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Research article

A Pontryagin maximum principle for optimal control problems involving generalized distributional-order derivatives

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Abstract: In this work, we extended fractional optimal control (OC) theory by proving a version of Pontryagin's maximum principle and establishing sufficient optimality conditions for an OC problem. The dynamical system constraint in the OC problem under investigation is described by a generalized fractional derivative: the left-sided Caputo distributed-order fractional derivative with an arbitrary kernel. This approach provides a more versatile representation of dynamic processes, accommodating a broader range of memory effects and hereditary properties inherent in diverse physical, biological, and engineering systems.

Keywords: fractional calculus; fractional optimal control; Pontryagin maximum principle
Mathematics Subject Classification: 26A33, 49K05

1. Introduction

Optimal control (OC) emerged in the 1950s as an extension of the calculus of variations. A significant breakthrough in this field was the development and proof of Pontryagin's maximum principle (PMP) by Pontryagin and his collaborators [34]. This principle established the necessary conditions for an optimal solution and has become one of the most powerful tools for addressing OC problems. This formalization of OC theory raised several new questions, particularly in the context of differential equations. It led to the introduction of new generalized solution concepts and produced significant results concerning the existence of trajectories. The applications of OC are extensive, encompassing fields such as mathematics [41], physics [9], engineering [17], robotics [11], biology [24], and economics [39], among others. The primary objective of an OC problem is to identify, from a set of possible solutions (referred to as the admissible set) the one that minimizes or maximizes a given functional. That is, the goal is to determine an OC trajectory and the corresponding state trajectory. The system's dynamics, modeled by state variables, are influenced by control variables that enter the system of equations, thereby affecting its behavior.

Fractional calculus (FC) is a well-known theory that extends the ideas of integration and differentiation to non-integer orders, providing a more flexible approach to modeling complex systems with non-local memory. Numerous publications have been devoted to exploring FC and fractional differential equations (FDEs) from various perspectives, covering both its fundamental definitions and principles (see, e.g., [22, 38] and the references therein), and its diverse applications in areas such as biology [6], medicine [23], epidemiology [28], electrochemistry [31], physics [33], mechanics [35], and economy [37]. Since the begining of fractional calculus in 1695, numerous definitions of fractional derivatives have been introduced, including the Riemann-Liouville, Caputo, Hadamard, Grünwald-Letnikov, Riesz, Erdélyi-Kober, and Miller-Ross derivatives, among others. Every form of fractional derivative comes with its own benefits and is selected based on the specific needs of the problem being addressed.

In this work, we will explore recent advancements in fractional derivatives as introduced by [10]. Specifically, we will focus on the novel concepts of distributed-order fractional derivatives with respect to an arbitrary kernel in the Riemann-Liouville and Caputo senses. In their study, the authors established necessary and sufficient conditions within the context of the calculus of variations and derived an associated Euler–Lagrange equation. Building upon their results, we extend the analysis to the setting of optimal control theory and derive a corresponding generalization of the Pontryagin maximum principle.

A key advantage of distributed-order fractional derivatives [8] lies in their ability to describe systems where the memory effect varies over time. Unlike classical or constant-order fractional derivatives, distributed-order derivatives are defined through an integral over a range of orders, weighted by a given distribution function. This provides a more flexible and realistic modeling tool for systems whose dynamic behavior cannot be adequately captured by a single, fixed fractional order. Distributed-order fractional derivatives have proven especially effective in modeling systems characterized by a broad spectrum of dynamic behaviors. This includes, for instance, viscoelastic materials exhibiting a wide range of relaxation times, transport phenomena influenced by multiple temporal and spatial scales, and control systems affected by diverse time delays. Notable applications span across fields such as viscoelasticity, anomalous diffusion, wave propagation, and the design of fractional-order Proportional-Integral-Derivative (PID) controllers [13, 18].

In parallel, the concept of fractional derivatives with respect to arbitrary kernels [3, 38] further improves the model. These operators generalize classical definitions (such as the Riemann–Liouville or Caputo derivatives) by introducing a kernel function that dictates the memory behavior of the system. By selecting an appropriate kernel, one can recover various classical forms or construct entirely new operators. This kernel-based approach allows for incorporating diverse memory effects, singularities, or fading influence in time, which are often observed in real-world processes but difficult to model with standard tools.

These two features—variable order and general kernels—collectively enable the modeling of complex phenomena with nonlocal and history-dependent behavior, particularly in contexts where the memory characteristics are not uniform or stationary. This is especially relevant in systems governed by internal friction, relaxation, hereditary stress-strain relations, or anomalous transport, where the effects of the past on the present are not constant and may depend on the nature of past events in intricate ways.

Building on these two ideas, we consider OC problems that involve a generalized fractional derivative constraint. OC provides a rigorous mathematical framework for determining the best possible strategy to steer a dynamical system from an initial state to a desired final state while minimizing (or maximizing) a given cost functional. The inclusion of fractional operators—particularly of distributed-order or arbitrary-kernel types—within this framework enhances the ability to model systems with complex dynamics and long-range temporal dependencies. It enables decision-makers to formulate strategies that are optimal with respect to cost, energy, time, or other performance metrics, all while satisfying dynamic constraints such as differential equations, control bounds, and state limitations. This is crucial for achieving efficiency, such as reducing fuel consumption in aerospace trajectories, minimizing energy use in industrial processes, or shortening recovery time in medical treatments.

