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Research article

An exploratory study on bivariate extended q-Laguerre-based Appell polynomials with some applications

Mohra Zayed[1,*], **Waseem Ahmad Khan**[2], **Cheon Seoung Ryoo**[3] **and Ugur Duran**[4]

[1] Mathematics Department, College of Science, King Khalid University, Abha 61413, Saudi Arabia
[2] Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Saudi Arabia
[3] Department of Mathematics, Hannam University, Daejeon 34430, South Korea
[4] Department of Basic Sciences of Engineering, Iskenderun Technical University, Hatay 31200, Turkiye

* **Correspondence:** Email: mzayed@kku.edu.sa.

Abstract: In this paper, we employed the q-Bessel Tricomi functions of zero-order to introduce bivariate extended q-Laguerre-based Appell polynomials. Then, the bivariate extended q-Laguerre-based Appell polynomials were established in the sense of quasi-monomiality. We examined some of their properties, such as q-multiplicative operator property, q-derivative operator property and two q-integro-differential equations. Additionally, we acquired q-differential equations and operational representations for the new polynomials. Moreover, we drew the zeros of the bivariate extended q-Laguerre-based Bernoulli and Euler polynomials, forming 2D and 3D structures, and provided a table including approximate zeros of the bivariate extended q-Laguerre-based Bernoulli and Euler polynomials.

Keywords: quasi monomiality; extension of monomiality principle; quantum calculus; q-Laguerre polynomials; q-Dilatation operator; q-Laguerre-based Appell polynomials
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1. Introduction

The French mathematician Edmond Nicolas Laguerre, born in 1885, is the name given to Laguerre polynomials, a system of complete and orthogonal polynomials with many mathematical properties and applications. There are multifarious ways to define them. Each definition emphasizes a different aspect and suggests extensions and connections to other mathematical structures in conjunction with the physical and numerical applications. Dattoli [9] investigated the theory of bivariate Laguerre polynomials and demonstrated that classical Laguerre polynomials can be expressed in terms of quasi-monomials. The significance of bivariate Laguerre polynomials lies in their mathematical properties, as they emerge in the treatment of radiation physics problems, including quantum beam lifetime in a storage ring and electromagnetic wave propagation [12] and are solutions to certain partial differential equations, such as the heat diffusion equation. The references [10, 11, 18] disclose detailed information on the Laguerre polynomials.

The q-calculus or quantum calculus, an extension of ordinary calculus, was created to investigate q-extensions of mathematical structures since the 18th century. One of the most important studied aspects of q-calculus is the q-special functions, which relate to this topic and serve as a connection between physics and mathematics. For mathematical physics, some q-special polynomials and functions have been framed and worked with the representations of quantum algebra [15]. Moreover, q-special polynomials have been considered, and many of their properties and applications have been given for a long time [1, 7, 16, 19–22]. Raza et al. [24] considered bivariate q-Hermite polynomials and explored several properties and applications. The bivariate q-Laguerre polynomials have been considered, and many applications, relations, and properties have been provided newly in [8]. For example, in [17], the theory of bivariate q-Laguerre polynomials by zeroth-order q-Bessel Tricomi functions was defined, and bivariate q-Laguerre polynomials from the context of quasi-monomiality were established. Then, q-integrodifferential equations and the operational representations for these polynomials were provided. In addition, mth-order bivariate q-Laguerre polynomials were introduced, and the quasi-monomiality characteristics of these polynomials were analyzed. Furthermore, in [17], several graphical representations of q-Laguerre polynomials were presented. In [25], bivariate q-Laguerre-Appell polynomials were considered by applying the q-monomiality principle methods, and their quasi-monomial properties and applications were studied and investigated. Several operational identities and quasi-monomial features were given. Furthermore, diverse q-differential equations of these polynomials were derived. As applications, utilizing the operational identity of the mentioned polynomials, specific presentations regarding several q-Laguerre-Appell polynomial families were drawn. Moreover, the family of q-Laguerre-Sheffer polynomials was introduced by an operational approach, and some of its fundamental properties were developed. In the current study, we implement the following notions of q-calculus.

The q-numbers and the q-factorial are respectively provided as follows

$$[\omega]_q = \frac{1-q^\omega}{1-q}, \quad 0<q<1, \quad \omega \in \mathbb{C} \tag{1.1}$$

and

$$[\omega]_q! = \begin{cases} \prod_{k=1}^{\omega} [k]_q\,, & 0<q<1, \quad \omega \geq 1 \\ 1, & \omega = 0. \end{cases} \tag{1.2}$$

The Gauss's q-binomial formula is provided as follows

$$(\zeta \pm a)_q^\omega = \sum_{k=0}^{\omega} \binom{\omega}{k}_q \zeta^k (\pm a)^{\omega-k} q^{\binom{\omega-k}{2}}. \tag{1.3}$$

For $0 < q < 1$, the two type of q-exponential functions are provided by [5, 13–15]

$$e_q(\zeta) = \sum_{\omega=0}^{\infty} \frac{\zeta^\omega}{[\omega]_q!}, \tag{1.4}$$

and

$$E_q(\zeta) = \sum_{\omega=0}^{\infty} \frac{q^{\binom{\omega}{2}}\zeta^\omega}{[\omega]_q!}, \tag{1.5}$$

which satisfy the following relations

$$e_q(\zeta)E_q(\eta) = \sum_{\omega=0}^{\infty} \frac{(\zeta+\eta)_q^\omega}{[\omega]_q!} \tag{1.6}$$

and

$$e_q(\zeta)E_q(-\zeta) = 1. \tag{1.7}$$

The q-derivative operator with respect to ζ is given as follows [5, 13–15]

$$\widehat{D}_{q,\zeta} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{\zeta\,(1-q)}, \quad 0 < q < 1, \quad \zeta \neq 0. \tag{1.8}$$

In particular, we have

$$\widehat{D}_{q,\zeta}\zeta^n = [n]_q \zeta^{n-1}, \tag{1.9}$$

and

$$\widehat{D}^k_{q,\zeta} e_q(\alpha\zeta) = \alpha^k e_q(\alpha\zeta), \ \ k \in \mathbb{N}, \ \alpha \in \mathbb{C}, \tag{1.10}$$

where $\widehat{D}^k_{q,\zeta}$ denotes the k^{th} order q-derivative operator.

It is noted that [17, 23]

$$\widehat{D}_{q,\zeta}(f(\zeta)g(\zeta)) = f(\zeta)\widehat{D}_{q,\zeta} g(\zeta) + g(q\zeta)\widehat{D}_{q,\zeta} f(\zeta). \tag{1.11}$$

Recently, Cao et al. [8] introduced the m^{th} order bivariate q-Laguerre polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ are considered as follows

$$C_{0,q}(-\xi\psi^m)e_q(\eta\psi) = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}, \tag{1.12}$$

the series definition, we have

$${}_{[m]}L_{\omega,q}(\xi,\eta) = [\omega]_q! \sum_{\phi=0}^{[\frac{\omega}{m}]} \frac{(-1)^\phi \xi^\phi \eta^{\omega-\phi}}{([\phi]_q!)^2[\omega-mk]_q!}, \tag{1.13}$$

where the symbol $C_{0,q}(\xi)$ is 0^{th} order q-Bessel Tricomi function defined by [8]:

$$C_{0,q}(\xi\psi) = e_q(-D^{-1}_{q,\xi}\psi)\{1\}, \tag{1.14}$$

also has the following series representation

$$C_{0,q}(\xi) = \sum_{\phi=0}^{\infty} \frac{(-1)^\phi \xi^\phi}{([\phi]_q!)^2}. \tag{1.15}$$

For all values of ξ, this series converges absolutely.

Equation (1.12) can be written as

$$e_q(\widehat{D}_{q,\xi}^{-1}\psi^m)e_q(\eta\psi)\{1\} = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{1.16}$$

It is noted that [8, 25]:

$$\widehat{D}_{q,\xi}^{-1}f(\xi) := \int_0^\xi f(\xi)d_q\xi, \tag{1.17}$$

which is the definite q-integral. Particularly, $\widehat{D}_{q,\xi}^{-1}\{1\} = \xi$, and for $r \in \mathbb{N}$ we have

$$\left(\widehat{D}_{q,\xi}^{-1}\right)^r\{1\} = \frac{\xi^r}{[r]_q!}, \qquad r \in \mathbb{N}. \tag{1.18}$$

The q-dilation operator T_ω performs on any function in the following form [6]

$$\mathrm{T}_\omega^\phi f(u) = f(q^\phi u), \qquad \phi \in \mathbb{R}, \tag{1.19}$$

satisfies the property

$$\mathrm{T}_u^{-1}\mathrm{T}_u^1 f(u) = f(u). \tag{1.20}$$

The following q-derivative rule is valid [8, 25]

$$\widehat{D}_{q,\psi}e_q(\xi\psi^m) = \xi\psi^{m-1}\mathrm{T}_{(\xi;m)}e_q(\xi\psi^m), \tag{1.21}$$

where

$$\mathrm{T}_{(\xi;m)} = \frac{1 - q^m\mathrm{T}_\xi^m}{1 - q\mathrm{T}_\xi} = 1 + q\mathrm{T}_\xi + \cdots + q^{m-1}\mathrm{T}_\xi^{m-1}. \tag{1.22}$$

In this paper, motivated by these potential uses, our presentation and examination of bivariate q-Laguerre-Appell polynomials involved using q-extension of the monomiality principle to explore their unique features. In addition, we present the usage of these newly discovered q-Laguerre-Appell polynomials to demonstrate their geometrical representations. Overall, our findings suggest that q-Bessel functions and q-Laguerre-Appell polynomials have promising applications in various fields.

2. Bivariate extended q-Laguerre-based Appell polynomials

In this part, we consider bivariate extended q-Laguerre-based Appell polynomials 2VgqLAP ${}_LA_{\omega,q}(\xi,\eta;m)$ utilizing the function in (1.20) and derive several properties and relations such as q-integro-differential equations, series definition of them, operational identities and q-quasi-monomiality characteristics.

