\sin(2\sqrt{-u_2}\eta)}{2u_2} - \frac{u_1}{2u_2}$.

Genre 3. If $u_0 > 0$, $u_2 > 0$ and $u_1 = 0$, then $\sigma(\eta) = \sqrt{\frac{u_0}{u_2}}\sin\left(\sqrt{u_2}\,\eta\right)$.

Genre 4. If $u_0 > 0$, $u_2 < 0$ and $u_1 = 0$, then $\sigma(\eta) = \sqrt{\frac{-u_0}{u_2}}\sin\left(\sqrt{-u_2}\,\eta\right)$.

Genre 5. If $u_2 > 0$ and $u_0 = \frac{u_1^2}{4u_2}$, then $\sigma(\eta) = \exp\left(\sqrt{u_2}\,\eta\right) - \frac{u_1}{2u_2}$.

Case 4: If $u_0 = u_1 = u_2 = u_6 = 0$, then the following genre is obtained:

Genre. If $u_3 < 0$ and $u_4 < 0$, then $\sigma(\eta) = \frac{4u_3}{u_3^2\eta^2 - 4u_4}$.

Case 5: If $u_2 = u_4 = u_6 = 0$, then the following genres is obtained:

Genre. If $u_3 > 0$, then $\sigma(\eta) = \xi\left(\frac{\sqrt{u_3}\eta}{2}; -\frac{4u_1}{u_3}, -\frac{4u_0}{u_3}\right)$.

Case 6: If $u_0 = u_1 = u_6 = 0$, then the following genres are obtained:

Genre 1. If $u_2 = \frac{u_3^2}{u_4} > 0$ and $u_2 > 0$, then $\sigma(\eta) = -\frac{u_2}{u_3}\left(\tanh\left(\frac{1}{2}\sqrt{u_2}\,\eta\right) + 1\right)$.

Genre 2. If $u_2 = \frac{u_3^2}{u_4} > 0$ and $u_2 > 0$, then $\sigma(\eta) = -\frac{u_2}{u_3}\left(\coth\left(\frac{1}{2}\sqrt{u_2}\,\eta\right) + 1\right)$.

Case 7: $u_0 = u_1 = u_3 = 0$, then the following genres are obtained:

Genre 1. If $u_2 > 0, u_4 > 0$ and $u_6 > 0$, then $\sigma(\eta) = \sqrt{\frac{2u_2\operatorname{sech}^2(\sqrt{u_2}\eta)}{2\sqrt{u_4^2-4u_6u_2} - (\sqrt{u_4^2-4u_6u_2}+u_4)\operatorname{sech}^2(\sqrt{u_2}\eta)}}$.

Genre 2. $u_2 > 0, u_4 > 0$ and $u_6 > 0$, then $\sigma(\eta) = \sqrt{\frac{2u_2\sec^2(\sqrt{-u_2}\eta)}{2\sqrt{u_4^2-4u_6u_2} - (\sqrt{u_4^2-4u_6u_2}-u_4)\sec^2(\sqrt{-u_2}\eta)}}$.

By replacing the established constants in the purported solution to Eq (2.3) with the conventional results of Eq (2.4), several solutions to Eq (2.1) can be obtained.

3. Solving methodology for the Zoomeron equation

In this section the MEDAM is applied on the ZE to seek analytic solutions for Eq (1.1) in the form below:

$$\varpi(x, y, t) = B(\eta), \eta = x + y - \mu t. \tag{3.1}$$

The real parameter is denoted by μ. Equation (1.1) is reduced to an ODE using the transformation given in Eq (3.1) as:

$$\mu^2\left(\frac{B''}{B}\right)'' - \left(\frac{B''}{B}\right)'' - 2\mu(B^2)'' = 0. \tag{3.2}$$

The result of integrating Eq (3.2) twice with regard to η is

$$(\mu^2-1)B''-2\mu B^3+\tau B=0, \tag{3.3}$$

where the constant is denoted by τ . One can get $l=1$ by balancing B'' and B^3. One way to express the solution to Eq (3.3) is

$$B(\eta)=b_0+b_1\sigma(\eta)+\frac{b_{-1}}{\sigma(\eta)}. \tag{3.4}$$

By executing Step 4 discussed in the preceding section and merging Eq (3.4) with Eq (2.4) into Eq (3.3), the below system of equations is yielded by equating the coefficient of similar power of $\sigma(\eta)$ to zero.

$\sigma^{-3}(\eta)$: $-2\mu b_{-1}^3-2b_{-1}u_0+2\mu^2 b_{-1}u_0=0$,
$\sigma^{-2}(\eta)$: $-6\mu b_{-1}^2 b_0-\frac{3}{2}b_{-1}u_1+\frac{3}{2}\mu^2 b_{-1}u_1=0$,
$\sigma^{-1}(\eta)$: $\tau b_{-1}-6\mu b_{-1}b_0^2-6\mu b_{-1}^2 b_1-b_{-1}u_2+\mu^2 b_{-1}u_2=0$,
$\sigma^{0}(\eta)$: $\tau a_0-2\mu b_0^3-12\mu b_{-1}b_0 b_1-\frac{1}{2}b_1u_1+\frac{1}{2}\mu^2 b_1u_1-\frac{1}{2}b_{-1}u_3+\frac{1}{2}\mu^2 b_{-1}u_3=0$,
$\sigma^{1}(\eta)$: $\tau b_1-6\mu b_0^2 b_1-6\mu b_{-1}b_1^2-b_1u_2+\mu^2 b_1u_2=0$,
$\sigma^{2}(\eta)$: $-6\mu b_0 b_1^2-\frac{3}{2}b_1u_3+\frac{3}{2}\mu^2 b_1u_3=0$,
$\sigma^{3}(\eta)$: $-2\mu b_1^3-2b_1t_4+2\mu^2 b_1u_4+b_{-1}u_6-\mu^2 b_{-1}u_6=0$,
$\sigma^{5}(\eta)$: $-3b_1u_6+3\mu^2 b_1u_6=0$.