Pioneering works, such as those in [1,2], developed necessary conditions for optimality with respect to the classical Caputo fractional derivative. These studies were further expanded in [14, 15] with the formulation of the fractional Noether's theorem. Since then, numerous studies have emerged on fractional OC problems, addressing various fractional operators such as the Caputo [5, 20], Riemann–Liouville [19], and distributed-order derivatives [30], as well as aspects like delays in systems [16] and fuzzy theory [32]. Numerical methods for solving fractional optimal control problems are available in the literature. For example, [21] uses a generalized differential transform method, [25, 42] employ second-order and third-order numerical integration methods, and [27] applies a Legendre orthonormal polynomial basis.

The primary objective of this paper is to establish a generalized fractional PMP applicable to OC problems involving the left-sided distributed-order Caputo fractional derivative with respect to an arbitrary smooth kernel. Additionally, we will present sufficient conditions for optimality based on PMP.

The rest of the paper is structured as follows. Section 2 introduces some fundamental concepts and results from fractional calculus required for our work. In Section 3, we present the OC problem $(\mathcal{P}_{OC})$ under study, some fundamental lemmas, PMP, and sufficient conditions for the global optimality of the problem $(\mathcal{P}_{OC})$. Section 4 provides two examples that illustrate the practical relevance of our research. Finally, in Section 5, we summarize our findings and present ideas for future work.

2. Preliminaries of fractional calculus

This section provides essential definitions and results with respect to the distributed-order Riemann-Liouville and Caputo derivatives depending on a given kernel [10]. We assume that readers are already acquainted with the definitions and properties of the classical fractional operators [22, 38].

Throughout the paper, we suppose that $a < b$ are two real numbers, $J = [a, b]$, $\gamma \in \mathbb{R}^+$, and $[\gamma]$ denotes the integer part of γ. First, we introduce essential concepts relevant to our work.

Definition 2.1. [38] (Fractional integrals in the Riemann–Liouville sense) Let $z \in L^1(J, \mathbb{R})$, and let $\sigma \in C^1(J, \mathbb{R})$ be a continuously differentiable function with a positive derivative.

For $t > a$, the left-sided σ-R-L fractional integral of the function z of order γ is given by

$$\mathrm{I}_{a^+}^{\gamma;\sigma} z(t) := \int_a^t \frac{\sigma'(\tau) z(\tau)}{\Gamma(\gamma)(\sigma(t) - \sigma(\tau))^{1-\gamma}} \, d\tau,$$

and for $t < b$, the right-sided σ-R-L fractional integral is

$$\mathrm{I}_{b^-}^{\gamma,\sigma} z(t) := \int_t^b \frac{\sigma'(\tau)z(\tau)}{\Gamma(\gamma)(\sigma(\tau)-\sigma(t))^{1-\gamma}}\, d\tau.$$

Definition 2.2. [38] (Fractional derivatives in the Riemann–Liouville sense) Let $z \in L^1(J,\mathbb{R})$, and let $\sigma \in C^n(J,\mathbb{R})$ be such that $\sigma'(t) > 0$, given $t \in J$.

The left-sided σ-R-L fractional derivative of the function z of order γ is given by

$$\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) := \left(\frac{1}{\sigma'(t)}\frac{d}{dt}\right)^n \mathrm{I}_{a^+}^{n-\gamma,\sigma} z(t),$$

and the right-sided σ-R-L fractional derivative is

$$\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) := \left(-\frac{1}{\sigma'(t)}\frac{d}{dt}\right)^n \mathrm{I}_{b^-}^{n-\gamma,\sigma} z(t),$$

where $n = [\gamma] + 1$.

Definition 2.3. [3] (Fractional derivatives in the Caputo sense) For a fixed $\gamma \in \mathbb{R}^+$, define $n \in \mathbb{N}$ as: $n = [\gamma] + 1$ if $\gamma \notin \mathbb{N}$, and $n = \gamma$ if $\gamma \in \mathbb{N}$. Moreover, consider $z, \sigma \in C^n(J,\mathbb{R})$ where $\sigma'(t) > 0$, $t \in J$. The left- and right-sided σ-C fractional derivatives of the function z of order γ are given by

$${}^C\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) := \mathrm{I}_{a^+}^{n-\gamma,\sigma} \left(\frac{1}{\sigma'(t)}\frac{d}{dt}\right)^n z(t),$$

and

$${}^C\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) := \mathrm{I}_{b^-}^{n-\gamma,\sigma} \left(-\frac{1}{\sigma'(t)}\frac{d}{dt}\right)^n z(t),$$

respectively.

We remark that, for $0 < \gamma < 1$:

$$\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) := \frac{1}{\Gamma(1-\gamma)}\left(\frac{1}{\sigma'(t)}\frac{d}{dt}\right)\int_a^t \sigma'(\tau)(\sigma(t)-\sigma(\tau))^{-\gamma}\, z(\tau)\, d\tau,$$

$$\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) := -\frac{1}{\Gamma(1-\gamma)}\left(\frac{1}{\sigma'(t)}\frac{d}{dt}\right)\int_t^b \sigma'(\tau)(\sigma(\tau)-\sigma(t))^{-\gamma}\, z(\tau)\, d\tau,$$

$${}^C\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) := \frac{1}{\Gamma(1-\gamma)}\int_a^t (\sigma(t)-\sigma(\tau))^{-\gamma}\, z'(\tau)\, d\tau,$$

and

$${}^C\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) := -\frac{1}{\Gamma(1-\gamma)}\int_t^b (\sigma(\tau)-\sigma(t))^{-\gamma}\, z'(\tau)\, d\tau.$$

If $\gamma = 1$, we get

$$\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) = {}^C\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) = \frac{z'(t)}{\sigma'(t)}$$

and

$$\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) = {}^C\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) = -\frac{z'(t)}{\sigma'(t)}.$$

Next, we recall some properties that will be useful in our proofs. For $\gamma \in (0, 1]$ and $z \in C^1(J, \mathbb{R})$, the following relations hold:

$${}^C\mathrm{D}_{a^+}^{\gamma,\sigma}\, \mathrm{I}_{a^+}^{\gamma,\sigma} z(t) = z(t)$$

and

$$\mathrm{I}_{a^+}^{\gamma,\sigma}\, {}^C\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) = z(t) - z(a).$$

For further details, we refer the reader to [3].