The q-Appell polynomials $\mathcal{A}_{n,q}(\zeta)$ have been studied and examined since their construction by Al-Salam [2, 3] and they are defined as follows:

$$\mathcal{A}_q(\psi)\, e_q(\zeta\psi) = \sum_{\omega=0}^{\infty} \mathcal{A}_{\omega,q}(\zeta)\frac{\psi^\omega}{[\omega]_q!} \tag{2.1}$$

with

$$\mathcal{A}_q(\psi) = \sum_{\omega=0}^{\infty} \mathcal{A}_{\omega,q} \frac{\psi^\omega}{[\omega]_q!}, \qquad \mathcal{A}_q(t) \neq 0, \qquad A_{0,q} = 1. \tag{2.2}$$

Zayed et al. [25] defined the generalized bivariate q-Laguerre polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ as follows

$$C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{2.3}$$

Considering the relations (2.1) and (2.2), we introduce bivariate extended q-Laguerre-based Appell polynomials 2VgqLAP ${}_LA_{\omega,q}(\xi,\eta;m)$ as follows

$$A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} {}_LA_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!}, \tag{2.4}$$

where the function $C_{0,q}(\xi\psi)$ is defined in (1.15).

Remark 2.1. *When $\xi = 0$, we have ${}_LA_{\omega,q}(0,\eta;m) := A_{\omega,q}(\eta;m)$ called the extended q-Appell polynomials. When $m = 1$, we have ${}_LA_{\omega,q}(\xi,\eta;1) := {}_LA_{\omega,q}(\xi,\eta)$ called the bivariate q-Laguerre-based Appell polynomials [17]. Also, when $\xi = 0$ and $m = 1$, we have ${}_LA_{\omega,q}(0,\eta;1) := A_{\omega,q}(\eta)$ called the q-Appell polynomials [2, 3].*

It can be seen from (1.16) and (2.4) that

$$e_q(-\widehat{D}_{q,\zeta}^{-1}\psi)\left\{A_{\omega,q}(\eta;m)\right\} = \sum_{\omega=0}^{\infty} {}_LA_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!}. \tag{2.5}$$

In [8, 15], the following relation is examined:

$$\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}C_{0,q}(\alpha\xi) = \frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}C_{0,q}(\alpha\xi) = -\alpha C_{0,q}(\alpha\xi). \tag{2.6}$$

The following q-derivative property can be given using (1.13):

$$\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}f(\xi) = (q\widehat{D}_{q,\xi} + \xi\widehat{D}_{q,\xi}^2)f(\xi). \tag{2.7}$$

We first provide the following identities.

Theorem 2.2. *The following operational identities of the new polynomials 2VgqLAP ${}_LA_{\omega,q}(\xi,\eta;m)$ are valid:*

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^m {}_LA_{\omega,q}(\xi,\eta;m) = \widehat{D}_{q,\eta}{}_LA_{\omega,q}(\xi,\eta;m), \tag{2.8}$$

and

$${}_LA_{\omega,q}(\xi,0) = [\omega]_q! \sum_{\phi=0}^{[\frac{\omega}{m}]} q^{\binom{\phi}{2}}{}_LA_{\omega-m\phi,q}(\xi,\eta;m)\frac{\eta^\phi}{[\phi]_q![\omega-m\phi]_q!}. \tag{2.9}$$

Proof. From Eq (2.6), we have

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^m A_q(\psi)e_q(\psi\widehat{D}_{q,\xi}^{-1})e_q(\eta\psi^m) = \psi^m A_q(\psi)C_{0,q}(-\xi\psi)e_q(\eta\psi^m). \tag{2.10}$$

Also, from Eq (1.10), we have

$$\widehat{D}_{q,\eta}^m A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \psi^m A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m). \tag{2.11}$$

From Eqs (2.10) and (2.11), we have

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^m A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \widehat{D}_{q,\eta}^m A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m), \tag{2.12}$$

which means the assertion (2.8).

Since, in view of the Eq (1.7), we have

$$E_q(-\eta\psi^m)\sum_{\omega=0}^{\infty} {}_LA_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!} = A_q(\psi)C_{0,q}(\xi\psi), \tag{2.13}$$

which means that

$$\sum_{\omega=0}^{\infty}\sum_{\phi=0}^{\infty}\frac{q^{\binom{\phi}{2}}{}_LA_{\omega,q}(\xi,\eta;m)\eta^\phi\psi^{\omega+m\phi}}{[\phi]_q![\omega]_q!} = \sum_{\omega=0}^{\infty}{}_LA_{\omega,q}(\xi,0)\frac{\psi^\omega}{[\omega]_q!}$$

$$\sum_{\omega=0}^{\infty}\left(\sum_{\phi=0}^{[\frac{\omega}{m}]}\frac{q^{\binom{\phi}{2}}{}_LA_{\omega-m\phi,q}(\xi,\eta)\eta^\phi\psi^\omega}{[\phi]_q![\omega-m\phi]_q!}\right) = \sum_{\omega=0}^{\infty}{}_LA_{\omega,q}(\xi,0)\frac{\psi^\omega}{[\omega]_q!}, \tag{2.14}$$

which gives the assertion (2.9). So, we complete the proofs of the theorem. □

Here we provide two different representations of the formula (2.3) as follows.

Remark 2.3. *The formula (2.4) can be rewritten using (2.6) as follows:*

$$\left(-\frac{\partial_q}{\partial_q D_{q,\xi}^{-1}}\right)^m {}_LA_{\omega,q}(\xi,\eta;m) = \widehat{D}_{q,\eta}\,{}_LA_{\omega,q}(\xi,\eta;m). \tag{2.15}$$

Remark 2.4. *The formula (2.4) can be rewritten using (2.7) as follows:*

$$\left(-\left(q\widehat{D}_{q,\xi}+\xi\widehat{D}_{q,\xi}^2\right)\right)^m {}_LA_{\omega,q}(\xi,\eta;m) = \widehat{D}_{q,\eta}\,{}_LA_{\omega,q}(\xi,\eta;m). \tag{2.16}$$

Theorem 2.5. *We have*

$${}_LA_{\omega,q}(\xi,\eta;m) = [\omega]_q!\sum_{k=0}^{\omega}\frac{\zeta^k(-1)^kA_{\omega-k,q}(\xi,\eta;m)}{([k]_q!)^2[\omega-k]_q!}. \tag{2.17}$$

Proof. We can write Eq (2.4) in operational form

$$\sum_{\omega=0}^{\infty}{}_LA_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!} = e_q\left(-D_{q,\xi}^{-1}t\right)\left\{A_{\omega,q}(\eta;m)\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-D_{q,\xi}^{-1})^k \psi^k}{[k]_q!} \left\{A_{\omega,q}(\eta; m)\right\}$$

$$= \sum_{\omega=0}^{\infty} \sum_{k=0}^{\omega} \frac{(-1)^k \zeta^k A_{\omega-k,q}(\xi, \eta; m)[\omega]_q!}{([k]_q!)^2 [\omega-k]_q!} \frac{\psi^\omega}{[\omega]_q!}, \tag{2.18}$$

which implies the desired result (2.17). □

Theorem 2.6. *The following summation formula is valid:*

$$_LA_{\omega,q}(\xi, \eta; m) = \sum_{k=0}^{\omega} \binom{\omega}{k}_q {}_{[m]}L_{\omega-k,q}(\xi, \eta) A_{k,q}, \tag{2.19}$$

where $A_{k,q}$ is given by Eq (2.2).

Proof. It is observed from (2.2) and (2.3) that

$$\sum_{\omega=0}^{\infty} {}_LA_{\omega,q}(\xi, \eta; m) \frac{\psi^\omega}{[\omega]_q!} = A_q(\psi) \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi, \eta) \frac{\psi^\omega}{[\omega]_q!}. \tag{2.20}$$

So, the assertion (2.19) can be obtained utilizing (2.2) and (2.20). □

By using a similar approach given in [25] and utilizing (2.4), the following determinant format for ${}_LA_{\omega,q}(\xi, \eta; m)$ is obtained.

Theorem 2.7. *The determinant representation of bivariate extended q-Laguerre-based Appell polynomials (q-LbAP)${}_LA_{\omega,q}(\xi, \eta; m)$ of degree n is*

$$_LA_{0,q}(\xi, \eta; m) = \frac{1}{\beta_{0,q}},$$

$$_LA_{w,q}(\xi, \eta; m) = \frac{(-1)^w}{(\beta_{0,q})^{w+1}} \begin{vmatrix} 1 & {}_{[m]}L_{1,q}(\xi,\eta) & {}_{[m]}L_{2,q}(\xi,\eta) & \cdots & {}_{[m]}L_{w-1,q}(\xi,\eta) & {}_{[m]}L_{w,q}(\xi,\eta) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{w-1,q} & \beta_{w,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_q \beta_{1,q} & \cdots & \binom{w-1}{1}_q \beta_{w-2,q} & \binom{w}{1}_q \beta_{w-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \binom{w-1}{1}_q \beta_{w-3,q} & \binom{w}{2}_q \beta_{w-2,q} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{w}{w-1}_q \beta_{1,q} \end{vmatrix}, \tag{2.21}$$

where

$$\beta_{w,q} = -\frac{1}{A_{0,q}} \left(\sum_{k=1}^{\omega} \binom{\omega}{k}_q A_{k,q} \beta_{\omega-k,q} \right), \quad \omega = 1, 2, 3, \ldots,$$

with $\beta_{0,q} = \frac{1}{A_{0,q}}$ and ${}_{[m]}L_{\omega,q}(\xi, \eta)$, $\omega = 0, 1, 2, \ldots$, are the polynomials in (2.3).