Solving this system of equations with Mathematica packages yields the following results:
Case 1: If $u_0=u_1=u_3=u_6=0$, this gives $b_0=b_{-1}=0, u_2=\frac{-\tau}{\mu^2-1}$, and $u_4=\frac{\mu b_1^2}{\mu^2-1}$, and then following three genres are prodeuced.
Genre 1. If $u_2>0$ and $u_4<0$, then these imply:

$$\varpi_{1,1}(x,y,t)=\sqrt{\frac{\tau}{\mu b_1^2}}\operatorname{sech}\Big((x-t\mu)\sqrt{\frac{-\tau}{\mu^2-1}}\Big). \tag{3.5}$$

Genre 2. If $u_2<0$ and $u_4>0$, then these imply:

$$\varpi_{1,2}(x,y,t)=\sqrt{\frac{\tau}{\mu b_1^2}}\sec\Big((x-t\mu)\sqrt{\frac{\tau}{\mu^2-1}}\Big). \tag{3.6}$$

Genre 3. If $u_2>0$ and $u_4>0$, then these imply:

$$\varpi_{1,3}(x,y,t)=\sqrt{\frac{\tau}{\mu b_1^2}}\csc\Big((x-t\mu)\sqrt{\frac{\tau}{\mu^2-1}}\Big). \tag{3.7}$$

Equation (3.5) presents a bright soliton solution while Eqs (3.6) and (3.7) present a singular periodic solution.
Case 2: If $u_0=\frac{u_2^2}{4u_4}$, $u_1=u_3=u_6=0$, this gives $b_0=b_{-1}=0$, $u_2=\frac{-\tau}{\mu^2-1}$, and $u_4=\frac{\mu b_1^2}{\mu^2-1}$, and then following three genres are produced.
Genre 1. If $u_2>0$ and $u_4>0$, then these imply:

$$\varpi_{2,1}(x,y,t)=\frac{\sqrt{\frac{\tau}{\mu b_1^2}}\tanh\big(\frac{(x-\mu t)\sqrt{\frac{\tau}{\mu^2-1}}}{\sqrt{2}}\big)}{\sqrt{2}}. \tag{3.8}$$

Genre 2. If $u_2 > 0$ and $u_4 > 0$, then these imply:

$$\varpi_{2,2}(x,y,t) = b_0 + \operatorname{csch}\left(\sqrt{2}(x - t\mu)\sqrt{\frac{-\tau + 6\mu u_0^2}{-1+\mu^2}}\right) b_1 \sqrt{-\frac{\tau - 6\mu u_0^2}{\mu b_1^2}}. \tag{3.9}$$

Genre 3. If $u_2 > 0$ and $u_4 > 0$, then these imply:

$$\varpi_{2,3}(x,y,t) = \frac{\sqrt{\frac{-\tau}{\mu b_1^2}}\tanh\left(\frac{(x-\mu t)\sqrt{\frac{-\tau}{\mu^2-1}}}{\sqrt{2}}\right)}{\sqrt{2}}. \tag{3.10}$$

Equation (3.8) presents a dark soliton solution, Eq (3.9) presents a singular soliton solution, and (3.10) presents a singular periodic solution.

Case 3: If $u_2 = u_4 = u_6 = 0$, this gives $b_1 = 0, u_1 = \frac{4\mu b_{-1} b_0}{\mu^2-1}$, and $u_2 = \frac{-\tau+6\mu b_0^2}{\mu^2-1}$, and then following five genres are produced.

Genre 1. If $u_2 < 0$ and $u_0 = 0$, then these imply:

$$\varpi_{3,1}(x,y,t) = b_0 + \frac{b_{-1}}{-\left(\frac{2\mu b_{-1} b_0}{-\tau+6\mu b_0^2}\right) + \frac{2\mu \sinh\left(2(x-t\mu)\sqrt{\frac{-\tau+6\mu b_0^2}{-1+\mu^2}}\right) b_{-1} b_0}{-\tau+6\mu b_0^2}}. \tag{3.11}$$

Genre 2. If $u_2 < 0$ and $u_0 = 0$, then these imply:

$$\varpi_{3,2}(x,y,t) = b_0 + \frac{b_{-1}}{\left(-\frac{2\mu b_{-1} b_0}{-\tau+6\mu b_0^2}\right) + \frac{\mu \sin\left(2(x-t\mu)\sqrt{\frac{\tau+6\mu b_0^2}{-1+\mu^2}}\right) b_{-1} b_0}{-\tau+6\mu b_0^2}}. \tag{3.12}$$

Genre 3. If $u_0 > 0$, $u_2 > 0$, and $u_1 = 0$, this gives $b_1 = 0, u_0 = \frac{w b_{-1}^2}{-1+\mu^2}$, and $u_2 = \frac{-\tau+6\mu b_0^2}{\mu^2-1}$, which imply:

$$\varpi_{3,3}(x,y,t) = b_0 + \frac{\operatorname{csch}\left((x - t\mu)\sqrt{\frac{-\tau+6\mu b_0^2}{-1+\mu^2}}\right) b_{-1}}{\sqrt{\frac{\mu b_{-1}^2}{-\tau+6\mu b_0^2}}}. \tag{3.13}$$

Genre 4. If $u_0 > 0$, $u_2 < 0$, and $u_1 = 0$, then these imply:

$$\varpi_{3,4}(x,y,t) = b_0 + \frac{\csc\left((x - t\mu)\sqrt{-\frac{-\tau+6\mu b_0^2}{-1+\mu^2}}\right) b_{-1}}{\sqrt{-\frac{\mu b_{-1}^2}{-\tau+6\mu b_0^2}}}. \tag{3.14}$$