Since the fractional OC problem examined in this paper involves a fractional derivative of order γ within the range (0, 1], we will henceforth assume $\gamma \in (0, 1]$.

To define the distributed-order derivatives, we need to fix the order-weighting function, denoted by Φ. Here, Φ is a continuous function defined on $[0, 1]$ such that $\Phi([0, 1]) \subseteq [0, 1]$ and $\int_0^1 \Phi(\gamma)d\gamma > 0$.

Definition 2.4. [10] (Distributional-order fractional derivatives in the Riemann–Liouville sense) Let $z \in L^1(J, \mathbb{R})$. The left- and right-sided σ-D-R-L fractional derivatives of a function z with respect to the distribution Φ are defined by

$$\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) \mathrm{D}_{a+}^{\gamma,\sigma} z(t) d\gamma \quad \text{and} \quad \mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) \mathrm{D}_{b-}^{\gamma,\sigma} z(t) d\gamma,$$

where $\mathrm{D}_{a^+}^{\gamma,\sigma}$ and $\mathrm{D}_{b^-}^{\gamma,\sigma}$ are the left- and right-sided σ-R-L fractional derivatives of order γ, respectively.

Definition 2.5. [10] (Distributional-order fractional derivatives in the Caputo sense) The left- and right-sided σ-D-C fractional derivatives of a function $z \in C^1(J, \mathbb{R})$ with respect to the distribution Φ are defined by

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) {}^C\mathrm{D}_{a^+}^{\gamma,\sigma} z(t) d\gamma \quad \text{and} \quad {}^C\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) {}^C\mathrm{D}_{b^-}^{\gamma,\sigma} z(t) d\gamma,$$

where ${}^C\mathrm{D}_{a^+}^{\gamma,\sigma}$ and ${}^C\mathrm{D}_{b^-}^{\gamma,\sigma}$ are the left- and right-sided σ-C fractional derivatives of order γ, respectively.

In what follows, we introduce the concepts of σ-distributional-order fractional integrals:

$$\mathrm{I}_{a^+}^{1-\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) \mathrm{I}_{a^+}^{1-\gamma,\sigma} z(t) d\gamma \quad \text{and} \quad \mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma} z(t) := \int_0^1 \Phi(\gamma) \mathrm{I}_{b^-}^{1-\gamma,\sigma} z(t) d\gamma,$$

where $\mathrm{I}_{a^+}^{1-\gamma,\sigma}$ and $\mathrm{I}_{b^-}^{1-\gamma,\sigma}$ are the left- and right-sided σ-R-L fractional integrals of order $1-\gamma$, respectively.

It is evident from the definitions that distributed-order operators are linear.

In the following, we present a generalized fractional integration by parts formula which is useful to demonstrate some of our results.

Lemma 2.6. *(Generalized fractional integration by parts)* [10] *Given $w \in C(J, \mathbb{R})$ and $z \in C^1(J, \mathbb{R})$, the following holds:*

$$\int_a^b w(t)\, {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} z(t) dt = \int_a^b z(t) \left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{w(t)}{\sigma'(t)} \right) \sigma'(t) dt + \left[z(t) \left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma} \frac{w(t)}{\sigma'(t)} \right) \right]_{t=a}^{t=b}.$$

We now present a general form of Gronwall's inequality, a crucial tool for comparing solutions of FDEs that involve σ-fractional derivatives.

Lemma 2.7. *(Generalized fractional Gronwall inequality)* [40] *Let* $u, v \in L^1(J, \mathbb{R})$, $\sigma \in C^1(J, \mathbb{R})$ *with* $\sigma'(t) > 0$ *for all* $t \in J$*, and* $h \in C(J, \mathbb{R})$*. Assume also that* u, v, h *are nonnegative and* h *is nondecreasing. If*

$$u(t) \leq v(t) + h(t) \int_a^t \sigma'(\tau)(\sigma(t) - \sigma(\tau))^{\gamma-1} u(\tau) d\tau, \quad t \in J,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{i=1}^{\infty} \frac{[\Gamma(\gamma)\, h(t)]^i}{\Gamma(i\gamma)} \sigma'(\tau)(\sigma(t) - \sigma(\tau))^{i\gamma-1} v(\tau) d\tau, \quad t \in J.$$

Remark 2.8. In Lemma 2.7, the assumption that h is nondecreasing is often satisfied in practical contexts where h represents quantities such as cumulative costs, energy consumption, or memory effects that naturally increase or remain constant over time.

Next, we recall the definition of the Mittag-Leffler function (with one parameter), which generalizes the standard exponential function and plays a fundamental role in the study of differential equations of fractional order.

Definition 2.9. The Mittag-Leffler function with parameter $\gamma > 0$ is defined by

$$E_\gamma(t) := \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\gamma i + 1)}, \quad t \in \mathbb{R}.$$

To conclude this section, we present the following result, which is a corollary of Lemma 2.7 and will be useful in the proof of Lemma 3.3.

Corollary 2.10. *(cf.* [40]*) Under the hypotheses of Lemma 2.7, if* v *is a nondecreasing function, then, for all* $t \in J$,

$$u(t) \leq v(t) \cdot E_\gamma\Big(\Gamma(\gamma)\, h(t)\, [\sigma(t) - \sigma(a)]^\gamma\Big).$$

3. Main results

The aim of this section is to establish necessary and sufficient optimality conditions for a fractional OC problem involving the left-sided Caputo distributed-order fractional derivative with respect to a smooth kernel σ.