Proof. By substituting the series representations of the bivariate extended q-Laguerre polynomials into the generating function of the generalized bivariate q-Laguerre-based Appell polynomials, we derive

$$A_q(\psi) \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi, \eta) \frac{\psi^\omega}{[\omega]_q!} = \sum_{\omega=0}^{\infty} {}_LA_{\omega,q}(\xi, \eta; m) \frac{\psi^\omega}{[\omega]_q!}. \tag{2.22}$$

Multiplying both sides by

$$\frac{1}{A_q(\psi)} = \sum_{\gamma=0}^{\infty} \beta_{\gamma,q} \frac{\psi^{\gamma}}{[\gamma]_q!}, \tag{2.23}$$

we obtain

$$\sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta) \frac{\psi^{\omega}}{[\omega]_q!} = \sum_{\gamma=0}^{\infty} \beta_{\gamma,q} \frac{\psi^{\gamma}}{[\gamma]_q!} \sum_{\omega=0}^{\infty} {}_{L}A_{\omega,q}(\xi,\eta;m) \frac{\psi^{\omega}}{[\omega]_q!}. \tag{2.24}$$

Applying Cauchy product in (2.24) gives

$$_{[m]}L_{\omega,q}(\xi,\eta) = \sum_{\gamma=0}^{\omega} \binom{\omega}{\gamma}_q \beta_{\gamma,q} {}_{L}A_{\omega-\gamma,q}(\xi,\eta;m). \tag{2.25}$$

This equality leads to a system of ω-equations with unknown ${}_{L}A_{\omega,q}(\xi,\eta;m), \quad \omega = 0,1,2,\cdots$. To solve this system, we employ Cramer's rule, leveraging the fact that the denominator corresponds to the determinant of a lower triangular matrix, which simplifies to $\left(\beta_{0,q}\right)^{\omega+1}$. By transposing the numerator and systematically shifting the i^{th} row by $(i+1)-th$ position, for $i = 1,2,\cdots,\omega-1$, we obtain the desired result in a structured and computationally efficient manner. □

3. Quasi-monomiality characteristics

In studying special functions and polynomials, along with their relations and properties, the monomiality principle is a valuable tool [1, 11, 12]. Steffensen made the initial proposal for this concept in the early 19th century, which was later refuted and established by Dattoli in 1996. Using the monomiality principle, novel hybrid special polynomials were introduced and studied by researchers in recent times [17]. The monomiality principle was extended to q-special polynomials by Cao et al. [8], which could lead to the creation of novel families of q-special polynomials and show that some q-special polynomials are quasi-monomial. Understanding special polynomials as specific solutions to extended versions of partial differential equations and integral differential equations can provide a framework by extension of the monomiality principle. The use of procedures specific to q operations can lead to the generation of many more classes of q-generating functions and various extensions of q-special polynomials. The q-operational process, one of these procedures, is more compatible with the techniques and typical mathematical tools employed to investigate solutions for q-differential equations.

For a q-polynomial set $p_{\omega,q}(\zeta)(\zeta \in \mathbb{C}, \omega \in \mathbb{N})$, the q-multiplicative operator denoted by $\widehat{M}_q$ and q-derivative operators denoted by $\widehat{P}_q$, are provided by [8]

$$\widehat{M}_q\{p_{\omega,q}(\zeta)\} = p_{\omega+1,q}(\zeta), \tag{3.1}$$

and

$$\widehat{P}_q\{p_{\omega,q}(\zeta)\} = [\omega]_q p_{\omega-1,q}(\zeta). \tag{3.2}$$

The following commutation relation is valid:

$$[\widehat{M}_q, \widehat{P}_q] = \widehat{P}_q\widehat{M}_q - \widehat{M}_q\widehat{P}_q. \tag{3.3}$$

The following properties hold for a q-polynomial set $p_{n,q}(\zeta)$:

$$\widehat{M}_q\widehat{P}_q\{p_{\omega,q}(\zeta)\} = [\omega]_q p_{\omega,q}(\zeta), \tag{3.4}$$

and

$$\widehat{P}_q\widehat{M}_q\{p_{\omega,q}(\zeta)\} = [\omega+1]_q p_{\omega,q}(\zeta). \tag{3.5}$$

It can be derived from (3.1)–(3.3) that

$$[\widehat{M}_q, \widehat{P}_q] = [\omega+1]_q - [\omega]_q \tag{3.6}$$

and

$$\widehat{M}_q^{\,r}\{p_{\omega,q}\} = p_{\omega+r,q}(\zeta). \tag{3.7}$$

In particular, we have

$$p_{\omega,q}(\zeta) = \widehat{M}_q^{\,\omega}\{p_{0,q}\} = \widehat{M}_q^{\,\omega}\{1\}, \tag{3.8}$$

where $p_{0,q}(\zeta) = 1$ is the q-sequel of polynomial $p_{\omega,q}(\zeta)$ provided by

$$e_q(\widehat{M}_q\psi)\{1\} = \sum_{\omega=0}^{\infty} p_{\omega,q}(\zeta)\frac{\psi^\omega}{[\omega]_q!}. \tag{3.9}$$

Now, we determine the quasi-monomial characteristics of the bivariate extended q-Laguerre-Appell polynomials ${}_LA_{\omega,q}(\xi,\eta;m)$ as follows.

Theorem 3.1. *The (2VGqLAP) ${}_LA_{\omega,q}(\xi,\eta;m)$ are quasi-monomials via the following q-multiplicative and q-derivative operators:*

$$\widehat{M}_{G2VqLAP} = \left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)^{m-1}\mathrm{T}_{(\eta;m)}T_\xi - \widehat{D}^{-1}_{q,\xi}\right)\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)} + \frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}, \tag{3.10}$$

or, equivalently

$$\widehat{M}_{G2VqLAP} = \left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)^{m-1}\mathrm{T}_{(\eta;m)} - \widehat{D}^{-1}_{q,\xi}T_\eta\right)\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)} + \frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}, \tag{3.11}$$

and

$$\widehat{P}_{G2VqLAP} = -\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}, \tag{3.12}$$

respectively.

Proof. Utilizing (1.11) and applying partial-q-derivative operator to the both sides of (2.4) with respect to ψ, by taking $f_q(\psi) = e_q(\eta\psi^m)$, and $g_q(\psi) = A_q(\psi)e_q(-\widehat{D}^{-1}_{q,\xi}\psi)$, we derive that

$$\sum_{\omega=1}^{\infty} {}_{L}A_{\omega,q}(\xi,\eta;m)\widehat{D}_{q,\psi}\frac{\psi^{\omega}}{[\omega]_q!} = A_q(q\psi)e_q(-\widehat{D}_{q,\xi}^{-1}q\psi)\widehat{D}_{q,\psi}(e_q(\eta\psi^m)) + e_q(\eta\psi^m)\widehat{D}_{q,\psi}\left(A_q(\psi)e_q(-\widehat{D}_{q,\xi}^{-1}\psi)\right), \tag{3.13}$$

which, on using Eq (1.11) by taking $f_q(\psi) = e_q(-\widehat{D}_{q,\xi}^{-1}\psi)$ and $g_q(\psi) = A_q(\psi)$ and then simplifying the resultant equation by using the Eqs (1.11), (1.19) and (1.21) on the right hand side, we have

$$\left(\left(\eta\psi^{m-1}\mathrm{T}_{(\eta;m)}\mathrm{T}_{\xi} - \widehat{D}_{q,\xi}^{-1}\right)\frac{A_q(q\psi)}{A_q(\psi)} + \frac{A_q'(\psi)}{A_q(\psi)}\right)A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m)$$

$$= \sum_{\omega=1}^{\infty} {}_{L}A_{\omega,q}(\xi,\eta;m)\frac{\psi^{\omega-1}}{[\omega-1]_q!}. \tag{3.14}$$

Therefore, by using Eqs (2.4) and (3.14), it gives

$$\sum_{\omega=0}^{\infty}\left(\left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{(\eta;m)}T_{\xi} - \widehat{D}_{q,\xi}^{-1}\right)\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)} + \frac{A_q'\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}\right){}_{L}A_{\omega,q}(\xi,\eta;m)\frac{\psi^{\omega}}{[\omega]_q!}$$

$$= \sum_{\omega=0}^{\infty} {}_{L}A_{\omega+1,q}(\xi,\eta;m)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{3.15}$$

which on using (3.15), we get

$${}_{L}A_{\omega+1,q}(\xi,\eta;m) = \left(\left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{(\eta;m)}T_{\xi} - \widehat{D}_{q,\xi}^{-1}\right)\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)} + \frac{A_q'\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}\right){}_{L}A_{\omega,q}(\xi,\eta;m) \tag{3.16}$$

which, in accordance with (3.1), we attain the assertion (3.10). Again, by utilizing Eq (3.13), by taking $f_q(\psi) = e_q(-\widehat{D}_{q,\xi}^{-1}\psi)$, and $g_q(\psi) = A_q(\psi)e_q(\eta\psi^m)$, and following the same proof of (3.10), we obtain

$${}_{L}A_{\omega+1,q}(\xi,\eta;m) = \left(\left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{(\eta;m)} - \widehat{D}_{q,\xi}^{-1}T_{\eta}\right)\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)} + \frac{A_q'\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)}\right){}_{L}A_{\omega,q}(\xi,\eta;m) \tag{3.17}$$

which, in accordance with (3.1), we attain the assertion (3.11). Operating $\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}$ on both sides of Eq (2.4) and using Eq (2.6), we have

$$\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty}\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}{}_{L}A_{\omega,q}(\xi,\eta;m)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{3.18}$$

or,

$$-\psi A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty}\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}{}_{L}A_{\omega,q}(\xi,\eta;m)\frac{\psi^{\omega}}{[\omega]_q!} = \sum_{\omega=0}^{\infty}\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}{}_{L}A_{\omega,q}(\xi,\eta;m)\frac{\psi^{\omega}}{[\omega]_q!}. \tag{3.19}$$

Using Eq (2.4) and comparing the coefficients of ψ on both sides of Eq (3.19), we have

$$-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi L}A_{\omega,q}(\xi,\eta;m) = -\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}{}_LA_{\omega,q}(\xi,\eta;m) = [\omega]_{qL}A_{\omega-1,q}(\xi,\eta;m), \tag{3.20}$$

which in view of Eqs (2.6) and (3.2), gives assertion (3.12). Therefore, we complete the proof. □

Remark 3.2. *In the special case $m = 1$, the operators in Theorem 3.1 become the q-multiplication operator and q-derivative operator for ${}_LA_{\omega,q}(\xi,\eta)$, cf. [17].*

Remark 3.3. *Since*

$$\widehat{D}_{q,\eta}A_q(\psi)e_q(-\widehat{D}^{-1}_{q,\xi}\psi)e_q(\eta\psi^m) = \left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)^m A_q(\psi)C_{0,q}(\xi\psi)e_q(\eta\psi^m), \tag{3.21}$$

therefore for $m = 1$, Eq (2.4) gives

$$\widehat{D}_{q,\eta}\left\{{}_LA_{\omega,q}(\xi,\eta;1)\right\} = -\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\left\{{}_LA_{\omega,q}(\xi,\eta;1)\right\}. \tag{3.22}$$

Now, the q-integro-differential equations for the bivariate extended q-Laguerre Appell polynomials ${}_LA_{\omega,q}(\xi,\eta;m)$ are provided as follows.

