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Research article

Stability and bifurcations in a delayed predator-prey system with prey-taxis and hunting cooperation functional response

Kimun Ryu and Wonlyul Ko*

Department of Mathematics Education, Cheongju University, Cheongju, Chungbuk 28503, Republic of Korea

* **Correspondence:** Email: kowl@cju.ac.kr.

Abstract: In this paper, we studied a diffusive predator-prey system that incorporated three ecological features under homogeneous Neumann boundary conditions. These features included a cooperative hunting functional response, prey-taxis, and a time delay effect in the predator growth rate, all of which influenced predator-prey interactions. These three features, respectively, indicated that predators cooperated while hunting their prey, that predators tended to move in the direction of an increasing prey density gradient, and that a certain amount of time was required for the conversion of captured prey into predator growth. We first analyzed the occurrence of steady-state bifurcation by examining the role of the prey-taxis rate χ. Furthermore, we examined the impact of the time delay effect on the occurrence of Hopf bifurcation, which involved the emergence of spatially homogeneous or nonhomogeneous periodic solutions, as well as stability switches at a positive constant steady state.

Keywords: predator-prey; prey-taxis; Hopf bifurcation; steady-state bifurcation; hunting cooperation; delay
Mathematics Subject Classification: 35B32, 35K57, 92D25

1. Introduction

In this paper, we study the following predator-prey system with a hunting cooperation functional response, prey-taxis, and time delay:

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = u(t,x)(1-u(t,x)) - \dfrac{\alpha u(t,x)(1+v(t,x))}{1+mu(t,x)(1+v(t,x))}v(t,x), & t>0,\ x\in\Omega,\\
\dfrac{\partial v(t,x)}{\partial t} - \rho\Delta v(t,x) + \chi\nabla(v(t,x)\nabla u(t,x)) & \\
\qquad = \beta v(t,x)\left(-\gamma + \dfrac{u(t-\tau,x)(1+v(t-\tau,x))}{1+mu(t-\tau,x)(1+v(t-\tau,x))}\right), & t>0,\ x\in\Omega,\\
\dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & t\geq 0,\ x\in\partial\Omega,\\
u(t,x) = u_0(t,x)\geq 0,\ \ v(t,x) = v_0(t,x)\geq 0, & t\in[-\tau,0],\ x\in\overline{\Omega},
\end{cases}
\tag{1.1}
$$

where $u(t,x)$ and $v(t,x)$ represent the population densities of the prey and predator at time t and spatial location x, respectively; $\Omega \subseteq \mathbb{R}^N$ denotes a bounded domain with a smooth boundary $\partial\Omega$; the given coefficients α, β, γ, ρ, χ, m, and τ are positive constants; ν denotes the outward normal derivative to $\partial\Omega$; and u_0, v_0 are continuous functions that are not identically zero.

System (1.1) models a predator-prey interaction that incorporates three ecological features. It is well-documented that many animals, such as lions and wolves, exhibit cooperative behavior during their hunting activities. In line with this observation, system (1.1) allows predators to collaborate while hunting their prey. To illustrate this dynamic, we introduce the following functional response, which characterizes the rate at which predators hunt and consume their prey:

$$
\frac{Ce_0u(1+av)}{1+h_0Ce_0u(1+av)},
\tag{1.2}
$$

where the positive constants a, e_0, h_0, and C represent the rate of cooperative hunting among predators, the encounter rate per predator per prey, the handling time per prey, and the portion of prey captured by each predator per encounter, respectively (see [1, 5, 8, 13, 29]). In fact, [5] introduced a generalized functional response based on the Holling type-II to explain predator hunting cooperation in predator-prey interactions, and (1.2) is an example of this functional response, as described in [1]. Functional response (1.2) increases simultaneously with the densities of both predators and prey and has an upper limit. This characteristic ecologically explains the cooperation among predators when hunting their prey. These properties set it apart from various existing functional responses: the Holling type-II and type-IV responses depend solely on prey density, while the ratio-dependent and Beddington-DeAngelis responses decrease with increasing predator density, which is interpreted as intraspecific competition [24] among predators for capturing prey. For a comprehensive overview of various functional responses, see [2–4, 6, 8, 10–12, 15, 17, 25, 39], and refer to their corresponding references. Furthermore, the hunting cooperation functional response, with such properties, induces a strong Allee effect in the predator population, enriching the dynamics of predator-prey interactions (e.g., [28]). In response, not only has (1.2) been introduced, but various hunting cooperation functional responses (e.g., [8, 17]) have also been incorporated into different forms of predator-prey models, including ordinary differential equations (ODEs), partial differential equations (PDEs) with constant diffusion rates, PDEs with cross-diffusion rates, and PDEs with prey-taxis. This has led to numerous analytical results [1, 13, 14, 17, 18, 26–29, 32, 35]. Alongside these findings, we anticipate that the study presented in this paper will further enhance the understanding of hunting cooperation dynamics in predator-prey interactions.

The second ecological feature of system (1.1) is that in a habitat with boundaries that species cannot cross (i.e., homogeneous Neumann boundary conditions), the prey species exhibit a constant self-diffusion rate, which represents random movement. In contrast, the predators not only have a constant self-diffusion rate but also demonstrate a tendency to aggregate in specific areas. Specifically, in system (1.1), predator species tend to move in the direction of the gradient of the prey density function. To explain this behavior, system (1.1) incorporates the term $\chi\nabla(v(t,x)\nabla u(t,x))$, known as prey-taxis, where χ is referred to as the prey-taxis rate. Since the work of [16], the introduction of prey-taxis (and predator-taxis) into predator-prey interactions has garnered significant interest for its role in enhancing the realism of predator-prey models, leading to intriguing results (see [17, 18, 30, 33, 34, 37, 38, 40] and their references). This topic remains a central focus in current research, particularly regarding the role of the prey-taxis rate χ, which can be observed in deriving results related to the existence of global positive solutions, pattern formation, and the stability of positive constant steady states in the studied PDEs. Furthermore, the authors in [17, 18] investigated the global well-posedness of solutions, as well as the existence and nonexistence of pattern formation in predator-prey PDEs with generalized hunting cooperation functional responses and (repulsive) prey-taxis. Building on these findings, we select χ as one of the primary variables to explore our research objectives in this paper, particularly the occurrence of steady-state and Hopf bifurcations.

The third ecological feature of system (1.1) is the incorporation of a time delay effect in the per capita growth rate of predators. Based on Caperon's experimental observations [7], predator-prey interaction models that include time delays have been proposed and studied. In these models, the functional response (e.g., Holling type-II/type-IV, ratio-dependent, or Beddington-DeAngelis) in the predator's growth rate is influenced by prey or predator density from τ time units earlier (see [4, 19–21, 23, 30, 36, 41, 42]). These models introduce a time delay τ in the functional response of the predator's growth rate, as described in system (1.1). Such a time delay can account for ecological scenarios where prey species require time to develop from larvae to adults, and predator species also need a similar duration to mature, in addition to considering nonzero gestation periods or reaction times to predation [4,19,22]. Consequently, incorporating time delays into predator-prey interactions provides a more realistic framework for understanding predator-prey dynamics [4]. In predator-prey models with time delays applicable to various ecological environments, the stability of positive constant steady states and the occurrence of various bifurcations, including Hopf bifurcation, have been extensively studied (see [4, 19, 20, 22, 30, 41–43] and their references). In deriving these research findings, the time delay was treated as a primary variable, and, particularly, the critical values of the time delay at which Hopf bifurcation occurs were identified. In alignment with this research trend, we also investigate the stability and Hopf bifurcation in relation to the time delay effect τ in system (1.1).

The PDEs in (1.1) represent a predator-prey model in which predators tend to move toward areas with high prey density, exhibit cooperative behavior while hunting, and experience a time delay in converting captured prey into their growth rate. System (1.1) without diffusion and delay, i.e., the corresponding ODEs without delay, has been analyzed in [13, 14, 29] concerning the stability of all possible nonnegative constant steady states and the occurrence of various bifurcations, including saddle-node and Hopf bifurcations. As previously mentioned, the occurrence of these bifurcations is attributed to the complex predator-prey dynamics induced solely by the functional response. Furthermore, in [13], a time delay was incorporated into the ODEs, and the impact of this time delay on the occurrence of Hopf bifurcation was investigated. It is important to note that the ODEs presented in [13] differ from those corresponding to system (1.1). In the ODEs associated with system (1.1), the term involving the time delay is multiplied by a nondelayed term v, whereas in the ODEs from [13], the delay is also included in this term v. Accordingly, our theoretical results on Hopf bifurcation in system (1.1) will encompass the occurrence of spatially homogeneous periodic solutions. In [14], Turing instability and the occurrence of Hopf bifurcation were examined in system (1.1) without time delay and prey-taxis, i.e., PDEs with the (rescaled) functional response given in (1.2) and constant diffusion rates. To compare with these findings, we will investigate Turing stability and instability in system (1.1) with $\tau = 0$, considering both the case of $\chi = 0$ and the role of $\chi > 0$. Furthermore, in situations where Hopf bifurcation cannot occur in system (1.1) with $\chi = 0$ and $\tau = 0$, we will observe the emergence of Hopf bifurcation induced by $\tau > 0$ and $\chi > 0$ through our research findings. Our investigation into the occurrence of Hopf bifurcation influenced by the roles of prey-taxis χ and time delay τ is motivated by the studies in [30, 33, 37]. Notably, in [33, 37], predator-prey models with spatial memory were explored by incorporating time delay into the taxis term.

By incorporating the basic mechanisms of predator-prey interactions, along with the predator's functional response presented in (1.2), and the ecologically meaningful prey-taxis and time delay mentioned earlier, we obtain the following predator-prey model, which serves as the origin of system (1.1):

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - d_1\Delta u(t,x) = ru(t,x)\left(1-\dfrac{u(t,x)}{K}\right) & \\ \qquad\qquad -\dfrac{Ce_0u(t,x)(1+av(t,x))}{1+h_0Ce_0u(t,x)(1+av(t,x))}v(t,x), & t>0,\ x\in\Omega, \\ \dfrac{\partial v(t,x)}{\partial t} - d_2\Delta v(t,x) + \chi\nabla(v(t,x)\nabla u(t,x)) & \\ \qquad = -\gamma v(t,x) + \dfrac{\epsilon Ce_0u(t-\tau,x)(1+av(t-\tau,x))}{1+h_0Ce_0u(t-\tau,x)(1+av(t-\tau,x))}v(t,x), & t>0,\ x\in\Omega, \\ \dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & t\geq 0,\ x\in\partial\Omega, \\ u(t,x)=u_0(t,x)\geq 0,\ \ v(t,x)=v_0(t,x)\geq 0, & t\in[-\tau,0],\ x\in\overline{\Omega}, \end{cases}$$

where the given coefficients are all positive constants; d_1 and d_2 denote the diffusion rate of the prey and predator species, respectively; r represents the intrinsic growth rate of the prey species and K denotes the carrying capacity of the prey in the absence of predators; ϵ represents the conversion rate of the prey into a predator; γ means the predator death rate; finally, χ is referred to as the intrinsic prey-taxis rate. For convenience, when studying the above PDEs, after introducing a new typical length scale and the following quantities and then dropping the upper bars, we can eventually obtain system (1.1):

$$\bar{t} = rt,\ \ \bar{x} = \sqrt{\frac{r}{d_1}}x,\ \ \bar{u} = \frac{u}{K},\ \ \bar{v} = av,\ \ m = Kh_0Ce_0,\ \ \alpha = \frac{Ce_0}{ar},\ \ \beta = \frac{\epsilon Ce_0K}{r},$$
$$\bar{\gamma} = \frac{\gamma}{\epsilon Ce_0K},\ \ \bar{\tau} = r\tau,\ \ \bar{\chi} = \frac{\chi K}{d_1},\ \ \rho = \frac{d_2}{d_1}.$$

Furthermore, the primary purposes of our study presented in this paper are as follows:

(i) We investigate the Turing stability and instability in system (1.1) with $\tau = 0$, focusing on the role of χ.

(ii) We find values of χ that allow steady-state bifurcation at positive constant steady states of system (1.1) exhibiting instability as investigated in (i).

(iii) We examine the range of χ for which system (1.1) undergoes homogeneous or nonhomogeneous Hopf bifurcation and identify all critical delay values at which this bifurcation occurs. Additionally, we analyze whether positive constant steady states of system (1.1) experience stability switches with respect to τ.

This paper is organized as follows. In Section 2, we establish the conditions that ensure the existence of positive constant steady states for system (1.1). In Section 3, we introduce the necessary notations and present simple calculations required for Section 4. Their proofs are provided in the Appendix. In Section 4, we investigate the stability and instability, as well as steady-state and Hopf bifurcations, of positive constant steady states in system (1.1), addressing the main objectives mentioned earlier.