In what follows, we use the standard notations: $PC(J, \mathbb{R})$ denotes the set of all real-valued piecewise continuous functions defined on J, and $PC^1(J, \mathbb{R})$ denotes the set of piecewise smooth functions defined on J. The fractional OC problem is defined by the following formulation:

Problem ($\mathcal{P}_{OC}$): Determine $x \in PC^1(J, \mathbb{R})$ and $u \in PC(J, \mathbb{R})$ that extremizes the functional

$$\mathcal{J}(x, u) = \int_a^b L\,(t, x(t), u(t))\, dt,$$

under the following restrictions: the FDE

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} x(t) = f(t, x(t), u(t)), \quad t \in J,$$

and the initial condition

$$x(a) = x_a,$$

where $L \in C^1(J \times \mathbb{R}^2, \mathbb{R})$ and $f \in C^2(J \times \mathbb{R}^2, \mathbb{R})$.

Remark 3.1. There are several equivalent ways to formulate an optimal control problem, notably in the Mayer, Lagrange, and Bolza forms. Through appropriate changes of variables and auxiliary state transformations, one can show that these formulations are theoretically equivalent (see, e.g., Chapter 3 in [26]). In this paper, we adopt the Lagrange form for consistency and simplicity.

We will now prove a result that establishes a relationship between an optimal state trajectory of the OC problem $(\mathcal{P}_{OC})$ and the state solution of a perturbed version of $(\mathcal{P}_{OC})$.

Lemma 3.2. *Let the pair $(\overline{u}, \overline{x})$ be an optimal solution to $(\mathcal{P}_{OC})$. For $t \in J$, consider a variation of $\overline{u}$ of the form $\overline{u} + \xi\eta$, where $\eta \in PC(J, \mathbb{R})$ and $\xi \in \mathbb{R}$. Denote by $u^\xi(t)$ such variation and x^ξ the solution of*

$$^C\mathrm{D}^{\Phi(\gamma),\sigma}_{a^+} y(t) = f(t, y(t), u^\xi(t)), \quad y(a) = x_a.$$

Then, as ξ tends to zero, $x^\xi \to \overline{x}$.

Proof. By hypothesis, we know that

$$^C\mathrm{D}^{\Phi(\gamma),\sigma}_{a^+} x^\xi(t) = f(t, x^\xi(t), u^\xi(t)) \quad \text{and} \quad ^C\mathrm{D}^{\Phi(\gamma),\sigma}_{a^+} \overline{x}(t) = f(t, \overline{x}(t), \overline{u}(t)), \quad t \in J,$$

where $x^\xi(a) = \overline{x}(a) = x_a$. From the linearity and the definition of the operator $^C\mathrm{D}^{\Phi(\gamma),\sigma}_{a^+}$, we may conclude that

$$\int_0^1 \Phi(\gamma)\, ^C\mathrm{D}^{\gamma,\sigma}_{a^+}(x^\xi(t) - \overline{x}(t)) d\gamma = f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t)).$$

Using the mean value theorem in the integral form, we conclude that there exists $\beta \in [0, 1]$ such that

$$^C\mathrm{D}^{\beta,\sigma}_{a^+}(x^\xi(t) - \overline{x}(t)) \int_0^1 \Phi(\gamma) d\gamma = f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t)).$$

Denoting $\omega := \int_0^1 \Phi(\gamma) d\gamma$, we conclude that

$$^C\mathrm{D}^{\beta,\sigma}_{a^+}(x^\xi(t) - \overline{x}(t)) = \frac{f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t))}{\omega}.$$

Thus,

$$x^\xi(t) - \overline{x}(t) = I^{\beta,\sigma}_{a^+}\left(\frac{f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t))}{\omega}\right).$$

Let k_1 and k_2 be two positive real numbers such that

$$|f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t))| \leq k_1|x^\xi(t) - \overline{x}(t)| + k_2|u^\xi(t) - \overline{u}(t)|, \quad t \in J,$$

and denote by k their maximum. Therefore,

$$|x^\xi(t) - \overline{x}(t)| \leq I^{\beta,\sigma}_{a^+}\left(\frac{|f(t, x^\xi(t), u^\xi(t)) - f(t, \overline{x}(t), \overline{u}(t))|}{\omega}\right)$$

$$\leq \frac{k}{\omega}\left(I_{a^+}^{\beta,\sigma}\left(|x^{\xi}(t)-\overline{x}(t)|+|u^{\xi}(t)-\overline{u}(t)|\right)\right),\ t\in J.$$

Since $u^{\xi}(t)-\overline{u}(t)=\xi\eta(t)$, we have

$$\begin{aligned}|x^{\xi}(t)-\overline{x}(t)| &\leq \frac{k}{\omega}\left(|\xi|I_{a^+}^{\beta,\sigma}(|\eta(t)|)+I_{a^+}^{\beta,\sigma}\left(|x^{\xi}(t)-\overline{x}(t)|\right)\right)\\ &= \frac{k}{\omega}|\xi|I_{a^+}^{\beta,\sigma}(|\eta(t)|)+\frac{k}{\omega}\frac{1}{\Gamma(\beta)}\int_a^t\sigma'(\tau)(\sigma(t)-\sigma(\tau))^{\beta-1}|x^{\xi}(\tau)-\overline{x}(\tau)|d\tau,\end{aligned}$$

for all $t\in J$.