Theorem 3.4. *The following q-integro-differential equations for ${}_LA_{\omega,q}(\xi,\eta;m)$ are valid:*

$$q\int_0^{\xi}\widehat{D}_{q,u}\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{}_LA_{\omega,q}(u,\eta;m)d_qu + \int_0^{\xi}u\widehat{D}^2_{q,u}\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{}_LA_{\omega,q}(u,\eta;m)d_qu$$

$$= \left(q\eta\frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}\widehat{D}_{q,\xi}\mathrm{T}_{\xi}\mathrm{T}_{(\eta;m)} + \xi\eta\frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}\widehat{D}^2_{q,\xi}\mathrm{T}_{\xi}\mathrm{T}_{(\eta;m)} - [\omega]_q\right){}_LA_{\omega,q}(\xi,\eta;m), \tag{3.23}$$

or, alternatively

$$q\int_0^{\xi}\widehat{D}_{q,u}T_{\eta}\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{}_LA_{\omega,q}(u,\eta;m)d_qu + \int_0^{\xi}u\widehat{D}^2_{q,u}T_{\eta}\frac{A_q\left(-q\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{}_LA_{\omega,q}(u,\eta;m)d_qu$$

$$= \left(q\eta\frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}\widehat{D}_{q,\xi}\mathrm{T}_{(\eta;m)} + \xi\eta\frac{A'_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}{A_q\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)}\widehat{D}^2_{q,\xi}\mathrm{T}_{(\eta;m)} - [\omega]_q\right){}_LA_{\omega,q}(\xi,\eta;m). \tag{3.24}$$

Proof. In view of Eqs (2.6), (2.7), (3.4), (3.10)–(3.12), the results are quickly established, so we do not need to include the details.

□

4. Some examples

Choosing an appropriate function $A_q(\psi)$ will result in a diverse Appell polynomial family that spans various members. The characteristics of each member are distinct, including their names, associated numerical properties, and generating functions. The versatility and rich properties of these polynomials enable them to be used in many mathematical domains. The specific polynomial within the family can be defined through the selection of $A_q(\psi)$, which can be customized for specific problems in mathematics and physics. To efficiently compute and analyze these polynomials, understanding the generating functions related to them is essential. In this part, we explore the details of the generating functions that support the diverse range of Appell polynomials, revealing their mathematical style and practical importance in a wide range of applications. The generating functions for the q-Bernoulli polynomials $B_{\omega,q}(\zeta)$, q-Euler polynomials $E_{\omega,q}(\zeta)$ and q-Genocchi polynomials $G_{\omega,q}(\zeta)$ are provided as follows [1, 3, 13, 17]

$$\frac{\psi}{e_q(\psi)-1}e_q(\zeta\psi)=\sum_{\omega=0}^{\infty}\mathbb{B}_{\omega,q}(\zeta)\frac{\psi^\omega}{[\omega]_q!},\quad |\,\psi\,|<2\pi,$$

$$\frac{2}{e_q(\psi)+1}e_q(\zeta\psi)=\sum_{\omega=0}^{\infty}\mathbb{E}_{\omega,q}(\zeta)\frac{\psi^\omega}{[\omega]_q!},\quad |\,\psi\,|<\pi,$$

and

$$\frac{2\psi}{e_q(\psi)+1}e_q(\zeta\psi)=\sum_{\omega=0}^{\infty}\mathbb{G}_{\omega,q}(\zeta)\frac{\psi^\omega}{[\omega]_q!},\quad |\,\psi\,|<\pi.$$

For $q\to 1$, these polynomials reduce to the usual Bernoulli, Euler and Genocchi polynomials (see [4, 18]).

Numerous fields of mathematics, like combinatorics, numerical analysis, and number theory, have relied heavily on Bernoulli polynomials and numbers, as well as Euler and Genocchi numbers and polynomials. Mathematicians can solve problems and investigate mathematical formulas through the practical applications of these polynomials and numbers. The sums of powers of natural numbers and the hyperbolic and trigonometric cotangent and tangent functions are just two instances of mathematical relations that include Bernoulli numbers. The role they play in number theory is crucial because they provide insights into patterns and relationships among integers. In the same way, the Euler numbers are closely linked to hyperbolic and trigonometric secant functions. Their applications aid in the analysis of structures and patterns in discrete mathematics in automata theory, graph theory, and the calculation of the number of up-down ascending sequences. Genocchi numbers can be advantageous in graph and automata theory, particularly when it comes to determining the number of ascending sequences, which involves analyzing the order and arrangement of elements in a sequence. Hence, the q-polynomials and numbers of Bernoulli, Euler, and Genocchi are of significant importance in miscellaneous mathematical disciplines, permitting the investigation of mathematical relationships, formulas, patterns, and structures.

From (2.4), the bivariate extended q-Laguerre-based Bernoulli ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)$, Euler ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)$, and Genocchi ${}_L\mathbb{G}_{\omega,q}(\xi,\eta;m)$ polynomials are provided as follows:

$$\frac{\psi}{e_q(\psi)-1}C_{0,q}(\xi\psi)e_q(\eta\psi^m)=\sum_{\omega=0}^{\infty}{}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!},\tag{4.1}$$

$$\frac{2}{e_q(\psi)+1}C_{0,q}(\xi\psi)e_q(\eta\psi^m)=\sum_{\omega=0}^{\infty}{}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!},\tag{4.2}$$

and

$$\frac{2\psi}{e_q(\psi)+1}C_{0,q}(\xi\psi)e_q(\eta\psi^m)=\sum_{\omega=0}^{\infty}{}_L\mathbb{G}_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!}.\tag{4.3}$$

Further, in view of expression (2.19), the polynomials ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)$, ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)$ and ${}_L\mathbb{G}_{\omega,q}(\xi,\eta;m)$ satisfy the following explicit form:

$${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)=\sum_{k=0}^{\omega}\binom{\omega}{k}_q\mathbb{B}_{k,q}\ {}_{[m]}L_{\omega-k,q}(\xi,\eta),\tag{4.4}$$

$${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)=\sum_{k=0}^{\omega}\binom{\omega}{k}_q\mathbb{E}_{k,q}\ {}_{[m]}L_{\omega-k,q}(\xi,\eta),\tag{4.5}$$

and

$${}_L\mathbb{G}_{\omega,q}(\xi,\eta;m)=\sum_{k=0}^{\omega}\binom{\omega}{k}_q\mathbb{G}_{k,q}\ {}_{[m]}L_{\omega-k,q}(\xi,\eta).\tag{4.6}$$

Furthermore, in view of expressions (2.21), the polynomials ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)$, ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)$ and ${}_L\mathbb{G}_{\omega,q}(\xi,\eta;m)$ satisfy the following determinant representations:

$${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)=(-1)^\omega\begin{vmatrix}1 & {}_{[m]}L_{1,q}(\xi,\eta) & {}_{[m]}L_{2,q}(\xi,\eta) & \cdots & {}_{[m]}L_{\omega-1,q}(\xi,\eta) & {}_{[m]}L_{\omega,q}(\xi,\eta)\\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[\omega]_q} & \frac{1}{[\omega+1]_q}\\ 0 & 1 & \binom{2}{1}_q\frac{1}{[2]_q} & \cdots & \binom{\omega-1}{1}_q\frac{1}{[\omega-1]_q} & \binom{\omega}{1}_q\frac{1}{[\omega]_q}\\ 0 & 0 & 1 & \cdots & \binom{\omega-1}{2}_q\frac{1}{[\omega-2]_q} & \binom{\omega}{2}_q\frac{1}{[\omega-1]_q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \binom{\omega}{\omega-1}_q\frac{1}{[2]_q}\end{vmatrix},\tag{4.7}$$

$$
{}_L\mathbb{E}_{\omega,q}(\xi,\eta;m) = (-1)^{\omega} \begin{vmatrix} 1 & {}_{[m]}L_{1,q}(\xi,\eta) & {}_{[m]}L_{2,q}(\xi,\eta) & \cdots & {}_{[m]}L_{\omega-1,q}(\xi,\eta) & {}_{[m]}L_{\omega,q}(\xi,\eta) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}_q \frac{1}{2} & \cdots & \binom{\omega-1}{1}_q \frac{1}{2} & \binom{\omega}{1}_q \frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{\omega-1}{2}_q \frac{1}{2} & \binom{\omega}{2}_q \frac{1}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{\omega}{\omega-1}_q \frac{1}{2} \end{vmatrix}, \tag{4.8}
$$

and

$$
{}_L\mathbb{G}_{\omega,q}(\xi,\eta;m) = (-1)^{\omega} \begin{vmatrix} 1 & {}_{[m]}L_{1,q}(\xi,\eta) & {}_{[m]}L_{2,q}(\xi,\eta) & \cdots & {}_{[m]}L_{\omega-1,q}(\xi,\eta) & {}_{[m]}L_{\omega,q}(\xi,\eta) \\ 1 & \frac{1}{2}\frac{1}{[2]_q} & \frac{1}{2}\frac{1}{[3]_q} & \cdots & \frac{1}{2}\frac{1}{[\omega]_q} & \frac{1}{2}\frac{1}{[\omega+1]_q} \\ 0 & 1 & \binom{2}{1}_q \frac{1}{2}\frac{1}{[2]_q} & \cdots & \binom{\omega-1}{1}_q \frac{1}{2}\frac{1}{[\omega-1]_q} & \binom{\omega}{1}_q \frac{1}{2}\frac{1}{[\omega]_q} \\ 0 & 0 & 1 & \cdots & \binom{\omega-1}{2}_q \frac{1}{2}\frac{1}{[\omega-2]_q} & \binom{\omega}{2}_q \frac{1}{2}\frac{1}{[\omega-1]_q} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{\omega}{\omega-1}_q \frac{1}{2}\frac{1}{[2]_q} \end{vmatrix}. \tag{4.9}
$$