Genre 5. If $u_2 > 0$ and $u_0 = \frac{u_1^2}{4u_2}$, then these imply:

$$\varpi_{3,5}(x,y,t) = b_0 + \frac{b_{-1}}{\exp\left((x - t\mu)\sqrt{\frac{-\tau+6\mu b_0^2}{-1+\mu^2}}\right) - \frac{2\mu b_{-1} b_0}{-\tau+6\mu b_0^2}}. \tag{3.15}$$

Equation (3.11) generates a periodic wave solution, Eqs (3.12) and (3.14) present periodic singular solutions, (3.13) presents a singular periodic solution, while Eq (3.15) presents a kink soliton solution.
Case 4: If $u_0 = u_1 = u_2 = u_6 = 0$, this gives $u_4 = \frac{\mu b_1^2}{\mu^2-1}$ and $u_3 = \frac{-2\tau u_0+4\mu u_0^3+24\mu u_{-1}u_0u_1}{(-1+\mu^2)u_{-1}}$, and then the following genre is produced.
Genre. If $u_3 > 0$ and $u_4 > 0$, then these imply:

$$\varpi_{4,1}(x,y,t) = b_0 + \frac{4b_1u_3}{(x-t\mu)^2u_3^2 - 4u_4} + \frac{1}{4u_3}b_{-1}\left((x-t\mu)^2u_3^2 - 4u_4\right). \tag{3.16}$$

Equation (3.16) presents a bright soliton solution.
Case 6: If $u_0 = u_1 = u_6 = 0$, this gives $u_4 = \frac{\mu b_1^2}{\mu^2-1}$ and $u_3 = \frac{-2\tau u_0+4\mu u_0^3+24\mu u_{-1}u_0u_1}{(-1+\mu^2)u_{-1}}$, and then the following two genres are produced.
Genre 1. If $u_2 = \frac{u_3^2}{u_4} > 0$ and $u_2 > 0$, then these imply:

$$\varpi_{6,1}(x,y,t) = b_0 + 4b_0\left(1 + \tanh\left(2(x-t\mu)\sqrt{\frac{\mu b_0^2}{-1+\mu^2}}\right)\right). \tag{3.17}$$

Genre 2. If $u_2 = \frac{u_3^2}{u_4} > 0$ and $u_2 > 0$, then these imply:

$$\varpi_{6,2}(x,y,t) = b_0 + 4b_0\left(1 + \coth\left(2(x-t\mu)\sqrt{\frac{\mu b_0^2}{-1+\mu^2}}\right)\right). \tag{3.18}$$

Equation (3.17) presents a kink soliton solution while Eq (3.18) presents a singular soliton solution.
Case 7: If $u_0 = u_1 = u_3 = 0$, this gives $u_2 = \frac{-\tau}{-1+\mu^2}$ and $u_4 = \frac{\mu b_1^2}{-1+\mu^2}$, and then following two genres are produced.
Genre 1. If $u_2 > 0, u_4 > 0$, and $u_6 > 0$, then these imply:

$$\varpi_{7,1}(x,y,t) = \sqrt{2}\,b_1\sqrt{\frac{\operatorname{sech}^2\left((x-t\mu)\sqrt{u_2}\right)u_2}{2\sqrt{u_4^2-4u_2u_6} - \operatorname{sech}^2\left((x-t\mu)\sqrt{u_2}\right)(u_4 + \sqrt{u_4^2-4u_2u_6})}}. \tag{3.19}$$

Genre 2. If $u_2 > 0, u_4 > 0$, and $u_6 > 0$, then these imply:

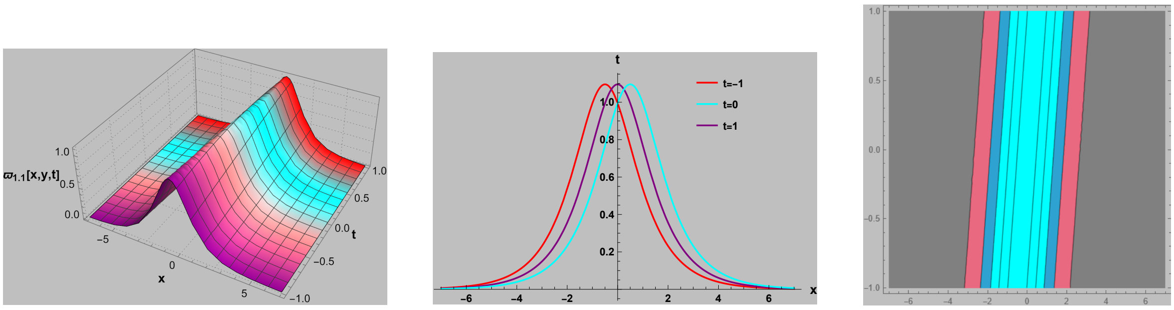
$$\varpi_{7,2}(x,y,t) = \sqrt{2}\,b_1\sqrt{\frac{\sec^2\left((x-t\mu)\sqrt{u_2}\right)u_2}{2\sqrt{u_4^2-4u_2u_6} - \sec^2\left((x-t\mu)\sqrt{-u_2}\right)(-u_4 + \sqrt{u_4^2-4u_2u_6})}}. \tag{3.20}$$

Equation (3.19) presents a bright soliton solution while Eq (3.20) presents a singular periodic solution.