2. Existence of positive equilibria

In this section, we investigate the existence of a positive constant steady state of system (1.1). Prior to this, it is easily established that system (1.1) has two nonnegative constant steady states, $(0, 0)$ and $(1, 0)$. Now, to achieve the purpose of this section, let (u, v) be a constant coexistence steady state of system (1.1). Obviously, (u, v) satisfies the equations

$$u\left(1-u-\frac{\alpha(1+v)v}{1+mu(1+v)}\right)=0,\quad v\left(-\gamma+\frac{u(1+v)}{1+mu(1+v)}\right)=0. \tag{2.1}$$

From these two equations, we obtain $\alpha\gamma v = u(1-u)$, that is,

$$v=\frac{1}{\alpha\gamma}u(1-u). \tag{2.2}$$

Furthermore, we have from the second equation in (2.1) that

$$u(1+v)=\frac{\gamma}{1-m\gamma}. \tag{2.3}$$

Thus, from (2.2) and (2.3), we deduce that for (u, v) to exist, the conditions $u < 1$ and $m\gamma < 1$ must be satisfied. Additionally, substituting (2.2) into (2.3), we obtain that u satisfies

$$F(u) := u^3-u^2-\alpha\gamma u+\frac{\alpha\gamma^2}{1-m\gamma}=0, \tag{2.4}$$

and subsequently, v is determined by (2.2). We now investigate all the conditions that guarantee that $F(u) = 0$ has a solution in the interval $(0, 1)$. Obviously, $F'(u) = 3u^2 - 2u - \alpha\gamma$, and it has only one positive root

$$u_d := \frac{1+\sqrt{1+3\alpha\gamma}}{3}.$$

Therefore, the existence of a constant u satisfying $F(u) = 0$ within the interval $(0, 1)$ is determined by whether $F(u_d) \le 0$ holds or not. By direct calculation, we can easily derive

$$F(u_d)=-\frac{2(1+3\alpha\gamma)u_d+\alpha\gamma}{9}+\frac{\alpha\gamma^2}{1-m\gamma},$$

so that when $m\gamma < 1$,

$$F(u_d) \le 0 \iff H(\alpha\gamma) := \frac{9\alpha\gamma}{2(1+3\alpha\gamma)u_d + \alpha\gamma} \le \frac{1-m\gamma}{\gamma}.$$

Additionally, since we need to find a $u \in (0,1)$, we further derive the following conditions regarding $F(1) < 0$ and $u_d < 1$:

$$F(1) = \alpha\gamma\left(-1 + \frac{\gamma}{1-m\gamma}\right) < 0 \iff \frac{\gamma}{1-m\gamma} < 1, \text{ and } u_d < 1 \iff \alpha\gamma < 1.$$

Obviously, if $F(u_d) < 0$, then $F(u) = 0$ has two positive roots, denoted by u_1^* and u_2^*, with $u_1^* < u_d < u_2^*$ (see Figure 1). The possible scenarios in which $F(u) = 0$ has a solution for $0 < u < 1$ are as follows:

(i) If $F(u_d) < 0$ and $F(1) < 0$, then $u_1^* < 1 < u_2^*$, so that (1.1) has only one positive constant steady state, (u_1^*, v_1^*);
(ii) If $F(u_d) < 0$, $F(1) = 0$, and $u_d < 1$, then $u_1^* < u_2^* = 1$, so that (1.1) has only one positive constant steady state, (u_1^*, v_1^*);
(iii) If $F(u_d) < 0$, $F(1) > 0$, and $u_d < 1$, then $u_1^* < u_2^* < 1$, so that (1.1) has two positive constant steady states, (u_1^*, v_1^*) and (u_2^*, v_2^*);
(iv) If $F(u_d) = 0$ and $u_d < 1$, then (1.1) has only one positive constant steady state, (u_d, v_d),

where

$$v_i^* = \frac{1}{\alpha\gamma}u_i^*(1-u_i^*) \text{ for } i = 1,2, \text{ and } v_d = \frac{1}{\alpha\gamma}u_d(1-u_d).$$

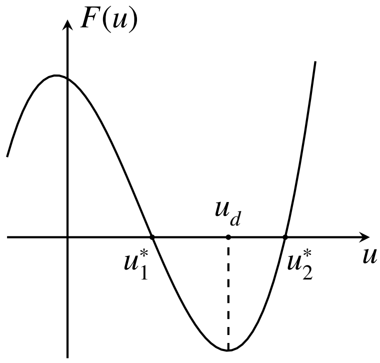


Figure 1. The graph of $F(u)$.

The above H function is increasing in the interval $(0,1)$ and decreasing in the interval $(1,\infty)$, satisfying $H(1) = 1$ and $\lim_{\alpha\gamma\to\infty} H(\alpha\gamma) = 0$. Thus, when $\frac{\gamma}{1-m\gamma} > 1$, there exists a unique $\alpha_1^* \in (0,1)$ such that $H(\alpha_1^*) = \frac{1-m\gamma}{\gamma}$, and so $H(\alpha\gamma) < \frac{1-m\gamma}{\gamma}$ (i.e., $F(u_d) < 0$) for $\alpha\gamma < \alpha_1^*$. Using the previously derived inequalities, the geometric properties of H, and the definition of α_1^*, we rewrite the sufficient conditions for positive constant coexistence given in (i)–(iv) above as follows, respectively:

$$\begin{aligned} &\text{(i) } 0 < \frac{\gamma}{1-m\gamma} < 1; \\ &\text{(ii) } \frac{\gamma}{1-m\gamma} = 1 \ \text{ and } \ \alpha\gamma < 1; \\ &\text{(iii) } \frac{\gamma}{1-m\gamma} > 1 \ \text{ and } \ \alpha\gamma < \alpha_1^*; \\ &\text{(iv) } \frac{\gamma}{1-m\gamma} > 1 \ \text{ and } \ \alpha\gamma = \alpha_1^*. \end{aligned} \tag{2.5}$$

Except for these four conditions, system (1.1) does not permit any positive constant steady states.

3. Notations and preliminary calculations

In this section, before providing our main results in the next section, we introduce some notations and computational results that are used throughout this article. The process of obtaining the results is straightforward but tedious, so the detailed calculations are left to the Appendix.

We denote the expressions observed in the linearization of system (1.1) at the nonnegative constant steady state $\mathbf{u}_* := (u_*, v_*)$ using the following symbols for simplicity:

$$\begin{aligned} f_1 &:= 1 - u_* - \frac{\alpha(1+v_*)v_*}{1+mu_*(1+v_*)} + u_*\left(-1 + \frac{\alpha m(1+v_*)^2 v_*}{(1+mu_*(1+v_*))^2}\right); \\ f_2 &:= -\alpha u_* \frac{(1+v_*)(1+mu_*(1+v_*)) + v_*}{(1+mu_*(1+v_*))^2}; \\ g_1 &:= \beta\overline{g}_1, \ \text{ where } \overline{g}_1 := \frac{(1+v_*)v_*}{(1+mu_*(1+v_*))^2}; \\ g_2 &:= \beta\overline{g}_2, \ \text{ where } \overline{g}_2 := -\gamma + \frac{u_*(1+v_*)}{1+mu_*(1+v_*)} + \frac{u_* v_*}{(1+mu_*(1+v_*))^2}; \\ T &:= f_1 + g_2; \\ D &:= f_1 g_2 - f_2 g_1. \end{aligned} \tag{3.1}$$

As mentioned earlier, system (1.1) has nonnegative constant solutions $(0,0)$, $(1,0)$, and (u_1^*, v_1^*), (u_2^*, v_2^*), or (u_d, v_d) when one of the conditions in (2.5) is satisfied. For convenience, we let $\mathbf{u}_1^* := (u_1^*, v_1^*)$, $\mathbf{u}_2^* := (u_2^*, v_2^*)$, and $\mathbf{u}_d^* := (u_d, v_d)$ when they exist.

Proposition 3.1. *(i) If* $\mathbf{u}_* = (0,0)$*, then* $T = 1 - \beta\gamma$ *and* $D = -\beta\gamma$*.*

(ii) If $\mathbf{u}_* = (1,0)$*, then*

$$T = -1 + \beta\left(\frac{1-\gamma-m\gamma}{1+m}\right) \ \text{ and } \ D = -\beta\left(\frac{1-\gamma-m\gamma}{1+m}\right).$$

(iii) If $\mathbf{u}_*$ *is a positive constant steady state of* (1.1) *(i.e.,* $\mathbf{u}_* = \mathbf{u}_1^*$*,* $\mathbf{u}_2^*$*, or* $\mathbf{u}_d^*$*), then*

$$T = m\gamma - (1+m\gamma)u_* + \beta\overline{g}_2 \ \text{and} \ D = -\frac{\beta v_*}{(1+mu_*(1+v_*))^2} F'(u_*).$$

Using the above calculation results, we can determine the signs of T and D as follows.

Proposition 3.2. *(i) If* $\mathbf{u}_* = \mathbf{u}_1^*$*, then*

$$T < 0 \iff \beta < -\frac{m\gamma - (1 + m\gamma)u_*}{\overline{g}_2} := \beta_1^* \text{ and } \frac{m\gamma}{1 + m\gamma} < u_*.$$

Moreover,

$$\frac{m\gamma}{1 + m\gamma} < u_* \iff \frac{\gamma}{1 - m\gamma} > \frac{m\gamma}{1 + m\gamma} \text{ and } \alpha\gamma > \frac{(m\gamma)^2}{(1 + m\gamma)^3} \cdot \frac{1}{\left(\frac{\gamma}{1-m\gamma} - \frac{m\gamma}{1+m\gamma}\right)} := \alpha_0^*.$$

(ii) If $\mathbf{u}_* = \mathbf{u}_1^*$*, then* $D > 0$*. On the other hand, if* $\mathbf{u}_* = \mathbf{u}_2^*$*, then* $D < 0$*, and if* $\mathbf{u}_* = \mathbf{u}_d^*$*, then* $D = 0$*.*

We present the fundamental calculation results used in the analysis of bifurcation in the next section.

Proposition 3.3. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1)*. Furthermore, assume that* $\beta_1^* > 0$ *and* $T < 0$ *(i.e.,* $\beta < \beta_1^*$*). Let*

$$Q_1(\mu) := \rho\mu^2 - (\rho f_1 + g_2 + f_2\chi v_1^*)\mu + D,$$
$$Q_2(\mu) := \rho\mu^2 - (\rho f_1 - g_2 + f_2\chi v_1^*)\mu - D.$$

(i) $Q_1(\mu) > 0$ *for all* $\mu \geq 0$ *if either of the following conditions holds:*

$$\begin{aligned} &(a)\, \beta \leq \beta_* := \left(\frac{\sqrt{-f_2\overline{g}_1} + \sqrt{f_1\overline{g}_2 - f_2\overline{g}_1}}{\overline{g}_2}\right)^2 \rho \text{ and } \chi > 0; \\ &(b)\, \beta > \beta_* \text{ and } \chi > \chi_* := \frac{\rho f_1 + g_2 - 2\sqrt{\rho D}}{-f_2 v_1^*}. \end{aligned} \tag{3.2}$$

(ii) $Q_1(\mu) = 0$ *has two distinct positive roots, which we denote as*

$$\mu_*^- \text{ and } \mu_*^+ \text{ with } \mu_*^- < \mu_*^+,$$

if

$$\beta > \beta_* \text{ and } \chi < \chi_*. \tag{3.3}$$

(iii) $Q_2(\mu) = 0$ *has only one positive root, which we denote as*

$$\widehat{\mu}_* := \frac{\rho f_1 - g_2 + f_2\chi v_1^* + \sqrt{(\rho f_1 - g_2 + f_2\chi v_1^*)^2 + 4\rho D}}{2\rho}.$$

For reference, the relationship between β_* and β_1^* observed above is given by

$$\beta_* < \beta_1^* \iff \rho < \rho^* := \left(\frac{\overline{g}_2}{\sqrt{f_1\overline{g}_2 - f_2\overline{g}_1} + \sqrt{-f_2\overline{g}_1}}\right)^2 \beta_1^*. \tag{3.4}$$

Proposition 3.4. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1). *Furthermore, assume that* $\beta_1^* > 0$ *and* $T < 0$ *(i.e.,* $\beta < \beta_1^*$*). Let*

$$P(\mu) := (\rho^2 + 1)\mu^2 + 2(f_2\chi v_1^* - f_1)\mu + f_1^2 - g_2^2.$$

Then, $P(\mu) \geq 0$ *for all* $\mu \geq 0$ *if*

$$\chi \leq \frac{-f_1 + \sqrt{(\rho^2 + 1)(f_1^2 - g_2^2)}}{-f_2 v_1^*} := \chi_0^*. \tag{3.5}$$

Here, $\chi_0^* > \chi_*$ *holds. On the other hand, if* $\chi > \chi_0^*$*, then* $P(\mu) = 0$ *has two distinct positive roots, which we denote as*

$$P_\pm := \frac{f_1 - f_2\chi v_1^* \pm \sqrt{(f_1 - f_2\chi v_1^*)^2 - (\rho^2 + 1)(f_1^2 - g_2^2)}}{\rho^2 + 1}.$$

From here on, for simplicity, let

$$Q(\mu) := Q_1(\mu)Q_2(\mu).$$

Now, using Propositions 3.3 and 3.4, we determine the signs of $P(\mu)$, $Q(\mu)$, and $(P(\mu))^2 - 4Q(\mu)$.

Proposition 3.5. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1). *Furthermore, assume that* $\beta_1^* > 0$ *and* $T < 0$ *(i.e.,* $\beta < \beta_1^*$*). Let*

$$M(\beta) := (f_1\overline{g}_2 - f_2\overline{g}_1)\beta - \frac{\rho^2 + \rho + 1}{\rho^2 + 1}(f_1^2 - (\overline{g}_2)^2\beta^2) + \frac{\rho f_1 + f_1 - \overline{g}_2\beta}{\sqrt{\rho^2 + 1}}\sqrt{f_1^2 - (\overline{g}_2)^2\beta^2}.$$

Then, there exists a unique $\beta_0^* \in (0, \beta_1^*)$ *such that* $M(\beta_0^*) = 0$*. In particular,* $M(\beta) \leq 0$ *for* $0 \leq \beta \leq \beta_0^*$*, and* $M(\beta) > 0$ *for* $\beta_0^* < \beta < \beta_1^*$*.*

Proposition 3.6. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1). *Furthermore, assume that* $\beta_1^* > 0$ *and* $\beta_0^* < \beta < \beta_1^*$*. Let*

$$P_0 := \frac{f_1 - f_2 v_1^*\chi}{\rho^2 + 1}.$$

If $\chi_0^* < \chi$*, then there exists a unique* $\chi_1^* > \chi_0^*$ *such that* $P_+ = \widehat{\mu}_*$ *at* $\chi = \chi_1^*$*, and furthermore, there exists a unique* $\chi_2^* > \chi_1^*$ *such that* $P_0 = \widehat{\mu}_*$ *at* $\chi = \chi_2^*$*. Thus, more specifically, if* $\chi_0^* < \chi \leq \chi_1^*$*, then* $P_+ \leq \widehat{\mu}_*$ *holds, and if* $\chi_1^* < \chi \leq \chi_2^*$*, then* $P_0 \leq \widehat{\mu}_* < P_+$ *holds.*

For a better understanding of the results in the above proposition, see Figure 2.