Now, applying Lemma 2.7, we obtain that

$$|x^{\xi}(t)-\overline{x}(t)| \leq \frac{k}{\omega}|\xi|I_{a^+}^{\beta,\sigma}(|\eta(t)|)+\int_a^t\sum_{i=1}^{\infty}\frac{(\frac{k}{\omega})^{i+1}}{\Gamma(i\beta)}\sigma'(\tau)(\sigma(t)-\sigma(\tau))^{i\beta-1}|\xi|I_{a^+}^{\beta,\sigma}(|\eta(\tau)|)d\tau,\quad t\in J.$$

Thus, by applying the mean value theorem once more, we deduce that there exists some $\overline{\tau}\in J$ such that

$$\begin{aligned}|x^{\xi}(t)-\overline{x}(t)| &\leq \frac{k}{\omega}|\xi|I_{a^+}^{\beta,\sigma}(|\eta(t)|)+|\xi|I_{a^+}^{\beta,\sigma}(|\eta(\overline{\tau})|)\sum_{i=1}^{\infty}\frac{(\frac{k}{\omega})^{i+1}}{\Gamma(i\beta+1)}(\sigma(t)-\sigma(a))^{i\beta}\\ &= \frac{k}{\omega}|\xi|\left[I_{a^+}^{\beta,\sigma}(|\eta(t)|)+I_{a^+}^{\beta,\sigma}(|\eta(\overline{\tau})|)\left(E_{\beta}\left(\frac{k}{\omega}(\sigma(t)-\sigma(a))^{\beta}\right)-1\right)\right],\quad t\in J.\end{aligned}$$

So, the proof is finished by taking the limit when $\xi\to 0$. □

Before presenting our next result, we extend the one presented in [12, Theorem 3.4] for FDEs with the σ-Caputo fractional derivative.

Lemma 3.3. *Consider the following two FDEs of order $\gamma\in(0,1]$:*

$${}^{C}\mathrm{D}_{a^+}^{\gamma,\sigma}x(t)=f(t,x(t))\quad\text{and}\quad {}^{C}\mathrm{D}_{a^+}^{\gamma,\sigma}x(t)=\tilde{f}(t,x(t)),\ t\in J,$$

with the same initial condition. Suppose the functions f and $\tilde{f}$ are Lipschitz continuous with respect to x, with Lipschitz constant $L>0$. If y and z are the unique solutions to the first and second equations, respectively, then there exists a constant $C>0$, independent of y and z, such that

$$\|y-z\|_{\infty}\leq C\,\|f-\tilde{f}\|_{\infty}.$$

Proof. Given $t\in J$, we have:

$$\begin{aligned}|y(t)-z(t)| &= \left|\mathrm{I}_{a^+}^{\gamma,\sigma}\left(f(t,y(t))-\tilde{f}(t,z(t))\right)\right|\\ &\leq \left|\mathrm{I}_{a^+}^{\gamma,\sigma}\left(f(t,y(t))-f(t,z(t))\right)\right|+\left|\mathrm{I}_{a^+}^{\gamma,\sigma}\left(f(t,z(t))-\tilde{f}(t,z(t))\right)\right|\\ &\leq \mathrm{I}_{a^+}^{\gamma,\sigma}(L|y(t)-z(t)|)+\|f-\tilde{f}\|_{\infty}\frac{1}{\Gamma(\gamma)}\int_a^t\sigma'(\tau)(\sigma(t)-\sigma(\tau))^{\gamma-1}d\tau\\ &\leq \frac{L}{\Gamma(\gamma)}\int_a^t\sigma'(\tau)(\sigma(t)-\sigma(\tau))^{\gamma-1}|y(\tau)-z(\tau)|d\tau+\|f-\tilde{f}\|_{\infty}\frac{(\sigma(b)-\sigma(a))^{\gamma}}{\Gamma(\gamma+1)}.\end{aligned}$$

Applying Corollary 2.10, with

$$u(t) := |y(t) - z(t)|, \quad v(t) := \|f - \tilde{f}\|_\infty \frac{(\sigma(b) - \sigma(a))^\gamma}{\Gamma(\gamma + 1)}, \quad h(t) := \frac{L}{\Gamma(\gamma)},$$

we obtain

$$|y(t) - z(t)| \leq \|f - \tilde{f}\|_\infty \frac{(\sigma(b) - \sigma(a))^\gamma}{\Gamma(\gamma + 1)} \cdot E_\gamma(L(\sigma(t) - \sigma(a))^\gamma), \quad t \in J.$$

Thus, we conclude that

$$\|y - z\|_\infty \leq C\,\|f - \tilde{f}\|_\infty, \quad \text{where } C = \frac{(\sigma(b) - \sigma(a))^\gamma}{\Gamma(\gamma + 1)} \cdot E_\gamma(L(\sigma(b) - \sigma(a))^\gamma),$$

which proves the desired result. □

Throughout the following, we use the standard Big-O notation: $a(x) = O(b(x))$ means that there exists a constant $C > 0$ such that $|a(x)| \leq C\,|b(x)|$ for all x sufficiently close to 0.