4. Graphical illustration and discussion

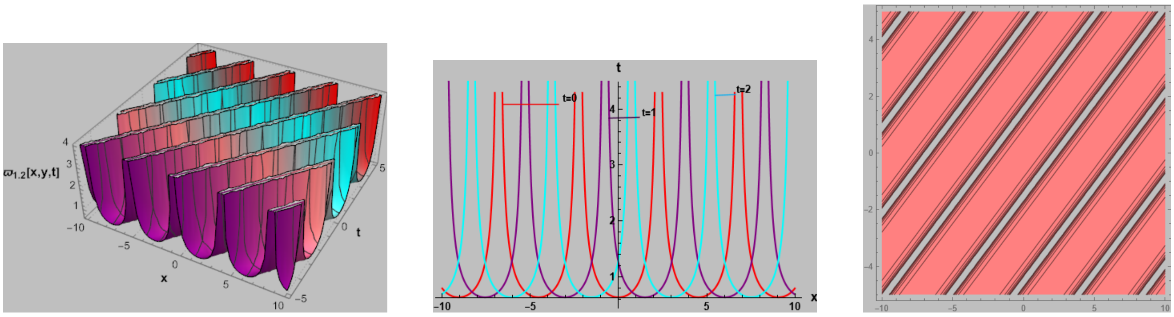
This section offers graphical representations that helps to explain how the outcomes of the ZE behave. Numerical simulations have been used to assess various families of solutions that have been derived using the MEDAM. The nature of solitons, periodic structures, and their propagation properties in nonlinear wave systems are highlighted in these visual aids. These solutions include kink, singular, singular periodic, bright, and dark solitons. The solution's propagation over space and time is depicted in 2D, contour, and 3D graphs. Insights into nonlinear wave dynamics are provided by these representations, which highlight important characteristics like amplitude, velocity, and structural stability. Various families of solutions were yielded from Eq (2.4) by assigning particular values to the parameters. The obtained solutions depend critically on the parameters μ, τ, b_0, b_1, and b_-1, which govern wave speed, amplitude, and stability. Wave speed parameter μ controls soliton propagation velocity. Nonlinearity coefficient τ determines the balance between dispersion and nonlinearity. The dynamic behavior of these solutions can be categorized as follows: Bright solitons in Eqs (3.5) and (3.19) are localized, bell-shaped waves that maintain their amplitude and shape during propagation, making them ideal for optical communication systems where signal integrity is crucial. Dark solitons in Eq (3.8) are characterized by intensity dips-these solutions model phenomena like quantum vortices in Bose-Einstein condensates. Singular and periodic solutions in Eq (3.6), (3.12), and (3.20) exhibit blow-up behavior or periodic oscillations, which can describe rogue waves in hydrodynamics or resonant modes in plasmas. Kink solitons in Eqs (3.15) and (3.17) are topological solutions that transition between two constant states, modeling phase transitions in ferromagnetic materials or domain walls in optics.

A bell shape or bright soliton of Eq (3.5) is presented in Figure 1 by inserting $\mu = 0.6$, $\tau = 0.6$, $b_1 = 0.6$, $b_{-1} = b_0 = 0$, and $y = 0$. A singular periodic soliton of Eq (3.6) is presented in Figure 2 by inserting $\mu = 1.5$, $\tau = 0.6$, $b_1 = 1$, $b_{-1} = b_0 = 0$, and $y = 0$. A dark soliton of Eq (3.8) is presented in Figure 3 by inserting $\mu = 1.2$, $\tau = 0.01$, $b_1 = 0.5$, $b_{-1} = b_0 = 0$, and $y = 0$. A singular soliton of Eq (3.9) is presented in Figure 4 by inserting $\mu = 1.5$, $\tau = -0.8$, $b_1 = 0.7$, $b_{-1} = b_0 = 0$, and $y = 0$. A periodic solution of Eq (3.11) is presented in Figure 5 by inserting $\mu = 0.5$, $\tau = -0.7$, $b_{-1} = 0.3$, $b_1 = 0$, $b_0 = 0.2$, and $y = 0$. A singular periodic solution of Eq (3.12) is presented in Figure 6 by inserting $\mu = 1.1$, $\tau = 0.7$, $b_{-1} = 0.3$, $b_1 = 0$, $b_0 = 0.1$, and $y = 0$. A kink soliton of Eq (3.15) is presented in Figure 7 by inserting $\mu = 0.5$, $\tau = 0.2$, $b_{-1} = 0.6$, $b_1 = 0$, $b_0 = 0.2$, and $y = 0$. A bright soliton of Eq (3.16) is presented in Figure 8 by inserting $\mu = -0.2$, $\tau = 0.8$, $b_{-1} = 0.7$, $b_1 = -0.4$, $b_0 = 0.3$, and $y = 0$. A kink soliton of Eq (3.17) is presented in Figure 9 by inserting $\mu = 1.5$, $\tau = 0.5$, $b_{-1} = -0.7$, $b_1 = 0$, $b_0 = 0.1$, and $y = 0$. A singular soliton of Eq (3.18) is presented in Figure 10 by inserting $\mu = -0.9$, $\tau = 0.2$, $b_{-1} = 0$, $b_1 = 0.3$, $b_0 = 0.01$, and $y = 0$. A bright soliton of Eq (3.19) is presented in Figure 11 by inserting $\mu = 1.2$, $\tau = -0.5$, $b_{-1} = 0.7$, $b_1 = 0.4$, $b_0 = 0$, $u_6 = 1.2$, and $y = 0$.A bright soliton of Eq (3.20) is presented in Figure 12 by inserting $\mu = 1.2$, $\tau = -0.5$, $b_{-1} = 0$, $b_1 = 0.4$, $b_0 = 0$, $u_6 = 1$, and $y = 0$.