5. Distribution of zeros and graphical representations

In this section, we demonstrate how numerical analysis can be employed to confirm theoretical predictions and uncover new and interesting patterns in the zeros of certain members of a recently introduced hybrid polynomial family. Specifically, we utilize computational methods to explore the "scattering" of the zeros of the bivariate extended q-Laguerre-Appell polynomials, denoted as ${}_LA_{\omega,q}(\xi,\eta;m)$, within the complex plane, which is a fascinating phenomenon to observe.

The bivariate extended q-Laguerre-based Bernoulli ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)$ polynomials are considered as follows:

$$
\frac{\psi C_{0,q}(\xi\psi)}{e_q(\psi)-1} e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} {}_L\mathbb{B}_{\omega,q}(\xi,\eta;m) \frac{\psi^{\omega}}{[\omega]_q!}. \tag{5.1}
$$

A few of them are

$$
\begin{aligned}
{}_L\mathbb{B}_{0,q}(\xi,\eta;5) &= 1,\\
{}_L\mathbb{B}_{1,q}(\xi,\eta;5) &= -\xi-\frac{1}{[2]_q!},\\
{}_L\mathbb{B}_{2,q}(\xi,\eta;5) &= \xi+\frac{1}{[2]_q!}+\frac{\xi^2}{[2]_q!}-\frac{[2]_q!}{[3]_q!},\\
{}_L\mathbb{B}_{3,q}(\xi,\eta;5) &= \xi+\frac{2}{[2]_q!}-\frac{\xi^3}{[3]_q!}-\frac{[3]_q!}{[2]_q!^3}-\frac{\xi^2[3]_q!}{[2]_q!^3}-\frac{\xi[3]_q!}{[2]_q!^2}-\frac{[3]_q!}{[4]_q!},\\
{}_L\mathbb{B}_{4,q}(\xi,\eta;5) &= \xi+\frac{2}{[2]_q!}+\frac{\xi^4}{[4]_q!}+\frac{[4]_q!}{[2]_q!^4}+\frac{\xi^2[4]_q!}{[2]_q!^4}+\frac{\xi[4]_q!}{[2]_q!^3}\\
&\quad+\frac{[4]_q!}{[3]_q!^2}+\frac{\xi^3[4]_q!}{[2]_q![3]_q!^2}-\frac{3[4]_q!}{[2]_q!^2[3]_q!}-\frac{\xi^2[4]_q!}{[2]_q!^2[3]_q!}-\frac{2\xi[4]_q!}{[2]_q![3]_q!}-\frac{[4]_q!}{[5]_q!},\\
{}_L\mathbb{B}_{5,q}(\xi,\eta;5) &= \xi+\frac{2}{[2]_q!}-\frac{\xi^5}{[5]_q!}+\eta[5]_q!-\frac{[5]_q!}{[2]_q!^5}-\frac{\xi^2[5]_q!}{[2]_q!^5}\\
&\quad-\frac{\xi[5]_q!}{[2]_q!^4}+\frac{\xi^3[5]_q!}{[3]_q!^3}-\frac{\xi[5]_q!}{[3]_q!^2}-\frac{\xi^3[5]_q!}{[2]_q!^2[3]_q!^2}-\frac{3[5]_q!}{[2]_q![3]_q!^2}.
\end{aligned}
$$

Here, we contribute to the field by giving the presentation of the first few values of the bivariate extended q-Laguerre-based Bernoulli ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;5)$ polynomials. These values are not only a practical reference but also help establish a foundation for further research and exploration.

We can develop the beautiful roots of the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m) = 0$, by making use of a math program on a computer. Thus, we draw these solutions for $m = 5, \eta = 7$ and $\omega = 30$ by the following Figure 1:

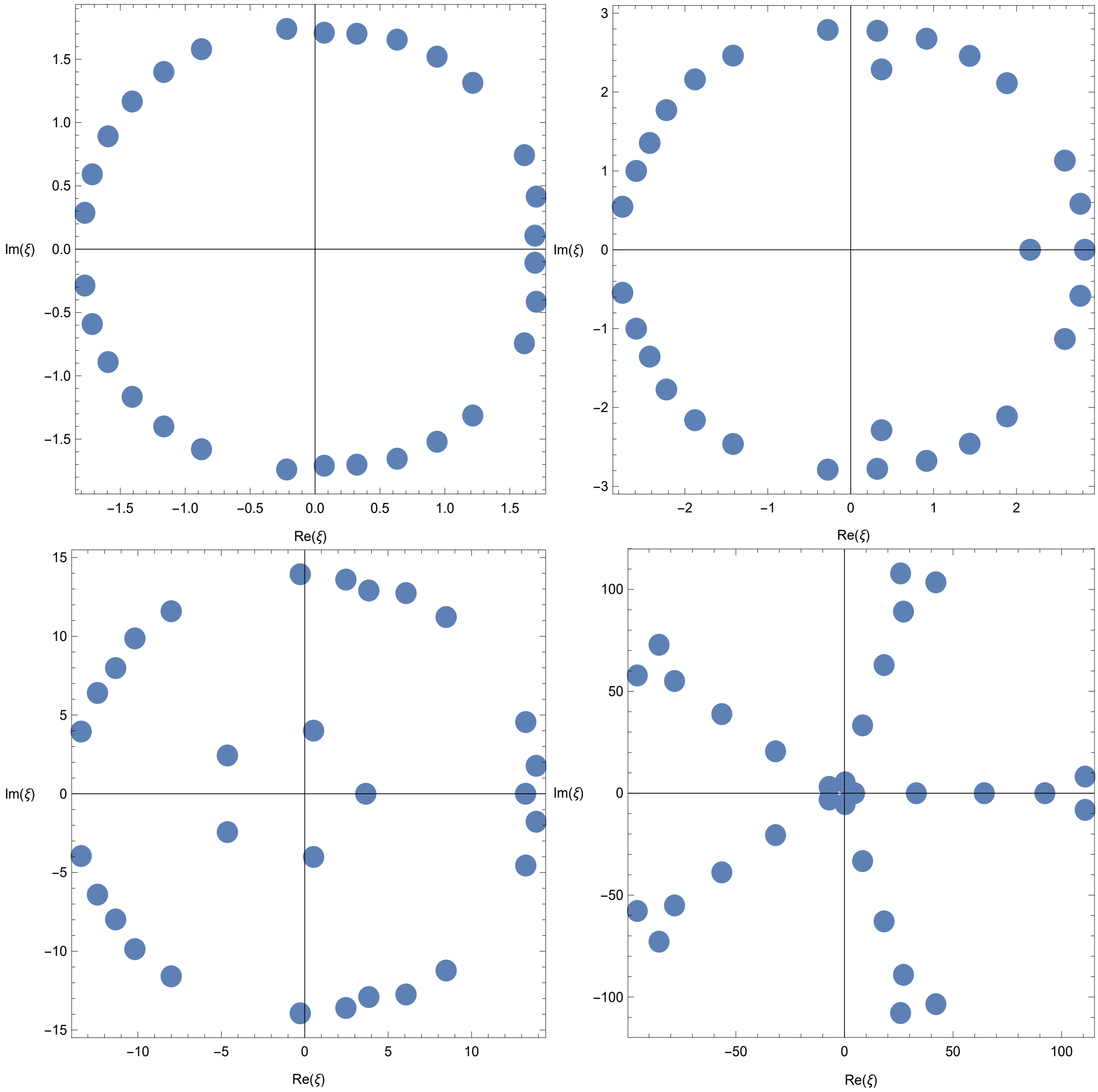

Figure 1. Zeros of the equality${}_L\mathbb{B}_{\omega,q}(\xi, \eta; m) = 0$.

Especially, we take $q = \frac{1}{10}$ (top-left), $q = \frac{3}{10}$ (top-right), $q = \frac{7}{10}$ (bottom-left) and $q = \frac{9}{10}$ (bottom-right) in Figure 1.