Lemma 3.4. *Suppose we are in the conditions of Lemma 3.2. Then, there exists a mapping $\nu : J \to \mathbb{R}$ such that*

$$x^\xi(t) = \overline{x}(t) + \xi\nu(t) + O(\xi^2).$$

Proof. Given that $f \in C^2$, the function f can be expanded as

$$\begin{aligned} f(t, x^\xi(t), u^\xi(t)) =& f(t, \overline{x}(t), \overline{u}(t)) + (x^\xi(t) - \overline{x}(t))\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial x} \\ &+ (u^\xi(t) - \overline{u}(t))\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial u} + O(|x^\xi(t) - \overline{x}(t)|^2, |u^\xi(t) - \overline{u}(t)|^2), \quad t \in J. \end{aligned}$$

For all $t \in J$, we have $u^\xi(t) - \overline{u}(t) = \xi\eta(t)$. Additionally, as proven in Lemma 3.2, $x^\xi(t) - \overline{x}(t) = \xi\chi(t)$, for some finite function χ. Thus, the term $O(|x^\xi(t) - \overline{x}(t)|^2, |u^\xi(t) - \overline{u}(t)|^2)$ simplifies to $O(\xi^2)$. Hence, we can express the fractional derivative of x^ξ as:

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} x^\xi(t) = {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} \overline{x}(t) + (x^\xi(t) - \overline{x}(t))\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial x} + \xi\eta(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial u} + O(\xi^2).$$

Therefore, for $\xi \neq 0$ and $t \in J$,

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} \frac{x^\xi(t) - \overline{x}(t)}{\xi} = \frac{x^\xi(t) - \overline{x}(t)}{\xi}\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial x} + \eta(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial u} + \frac{O(\xi^2)}{\xi}.$$

Consider the following two FDEs:

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} y(t) = y(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial x} + \eta(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial u} + \frac{O(\xi^2)}{\xi} \tag{3.1}$$

and

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} y(t) = y(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial x} + \eta(t)\frac{\partial f(t, \overline{x}(t), \overline{u}(t))}{\partial u}, \tag{3.2}$$

subject to $y(a) = 0$. The existence and uniqueness of solutions for distributed-order FDEs can be ensured using standard results for FDEs involving the σ-Caputo fractional derivative (see, e.g., [4]). This is achieved by using the relation

$$^{C}\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} y(t) = \omega \cdot {}^{C}\mathrm{D}_{a^+}^{\overline{\gamma},\sigma} y(t),$$

for some $\overline{\gamma} \in [0,1]$ and with $\omega = \int_0^1 \Phi(\gamma)\, d\gamma$. Equation (3.1) has the solution $y_1 := \frac{x^\xi - \overline{x}}{\xi}$, and let y_2 be the solution of Eq (3.2). By Lemma 3.3, we obtain that $y_2(t) = \lim_{\xi \to 0} y_1(t)$, proving the desired result with $\nu := y_2$. □

We are now ready to present the main result of our work:

Theorem 3.5. *(PMP for ($\mathcal{P}_{OC}$)) Let $(\overline{x}, \overline{u})$ be an optimal pair to problem ($\mathcal{P}_{OC}$). Then, there exists a mapping $\lambda \in PC^1(J, \mathbb{R})$ such that:*

- *The two following FDEs hold:*

$$\frac{\partial L}{\partial u}(t, \overline{x}(t), \overline{u}(t)) + \lambda(t) \frac{\partial f}{\partial u}(t, \overline{x}(t), \overline{u}(t)) = 0, \quad t \in J; \tag{3.3}$$

and

$$\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)} \right) \sigma'(t) = \frac{\partial L}{\partial x}(t, \overline{x}(t), \overline{u}(t)) + \lambda(t) \frac{\partial f}{\partial x}(t, \overline{x}(t), \overline{u}(t)), \quad t \in J; \tag{3.4}$$

- *The transversality condition*

$$I_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\lambda(b)}{\sigma'(b)} = 0. \tag{3.5}$$

Proof. Suppose that $(\overline{x}, \overline{u})$ is a solution to problem ($\mathcal{P}_{OC}$). Consider a variation of $\overline{u}$ of the form $\overline{u} + \xi\eta$, denoted by u^ξ, where $\eta \in PC(J, \mathbb{R})$ and $\xi \in \mathbb{R}$. Also, let x^ξ be the state variable satisfying

$$\begin{cases} {}^{C}\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} x^\xi(t) = f(t, x^\xi(t), u^\xi(t)), \quad t \in J, \\ x^\xi(a) = x_a. \end{cases} \tag{3.6}$$

Observe that, as ξ goes to zero, u^ξ goes to $\overline{u}$ on J, and that

$$\left.\frac{\partial u^\xi(t)}{\partial \xi}\right|_{\xi=0} = \eta(t). \tag{3.7}$$

By Lemma 3.2 we can conclude that $x^\xi \to \overline{x}$ on J when $\xi \to 0$. Furthermore, by Lemma 3.4, the partial derivative $\left.\frac{\partial x^\xi(t)}{\partial \xi}\right|_{\xi=0} = \nu(t)$ exists for each t. We are considering the following functional

$$\mathcal{J}(x^\xi, u^\xi) = \int_a^b L\left(t, x^\xi(t), u^\xi(t)\right) dt.$$

Let $\lambda \in PC^1(J, \mathbb{R})$, to be explained later. From Lemma 2.6, we conclude that

$$\int_a^b \lambda(t) {}^{C}\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} x^\xi(t) dt - \int_a^b x^\xi(t) \left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)} \right) \sigma'(t) dt - \left[x^\xi(t) \left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)} \right) \right]_{t=a}^{t=b} = 0.$$

Then,

$$\mathcal{J}(x^\xi,u^\xi)=\int_a^b\left[L(t,x^\xi(t),u^\xi(t))+\lambda(t)f(t,x^\xi(t),u^\xi(t))-x^\xi(t)\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)\right]dt$$
$$-x^\xi(b)\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}\right)+x_a\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(a)}{\sigma'(a)}\right).$$