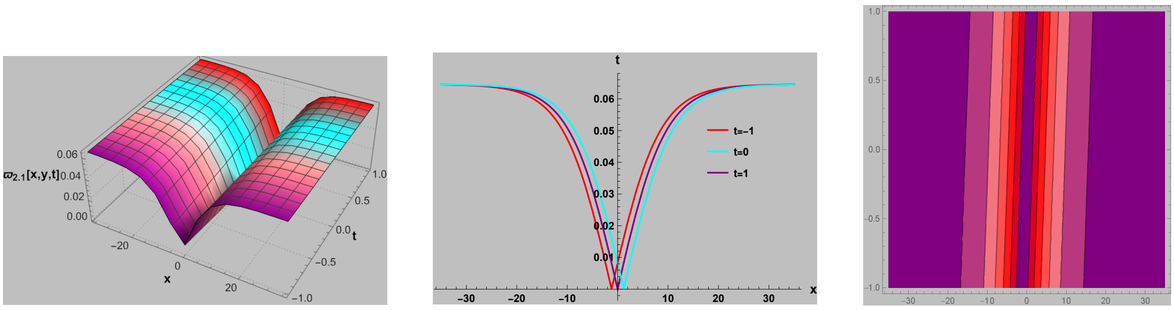
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 1. Representation of Eq (3.5) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 0.6$, $\tau = 0.6$, $b_1 = 0.6$, $b_{-1} = b_0 = 0$, and $y = 0$.



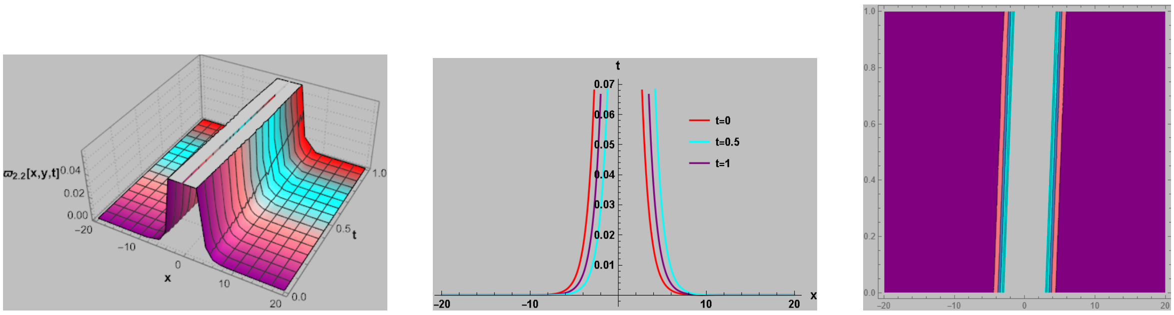
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 2. Representation of Eq (3.6) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.5$, $\tau = 0.6$, $b_1 = 1$, $b_{-1} = b_0 = 0$, and $y = 0$.



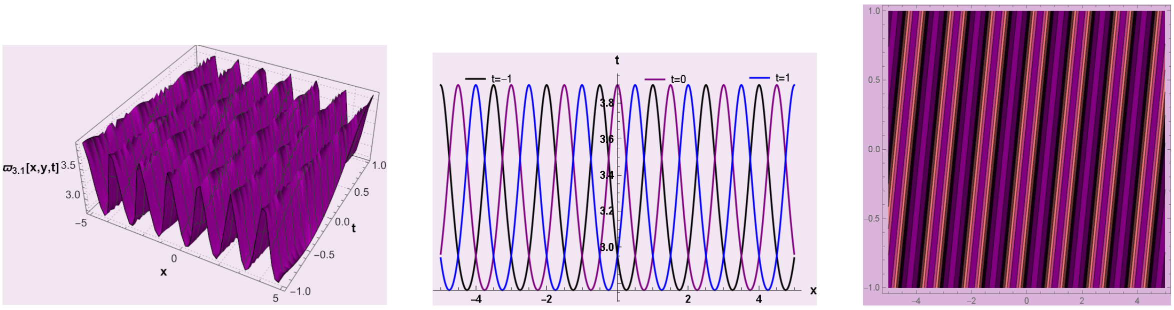
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 3. Representation of Eq (3.8) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.2$, $\tau = 0.01$, $b_1 = 0.5$, $b_{-1} = b_0 = 0$, and $y = 0$.



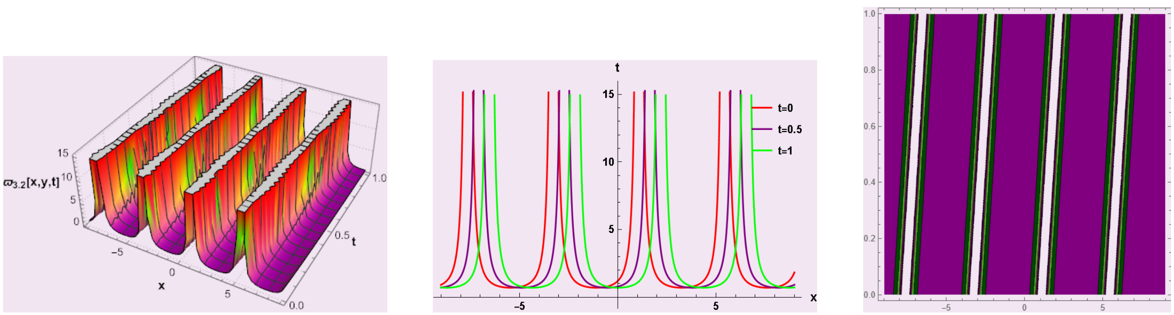
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 4. Representation of Eq (3.9) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.5$, $\tau = -0.8$, $b_1 = 0.7$, $b_{-1} = b_0 = 0$, and $y = 0$.