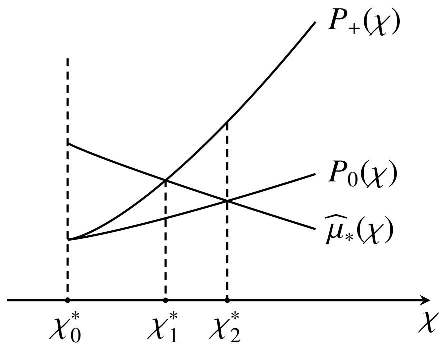
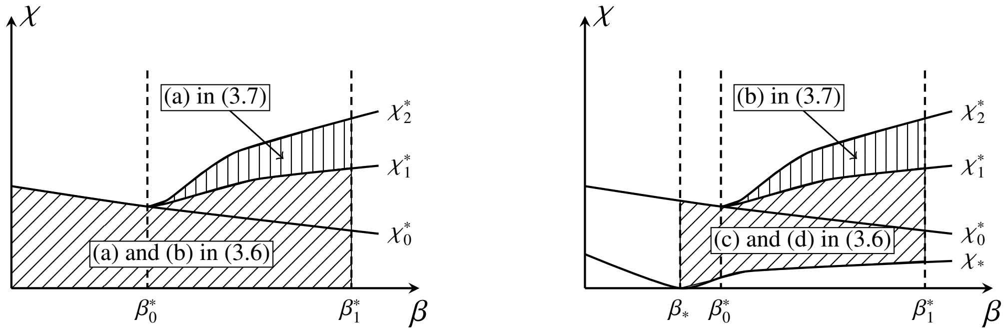


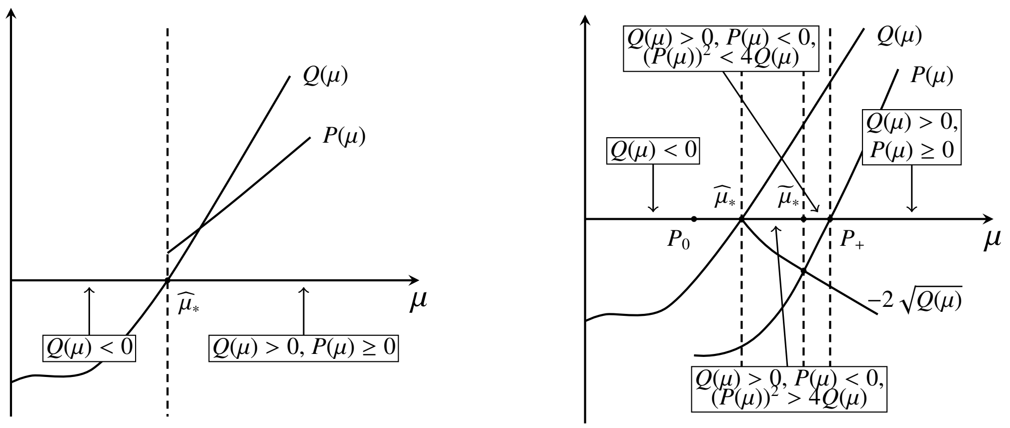
Figure 2. The graphs of $P_+(\chi)$, $P_0(\chi)$, and $\widehat{\mu}_*(\chi)$ when $\beta_0^* < \beta < \beta_1^*$.

Now, using Propositions 3.5 and 3.6, we obtain the following results, which are summarized in Figures 3 and 4.



(a) When $\rho \geq \rho_*$ (b) When $\rho < \rho_*$ and $\beta_0^* > \beta_*$

Figure 3. The β-χ region satisfying (3.6) and (3.7).



(a) When one of (3.6) holds (b) When one of (3.7) holds

Figure 4. The diagram of the results of Proposition 3.7.

Proposition 3.7. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1). *Furthermore, assume that* $\beta_1^* > 0$.

(i) Assume that one of the following conditions is satisfied:

$$\begin{aligned}
&(a)\ \rho \geq \rho^*,\ \beta < \beta_1^* \text{ and } \chi \leq \chi_0^*;\\
&(b)\ \rho \geq \rho^*,\ \beta_0^* < \beta < \beta_1^* \text{ and } \chi_0^* < \chi \leq \chi_1^*;\\
&(c)\ \rho < \rho^*,\ \beta_* < \beta < \beta_1^* \text{ and } \chi_* < \chi \leq \chi_0^*;\\
&(d)\ \rho < \rho^*,\ \max\{\beta_0^*, \beta_*\} < \beta < \beta_1^* \text{ and } \chi_0^* < \chi \leq \chi_1^*.
\end{aligned} \tag{3.6}$$

Then,

$$\begin{aligned}
&Q(\mu) < 0 \text{ for } 0 \leq \mu < \widehat{\mu}_*;\\
&Q(\mu) > 0 \text{ and } P(\mu) \geq 0 \text{ for } \mu > \widehat{\mu}_*.
\end{aligned}$$

(ii) Assume that one of the following conditions holds:

$$\begin{aligned}
&(a)\ \rho \geq \rho^*,\ \beta_0^* < \beta < \beta_1^* \text{ and } \chi_1^* < \chi \leq \chi_2^*;\\
&(b)\ \rho < \rho^*,\ \max\{\beta_0^*, \beta_*\} < \beta < \beta_1^* \text{ and } \chi_1^* < \chi \leq \chi_2^*.
\end{aligned} \tag{3.7}$$

Then,

$$Q(\mu) < 0 \text{ for } 0 \le \mu < \widehat{\mu}_*;$$
$$Q(\mu) > 0 \text{ and } P(\mu) < 0 \text{ for } \widehat{\mu}_* < \mu < P_+;$$
$$Q(\mu) > 0 \text{ and } P(\mu) \ge 0 \text{ for } \mu \ge P_+.$$

Moreover, there exists a unique $\widetilde{\mu}_* \in (\widehat{\mu}_*, P_+)$ *such that* $P(\widetilde{\mu}_*) = -2\sqrt{Q(\widetilde{\mu}_*)}$*, and thus,*

$$Q(\mu) > 0,\ P(\mu) < 0 \text{ and } (P(\mu))^2 > 4Q(\mu) \text{ for } \widehat{\mu}_* < \mu < \widetilde{\mu}_*;$$
$$Q(\mu) > 0,\ P(\mu) < 0 \text{ and } (P(\mu))^2 < 4Q(\mu) \text{ for } \widetilde{\mu}_* < \mu < P_+;$$
$$Q(\mu) > 0,\ P(\mu) \ge 0 \text{ for } \mu \ge P_+.$$

Remark 3.8. *(i) We know from Proposition 3.5 that* $M(\beta) \to 0$ *as* $\beta \to \beta_0^*+$*. As a result, from the proof of Proposition 3.6, we obtain that* χ_1^*, $\chi_2^* \to \chi_0^*$ *as* $\beta \to \beta_0^*+$ *(see Figure 2). Based on this, the* β-χ *region satisfying* (3.6) *and* (3.7) *is depicted in Figure 3. Additionally, as* $\rho \to \rho^*-$*, we have* $\beta_* > \beta_0^*$*, whereas as* $\rho \to 0+$*, we obtain* $\beta_* < \beta_0^*$*.*

(ii) In Proposition 3.7(ii), the values of μ *satisfying* $(P(\mu))^2 > 4Q(\mu)$ *were investigated. To this end, in Proposition 3.6,* $P_0 \in (P_-, P_+)$ *is considered. This consideration arises because, for* $\mu \in (P_-, P_0)$, $-2\sqrt{Q(\mu)}$ *is concave upward, complicating the identification of a* μ *that fulfills the condition* $(P(\mu))^2 > 4Q(\mu)$*. In other words, by ensuring that* $\widehat{\mu}$ *lies between* P_- *and* P_0*, the* χ *given in* (3.7) *can be further extended. As a result, one of our main results in the following section, Theorem 4.8, can also be extended accordingly.*

The following result will later be used in deriving the critical values of τ associated with Hopf bifurcation.

Proposition 3.9. *Assume that* $\mathbf{u}_1^*$ *exists, and let* $\mathbf{u}_* = \mathbf{u}_1^*$ *in* (3.1)*. Furthermore, assume that* $\beta_1^* > 0$ *and* $T < 0$ *(i.e.,* $\beta < \beta_1^*$*). Let*

$$S(\mu) := g_2((\rho - 1)g_1 + \chi v_1^* g_2)\mu + g_1 D.$$

Then, $S(\mu) \ge 0$ *for* $0 \le \mu \le \widehat{\mu}_*$*.*

4. Stability and bifurcations

In this section, we examine the delay-induced instability and stability of a positive constant steady state of system (1.1) by selecting either the time delay τ or the taxis rate χ as the primary parameter. Additionally, we investigate the occurrence of Hopf and steady-state bifurcations.

We let

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$$

be the eigenvalues of $-\Delta$ in Ω under the homogeneous Neumann boundary condition on $\partial\Omega$. We further assume that

(H1) For all $n \in \mathbb{N} \cup \{0\}$, μ_n is a simple eigenvalue with a one-dimensional eigenspace.

We note that this assumption holds at least when Ω is a one-dimensional interval. When $\mathbf{u}_* = (u_*, v_*)$ is a positive constant steady state of (1.1), the linearization of (1.1) at $\mathbf{u}_*$ can be expressed as follows:

$$\begin{aligned} &\begin{pmatrix}\phi_t(t,x)\\ \psi_t(t,x)\end{pmatrix} = \mathcal{D}\begin{pmatrix}\phi(t,x)\\ \psi(t,x)\end{pmatrix} + \mathcal{L}_0\begin{pmatrix}\phi(t,x)\\ \psi(t,x)\end{pmatrix} + \mathcal{L}_\tau\begin{pmatrix}\phi(t-\tau,x)\\ \psi(t-\tau,x)\end{pmatrix}, \ t>0, \ \ x\in\Omega,\\ &\frac{\partial\phi(t,x)}{\partial\nu} = 0, \ \ \frac{\partial\psi(t,x)}{\partial\nu} = 0, \ \ t>0, \ \ x\in\partial\Omega, \end{aligned} \tag{4.1}$$

where

$$\mathcal{D} := \begin{pmatrix}\Delta & 0\\ -\chi v_*\Delta & \rho\Delta\end{pmatrix}, \ \ \mathcal{L}_0 := \begin{pmatrix} f_1 & f_2\\ 0 & 0\end{pmatrix}, \ \ \mathcal{L}_\tau := \begin{pmatrix}0 & 0\\ g_1 & g_2\end{pmatrix}.$$

Using the Fourier expansions in (4.1), we can obtain that the characteristic equation of (4.1) is

$$\det\left(\lambda I + \mathcal{D}_n - \mathcal{L}_0 - \mathcal{L}_\tau e^{-\lambda\tau}\right) = 0, \tag{4.2}$$

where I is the identity matrix of size 2×2, and

$$\mathcal{D}_n := \begin{pmatrix}\mu_n & 0\\ -\chi v_*\mu_n & \rho\mu_n\end{pmatrix}.$$

Then, we see that all the characteristic roots of (4.2) are determined by the following sequence of characteristic equations:

$$\Delta_n(\lambda,\tau) := \lambda^2 + A_n\lambda + B_n - e^{-\lambda\tau}(g_2\lambda + C_n) = 0, \ \ n\in\mathbb{N}\cup\{0\}, \tag{4.3}$$

where

$$\begin{aligned} A_n &:= (\rho+1)\mu_n - f_1,\\ B_n &:= \mu_n\left(\rho\mu_n - \rho f_1 - f_2\chi v_*\right),\\ C_n &:= g_2\mu_n - D. \end{aligned}$$

We first consider the case where $\tau = 0$ and $n = 0$ in (4.3). In this case, (4.3) simplifies to

$$\lambda^2 - T\lambda + D = 0. \tag{4.4}$$

Thus, if both $T < 0$ and $D > 0$ are satisfied, then (4.4) has only roots with negative real parts. Therefore, based on (2.5) and Proposition 3.2, we derive the following stability result.

Theorem 4.1. *The positive constant steady state* $\mathbf{u}_1^*$ *of the ODE system corresponding to* (1.1) *with* $\tau = 0$ *is stable if any one of the following conditions is satisfied:*

$$\begin{aligned} &\textit{(i)} \ \ \frac{m\gamma}{1+m\gamma} < \frac{\gamma}{1-m\gamma} < 1, \ \ \alpha\gamma > \alpha_0^*, \ \ \textit{and} \ \ \beta < \beta_1^*;\\ &\textit{(ii)} \ \ \frac{\gamma}{1-m\gamma} = 1, \ \ \left(\frac{m\gamma}{1+m\gamma}\right)^2 < \alpha\gamma < 1, \ \ \textit{and} \ \ \beta < \beta_1^*;\\ &\textit{(iii)} \ \ \frac{\gamma}{1-m\gamma} > 1, \ \ \alpha_0^* < \alpha\gamma < \alpha_1^*, \ \ \textit{and} \ \ \beta < \beta_1^*. \end{aligned} \tag{4.5}$$

Remark 4.2. *(i) In the inequality $\alpha_0^* < \alpha\gamma < \alpha_1^*$, which is presented in sufficient condition (iii) of the aforementioned theorem regarding the stability of $\mathbf{u}_1^*$, we note that the following condition is satisfied:*

$$\alpha_1^* \to 1, \;\; \alpha_0^* \to \left(\frac{1-\gamma}{2-\gamma}\right)^2, \;\; \textit{as } m\gamma \to (1-\gamma)^+.$$

Thus, when β is small, the set $\{(\alpha, \beta, \gamma, m) :$ (iii) in (4.5) holds$\}$ is nonempty.

(ii) In the ODE system corresponding to (1.1) with $\tau = 0$, we can further derive the stability results presented below. By applying Proposition 3.1(ii) and employing a discussion analogous to that in the preceding, we can ascertain the stability results at $(1,0)$:

$$(1,0) \textit{ is stable if } 1 - m\gamma < \gamma.$$

Therefore, according to this result and the aforementioned theorem, if (iii) given in (4.5) is satisfied, we can observe bistability, where both the predator-free state $(1,0)$ and the coexistence state $\mathbf{u}_1^*$ are stable simultaneously. However, when $\mathbf{u}_* = (1,0)$ or $\mathbf{u}_2^*$, Proposition 3.1(i) and 3.2(ii) indicate that $D < 0$. Consequently, the states $(0,0)$ and $\mathbf{u}_2^*$ (if it exists) are always unstable. When $\mathbf{u}_* = \mathbf{u}_d^*$, $D = 0$, which implies that one eigenvalue has a zero real part. Thus, a saddle-node bifurcation occurs at a critical value of the parameter $\alpha\gamma = \alpha_1^*$ (e.g., see [28] and the references therein).