Since $(\overline{x},\overline{u})$ is a solution to problem $(\mathcal{P}_{OC})$, we conclude that

$$\begin{aligned}0 &= \frac{d}{d\xi}\mathcal{J}(x^\xi,u^\xi)\Big|_{\xi=0}\\ &= \int_a^b\left[\frac{\partial L}{\partial x}\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\frac{\partial L}{\partial u}\frac{\partial u^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\lambda(t)\left(\frac{\partial f}{\partial x}\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\frac{\partial f}{\partial u}\frac{\partial u^\xi(t)}{\partial\xi}\Big|_{\xi=0}\right)\right]dt\\ &\quad-\int_a^b\left[\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}\right]dt-\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}\right)\frac{\partial x^\xi(b)}{\partial\xi}\Big|_{\xi=0},\end{aligned}$$

where the partial derivatives of L and f are evaluated at the point $(t,\overline{x}(t),\overline{u}(t))$. Now replacing (3.7), we get

$$\begin{aligned}0 &= \int_a^b\left[\frac{\partial L}{\partial x}\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\frac{\partial L}{\partial u}\eta(t)+\lambda(t)\left(\frac{\partial f}{\partial x}\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\frac{\partial f}{\partial u}\eta(t)\right)\right]dt\\ &\quad-\int_a^b\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}dt-\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}\right)\frac{\partial x^\xi(b)}{\partial\xi}\Big|_{\xi=0}.\end{aligned}$$

Rearranging the terms we have

$$0=\int_a^b\left[\left(\frac{\partial L}{\partial x}+\lambda(t)\frac{\partial f}{\partial x}-\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)\right)\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\left(\frac{\partial L}{\partial u}+\lambda(t)\frac{\partial f}{\partial u}\right)\eta(t)\right]dt$$
$$-\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}\right)\frac{\partial x^\xi(b)}{\partial\xi}\Big|_{\xi=0}.$$

Introducing the Hamiltonian function

$$H(t,x(t),u(t),\lambda(t))=L(t,x(t),u(t))+\lambda(t)f(t,x(t),u(t)),\quad t\in J,$$

the expression simplifies to:

$$0=\int_a^b\left[\left(\frac{\partial H}{\partial x}-\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)\right)\frac{\partial x^\xi(t)}{\partial\xi}\Big|_{\xi=0}+\frac{\partial H}{\partial u}\eta(t)\right]dt$$
$$-\left(\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}\right)\frac{\partial x^\xi(b)}{\partial\xi}\Big|_{\xi=0},$$

where $\partial H/\partial x$ and $\partial H/\partial u$ are evaluated at $(t,\overline{x}(t),\overline{u}(t),\lambda(t))$. Choosing the function λ as the solution of the system

$$\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t)=\frac{\partial H}{\partial x},\quad\text{with}\quad\mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma}\frac{\lambda(b)}{\sigma'(b)}=0,$$

we get

$$\int_a^b \frac{\partial H}{\partial u}(t, \overline{x}(t), \overline{u}(t), \lambda(t))\eta(t)dt = 0.$$

Since η is arbitrary, from the Du Bois-Reymond lemma (see, e.g., [7]), we obtain

$$\frac{\partial H}{\partial u}(t, \overline{x}(t), \overline{u}(t), \lambda(t)) = 0,$$

for all $t \in J$, proving the desired result. □

Definition 3.6. The Hamiltonian is defined as $H(t, x(t), u(t), \lambda(t)) = L(t, x(t), u(t)) + \lambda(t) f(t, x(t), u(t))$, where L is the Lagrange function, f is the system dynamics, and λ is the adjoint variable.

Remark 3.7. We observe that:

(1) Taking σ as the identity function, we recover the results presented in [29], while also correcting minor inaccuracies in their proofs. This allows us to refine and extend the results previously established for the classical distributed-order fractional derivative. Furthermore, by introducing a general kernel, we not only broaden the scope of the original framework but also provide a more general formulation that encompasses and extends the findings of the aforementioned study.

(2) If $f(t, x(t), u(t)) = u(t)$, $t \in J$, our problem $(\mathcal{P}_{OC})$ reduces to the following fractional problem of the calculus of variations:

$$\mathcal{J}(x) = \int_a^b L\left(t, x(t), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} x(t)\right) dt \to extr,$$

subject to the initial condition $x(a) = x_a$, where $L \in C^1$. From Theorem 3.5 we deduce that if $\overline{x}(t)$ is an extremizer, then there exists $\lambda \in PC^1(J, \mathbb{R})$ such that:

- $\lambda(t) = -\partial_3 L(t, \overline{x}(t), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma}\overline{x}(t)), \quad t \in J,$
- $\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t) = \partial_2 L(t, \overline{x}(t), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma}\overline{x}(t)), \quad t \in J,$
- $I_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\lambda(b)}{\sigma'(b)} = 0,$

where $\partial_i L$ denotes the partial derivative of L with respect to its ith coordinate. Hence, we obtain the Euler-Lagrange equation:

$$\partial_2 L(t, \overline{x}(t), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma}\overline{x}(t)) + \left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\partial_3 L(t, \overline{x}(t), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma}\overline{x}(t))}{\sigma'(t)}\right)\sigma'(t) = 0, \quad t \in J,$$

and the tranversality condition:

$$I_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\partial_3 L(b, \overline{x}(b), {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma}\overline{x}(b))}{\sigma'(b)} = 0,$$

proved in [10].

We conclude this section by establishing sufficient optimality conditions for our OC problem.

Definition 3.8. We say that a triple $(\overline{x}, \overline{u}, \lambda)$ is a Pontryagin extremal of Problem $(\mathcal{P}_{OC})$ if it satisfies conditions (3.3)–(3.5).

Theorem 3.9. *(Sufficient conditions for global optimality I) Let* $(\overline{x}, \overline{u}, \lambda)$ *be a Pontryagin extremal of Problem* $(\mathcal{P}_{OC})$ *with* $\lambda(t) \geq 0$*, for all* $t \in J$.