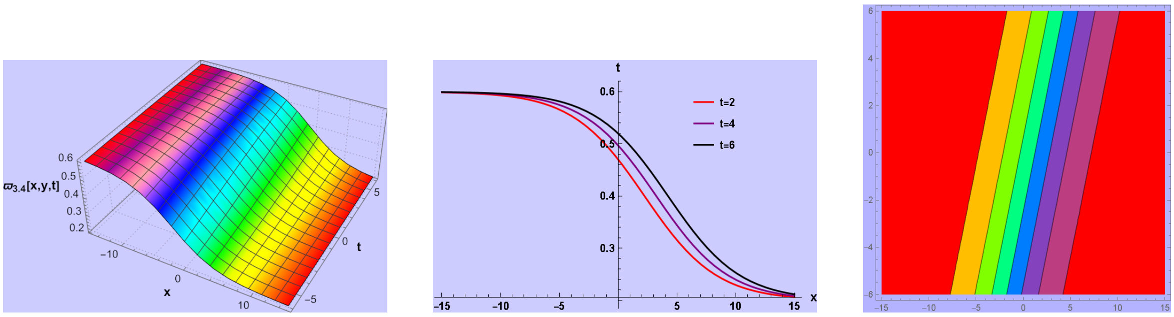
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 5. Representation of Eq (3.11) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 0.5$, $\tau = -0.7$, $b_{-1} = 0.3$, $b_1 = 0$, $b_0 = 0.2$, and $y = 0$.



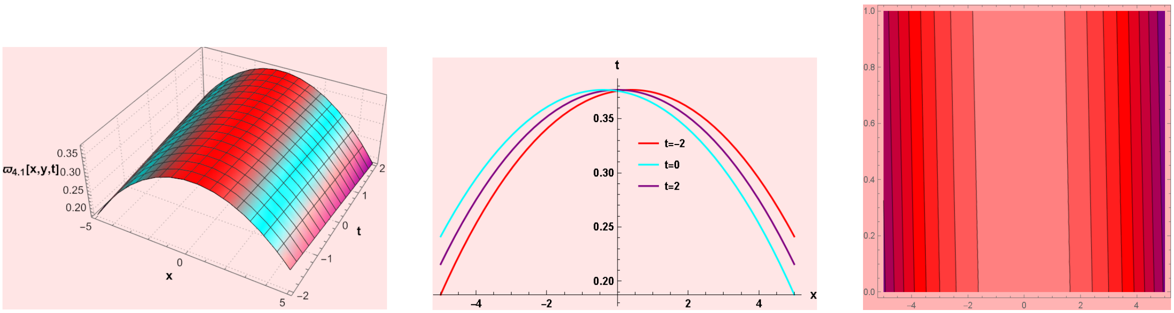
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 6. Representation of Eq (3.12) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.1$, $\tau = 0.7$, $b_{-1} = 0.3$, $b_1 = 0$, $b_0 = 0.1$, and $y = 0$.



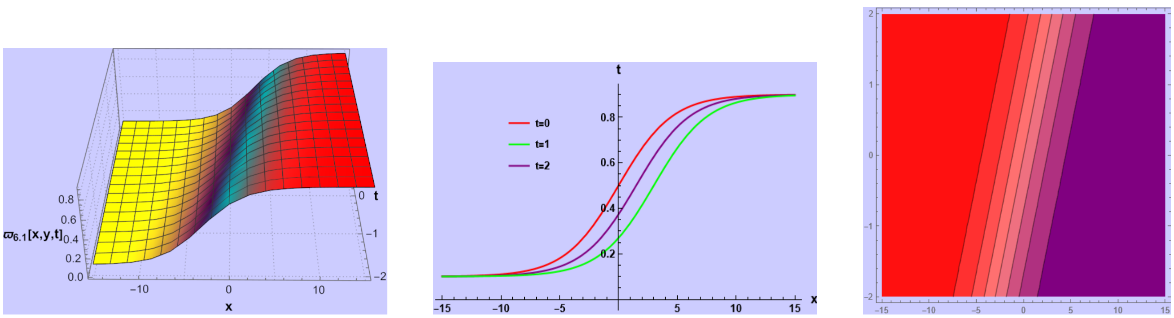
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 7. Representation of Eq (3.15) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 0.5$, $\tau = 0.2$, $b_{-1} = 0.6$, $b_1 = 0$, $b_0 = 0.2$, and $y = 0$.



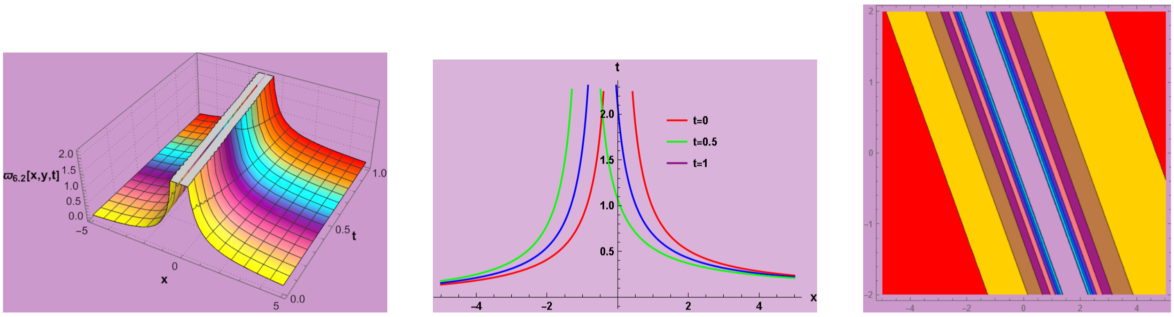
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 8. Representation of Eq (3.16) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = -0.2$, $\tau = 0.8$, $b_{-1} = 0.7$, $b_1 = -0.4$, $b_0 = 0.3$, and $y = 0$.