From here on, to investigate the instability at $\mathbf{u}_1^*$ induced by diffusion, taxis rate, or time delay, we set $\mathbf{u}_* = \mathbf{u}_1^*$ in (3.1). Furthermore, we will only consider the scenario where $T < 0$. As previously established, we reiterate that $D > 0$ always holds at $\mathbf{u}_* = \mathbf{u}_1^*$ (see Proposition 3.2). Since the conditions provided in (4.5) ensure the existence of $\mathbf{u}_1^*$ and further guarantee that $T < 0$ and $D > 0$ at $\mathbf{u}_1^*$, we will only consider cases where at least one of the conditions in (4.5) is satisfied.

By substituting $\tau = 0$ into (4.3), we derive that

$$\lambda^2 + (A_n - g_2)\lambda + B_n - C_n = 0, \quad n \in \mathbb{N} \cup \{0\}.$$

Through previous discussions, we have established that if $T < 0$, $D > 0$, and one of the conditions in (3.2) is satisfied, then for all $n \geq 0$, the inequalities $A_n - g_2 > 0$ and $B_n - C_n = Q_1(\mu_n) > 0$ hold. However, in the same situation, if (3.3) is given instead of (3.2), then $B_n - C_n < 0$ for n such that $\mu_*^- < \mu_n < \mu_*^+$. Therefore, by connecting this discussion with Theorem 4.1 and Remark 4.2(ii), we can derive the following straightforward result.

Theorem 4.3. *(i) Assume that one of the conditions in (4.5) is satisfied. Then, we obtain the following results regarding Turing stability and instability in system (1.1) with $\tau = 0$:*

(a) If one of the conditions in (3.2) is satisfied, then $\mathbf{u}_1^*$ is locally asymptotically stable, so that Turing instability does not occur at $\mathbf{u}_1^*$.

(b) If (3.3) is satisfied and there exists an n such that $\mu_*^- < \mu_n < \mu_*^+$, then Turing instability occurs at $\mathbf{u}_1^*$.

(ii) Assume that the third condition in (4.5) is satisfied. Then, $\mathbf{u}_2^*$ is unstable.

Next, consider the case when $\lambda = 0$ in (4.3). Then, we have

$$0 = B_n - C_n = Q_1(\mu_n). \tag{4.6}$$

According to Proposition 3.3(i), if either of the two options in (3.2) is satisfied, then (4.6) does not hold. In other words, $\lambda = 0$ cannot be a solution of (4.3), which means that if a condition in (4.5) and a condition in (3.2) are both satisfied, then the steady-state bifurcation at $\mathbf{u}_1^*$ does not occur in system (1.1).

To investigate the steady-state bifurcation at $\mathbf{u}_1^*$ (if it exists), we treat the taxis rate χ as the main parameter. We consider the case when (3.3) is satisfied. According to Proposition 3.3(ii), we know that $Q_1(\mu) = 0$ has two distinct positive roots, μ_*^- and μ_*^+. If $\mu_{n_1} = \mu_*^-$ for some $n_1 \in \mathbb{N}$, in other words,

$$\chi = \frac{\rho\mu_{n_1}^2 - (\rho f_1 + g_2)\mu_{n_1} + D}{f_2 v_1^* \mu_{n_1}} := \chi_{n_1}^*,$$

then it follows that $B_{n_1} = C_{n_1}$. Similarly, if $\mu_{n_2} = \mu_*^+$ for some $n_2 \in \mathbb{N}$ with $n_2 > n_1$, i.e., $\chi = \chi_{n_2}^*$, then $B_{n_2} = C_{n_2}$ also holds. Here, using the definitions of μ_{n_1} and μ_{n_2}, we note that $0 < \chi_{n_1}^*, \chi_{n_2}^* < \chi_*$ holds. Thus, if such an n_1 or n_2 exists, then when $\chi = \chi_{n_1}^*$ or $\chi_{n_2}^*$, (4.3) has $\lambda = 0$ as a root. Furthermore, since $T < 0$ and $C_{n_i} = B_{n_i} > 0$, we obtain

$$\left.\frac{\partial \Delta_n}{\partial \lambda}\right|_{\lambda=0} = A_{n_i} - g_2 + \tau C_{n_i} > 0,$$

which implies that $\lambda = 0$ is a simple eigenvalue of (4.3). When we denote $\lambda(\chi)$ as the root of (4.3) such that $\lambda(\chi_{n_i}^*) = 0$ for $i = 1$ or 2, the transversality condition

$$\left.\frac{d\lambda(\chi)}{d\chi}\right|_{\chi=\chi_{n_i}^*} = \frac{f_2 v_1^* \mu_{n_i}}{A_{n_i} - g_2 + \tau C_{n_i}} < 0$$

follows from (4.3), the fact that $f_2 < 0$ and $C_{n_i} = B_{n_i} > 0$, and the assumption that $T < 0$. Hence, we can apply the simple eigenvalue theorem [9] (for an example of its application, see [34,38]) to establish the following steady-state bifurcation result. The detailed proof process is omitted here.

Theorem 4.4. *Assume that (H1), one of the conditions in* (4.5)*, and* (3.3) *hold. If, for some* $n_1, n_2 \in \mathbb{N}$,

$$\textit{either } \mu_*^- = \mu_{n_1} \textit{ or } \mu_*^+ = \mu_{n_2}$$

holds, then system (1.1) undergoes a steady-state bifurcation near $\mathbf{u}_1^*$ *at* $\chi = \chi_{n_1}^*$ *or* $\chi_{n_2}^*$*. More specifically, when* $n = n_1$ *or* n_2*, there exists a positive constant* ϵ *such that positive nonconstant steady-state solutions of* (1.1) *lie on a curve*

$$\Gamma_n = \{(\chi_n(s), u_n(s), v_n(s)) : -\epsilon < s < \epsilon\},$$

where $\chi_n(s)$ *is a continuous function such that* $\chi_n(0) = \chi_{n_1}^*$ *or* $\chi_{n_2}^*$*;* $u_n(s) = u_1^* + a_1 s\varphi_n + sU_n(s)$ *and* $v_n(s) = v_1^* + a_2 s\varphi_n + sV_n(s)$ *for* $a_1, a_2 \in \mathbb{R}$*, continuous functions* $U_n(s)$ *and* $V_n(s)$ *satisfying* $U_n(0) = V_n(0) = 0$*, and the eigenvector* φ_n *corresponding to the eigenvalue* μ_n*. Here,* Γ_n *bifurcates from* $(\chi, u, v) = (\chi_{n_1}^*, \mathbf{u}_1^*)$ *or* $(\chi_{n_2}^*, \mathbf{u}_1^*)$.

Remark 4.5. *From Theorems 4.3 and 4.4, if one of the conditions in* (4.5) *holds, along with* $\beta_* < \beta < \beta_1^*$ *and* $\rho < \rho^*$*, then we observe that as* χ *decreases,* $\mathbf{u}_1^*$ *loses its stability. Furthermore, if* χ *decreases past the critical value* χ_**, we observe a steady-state bifurcation in* (1.1)*, giving rise to spatially nonhomogeneous steady-state solutions.*

We now focus on the stability changes of $\mathbf{u}_1^*$ induced solely by the time delay τ. Hence, based on Theorem 4.3(i), we assume that Turing instability does not occur, i.e., one of the conditions in (3.2) is satisfied. We know that this stability change occurs only when $\Delta_n = 0$ in (4.3) has purely imaginary roots. To this end, suppose that $\lambda = \pm wi$ ($w > 0$) is a root of $\Delta_n = 0$. Then, we have

$$-w^2 + A_n wi + B_n - e^{-iw\tau}(g_2 wi + C_n) = 0,$$

from which we further obtain

$$\begin{aligned} -w^2 + B_n - C_n \cos(\tau w) - g_2 w \sin(\tau w) &= 0, \\ A_n w - g_2 w \cos(\tau w) + C_n \sin(\tau w) &= 0. \end{aligned} \tag{4.7}$$

Using these results, the following expressions can be easily derived:

$$\begin{aligned} \sin(\tau w) &= -\frac{(g_2 w^2 - g_2 B_n + A_n C_n) w}{(g_2 w)^2 + C_n^2}, \\ \cos(\tau w) &= \frac{(g_2 A_n - C_n) w^2 + B_n C_n}{(g_2 w)^2 + C_n^2}. \end{aligned} \tag{4.8}$$

Furthermore, we obtain

$$W(w) := w^4 + (A_n^2 - g_2^2 - 2B_n) w^2 + B_n^2 - C_n^2 = 0. \tag{4.9}$$

Obviously,

$$A_n^2 - g_2^2 - 2B_n = P(\mu_n) \text{ and } B_n^2 - C_n^2 = Q(\mu_n).$$

According to Proposition 3.3(i), we know that if any of the conditions in (3.2) is satisfied, then $B_n - C_n = Q_1(\mu_n) > 0$ holds for all $n \geq 0$. However, through a simple calculation, we find that $B_n + C_n = Q_2(\mu_n)$, where $Q_2(\mu_n) < 0$ for $\mu_n < \widehat{\mu}_*$ according to Proposition 3.3(iii). Here, we additionally assume that

(H2) There exists an $N \geq 0$ such that $\mu_N < \widehat{\mu}_* < \mu_{N+1}$.

Then, we see that for $0 \leq n \leq N$, $Q(\mu_n) < 0$; otherwise, $Q(\mu_n) > 0$.

We first consider the case where $0 \leq n \leq N$. In this case, (4.9) has a unique positive root, which we denote as w_n^+, and it can be expressed as follows:

$$(w_n^+)^2 = \frac{-P(\mu_n) + \sqrt{(P(\mu_n))^2 - 4Q(\mu_n)}}{2}. \tag{4.10}$$

In this case, $\Delta_n = 0$ has a pair of purely imaginary roots, $\pm w_n^+ i$. To find the critical value of τ corresponding to w_n^+, we examine the sign of $g_2(w_n^+)^2 - g_2 B_n + A_n C_n$ for $0 \leq n \leq N$. Since $T < 0$, $-f_1 + g_2 > 0$ and $f_2 < 0$ hold, and according to Proposition 3.9, $S(\mu) \geq 0$ for $\mu_n \leq \widehat{\mu}_*$, we obtain that

$$W\left(\frac{g_2 B_n - A_n C_n}{g_2}\right) = \frac{(-f_2)((\rho+1)\mu_n - T)((\rho+1)\mu_n - f_1 + g_2) S(\mu_n)}{g_2^2} \geq 0$$

for all $0 \leq n \leq N$. Thus, it follows from the geometric property of $W(w)$ that $(w_n^+)^2 \leq \frac{g_2 B_n - A_n C_n}{g_2}$ for all $0 \leq n \leq N$. As a consequence, (4.8) gives that $\sin(\tau w_n^+) \geq 0$ for all $0 \leq n \leq N$. Correspondingly, for each $0 \leq n \leq N$, we define

$$\tau_n^{j+} := \frac{1}{w_n^+}\left[\arccos\left(\frac{(g_2 A_n - C_n)(w_n^+)^2 + B_n C_n}{(g_2 w_n^+)^2 + C_n^2}\right) + 2j\pi\right], \quad j = 0, 1, 2, \cdots .$$

Through this discussion and Theorem 4.3, we obtain the following result concerning the stability and the occurrence of Hopf bifurcation at $\mathbf{u}_1^*$.

Theorem 4.6. *Assume that (H1), (H2), one of the conditions in* (4.5)*, and one of the conditions in* (3.6) *hold. Let*

$$\tau^* := \min\{\tau_n^{0+} : 0 \leq n \leq N\}.$$

Then, the following statements hold true:

(i) If $0 \leq \tau < \tau^*$*, then the positive constant steady state* $\mathbf{u}_1^*$ *of* (1.1) *is locally asymptotically stable.*

(ii) If $\tau > \tau^*$*, then* $\mathbf{u}_1^*$ *is unstable.*

(iii) For $0 \leq n \leq N$*, when* $\tau = \tau_n^{j+}$ *(for* $j = 0, 1, 2, \cdots$*), system* (1.1) *experiences a spatially homogeneous or nonhomogeneous Hopf bifurcation (i.e., appearance of a periodic solution) at* $\mathbf{u}_1^*$.

Proof. We first note that $T < 0$ and $D > 0$ at $\mathbf{u}_* = \mathbf{u}_1^*$, since one of the conditions in (4.5) holds. Moreover, due to Proposition 3.7(i) and the assumption that one of the conditions in (3.6) is satisfied, we see that $Q(\mu_n) < 0$ for $0 \leq n \leq N$; $Q(\mu_n) > 0$ and $P(\mu_n) \geq 0$ for $n \geq N+1$. Thus, for $n \geq N+1$, (4.9) does not have any positive roots, but for $0 \leq n \leq N$, (4.9) has a unique positive root w_n^+, so that the critical delay sequence $\{\tau_n^{j+}\}_{j=0}^{\infty}$ exists only for $0 \leq n \leq N$. Moreover, the conditions in (3.6) imply (3.2). Thus, according to Theorem 4.3(i), the positive constant steady state $\mathbf{u}_1^*$ in (1.1) without time delay is locally asymptotically stable. We also note that it is obvious that for $n = 0, 1, 2, \cdots, N$, the sequence $\{\tau_n^{j+}\}_{j=0}^{\infty}$ is increasing with respect to j, so that $\tau^* = \min\{\tau_n^{j+} : n = 0, 1, 2, \cdots, N, \ j = 0, 1, 2, \cdots\}$.