We provide, forming a 3D structure, the stacks of zeros for the equality ${}_L\mathbb{B}_{\omega,q}(\xi, \eta; m) = 0$ for $m = 5, \eta = 7$, and $1 \leq \omega \leq 50$ by the following Figure 2:

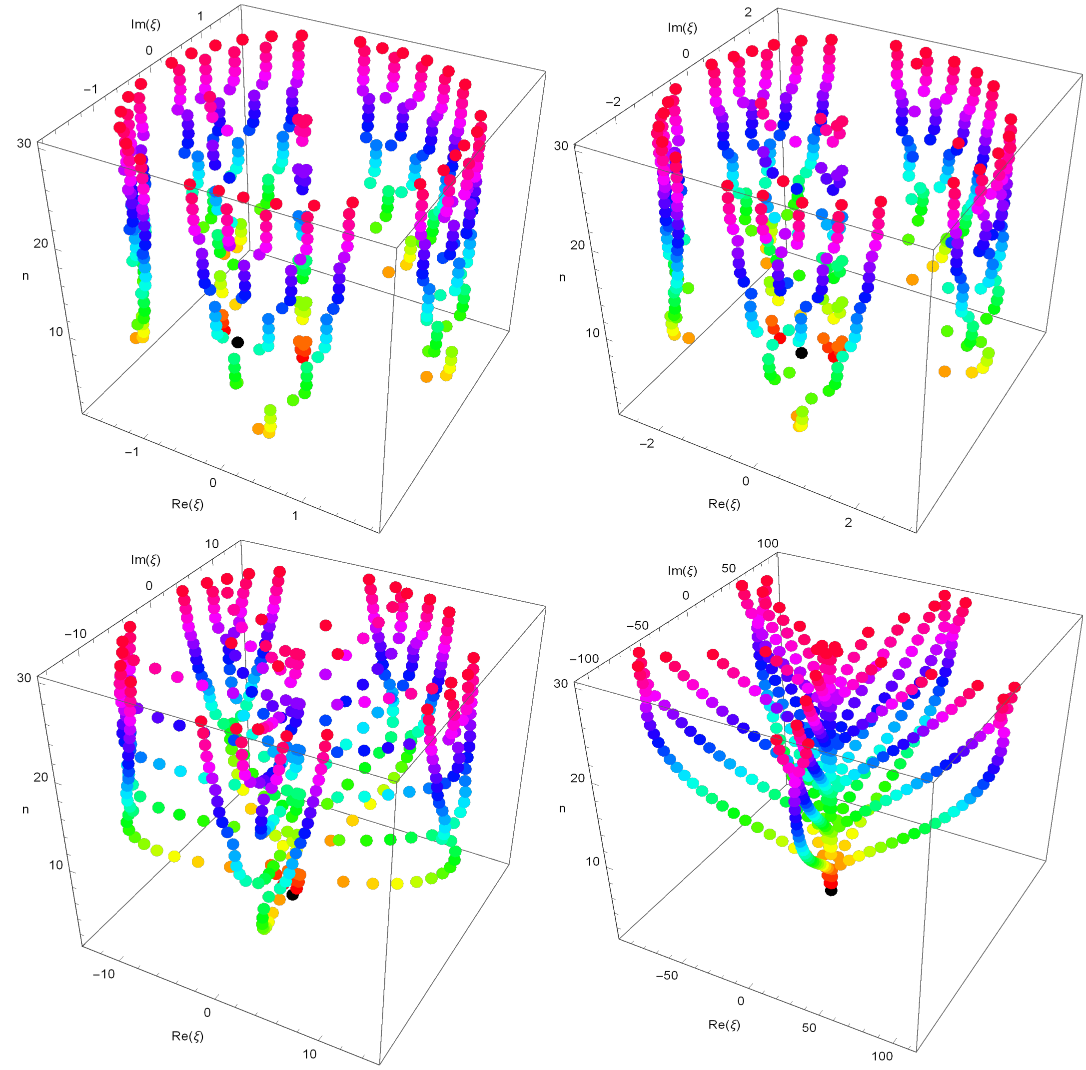

Figure 2. Zeros of the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m) = 0$

Here, we take $q = \frac{1}{10}$ (top-left), $q = \frac{3}{10}$ (top-right), $q = \frac{7}{10}$ (bottom-left) and $q = \frac{9}{10}$ (bottom-right) in Figure 2.

We give, forming a 2D structure, the stacks of real zeros for the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m) = 0$ for $m = 5, \eta = 7$, and $1 \leq \omega \leq 30$ by the following Figure 3:

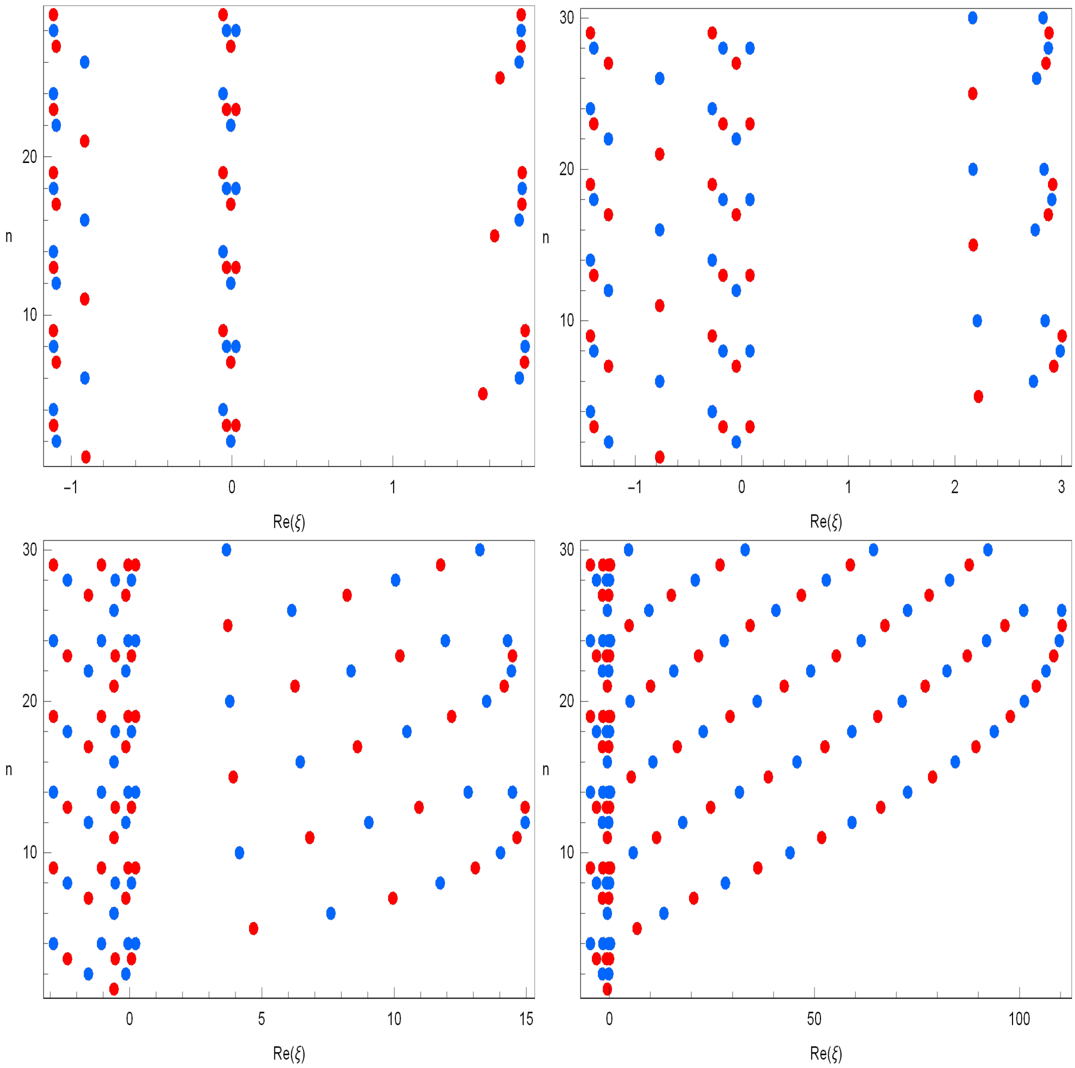

Figure 3. Real zeros of the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m) = 0$.

Here, we take $q = \frac{1}{10}$ (top-left), $q = \frac{3}{10}$ (top-right), $q = \frac{7}{10}$ (bottom-left) and $q = \frac{9}{10}$ (bottom-right) in Figure 3.

Here, we provide the graphical representations in Figures 1–3, enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed.

Now, we compute an approximate solution fulfilling the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)=0$ for $m=5,\eta=7$, and $q=\frac{9}{10}$ provided by the following Table 1.

Table 1. Approximate solutions of the equality ${}_L\mathbb{B}_{\omega,q}(\xi,\eta;m)=0$.

degree ω	ξ
1	-0.5263158
2	-1.726921, -0.1730785
3	-3.166321, -0.7240529, 0.02505760
4	-4.650997, -1.619159, -0.2099124, 0.2554790
5	$-8.694437-4.532775i$, $-8.694437+4.532775i$, $0.914656-7.484820i$, $0.914656+7.484820i$, 6.733328
6	$-14.75516-8.71538i$, $-14.75516+8.71538i$, -0.5263300, $2.60341-14.15998i$, $2.60341+14.15998i$, 13.27469
7	$-21.15158-13.16876i$, $-21.15158+13.16876i$, -1.726952, -0.1730809, $4.65617-21.34268i$, $4.65617+21.34268i$, 20.56590
8	$-27.68021-17.79298i$, $-27.68021+17.79298i$, -3.166338, -0.7240602, 0.02505676, $6.93241-28.81735i$, $6.93241+28.81735i$, 28.28895
9	$-34.18230-22.46667i$, $-34.18230+22.46667i$, -4.651001, -1.619170, $,-0.2099148$, 0.2554785, $9.32728-36.37663i$, $9.32728+36.37663i$, 36.18445
10	$-40.54104-27.09138i$, $-40.54104+27.09138i$, $-7.939087-3.901462i$, $-7.939087+3.901462i$, $0.634581-6.512172i$, $0.634581+6.512172i$, 5.783184, $11.76238-43.85789i$, $11.76238+43.85789i$, 44.05579

The extended bivariate q-Laguerre-based Euler ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)$ polynomials are considered as follows:

$$\frac{2}{e_q(\psi)+1}C_{0,q}(\xi\psi)e_q(\eta\psi^m)=\sum_{\omega=0}^{\infty}{}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)\frac{\psi^\omega}{[\omega]_q!}. \tag{5.2}$$