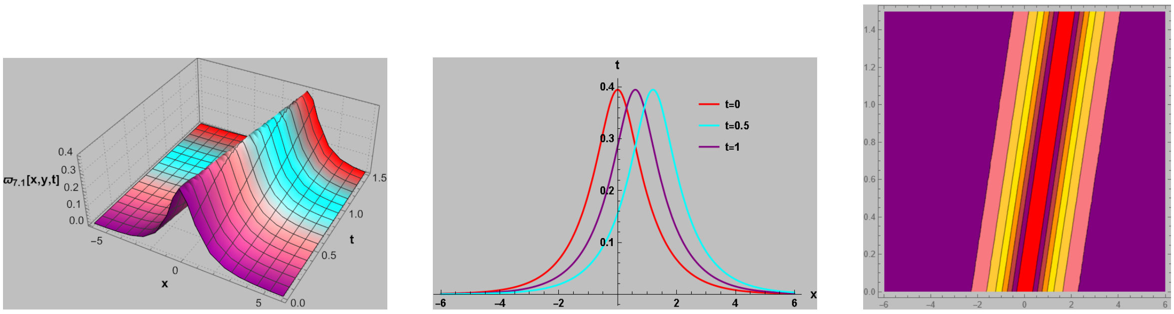
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 9. Representation of Eq (3.17) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.5$, $\tau = 0.5$, $b_{-1} = -0.7$, $b_1 = 0$, $b_0 = 0.1$, and $y = 0$.



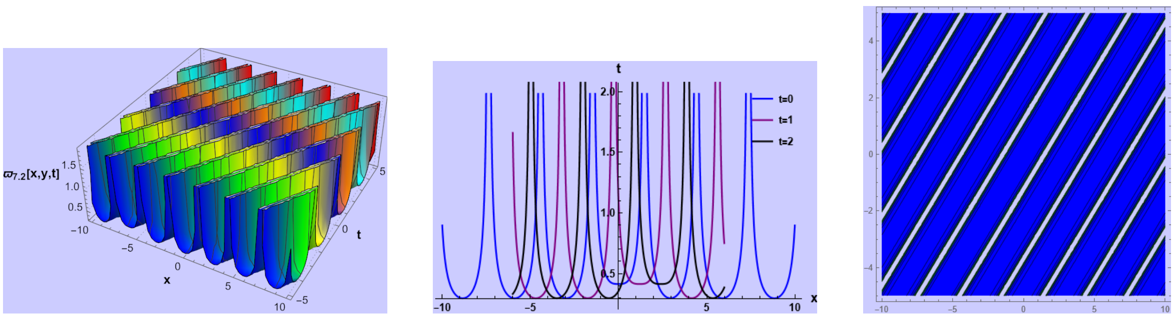
(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 10. Representation of Eq (3.18) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = -0.9$, $\tau = 0.2$, $b_{-1} = 0$, $b_1 = 0.3$, $b_0 = 0.01$, and $y = 0$.



(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 11. Representation of Eq (3.19) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.2$, $\tau = -0.5$, $b_{-1} = 0.7$, $b_1 = 0.4$, $b_0 = 0$, $u_6 = 1.2$, and $y = 0$.



(a) 3D Graph (b) 2D Graph (c) Contour Graph

Figure 12. Representation of Eq (3.20) using different patterns of graphs such as a 3D graph (top left), line graph (top right), and contour graph (bottom) with $\mu = 1.2$, $\tau = -0.5$, $b_{-1} = 0$, $b_1 = 0.4$, $b_0 = 0$, $u_6 = 1$, and $y = 0$.

5. Physical interpretation and applications

The ZE was derived by Calogero and Degasperis (1976) to describe "boomeron" and "trappon" solitons—special types of solitons with variable speeds and non-trivial interactions. Boomerons are solitons that reverse direction due to nonlinearity. Trappons are solitons that become trapped in a localized region, resembling bound states. For the analysis of the behavior of nonlinear waves in some physical systems, including optical media, fluids, and plasma, the (2+1)-dimensional ZE must be understood. Each term in Eq (1.1) determines various elements of wave motion.

- $(\frac{\varpi_{xy}}{\varpi})_{tt}$ describes wave dispersion, revealing how the wave profile alters over time.
- By presenting spatial diffusion, the expression $(\frac{\varpi_{xy}}{\varpi})_{xx}$ implies whether the wave widens or sharpens during propagation.
- The nonlinear term $2(\varpi^2)_{xt}$, connecting spatial and temporal fluctuations, illustrates energy exchange and wave interactions [33].

Several kinds of soliton solutions, such as bright, dark, singular, periodic singular, kink, and periodic solitons, have been produced by applying the MEDA technique. In addition to illuminating wave dynamics in higher-dimensional systems, plasma waves, and optical media, these solutions offer greater insights into the complex relationship between dispersion, diffusion, and nonlinearity.

Real-world applications

The (2+1)-dimensional ZE arises in mathematical physics to describe the behavior of nonlinear and dispersive waves in two spatial and one temporal dimensions. It models phenomena such as shallow water waves, ion-acoustic waves in plasma, and spatiotemporal optical solitons in nonlinear media. This equation is significant in fields like fluid dynamics, plasma physics, and nonlinear optics, where it helps capture complex wave interactions and localized structures such as solitons and breathers. Additionally, it serves as a useful model for testing analytical techniques in solving nonlinear partial differential equations. There are vital implications for these soliton solutions in many distinct fields:

- Bright solitons, which are distinguished by isolated peaks, are essential for optical fiber networks since they ensure fast data transfer with minimum signal deterioration.
- The study of quantum processes is greatly aided by the observation of dark solitons, which manifest as intensity dips in Bose-Einstein condensates [34].
- Singular solitons are used to forecast and mitigate extreme wave occurrences, including shock waves in plasma physics and rogue waves in oceanography [35].
- Periodic and kink solitons are used in condensed matter physics (e.g., domain walls in ferromagnetic materials) and biophysics (e.g., nerve impulse propagation) to characterize phase transitions and recurrent wave patterns.

In addition to their theoretical importance, these results have applications in the development of contemporary technology. Kink solitons are essential to magnetic storage developments, bright solitons boost optical communication networks [36], singular solitons are involved in maritime and aircraft safety, and dark solitons play a key role in all-optical signal processing in optical and quantum technologies. These uses highlight how nonlinear wave research has a wider impact on solving current scientific and engineering problems.