Now, to verify whether the transversality condition holds, we calculate $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n^{j+}}$. Differentiating both sides of (4.3) with respect to τ, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{\tau}{\lambda} - \frac{e^{\lambda\tau}(2\lambda + A_n) - g_2}{\lambda(g_2\lambda + C_n)}.$$

At $\tau = \tau_n^{j+}$, we have $\lambda = w_n^+ i$, thus we obtain the following:

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_n^{j+}} &= \frac{\tau_n^{j+}}{w_n^+} i + \frac{e^{\tau_n^{j+} w_n^+ i}(2w_n^+ i + A_n) - g_2}{w_n^+(g_2 w_n^+ - C_n i)} \\
&= \frac{\tau_n^{j+}}{w_n^+} i + \left[\frac{A_n \cos(\tau_n^{j+} w_n^+) - 2w_n^+ \sin(\tau_n^{j+} w_n^+) - g_2}{w_n^+((g_2 w_n^+)^2 + (C_n)^2)}\right. \\
&\qquad \left. + i\frac{2w_n^+ \cos(\tau_n^{j+} w_n^+) + A_n \sin(\tau_n^{j+} w_n^+)}{w_n^+((g_2 w_n^+)^2 + (C_n)^2)}\right](g_2 w_n^+ + C_n i).
\end{aligned}$$

From this, we can easily derive

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_n^{j+}} = \frac{\Lambda}{w_n^+((g_2 w_n^+)^2 + (C_n)^2)},$$

where

$$\Lambda = (A_n \cos(\tau_n^{j+} w_n^+) - 2w_n^+ \sin(\tau_n^{j+} w_n^+) - g_2)g_2 w_n^+ - (2w_n^+ \cos(\tau_n^{j+} w_n^+) + A_n \sin(\tau_n^{j+} w_n^+))C_n.$$

Here, using (4.7), which states that

$$\begin{aligned} g_2 w_n^+ \cos(\tau_n^{j+} w_n^+) &= A_n w_n^+ + C_n \sin(\tau_n^{j+} w_n^+), \\ C_n \cos(\tau_n^{j+} w_n^+) &= -(w_n^+)^2 + B_n - g_2 w_n^+ \sin(\tau_n^{j+} w_n^+), \end{aligned}$$

and using (4.10), we obtain

$$\Lambda = (A_n^2 - g_2^2 - 2(-(w_n^+)^2 + B_n))w_n^+ = w_n^+ \sqrt{(P(\mu_n))^2 - 4Q(\mu_n)} > 0.$$

Thus, it follows that $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n^{j+}} > 0$, which implies

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Bigg|_{\tau=\tau_n^{j+}} > 0. \tag{4.11}$$

Hence, this proves the transversality condition.

Due to (4.11), at each $\tau = \tau_n^{j+}$, a pair of complex eigenvalues crosses the imaginary axis from left to right, thereby increasing the number of eigenvalues with positive real parts by two (e.g., [19, 31]). As mentioned earlier, for $n \geq N + 1$, (4.9) has no positive roots. This yields that as τ increases, instability at $\mathbf{u}_1^*$ persists after passing τ_*. From this, conclusion (ii) follows. On the other hand, by applying Rouché's lemma, and using Theorem 4.3(i), along with the definition of τ^* and the fact that (4.9) does not possess any positive roots for $n \geq N + 1$, we can conclude that all roots of $\Delta_n = 0$ in (4.3) have negative real parts when $0 \leq \tau < \tau^*$. Therefore, conclusion (i) holds. Finally, since (4.11) holds, and for $n = 0, 1, 2, \cdots, N$ and $j = 0, 1, 2, \cdots$, at the critical values τ_n^{j+}, (4.3) has purely imaginary roots $\pm w_n^+ i$, and conclusion (iii) is also established. □

Remark 4.7. *(i) In Theorem 4.6(iii), we note that when $\tau = \tau_0^{j+}$, the Hopf bifurcation at $\mathbf{u}_1^*$ is spatially homogeneous, whereas when $\tau = \tau_n^{j+}$ with $n \neq 0$, this bifurcation is spatially nonhomogeneous.*

(ii) When the assumptions in Theorem 4.6 are satisfied, as τ increases, the eigenvalues in (4.3) *cross the imaginary axis only from left to right. Consequently, once $\mathbf{u}_1^*$ loses its stability, it cannot regain stability.*

Through the discussion preceding Theorem 4.6, we know that if one of the conditions in (3.2) holds, then $Q(\mu_n) > 0$ for $n \geq N+1$. Below, we consider only the case where $n \geq N+1$ satisfies the following:

$$P(\mu_n) < -2\sqrt{Q(\mu_n)}. \tag{4.12}$$

As a reference, since $\lim_{n\to\infty} \mu_n = \infty$, it follows that as $n \to \infty$, $P(\mu_n) \to \infty$. From this, we can infer that the number of n satisfying (4.12) is finite. Apparently, when (4.12) holds, $W(w) = 0$ in (4.9) has two distinct positive roots, which we denote as w_n^- and w_n^+. Indeed, $(w_n^+)^2$ satisfies (4.10) and

$$(w_n^-)^2 = \frac{-P(\mu_n) - \sqrt{(P(\mu_n))^2 - 4Q(\mu_n)}}{2}.$$

In turn, $\Delta_n = 0$ in (4.3) has two pairs of purely imaginary roots, $\pm w_n^{\pm} i$. We need to find the critical values of τ corresponding to these roots. However, when (4.12) holds, we cannot easily determine the signs of the trigonometric functions in (4.8) as we did prior to Theorem 4.6. In fact, determining these signs requires additional complex conditions. To circumvent this issue, we define the critical values of τ as follows, categorizing them into two cases:

(i) If $g_2(w_n^{\pm})^2 - g_2 B_n + A_n C_n \leq 0$,

$$\overline{\tau}_n^{j\pm} := \frac{1}{w_n^{\pm}}\left[\arccos\left(\frac{(g_2 A_n - C_n)(w_n^{\pm})^2 + B_n C_n}{(g_2 w_n^{\pm})^2 + C_n^2}\right) + 2j\pi\right] \quad \text{for } j = 0, 1, 2, \cdots;$$

(ii) If $g_2(w_n^{\pm})^2 - g_2 B_n + A_n C_n > 0$,

$$\overline{\tau}_n^{j\pm} := \frac{1}{w_n^{\pm}}\left[2\pi - \arccos\left(\frac{(g_2 A_n - C_n)(w_n^{\pm})^2 + B_n C_n}{(g_2 w_n^{\pm})^2 + C_n^2}\right) + 2j\pi\right] \quad \text{for } j = 0, 1, 2, \cdots.$$

We are now ready to present our final theorem, focusing on the case where one of the conditions in (3.7) is satisfied. In this case, we know from Proposition 3.7(ii) and the definition of P_+ provided in Proposition 3.4 that there exists the unique $\widetilde{\mu}_* \in (\widehat{\mu}_*, P_+)$ that satisfies $P(\widetilde{\mu}_*) = -2\sqrt{Q(\widetilde{\mu}_*)}$. Accordingly, we assume that

(H3) There exists an $N_* > 0$ such that $\mu_{N_*} < \widetilde{\mu}_* < \mu_{N_*+1}$.

Thus, through Proposition 3.7(ii), we arrive at the following result (see Figure 4):

$$\begin{aligned} &Q(\mu_n) < 0 \text{ for } 0 \leq n \leq N; \\ &Q(\mu_n) > 0,\ P(\mu_n) < 0 \text{ and } (P(\mu_n))^2 > 4Q(\mu_n) \text{ for } N+1 \leq n \leq N_*; \\ &Q(\mu_n) > 0,\ P(\mu_n) < 0 \text{ and } (P(\mu_n))^2 < 4Q(\mu_n) \text{ for } n \text{ satisfying } \mu_{N_*+1} \leq \mu_n < P_+; \\ &Q(\mu_n) > 0 \text{ and } P(\mu_n) \geq 0 \text{ for } n \text{ satisfying } \mu_n \geq P_+. \end{aligned} \tag{4.13}$$

Therefore, for $0 \leq n \leq N$, (4.9) has only one positive root w_n^+; for $N+1 \leq n \leq N_*$, as mentioned before, it has two positive roots $w_n^{\pm}$; for n satisfying $n \geq N_*+1$, it has no positive roots. Accordingly, the critical delay sequences $\{\tau_n^{j+}\}_{j=0}^{\infty}$ and $\{\overline{\tau}_n^{j\pm}\}_{j=0}^{\infty}$ exist only for $0 \leq n \leq N$ and $N+1 \leq n \leq N_*$, respectively.

Theorem 4.8. *Assume that (H1), (H2), and (H3), along with one of the conditions in* (4.5) *and one of the conditions in* (3.7)*, hold. Let*

$$\overline{\tau}^* := \min\{\{\tau_n^{0+} : 0 \leq n \leq N\} \cup \{\overline{\tau}_n^{0+} : N+1 \leq n \leq N_*\}\}.$$

Then, the following statements hold true:

(i) If $0 \leq \tau < \overline{\tau}^*$, then the positive constant steady state $\mathbf{u}_1^*$ of (1.1) *is locally asymptotically stable.*

(ii) There exists a $\widehat{\tau} \geq \overline{\tau}^*$ such that $\mathbf{u}_1^*$ is unstable for $\tau > \widehat{\tau}$.

(iii) For $0 \leq n \leq N$, when $\tau = \tau_n^{j+}$ (where $j = 0, 1, 2, \cdots$), and for $N+1 \leq n \leq N_*$, when $\tau = \overline{\tau}_n^{j\pm}$ (where $j = 0, 1, 2, \cdots$), system (1.1) *undergoes a spatially homogeneous or nonhomogeneous Hopf bifurcation at $\mathbf{u}_1^*$. Furthermore, there exists an integer $n_* \geq 0$ and a corresponding set $\{\tau_*^k\}_0^{n_*}$ such that*

$$\begin{aligned} &\mathbf{u}_1^* \textit{ is locally asymptotically stable for } \tau \in [0, \tau_*^0) \cup (\tau_*^1, \tau_*^2) \cup \cdots \cup (\tau_*^{n_*-1}, \tau_*^{n_*}); \\ &\mathbf{u}_1^* \textit{ is unstable for } \tau \in (\tau_*^0, \tau_*^1) \cup (\tau_*^2, \tau_*^3) \cup \cdots \cup (\tau_*^{n_*}, \infty). \end{aligned}$$

Here, $\tau_*^0 = \overline{\tau}^*$ and $\tau_*^{n_*} \leq \widehat{\tau}$.

Proof. As mentioned before, if the given assumptions are satisfied, then there exists the sequence $\{\tau_n^{j+}\}_{j=0}^{\infty}$ for $0 \le n \le N$ and the sequence $\{\overline{\tau}_n^{j\pm}\}_{j=0}^{\infty}$ for $N+1 \le n \le N_*$; no critical delay sequence exists for $n \ge N_* + 1$. Furthermore, we know that (4.11) holds. Similar to the proof of Theorem 4.6, we can deduce that

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\overline{\tau}_n^{j\pm}} = \pm\frac{\sqrt{(P(\mu_n))^2 - 4Q(\mu_n)}}{(g_2 w_n^{\pm})^2 + (C_n)^2} \text{ for } N+1 \le n \le N_*,$$

which leads to

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Bigg|_{\tau=\overline{\tau}_n^{j+}} > 0 \text{ and } \mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Bigg|_{\tau=\overline{\tau}_n^{j-}} < 0 \text{ for } N+1 \le n \le N_*. \tag{4.14}$$

Due to (4.11) and (4.14), at each $\tau = \tau_n^{j+}$ and $\tau = \overline{\tau}_n^{j+}$, a pair of complex eigenvalues of (4.9) crosses the imaginary axis from left to right, resulting in an increase of two in the number of eigenvalues with positive real parts. Conversely, at each $\tau = \overline{\tau}_n^{j-}$, a pair of complex eigenvalues crosses the imaginary axis from right to left, leading to a decrease of two in the number of eigenvalues with positive real parts. Furthermore, since each of the conditions specified in (3.7) implies (3.2), the assumptions in this theorem ensure that Theorem 4.3(i) is applicable, confirming that $\mathbf{u}_1^*$ is locally asymptotically stable when $\tau = 0$.

We know that both sequences $\{\tau_n^{j+}\}_{j=0}^{\infty}$ and $\{\overline{\tau}_n^{j+}\}_{j=0}^{\infty}$ are increasing with respect to j. Therefore, based on the preceding discussion derived from (4.11) and (4.14), along with the definition of $\overline{\tau}^*$, we can readily conclude that result (i) holds.

Not only do we already know that at $\tau = \tau_n^{j+}$, a pair of complex eigenvalues crosses the imaginary axis from left to right, but we can also observe, based on the definition of $\overline{\tau}_n^{j\pm}$, that

$$\overline{\tau}_n^{j+1+} - \overline{\tau}_n^{j+} = \frac{2\pi}{w_n^+} < \frac{2\pi}{w_n^-} = \overline{\tau}_n^{j+1-} - \overline{\tau}_n^{j-} \text{ for } N+1 \le n \le N_*.$$

Thus, as τ increases, delays with a positive jump occur more frequently than those with a negative jump. Consequently, after a certain finite value of τ, $\mathbf{u}_1^*$ remains unstable (for a similar discussion, refer to [19, 31, 37]). Therefore, conclusion (ii) follows.

To characterize the possible stability switches at $\{\overline{\tau}_n^{j\pm}\}_{j=0}^{\infty}$, we define

$$\sigma_n^j(s) := \begin{cases} +1, & \text{if } s = \tau_n^{j+} \text{ or } \overline{\tau}_n^{j+}, \\ -1, & \text{if } s = \overline{\tau}_n^{j-}, \end{cases} \qquad \sigma_n(\tau) = \sigma_* + 2 \cdot \sum_{s \in \{\overline{\tau}_n^{j\pm} : \overline{\tau}_n^{j\pm} < \tau\}} \sigma_n^j(s),$$

where $\sigma_* = 0$, which is chosen to ensure the stability of $\mathbf{u}_1^*$ (i.e., the number of characteristic roots with positive real parts is zero) when $\tau = 0$. Moreover, $\sigma_n(\tau)$ is a piecewise constant function defined on $[0, \infty)$.