A few of them are

$$\begin{aligned}
{}_L\mathbb{E}_{0,q}(\xi,\eta;4) &= 1,\\
{}_L\mathbb{E}_{1,q}(\xi,\eta;4) &= -\frac{1}{2}-\xi,\\
{}_L\mathbb{E}_{2,q}(\xi,\eta;4) &= -\frac{1}{2}+\frac{\xi^2}{[2]_q!}+\frac{1}{4}[2]_q!+\frac{1}{2}\xi[2]_q!,\\
{}_L\mathbb{E}_{3,q}(\xi,\eta;4) &= -\frac{1}{2}-\frac{\xi^3}{[3]_q!}-\frac{1}{8}[3]_q!-\frac{1}{4}\xi[3]_q!-\frac{\xi^2[3]_q!}{2[2]_q!^2}+\frac{[3]_q!}{2[2]_q!}+\frac{\xi[3]_q!}{2[2]_q!},\\
{}_L\mathbb{E}_{4,q}(\xi,\eta;4) &= -\frac{1}{2}+\frac{\xi^4}{[4]_q!}+\frac{1}{16}[4]_q!+\eta[4]_q!+\frac{1}{8}\xi[4]_q!\\
&\quad -\frac{\xi^2[4]_q!}{2[2]_q!^3}+\frac{[4]_q!}{4[2]_q!^2}+\frac{\xi^2[4]_q!}{4[2]_q!^2}-\frac{3[4]_q!}{8[2]_q!}-\frac{\xi[4]_q!}{2[2]_q!}\\
&\quad +\frac{\xi^3[4]_q!}{2[3]_q!^2}+\frac{[4]_q!}{2[3]_q!}+\frac{\xi[4]_q!}{2[3]_q!},\\
{}_L\mathbb{E}_{5,q}(\xi,\eta;4) &= -\frac{1}{2}-\frac{\xi^5}{[5]_q!}-\frac{1}{32}[5]_q!-\frac{1}{2}\eta[5]_q!-\frac{1}{16}\xi[5]_q!-\eta\xi[5]_q!\\
&\quad +\frac{\xi^2[5]_q!}{2[2]_q!^3}-\frac{3[5]_q!}{8[2]_q!^2}-\frac{\xi[5]_q!}{4[2]_q!^2}-\frac{\xi^2[5]_q!}{8[2]_q!^2}+\frac{[5]_q!}{4[2]_q!}+\frac{3\xi[5]_q!}{8[2]_q!}\\
&\quad -\frac{\xi^3[5]_q!}{4[3]_q!^2}+\frac{\xi^3[5]_q!}{2[2]_q![3]_q!^2}-\frac{3[5]_q!}{8[3]_q!}-\frac{\xi[5]_q!}{2[3]_q!}-\frac{\xi^2[5]_q!}{2[2]_q!^2[3]_q!}\\
&\quad +\frac{[5]_q!}{2[2]_q![3]_q!}-\frac{\xi^4[5]_q!}{2[4]_q!^2}+\frac{[5]_q!}{2[4]_q!}+\frac{\xi[5]_q!}{2[4]_q!}.
\end{aligned}$$

Here, we contribute to the field by giving the presentation of the first few values of the bivariate extended q-Laguerre-based Euler ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;4)$ polynomials, which are not only a practical reference but also help to establish a foundation for further research and exploration.

We research the solutions of the equality ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m) = 0$, utilizing a computer programme. So, we draw these solutions for $m = 4, \eta = 6$ and $\omega = 30$ by the following Figure 4:

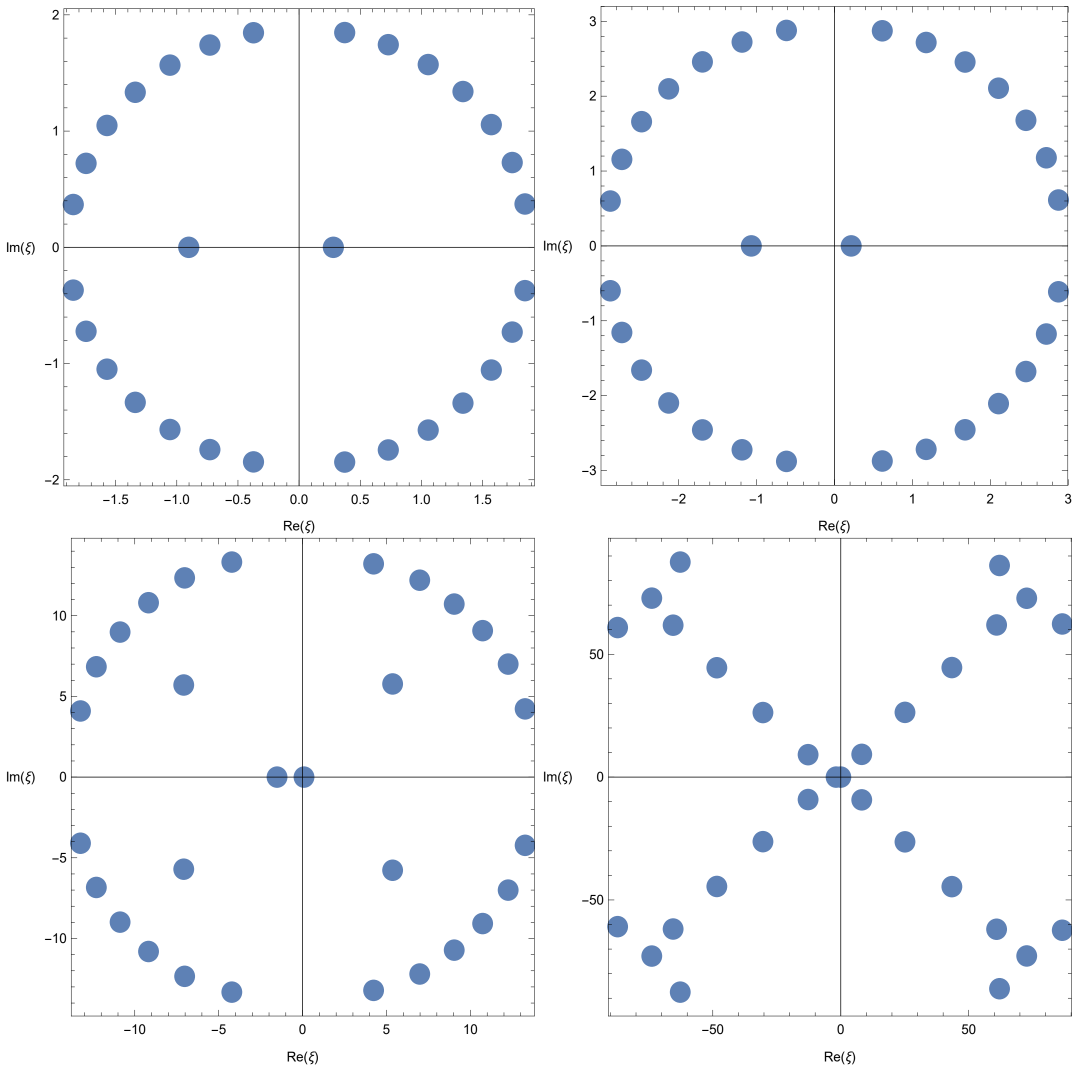

Figure 4. Zeros of the equality ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m) = 0$.

Especially, we take $q = \frac{1}{10}$ (top-left), $q = \frac{3}{10}$ (top-right), $q = \frac{7}{10}$ (bottom-left) and $q = \frac{9}{10}$ (bottom-right) in Figure 4.

We provide, forming a 3D structure, the stacks of zeros for the equality ${}_{L}\mathbb{E}_{\omega,q}(\xi,\eta;m)=0$ for $m=4,\eta=6$, and $1\leq\omega\leq 50$ by the following Figure 5:

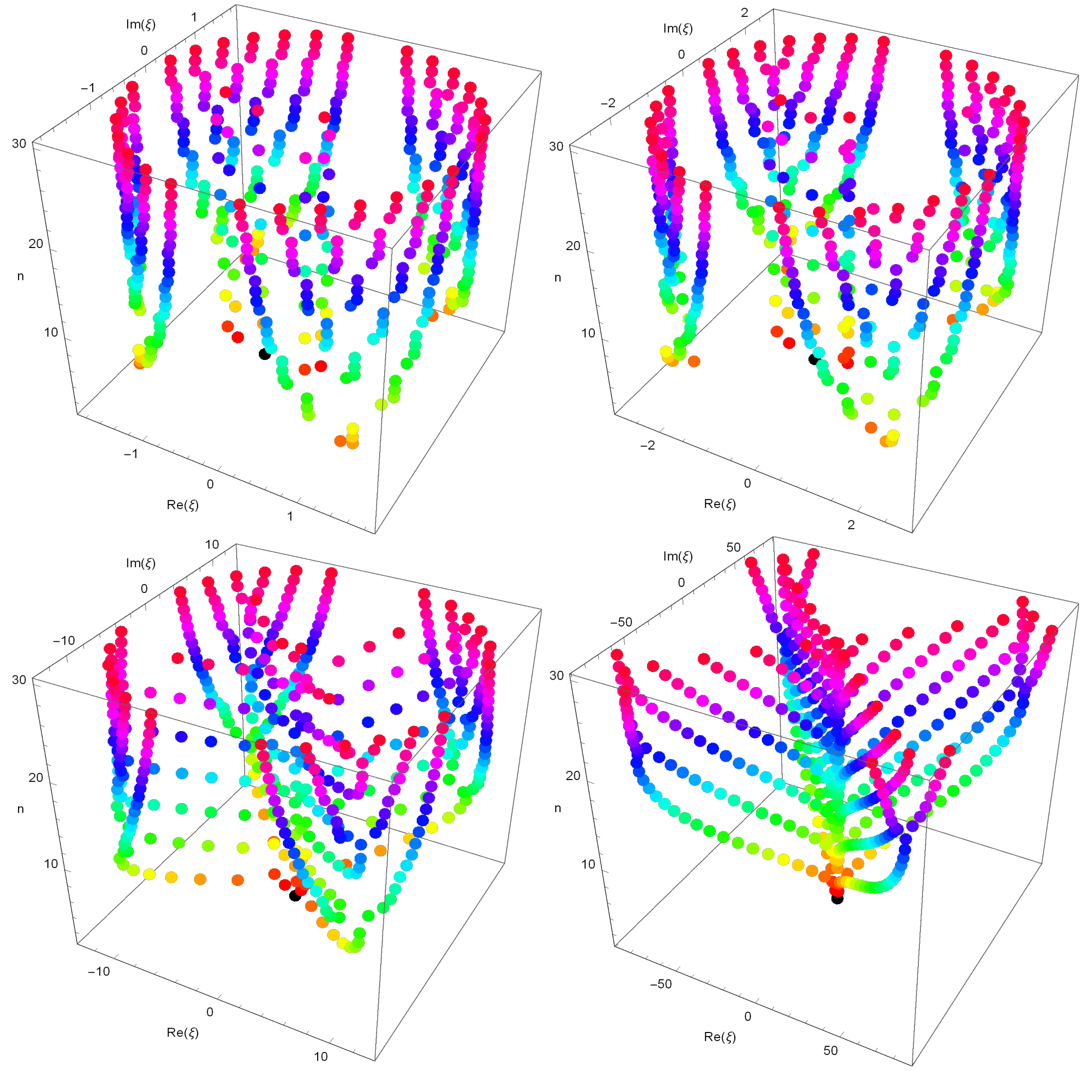

Figure 5. Zeros of the equality ${}_{L}\mathbb{E}_{\omega,q}(\xi,\eta;m)=0$.

Here, we take $q=\frac{1}{10}$ (top-left), $q=\frac{3}{10}$ (top-right), $q=\frac{7}{10}$ (bottom-left) and $q=\frac{9}{10}$ (bottom-right) in Figure 5.

We give, forming a 2D structure, the stacks of zeros for the equality ${}_L\mathbb{E}_{\omega,q}(\xi, \eta; m) = 0$ for $m = 4, \eta = 6$, and $1 \leq \omega \leq 50$ by the following Figure 6:

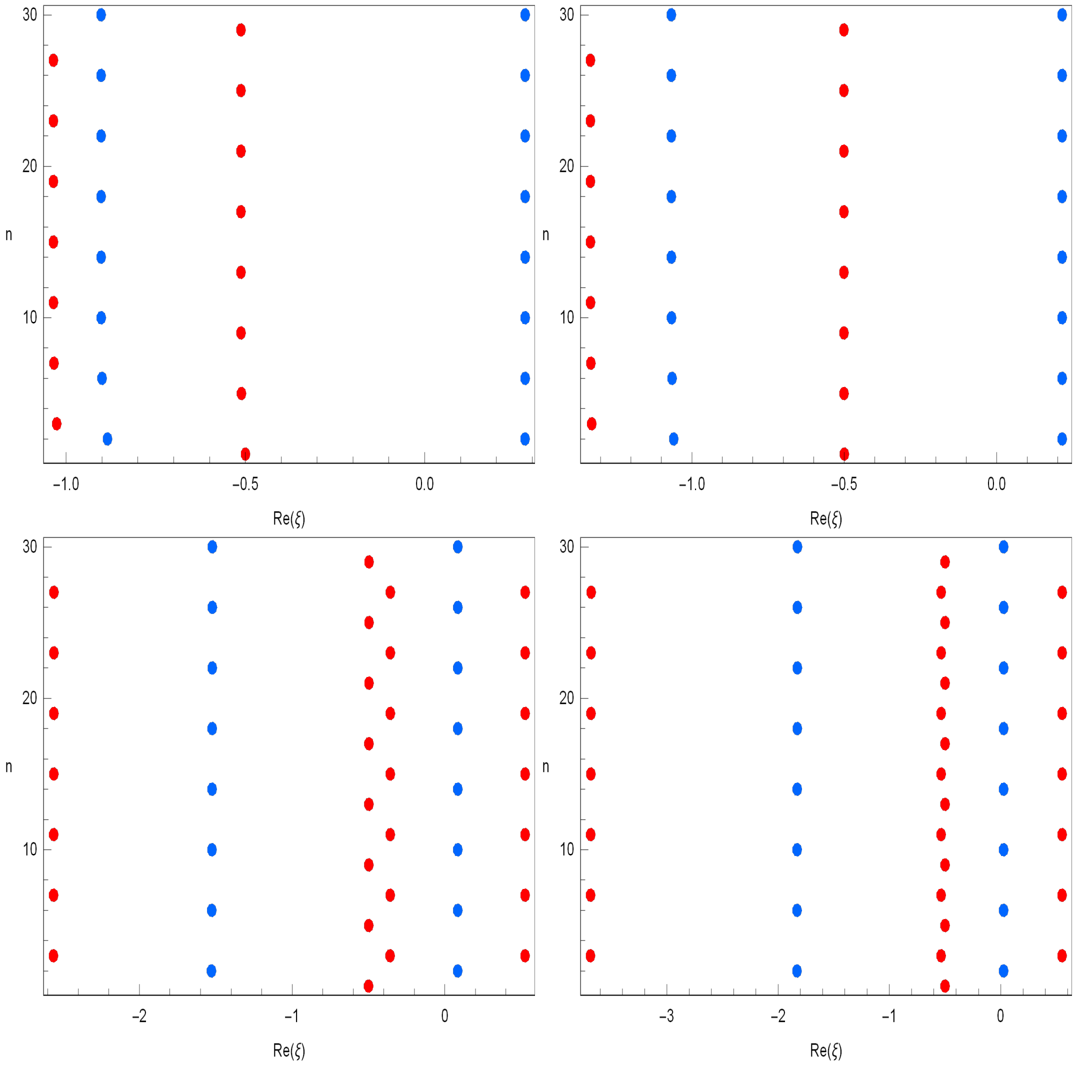

Figure 6. Real zeros of the equality ${}_L\mathbb{E}_{\omega,q}(\xi, \eta; m) = 0$.

Here, we take $q = \frac{1}{10}$ (top-left), $q = \frac{3}{10}$ (top-right), $q = \frac{7}{10}$ (bottom-left) and $q = \frac{9}{10}$ (bottom-right) in Figure 6.

Here, we contribute to the field by giving the stacks of zeros of ${}_L\mathbb{E}_{\omega,q}(\xi, 6; 4)$ in conjunction with the graphical representations enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed.

Now, we compute an approximate solution fulfilling the equality ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)=0$ for $m=4,\eta=6$, and $q=\frac{9}{10}$ provided by the following Table 2.

Table 2. Approximate solutions of ${}_L\mathbb{E}_{\omega,q}(\xi,\eta;m)=0$.

degree ω	ξ
1	-0.50000
2	$-1.8309,\quad 0.025943$
3	$-3.6866,\quad -0.53612,\quad 0.55067$
4	$-6.5056-4.1632i,\quad -6.5056+4.1632i,$ $3.5489-4.3542i,\quad 3.5489+4.3542i$
5	$-11.790-8.983i,\quad -11.790+8.983i,\quad -0.49994,$ $7.8471-9.0689i,\quad 7.8471+9.0689i$
6	$-17.542-14.324i,\quad -17.542+14.324i,\quad -1.8301,$ $0.026081,\quad 12.955-14.386i,\quad 12.955+14.386i$
7	$-23.516-20.017i,\quad -23.516+20.017i,\quad -3.6852,$ $-0.53586,\quad 0.55097,\quad 18.547-20.074i,\quad 18.547+20.074i$
8	$-29.532-25.874i,\quad -29.532+25.874i,\quad -5.8712-3.3574i,$ $-5.8712+3.3574i,\quad 2.9169-3.6244i,\quad 2.9169+3.6244i,$ $24.377-25.929i,\quad 24.377+25.929i$
9	$-35.440-31.722i,\quad -35.440+31.722i,\quad -10.4555-7.5110i,$ $-10.4555+7.5110i,\quad -0.49988,\quad 6.5192-7.6291i,$ $6.5192+7.6291i,\quad 30.245-31.777i,\quad 30.245+31.777i$
10	$-41.116-37.414i,\quad -41.116+37.414i,\quad -15.443-12.094i,$ $-15.443+12.094i,\quad -1.8293,\quad 0.026207,\quad 10.864-12.176i,$ $10.864+12.176i,\quad 35.991-37.468i,\quad 35.991+37.468i$

6. Conclusions

In the present paper, the bivariate extended q-Laguerre-based Appell polynomials were considered utilizing the q-Bessel Tricomi functions of zero-order, and then, some of their properties were investigated. Some operational identities in Theorem 2.2, two summation formulas in Theorems 2.5 and 2.6, and a determinant representation in Theorem 2.7 were provided with their proofs.

Also, it is shown that the bivariate extended q-Laguerre-based Appell polynomials are quasi-monomials by the q-multiplicative operators in (3.10) and (3.11) and q-derivative operator in (3.12). Theorem 3.4 includes q-integro-differential equations for the bivariate extended q-Laguerre-based Appell polynomials. Section 4 examines some special cases for the bivariate extended q-Laguerre-based Appell polynomials, choosing the bivariate extended q-Laguerre-based Bernoulli, Euler, and Genocchi polynomials. In Section 5, distributions of the zeros and graphical representations for the bivariate extended q-Laguerre-based Bernoulli and Euler polynomials were analyzed in detail.

In conclusion, the introduction and investigation of q-hybrid polynomials represent a significant milestone in the field of mathematics and science, promoting novel examination avenues and applications in different disciplines. It is crucial to continue exploring and collaborating to fully realize their potential and understand their broader implications.

Author contributions

All authors contribute equally to this study. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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