6. Stability analysis

To examine its stability small perturbations are inserted into the base solution. It assumes a constant equilibrium solution. Taking into account a constant equilibrium solution $\varpi_0 = a$, where a is a constant, a renowned steady-state solution $\varpi_0(x, y, t)$ is anticipated. A small perturbation $\gamma(x, y, t)$ is inserted as:

$$\varpi(x, y, t) = \varpi_0 + \rho\gamma(x, y, t), \tag{6.1}$$

where *rho* << 1 is a minimal parameter. The linearized equation is generated after putting Eq (6.1) into Eq (1.1) and extending terms up to the first order in ρ and ignoring higher-order terms $O(\rho^2)$.

$$(\frac{\gamma_{xy}}{\varpi_0})_{tt} - (\frac{\gamma_{xy}}{\varpi_0})_{xx} + 4(\varpi_0)\gamma_{xt} = 0. \tag{6.2}$$

As mentioned above ϖ_0 is a constant, and Eq (6.2) yields:

$$\gamma_{xytt} - \gamma_{xyxx} + 4\varpi_0\gamma_{xt} = 0. \tag{6.3}$$

Taking into account Eq (6.3) the initial solution is as follows:

$$\gamma(x, y, t) = \lambda \exp^{i(l_1 x + l_2 y - \upsilon t)}, \tag{6.4}$$

where λ is the perturbation amplitude, υ is the frequency, and l_1, l_2 are spatial wave numbers. Putting derivatives of Eq (6.4) into Eq (6.3) and simplifying yields

$$\upsilon = \frac{l_1^2 l_2}{l_1 l_2 - 4i\varpi_0 l_1}. \tag{6.5}$$

The imaginary part of υ must be non-positive in order to maintain stability.

- If $\mathrm{Im}(\upsilon) > 0$, the perturbation grows exponentially, representing instability.
- If $\mathrm{Im}(\upsilon) \leq 0$, the stable solution is obtained.

The term $-4i\varpi_0 l_1$ adds to an imaginary component when looking at the denominator $l_1 l_2 - 4i\varpi_0 l_1$. The key factor governing the stability of the perturbation is ϖ_0 which determines whether the perturbation grows or stays under control.

- If ϖ_0 is large and positive, the imaginary component of υ becomes positive, leading to exponential growth of perturbations and indicating instability in the system.
- If ϖ_0 is negative or sufficiently small, the imaginary component of υ does not contribute to unbounded growth, implying stability of the perturbations.
- While l_1 and l_2 shape the wave behavior, the dominant factor influencing stability is ϖ_0.

Consequently, conditional stability is demonstrated by Eq (1.1), which is stable for certain values of ϖ_0 but unstable for greater positive values. This finding raises the possibility that soliton-like structures in this model could become unstable in specific situations, especially when ϖ_0 is noticeably positive.

7. Conclusions

In this study, we successfully applied the MEDAM to construct a wide range of exact solutions for the (2+1)-dimensional ZE. Through a systematic algebraic approach and the use of appropriate wave transformations, we derived multiple families of soliton and wave solutions, including bright solitons, dark solitons, kink solitons, singular and periodic waves, as well as singular periodic and kink-periodic structures. These solutions reveal diverse nonlinear dynamics and exhibit distinct physical features, with detailed 2D, 3D, and contour plots demonstrating their spatial and temporal behaviors.

The novelty of our work lies in the breadth and complexity of the obtained solutions, many of which have not been previously reported in literature. The conditional stability analysis further supports the physical relevance and structural integrity of these solutions under small perturbations. In addition, we discussed potential real-world applications of these wave structures in nonlinear optics, plasma physics, and fluid dynamics, linking the theoretical findings to practical phenomena.

This research highlights the strength of the MEDAM in solving high-dimensional nonlinear PDEs and generating rich solution structures. Future work may explore the extension of this method to fractional, perturbed, or stochastic versions of the Zoomeron equation, study interactions among the obtained wave solutions, and validate the theoretical predictions through numerical simulations or experimental models.

Despite its effectiveness in generating diverse exact solutions for nonlinear partial differential equations, the MEDAM has several limitations. First, the method relies heavily on specific forms of wave transformations and auxiliary functions, which may not always be suitable for highly complex or strongly non-integrable systems. Its applicability is constrained to equations that can be reduced to solvable ordinary differential equations through such transformations. Additionally, the MEDAM typically assumes smooth and continuous solution structures, making it less effective in capturing discontinuous or shock-type solutions. The method also depends on symbolic computation software for solving resulting nonlinear algebraic systems, which can become computationally intensive and challenging for high-order equations or coupled systems. Moreover, while the MEDAM provides exact solutions, it offers limited insight into the global dynamics or long-time behavior of the solutions compared to numerical simulations. Lastly, the stability analysis performed through linear perturbation techniques is local in nature and may not fully account for nonlinear instabilities in broader parameter regimes.

Future research can extend this method to other nonlinear evolution equations and higher-dimensional or coupled systems to explore a broader range of wave dynamics. Further investigation into the stability, physical relevance, and potential real-world applications of the obtained solutions is essential. Incorporating perturbation or damping effects could enhance the physical realism of the models. Additionally, numerical simulations, parameter sensitivity analysis, and hybrid analytical-numerical approaches can be employed to validate and generalize the results. These directions will contribute to a deeper understanding of nonlinear wave phenomena across diverse scientific and engineering domains.

Author contributions

Maheen Waqar: Methodology, Investigation, Writing–original draft. **Khaled M. Saad**: Software, Investigation, Writing–review and editing **Muhammad Abbas**: Supervision, Writing–original draft, Writing–review and editing. **Miguel Vivas-Cortez**: Supervision, Writing–original draft, Writing–review and editing. **Waleed M. Hamanah**: Software, Investigation, Writing–review and editing. All authors have read and agreed to publish the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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