Now, focusing on deriving the result of (iii), the occurrence of Hopf bifurcation at $\mathbf{u}_1^*$ can be easily verified as in the proof of Theorem 4.6. Note that $\sigma_n(0) = 0$. For each fixed $N+1 \le n \le N_*$, as τ increases from 0, the first value of τ at which a stability switch occurs, causing $\mathbf{u}_1^*$ to change from stable to unstable, is defined using $\sigma_n(\tau)$ as follows:

$$\tau_n^0 := \max\{\tau \in T_n : \sigma_n(\tau) = 0 \text{ on } [0, \tau)\},$$

where $T_n := \{\overline{\tau}_n^{j\pm} : j = 0, 1, 2, \cdots\}$. The value τ_n^0 clearly exists due to the existence of $\overline{\tau}^*$ in (i) and $\widehat{\tau}$ in (ii). Additionally, τ_n^0 is one of the values $\overline{\tau}_n^{j+}$. As τ increases from τ_n^0, the first stability switch, where $\mathbf{u}_1^*$ changes from unstable to stable, may occur at

$$\tau_n^1 := \max\{\tau \in T_n : \sigma_n(\tau) > 0 \text{ on } (\tau_n^0, \tau)\}.$$

Clearly, τ_n^1 occurs among the $\overline{\tau}_n^{j-}$ values. We note that the value τ_n^1 may not exist, depending on the positions of $\overline{\tau}_n^{j+}$ and $\overline{\tau}_n^{j-}$. Similarly, we can consecutively define

$$\tau_n^2 := \max\{\tau \in T_n : \sigma_n(\tau) = 0 \text{ on } (\tau_n^1, \tau)\},$$

which is one of the values $\overline{\tau}_n^{j+}$. Furthermore, this process (e.g., see [31, 37] for a similar discussion) can be repeated to generate a set

$$S_n := \{\tau_n^0, \tau_n^1, \cdots, \tau_n^{k_n-1}, \tau_n^{k_n}\}$$

of delay values that satisfy $0 < \tau_n^0 < \tau_n^1 < \tau_n^2 < \cdots < \tau_n^{k_n-1} < \tau_n^{k_n}$;

$$\begin{aligned} &\sigma_n(\tau) = 0 \text{ for } \tau \in [0, \tau_n^0) \cup (\tau_n^1, \tau_n^2) \cup \cdots \cup (\tau_n^{k_n-1}, \tau_n^{k_n}); \\ &\sigma_n(\tau) > 0 \text{ for } \tau \in (\tau_n^0, \tau_n^1) \cup (\tau_n^2, \tau_n^3) \cup \cdots \cup (\tau_n^{k_n}, \infty). \end{aligned}$$

Recalling that at each $\tau = \tau_n^{j+}$ with $0 \le n \le N$, a pair of complex eigenvalues of (4.9) crosses the imaginary axis from left to right, and we know that after τ^* (which was defined in Theorem 4.6), $\mathbf{u}_1^*$ is unstable. Using S_n and σ_n, we define $\{s_*^k\}$ sequentially as follows:

$$\begin{aligned} &s_*^0 := \overline{\tau}^*, \\ &s_*^1 := \max\left\{\tau \in \cup_{n=N+1}^{N_*} S_n : \prod_{n=N+1}^{N_*} \sigma_n(\tau) = 0 \text{ on } (s_*^0, \tau)\right\}, \\ &s_*^2 := \max\left\{\tau \in \cup_{n=N+1}^{N_*} S_n : \prod_{n=N+1}^{N_*} \sigma_n(\tau) > 0 \text{ on } (s_*^1, \tau)\right\}, \\ &s_*^3 := \max\left\{\tau \in \cup_{n=N+1}^{N_*} S_n : \prod_{n=N+1}^{N_*} \sigma_n(\tau) = 0 \text{ on } (s_*^2, \tau)\right\}, \\ &\qquad\qquad\vdots \end{aligned}$$

Obviously, $s_*^0 = \overline{\tau}^* \le \tau^*$. If $\tau^* \le s_*^1$, then $\mathbf{u}_1^*$ is locally asymptotically stable for $\tau \in [0, s_*^0)$ and unstable for $\tau \in (s_*^0, \infty)$. In this case, we let $\tau_*^0 = s_*^0$, so that we have the stability switch delay value $\{\tau_*^0\}$. If $s_*^1 < \tau^* \le s_*^2$, then due to (ii), $\mathbf{u}_1^*$ is locally asymptotically stable for $\tau \in [0, s_*^0) \cup (s_*^1, \tau^*)$ and unstable for $\tau \in (s_0^*, s_*^1) \cup (\tau^*, \infty)$. In this case, we let $\tau_*^0 = s_*^0$, $\tau_*^1 = s_*^1$, and $\tau_*^2 = \tau^*$, so that we obtain the stability switch delay values $\{\tau_*^k\}_0^2$. By repeating this process, we can obtain the desired conclusions regarding stability switches. Here, we easily see that $\tau_*^{n_*} \le \tau^* \le \widehat{\tau}$ due to the definition of $\tau_*^{n_*}$ and the existence of $\widehat{\tau}$ in (ii). Therefore, the proof of (iii) is completed. □

Remark 4.9. *(i) Comparing Theorems 4.6 and 4.8, we can see that as the taxis rate χ increases, more complex Hopf bifurcations occur in system* (1.1).

In Theorem 4.8(iii), we observe that $\mathbf{u}_1^*$ *undergoes* $n_* + 1$ *stability switches. However, when* $n_* = 0$*, the stability changes only once, as in Theorem 4.6, and in this case,* $\tau_*^0 = \overline{\tau}^* = \widehat{\tau}$*. According to the proof of Theorem 4.8(iii), we know that stability switches with* $n_* \geq 2$ *can occur in exceptional cases, when* τ^* *is sufficiently large. The values of* n *in the range* $0 \leq n \leq N$ *and those in the range* $N+1 \leq n \leq N_*$ *are closely adjacent, and the functions used to determine* τ_n^{j+} *and* $\overline{\tau}_n^{j\pm}$ *are continuous. For these reasons, it may seem theoretically plausible that* τ_n^{j+} *for* $0 \leq n \leq N$ *could be much larger than* $\overline{\tau}_n^{j\pm}$ *for* $N+1 \leq n \leq N_*$*. However, we note that it is difficult to present concrete simulation examples that satisfy such cases. On the other hand, in time-delay PDEs where* τ_n^{j+} *(for* $0 \leq n \leq N$*) do not occur, it is possible to find concrete simulation examples of stability switches with* $n* \geq 2$*: for instance, see [33].*

(ii) Under the assumptions given in Theorems 4.6 and 4.8, the positive constant steady state $\mathbf{u}_1^*$ *of system* (1.1) *with* $\tau = 0$ *and the corresponding ODE system, was stable (see Theorems 4.1 and 4.3(i)). However, it can be observed that the introduction of the time-delay* τ *in* (1.1) *changes the stability of* $\mathbf{u}_1^*$ *in its neighborhood, facilitating the emergence of periodic solutions. Therefore, it can be concluded that the time-delay* τ *plays a crucial role in inducing a Hopf bifurcation in* (1.1)*.*

(iii) When considering the conditions (3.6) *and* (3.7) *assumed in Theorems 4.6 and 4.8, particularly when the diffusion rate* ρ *of the predator is small (i.e.,* $\rho < \rho^*$*), it can be observed that if prey-taxis is not introduced in* (1.1) *(i.e.,* $\chi = 0$*), the occurrence of a Hopf bifurcation, as seen in the previous theorems, cannot be expected. From this viewpoint, we can observe the role of* χ *in creating a Hopf bifurcation in* (1.1)*.*

(iv) When condition (iii) in (4.5) *is satisfied,* $\mathbf{u}_2^*$ *also exists. At this positive constant steady state, one can obtain critical delay sequences corresponding to the Hopf bifurcations described in Theorems 4.6 and 4.8. In particular, by Theorem 4.3(iii),* $\mathbf{u}_2^*$ *is unstable for all* $\tau \geq 0$*, and thus, stability switches do not occur.*

We aim to present the previously analytical results obtained in Theorems 4.6 and 4.8 through numerical simulations: similar simulation results for other population dynamics models can also be found in [30, 33, 35, 37, 43].

Remark 4.10. *(i) To verify the results of Theorem 4.6, we conduct numerical simulations under the following parameter settings:*

$$m = 0.6,\ \gamma = 0.5,\ \alpha = 0.5,\ \beta = 1,\ \rho = 0.08,\ \chi = 1,\ \Omega = (0, 2\pi). \tag{4.15}$$

Under these settings, system (1.1) *admits a unique positive constant steady state given by* $(u_1^*, v_1^*) \doteq (0.3697, 0.9321)$*. The initial functions are chosen as follows:*

$$u_0(t,x) = u_1^* + 10^{-3}\sin(x),\ v_0(t,x) = v_1^* - 10^{-3}\sin(x). \tag{4.16}$$

Under these conditions, it can be confirmed that part (i) of (4.5) *and part (a) of* (3.6) *are satisfied. According to Proposition 3.3, we have* $\widehat{\mu}_* \doteq 0.4382$*, and the Neumann eigenvalues* $\mu_n = (\frac{n}{2})^2$ *on* Ω *satisfy* $\mu_0 = 0 < \mu_1 = 0.25 < \widehat{\mu}_* < \mu_2 = 1 < \cdots$*, which verifies that* $N = 1$ *satisfies assumption (H2). Based on the definition for* τ_n^{j+}*, we obtain*

$$\tau_0^{0+} \doteq 0.7737 < \tau_1^{0+} \doteq 3.581 < \tau_0^{1+} \doteq 14.8794 < \tau_1^{1+} \doteq 22.4927 < \cdots .$$

Thus, the value of τ^* defined in Theorem 4.6 is determined as $\tau^* = \tau_0^{0+}$, which corresponds to the first Hopf bifurcation value. In Figures 5(a) and 5(b), we observe that the positive equilibrium (u_1^*, v_1^*) is asymptotically stable when $\tau = 0$ and $\tau = 0.3 < \tau^*$, respectively. In contrast, Figure 5(c) shows that (u_1^*, v_1^*) becomes unstable when $\tau = 1 > \tau^*$. Specifically, spatially homogeneous periodic oscillations of both prey and predator populations are observed in this case.

(ii) To validate the results presented Theorem 4.8, we conduct numerical simulations under the following parameter values:

$$m = 0.6,\ \gamma = 0.5,\ \alpha = 0.17,\ \beta = 0.4,\ \rho = 0.037,\ \chi = 1.6824,\ \Omega = (0, 2\pi). \tag{4.17}$$

Under this setting, system (1.1) *possesses a unique positive constant steady state given by* $(u_1^*, v_1^*) \doteq (0.2311, 2.0906)$. *The initial functions are chosen as*

$$u_0(t, x) = u_1^* + 10^{-4}\sin(\pi x),\ v_0(t, x) = v_1^* - 10^{-4}\sin(\pi x). \tag{4.18}$$

Under these conditions, it can be checked that part (i) of (4.5) *and part (a) of* (3.7) *are satisfied. According to Propositions 3.3 and 3.7, we have $\widehat{\mu}_* \doteq 0.2499$ and $\widetilde{\mu}_* \doteq 0.2502$. Thus, the Neumann eigenvalues $\mu_n = (\frac{n}{2})^2$ on Ω satisfy $\mu_0 = 0 < \widehat{\mu}_* < \mu_1 = 0.25 < \widehat{\mu}_* < \mu_2 = 1 < \cdots$, confirming that assumption (H2) holds with $N = 0$, and assumption (H3) holds with $N_* = 1$. Consequently, the critical delay sequences $\{\tau_0^{j+}\}_{j=0}^{\infty}$ and $\{\overline{\tau}_1^{j\pm}\}_{j=0}^{\infty}$ are well-defined, and system* (1.1) *undergoes Hopf bifurcations at these values. More specifically, using the definitions for τ_n^{j+} and $\overline{\tau}_1^{j\pm}$, we obtain*

$$\tau_0^{0+} \doteq 0.935 < \tau_0^{1+} \doteq 19.0415 < \overline{\tau}_1^{0+} \doteq 24.978 < \tau_0^{2+} \doteq 37.148 < \cdots < \overline{\tau}_1^{0-} \doteq 134.976 < \cdots .$$

Accordingly, the critical value of $\overline{\tau}^*$ introduced in Theorem 4.8 is determined as $\overline{\tau}^* = \tau_0^{0+}$. In Figures 6(a) and 6(b), we confirm that the positive equilibrium (u_1^*, v_1^*) remains asymptotically stable for $\tau = 0$ and $\tau = 0.3 < \tau^*$, respectively. On the other hand, Figure 6(c) illustrates the loss of stability when $\tau = 1 > \tau^*$. In this regime, both prey and predator populations exhibit clear spatially homogeneous periodic oscillations.

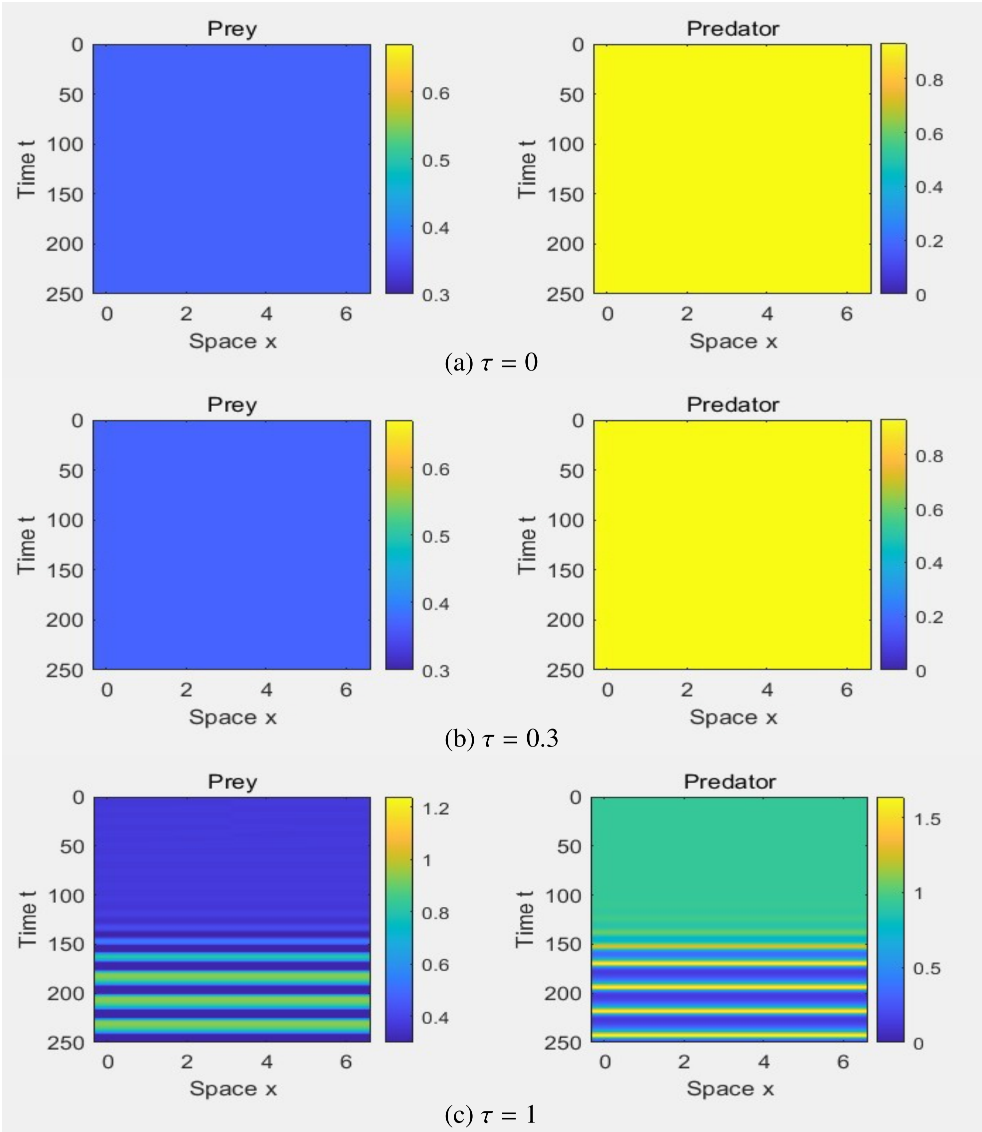
(iii) Finally, under a set of conditions not covered by Theorems 4.6 and 4.8, we investigate the emergence of spatiotemporal patterns in system (1.1) *as the delay parameter τ increases, using the following parameter settings:*

$$m = 0.6,\ \gamma = 0.5,\ \alpha = 0.5,\ \beta = 1.06,\ \rho = 1.3,\ \chi = 134,\ \Omega = (0, 2\pi). \tag{4.19}$$

With these parameters, we obtain the positive steady state $(u_1^*, v_1^*) \doteq (0.3697, 0.9321)$. *The initial functions are then chosen as*

$$u_0(t, x) = u_1^* + 10^{-6}\cos(0.5x),\ v_0(t, x) = v_1^* + 10^{-6}\cos(0.5x), \tag{4.20}$$

The selected parameters are verified to satisfy condition (i) in (4.5) *and the first two inequalities in part (a) of* (3.6), *as in (i) of this remark. Since χ is very large in* (4.19), *none of the conditions given in* (3.6) *or* (3.7) *are satisfied. As noted in Remark 3.8(ii), determining the sign of $(P(\mu))^2 - 4Q(\mu)$ with respect to μ is typically nontrivial. However, numerical simulations enable us to determine the sign of $(P(\mu_n))^2 - 4Q(\mu_n)$ as well as those of $P(\mu_n)$ and $Q(\mu_n)$ for each eigenvalue μ_n: $P(\mu_0) > 0$ and $Q(\mu_0) < 0$; $(P(\mu_n))^2 < 4Q(\mu_n)$ for $1 \le n \le 86$; $P(\mu_n) > 0$ and $Q(\mu_n) > 0$ for $n \ge 87$. This sign* *determination becomes feasible due to the asymptotic behavior of the positive roots of the polynomials $P(\mu)$, $Q(\mu)$, and $(P(\mu))^2 - 4Q(\mu)$ in the regime where χ is sufficiently large. As a consequence, we see that* (4.9) *possesses a unique positive root only in the case $n = 0$, which was denoted by w_0^+. Moreover, following an argument similar to the proof of Theorem 4.6, we obtain $\tau^* = \tau_0^{0+} \doteq 0.6823$, and thus, the conclusion of Theorem 4.6 with $N = 0$ remains valid. The corresponding numerical results are presented in Figure 7. In particular, when $\tau = 0.7 > \tau^*$, a spatially inhomogeneous periodic pattern is observed, in contrast to the spatially homogeneous oscillations shown in Figures 5(c) and 6(c).*

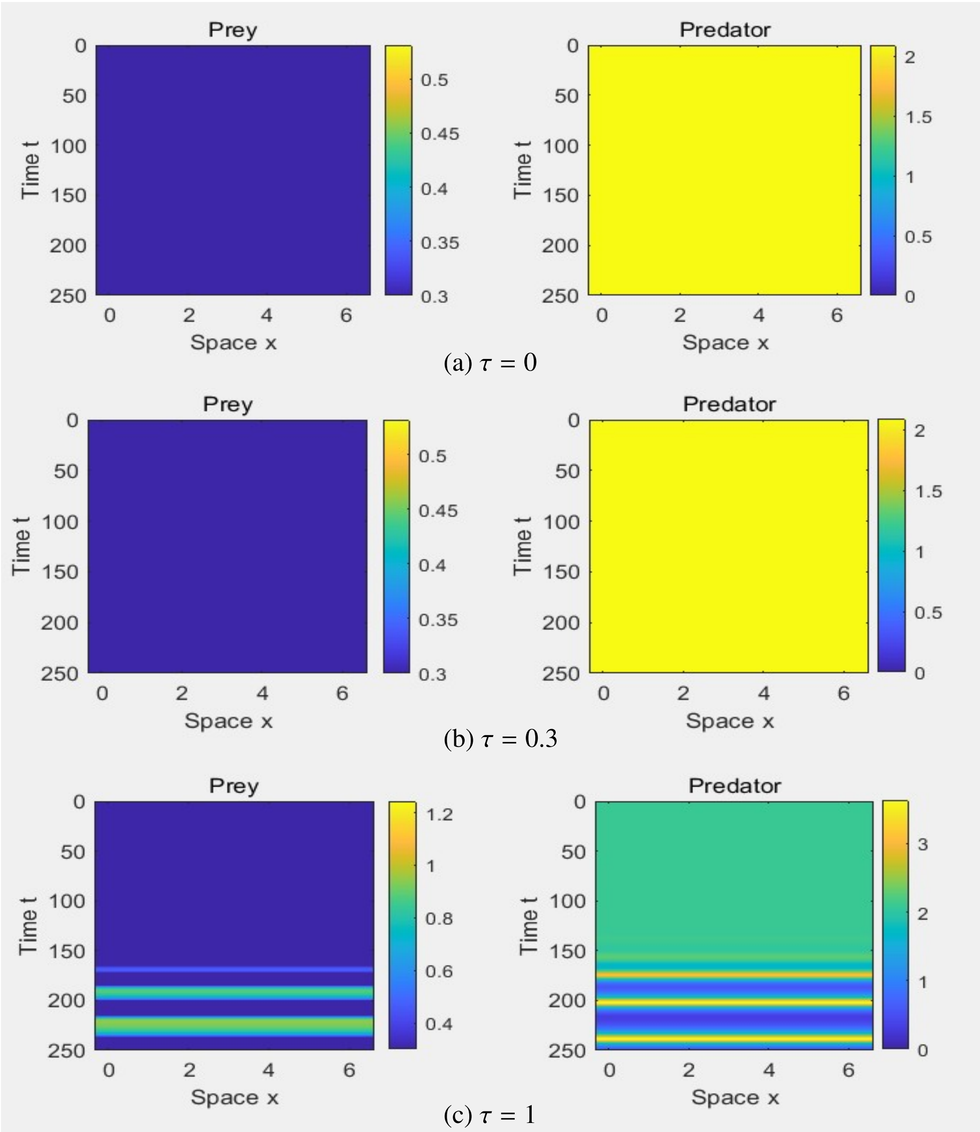


(a) $\tau = 0$

(b) $\tau = 0.3$

(c) $\tau = 1$

Figure 5. The spatiotemporal diagrams of system (1.1) with parameters given in (4.15) and initial functions in (4.16).



(a) $\tau = 0$

(b) $\tau = 0.3$

(c) $\tau = 1$

Figure 6. The spatiotemporal diagrams of system (1.1) with parameters given in (4.17) and initial functions in (4.18).

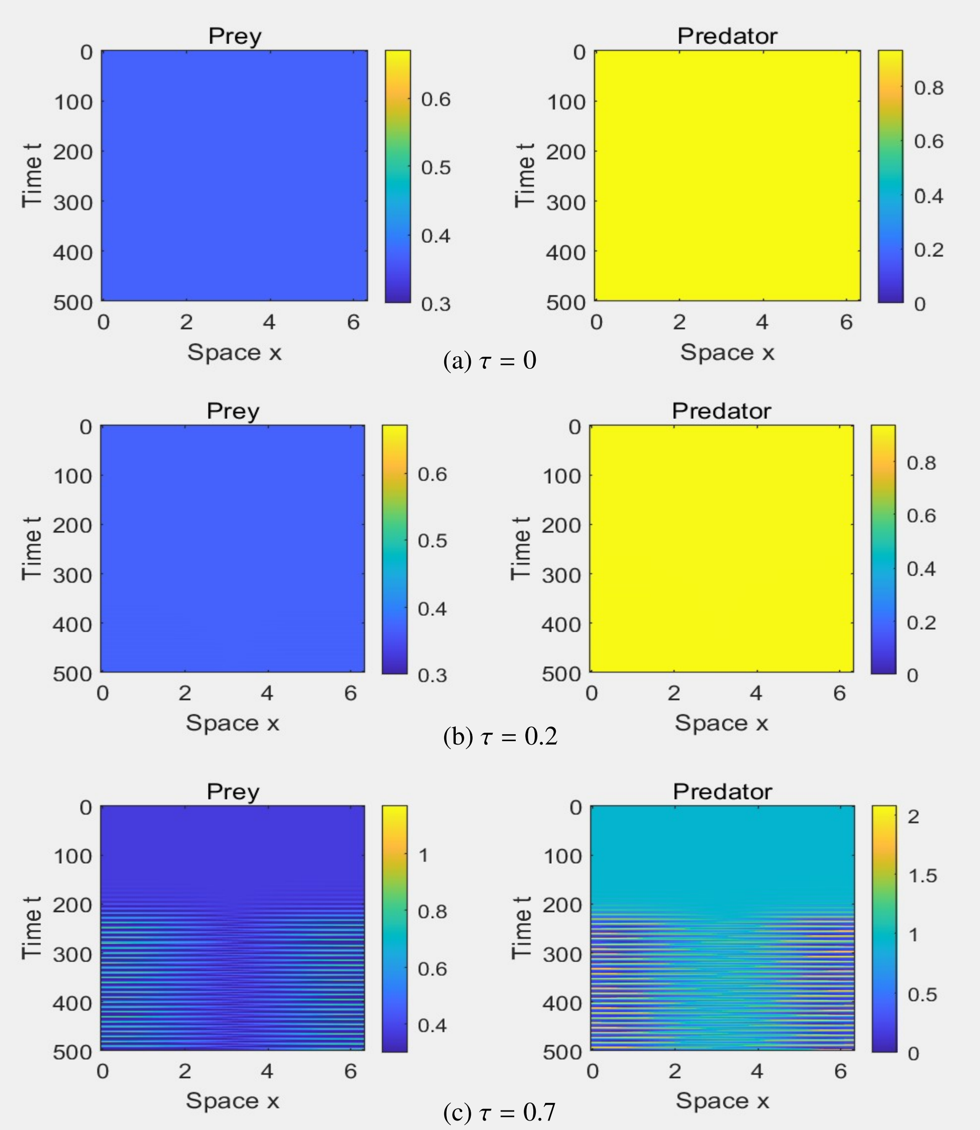


Figure 7. The spatiotemporal diagrams of system (1.1) with parameters given in (4.19) and initial functions in (4.20).

5. Conclusions

In this study, we investigated a delayed predator-prey system incorporating prey-taxis and hunting cooperation functional response. Our analysis focused on the stability of equilibria and the occurrence of steady-state and Hopf bifurcations induced by both the prey-taxis rate χ and the time-delay τ. By examining the interaction between these factors and the remaining parameters of the system, we established explicit conditions for stability changes and bifurcation phenomena, which offer insights into how predator movement and time delays govern population dynamics.

A key finding of our study is that increasing the prey-taxis rate χ introduces more complex Hopf bifurcations, leading to multiple stability switches. Furthermore, when the diffusion rate ρ of the predator is small, introducing prey-taxis χ is necessary for the occurrence of Hopf bifurcations.

This highlights the crucial role of prey-taxis in driving oscillatory behavior and shaping population dynamics. Additionally, we analyzed the Turing stability and instability conditions for the system when $\tau = 0$ and identified a critical χ-threshold at which steady-state bifurcation occurs. Our results indicate that as χ decreases below the threshold value, a steady-state bifurcation leads to the emergence of spatially nonhomogeneous equilibrium patterns, demonstrating the influence of prey-taxis not only on inducing Hopf bifurcations but also on shaping spatial structures in the system.

These findings provide a comprehensive understanding of how prey-taxis and time delay influence ecological dynamics. Future research directions could explore extensions of this model by incorporating spatial heterogeneity or nonlocal interactions, further enhancing its applicability to real-world predator-prey systems.

Author contributions

Kimun Ryu: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing–original draft; Wonlyul Ko: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing–original draft, Writing–review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

Proof of Proposition 3.1. We will only verify (iii). Using the fact that $\mathbf{u}_*$ satisfies (2.2)–(2.4), we can easily derive that

$$\begin{aligned} T &= -u_* + \alpha m \cdot \left(\frac{u_*(1+v_*)}{1+mu_*(1+v_*)}\right)^2 \cdot \frac{v_*}{u_*} + \beta\overline{g}_2 \\ &= -u_* + \alpha m \cdot \gamma^2 \cdot \frac{1-u_*}{\alpha\gamma} + \beta\overline{g}_2 \\ &= m\gamma - (1+m\gamma)u_* + \beta\overline{g}_2 \end{aligned}$$

and

$$
\begin{aligned}
D = & \frac{\beta u_* v_*}{(1+mu_*(1+v_*))^2}\left[-u_* + \alpha m \cdot \left(\frac{u_*(1+v_*)}{1+mu_*(1+v_*)}\right)^2 \cdot \frac{v_*}{u_*}\right. \\
& \qquad + \alpha \cdot \left(\frac{u_*(1+v_*)}{1+mu_*(1+v_*)}\right)^2 \cdot \frac{1}{(u_*)^2} \cdot (1+mu_*(1+v_*)) \\
& \qquad \left. +\alpha \cdot \frac{v_*}{u_*} \cdot \frac{u_*(1+v_*)}{1+mu_*(1+v_*)} \cdot \frac{1}{1+mu_*(1+v_*)}\right] \\
= & \frac{\beta u_* v_*}{(1+mu_*(1+v_*))^2}\left[-u_* + \alpha m \cdot \gamma^2 \cdot \frac{1-u_*}{\alpha\gamma} + \alpha \cdot \gamma^2 \cdot \frac{1}{(u_*)^2} \cdot \frac{1}{1-m\gamma}\right. \\
& \qquad \left. +\alpha \cdot \frac{1-u_*}{\alpha\gamma} \cdot \gamma \cdot (1-m\gamma)\right] \\
= & \frac{\beta v_*}{(1+mu_*(1+v_*))^2}\left(-2(u_*)^2 + u_* + \frac{\alpha\gamma^2}{(1-m\gamma)u_*}\right) \\
= & \frac{\beta v_*}{(1+mu_*(1+v_*))^2}\left(-2(u_*)^2 + u_* + \alpha\gamma - (u_*)^2 + u_*\right) \\
= & -\frac{\beta v_*}{(1+mu_*(1+v_*))^2}(3(u_*)^2 - 2u_* - \alpha\gamma) = -\frac{\beta v_*}{(1+mu_*(1+v_*))^2}F'(u_*).
\end{aligned}
$$

□

Proof of Proposition 3.2. (i) We first note that, since $\mathbf{u}_1^*$ satisfies (2.3), $u_1^* < \frac{\gamma}{1-m\gamma}$ must be satisfied, and that u_1^* does not depend on β. Using this in Proposition 3.1(iii), we can easily obtain our first result. Thus, we see that

$$\beta_1^* > 0 \;\Leftrightarrow\; u_1^* > \frac{m\gamma}{1+m\gamma}.$$

In addition, because $u_1^*, \frac{m\gamma}{1+m\gamma} < 1$,

$$u_1^* > \frac{m\gamma}{1+m\gamma} \;\Leftrightarrow\; F\left(\frac{m\gamma}{1+m\gamma}\right) = -\frac{(m\gamma)^2}{(1+m\gamma)^3} + \alpha\gamma\left(\frac{\gamma}{1-m\gamma} - \frac{m\gamma}{1+m\gamma}\right) > 0,$$

which gives the second result.

(ii) If $\mathbf{u}_1^*$ exists, then $u_1^* < u_d$ always holds. Thus, $F'(u_1^*) < 0$, along with Proposition 3.1(iii), leads to $D > 0$ for $\mathbf{u}_* = \mathbf{u}_1^*$. Similarly, since $F'(u_2^*) > 0$ (when $\mathbf{u}_2^*$ exists) holds, we have $D < 0$ for $\mathbf{u}_* = \mathbf{u}_2^*$, and since $F'(u_d) = 0$ (when $\mathbf{u}_d^*$ exists) holds, we obtain $D = 0$ for $\mathbf{u}_* = \mathbf{u}_d^*$. □

Proof of Proposition 3.3. (i) Through some calculations, we can easily see that $Q_1(\mu) > 0$ holds for all $\mu \geq 0$ if any one of the following conditions is satisfied, due to the fact that Proposition 3.2 gives $D > 0$ at $\mathbf{u}_* = \mathbf{u}_1^*$:

$\rho f_1 + g_2 + f_2\chi v_1^* \leq 0$;
$\rho f_1 + g_2 + f_2\chi v_1^* > 0$ and $(\rho f_1 + g_2 + f_2\chi v_1^*)^2 - 4\rho D < 0$.

Noting that $f_2 < 0$, and that $T < 0$ and $g_2 > 0$ imply $f_1 < 0$, we can see that if

$$\rho f_1 + g_2 \leq 0 \text{ (i.e., } \beta \leq -\frac{\rho f_1}{g_2}) \text{ and } \chi > 0, \tag{A1}$$

then $Q_1(\mu) > 0$ holds for all $\mu \geq 0$. Considering the case where $\rho f_1 + g_2 > 0$ (i.e., $\beta > -\frac{\rho f_1}{\overline{g}_2}$), the above two options can be rewritten as follows, respectively:

$$\chi \geq \frac{\rho f_1 + g_2}{-f_2 v_1^*} \text{ and } \chi_* < \chi < \frac{\rho f_1 + g_2}{-f_2 v_1^*},$$

and combining these inequalities gives $\chi > \chi_*$. The sign of the numerator of χ_* is easily determined as follows:

$$\text{if } \beta \leq \beta_*, \text{ then } \rho f_1 + g_2 - 2\sqrt{\rho D} \leq 0; \text{ otherwise, } \rho f_1 + g_2 - 2\sqrt{\rho D} > 0.$$

Thus, noting that $-\frac{\rho f_1}{\overline{g}_2} < \beta_*$, we see from this discussion that $Q_1(\mu) > 0$ holds for all $\mu \geq 0$ if

$$\text{either } \beta_* \geq \beta > -\frac{\rho f_1}{\overline{g}_2} \text{ and } \chi > 0 \text{ or } \beta_* < \beta \text{ and } \chi > \chi_*.$$

Therefore, this result, along with (A1), leads to the conclusion that $Q_1(\mu) > 0$ for all $\mu \geq 0$ when one of the conditions given in (3.2) is satisfied.

(ii) We consider the case where (3.3) is given. According to the discussion in (i), we know that $Q_1(\mu) = 0$ has two distinct positive roots.

(iii) Since $D > 0$ at $\mathbf{u}_* = \mathbf{u}_1^*$, this result follows immediately. □

Proof of Proposition 3.4. Clearly, $T < 0$ and $g_2 > 0$ give $f_1^2 - g_2^2 > 0$. Thus, if one of the following conditions is satisfied, then $P(\mu) \geq 0$ for all $\mu \geq 0$:

$f_2\chi v_1^* - f_1 \geq 0$;
$f_2\chi v_1^* - f_1 < 0$ and $(f_2\chi v_1^* - f_1)^2 - (\rho^2 + 1)(f_1^2 - g_2^2) \leq 0$.

By combining these two conditions based on χ, we obtain (3.5). Furthermore, we note that the assumption $T < 0$ directly gives $\chi_0^* > \chi_*$.

If $\chi > \chi_0^*$, then $f_2\chi v_1^* - f_1 < 0$ and $(f_2\chi v_1^* - f_1)^2 - (\rho^2 + 1)(f_1^2 - g_2^2) > 0$ are derived, leading to the conclusion that the desired final assertion holds. □

Proof of Proposition 3.5. We see that M is a continuous function of $\beta \in [0, \beta_1^*]$, and that $\beta_1^* > 0$ and $D > 0$ give $f_1 < 0$ and $f_1\overline{g}_2 - f_2\overline{g}_1 > 0$, respectively. Moreover, $M = 0$ can be rewritten as

$$\frac{\frac{\rho^2+\rho+1}{\rho^2+1}(\overline{g}_2)^2\beta^2 + (f_1\overline{g}_2 - f_2\overline{g}_1)\beta - \frac{\rho^2+\rho+1}{\rho^2+1}f_1^2}{\overline{g}_2\beta - (\rho+1)f_1} = \frac{\sqrt{f_1^2 - (\overline{g}_2)^2\beta^2}}{\sqrt{\rho^2+1}}.$$

By the geometric properties of the functions in β on both the left-hand side and righthand side of this equation, we can easily establish the existence of $\beta_0^* \in (0, \beta_1^*)$, and the remaining results can also be derived. □

Proof of Proposition 3.6. We first note that since $\beta_0^* < \beta < \beta_1^*$ is given, $T < 0$ holds at $\mathbf{u}_* = \mathbf{u}_1^*$, and furthermore, $D > 0$ holds (see Proposition 3.2).

Obviously, P_0, P_+, and $\widehat{\mu}_*$ exist for $\chi > \chi_0^*$. When we treat P_0, P_+, and $\widehat{\mu}_*$ as functions of χ, we see that these functions are differentiable with respect to $\chi > \chi_0^*$. In particular, we can easily check that

$\frac{dP_0}{d\chi} > 0$, $\frac{dP_+}{d\chi} > 0$, and $\frac{d\widehat{\mu}_*}{d\chi} < 0$ for $\chi > \chi_0^*$; $P_0(\chi) < P_+(\chi)$ for $\chi > \chi_0^*$; $P_0(\chi_0^*) = P_+(\chi_0^*)$; $P_+(\chi) \to \infty$, $P_0(\chi) \to \infty$, and $\widehat{\mu}_*(\chi) \to 0$ as $\chi \to \infty$. Thus, if

$$P_+(\chi_0^*) < \widehat{\mu}_*(\chi_0^*),$$

then the existence of a unique $\chi_1^* \in (\chi_0^*, \infty)$ satisfying $P_+(\chi_1^*) = \widehat{\mu}_*(\chi_1^*)$ follows, and furthermore, the existence of a unique $\chi_2^* \in (\chi_1^*, \infty)$ satisfying $P_0(\chi_2^*) = \widehat{\mu}_*(\chi_2^*)$ follows. Thus, $P_+ \leq \widehat{\mu}_*$ holds for $\chi_0^* < \chi \leq \chi_1^*$ and $P_0 \leq \widehat{\mu}_* < P_+$ for $\chi_1^* < \chi \leq \chi_2^*$. In fact, with some calculations, the following can be derived:

$$P_+(\chi_0^*) < \widehat{\mu}_*(\chi_0^*) \Leftrightarrow M(\beta) > 0,$$

where M was defined in Proposition 3.5. Therefore, since we see from Proposition 3.5 that $M > 0$ for $\beta_0^* < \beta < \beta_1^*$, we obtain the desired result. □

Proof of Proposition 3.7. (i) We first consider the case where (a) in (3.6) holds. In this case, the first condition of (3.2) and (3.5) are satisfied. Thus, according to Propositions 3.3 and 3.4, $P(\mu) \geq 0$ for all $\mu \geq 0$; $Q(\mu) > 0$ for all $\mu > \widehat{\mu}_*$; and $Q(\mu) < 0$ for all $\mu < \widehat{\mu}_*$. Therefore, the desired result follows.

In the case where (b) in (3.6) holds, the first condition of (3.2) holds. Due to (3.4), $\beta_* \geq \beta_1^*$, and since $\chi > \chi_0^*$, P_+ exists. Furthermore, it follows from Proposition 3.6 that $P_+ \leq \widehat{\mu}_*$. Thus, along with this result, the definitions of P_+ and $\widehat{\mu}_*$ yield our desired result.

In the case where (c) in (3.6) holds, the second condition of (3.2) and (3.5) are satisfied. Here, we note that (3.4) provides the existence of β satisfying $\beta_* < \beta < \beta_1^*$, and that Proposition 3.4 gives $\chi_* < \chi_0^*$ to ensure the existence of χ satisfying $\chi_* < \chi \leq \chi_0^*$. Thus, the same result is obtained as when (a) of (3.6) is satisfied.

In the final case where (d) in (3.6) holds, the second condition of (3.2) and $\beta_* < \beta_1^*$ hold, and since $\chi > \chi_0^*$, P_+ exists. Therefore, by Proposition 3.6, the definitions of β_0^* and χ_1^* lead to our desired result.

(ii) According to Proposition 3.6, $P_0 \leq \widehat{\mu}_* < P_+$ holds when (3.7) is given. Thus, similar to (i), our first assertion can be easily proved. In particular, for $\mu \geq \widehat{\mu}_*$, P is strictly increasing, whereas $-2\sqrt{Q}$ is strictly decreasing. Moreover, since $P(\widehat{\mu}_*) < 0$ and $P(\mu) \geq 0$ for $\mu \geq P_+$, the existence of $\widetilde{\mu}_*$ is ensured. As a consequence, for $\widehat{\mu}_* < \mu < \widetilde{\mu}_*$, $P(\mu) < -2\sqrt{Q(\mu)}$ holds, yielding $(P(\mu))^2 > 4Q(\mu)$. The remaining signs can also be easily checked. □

Proof of Proposition 3.9. Obviously, $g_1 > 0$ and $g_2 > 0$. If $(\rho - 1)g_1 + \chi v_1^* g_2 \geq 0$ holds, then the assertion is clearly valid. Conversely, in the case that $(\rho - 1)g_1 + \chi v_1^* g_2 < 0$ (i.e., $\chi < \frac{(1-\rho)g_1}{v_1^* g_2}$), we can see that the same assertion is valid, provided that

$$\widehat{\mu}_* \leq -\frac{g_1 D}{g_2((\rho - 1)g_1 + \chi v_1^* g_2)}.$$

Thus, through the geometric property of the quadratic function $Q_2(\mu)$, we obtain our desired result by showing

$$0 \leq Q_2\left(-\frac{g_1 D}{g_2((\rho - 1)g_1 + \chi v_1^* g_2)}\right) = -\frac{D}{g_2^2((\rho - 1)g_1 + \chi v_1^* g_2)^2}\widetilde{Q}_2(\chi),$$

where

$$\begin{aligned}\widetilde{Q}_2(\chi) = (g_2^2 - f_2 g_1) g_2^2 (v_1^*)^2 \chi^2 + g_1 g_2 v_1^* \left[g_2(g_2 - f_1)\rho + (g_2^2 - f_2 g_1)(\rho - 1)\right]\chi \\ - g_1^2 \rho \left[g_2(g_2 - f_1)(1 - \rho) + D\right].\end{aligned}$$

Additionally, we see that $f_2 < 0$, $D > 0$, and furthermore, $T < 0$ and $(\rho - 1)g_1 + \chi v_1^* g_2 < 0$, respectively, imply $f_1 < 0$ and $\rho < 1$. Thus, because of the geometric property of the quadratic function $\widetilde{Q}_2(\chi)$ and the calculation result $\widetilde{Q}_2(\frac{(1-\rho)g_1}{v_1^* g_2}) = -g_1^2 D\rho < 0$, we conclude that $\widetilde{Q}_2(\chi) < 0$ is satisfied for $\chi < \frac{(1-\rho)g_1}{v_1^* g_2}$, yielding the desired assertion. □