- *If* L *and* f *are convex functions, then* $(\overline{x}, \overline{u})$ *is a global minimizer of functional* $\mathcal{J}$*;*
- *If* L *and* f *are concave functions, then* $(\overline{x}, \overline{u})$ *is a global maximizer of functional* $\mathcal{J}$.

Proof. We will only prove the case where L and f are convex; the other case is analogous. If L is a convex function, then (see, e.g., [36])

$$L(t, \overline{x}(t), \overline{u}(t)) - L(t, x(t), u(t)) \leq \frac{\partial L}{\partial x}(t, \overline{x}(t), \overline{u}(t))(\overline{x} - x)(t) + \frac{\partial L}{\partial u}(t, \overline{x}(t), \overline{u}(t))(\overline{u} - u)(t),$$

for any control u and associate state x. Hence,

$$\begin{aligned}
\mathcal{J}(\overline{x}, \overline{u}) - \mathcal{J}(x, u) &= \int_a^b \Big(L(t, \overline{x}(t), \overline{u}(t)) - L(t, x(t), u(t))\Big) dt \\
&\leq \int_a^b \left(\frac{\partial L}{\partial x}(t, \overline{x}(t), \overline{u}(t))(\overline{x} - x)(t) + \frac{\partial L}{\partial u}(t, \overline{x}(t), \overline{u}(t))(\overline{u} - u)(t)\right) dt.
\end{aligned}$$

Using the adjoint equation (3.4) and the optimality condition (3.3), we obtain

$$\begin{aligned}
\mathcal{J}(\overline{x}, \overline{u}) - \mathcal{J}(x, u) \quad &\leq \int_a^b \left[\left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)}\right) \sigma'(t) - \lambda(t) \frac{\partial f}{\partial x}(t, \overline{x}(t), \overline{u}(t))\right] (\overline{x} - x)(t) dt \\
&\quad - \int_a^b \lambda(t) \frac{\partial f}{\partial u}(t, \overline{x}(t), \overline{u}(t))(\overline{u} - u)(t) dt.
\end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}
\mathcal{J}(\overline{x}, \overline{u}) - \mathcal{J}(x, u) &\leq \int_a^b (\overline{x} - x)(t) \left(\mathrm{D}_{b^-}^{\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)}\right) \sigma'(t) dt \\
&\quad - \int_a^b \left(\lambda(t) \frac{\partial f}{\partial x}(t, \overline{x}(t), \overline{u}(t))(\overline{x} - x)(t) + \lambda(t) \frac{\partial f}{\partial u}(t, \overline{x}(t), \overline{u}(t))(\overline{u} - u)(t)\right) dt.
\end{aligned}$$

Using the generalized integration by parts formula (Lemma 2.6) in the first integral, we obtain

$$\begin{aligned}
\mathcal{J}(\overline{x}, \overline{u}) - \mathcal{J}(x, u) &\leq \int_a^b \lambda(t) {}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} (\overline{x} - x)(t) dt - \left[(\overline{x} - x)(t) \mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)}\right]_{t=a}^{t=b} \\
&\quad - \int_a^b \left(\lambda(t) \frac{\partial f}{\partial x}(t, \overline{x}(t), \overline{u}(t))(\overline{x} - x)(t) + \lambda(t) \frac{\partial f}{\partial u}(t, \overline{x}(t), \overline{u}(t))(\overline{u} - u)(t)\right) dt.
\end{aligned}$$

Since

$$\left[(\overline{x} - x)(t) \mathrm{I}_{b^-}^{1-\Phi(\gamma),\sigma} \frac{\lambda(t)}{\sigma'(t)}\right]_{t=a}^{t=b} = 0 \ \Big(\text{by the transversally condition (3.5) and } \overline{x}(a) = x(a) = x_a\Big)$$

and

$${}^C\mathrm{D}_{a^+}^{\Phi(\gamma),\sigma} (\overline{x} - x)(t) = f(t, \overline{x}(t), \overline{u}(t)) - f(t, x(t), u(t)),$$

we get

$$\begin{aligned}\mathcal{J}(\overline{x},\overline{u})-\mathcal{J}(x,u) \ \leq\ & \int_a^b \lambda(t)\Big(f(t,\overline{x}(t),\overline{u}(t))-f(t,x(t),u(t))\Big)dt \\ & -\int_a^b \lambda(t)\Big(\frac{\partial f}{\partial x}(t,\overline{x}(t),\overline{u}(t))(\overline{x}-x)(t)+\frac{\partial f}{\partial u}(t,\overline{x}(t),\overline{u}(t))(\overline{u}-u)(t)\Big)dt.\end{aligned}$$

Since f is convex, we have

$$f(t,\overline{x}(t),\overline{u}(t))-f(t,x(t),u(t))\leq \frac{\partial f}{\partial x}(t,\overline{x}(t),\overline{u}(t))(\overline{x}-x)(t)+\frac{\partial f}{\partial u}(t,\overline{x}(t),\overline{u}(t))(\overline{u}-u)(t),$$

for any admissible control u and its associated state x. Moreover, since $\lambda(t)\geq 0$, for all $t\in J$, it follows that $\mathcal{J}(\overline{x},\overline{u})-\mathcal{J}(x,u)\leq 0$, proving the desired result. □

Similarly, we can prove the following result.

Theorem 3.10. *(Sufficient conditions for global optimality II) Let $(\overline{x},\overline{u},\lambda)$ be a Pontryagin extremal of Problem $(\mathcal{P}_{OC})$ with $\lambda(t)<0$, for all $t\in J$.*

- *If L and f are convex functions, then $(\overline{x},\overline{u})$ is a global maximizer of functional $\mathcal{J}$;*
- *If L and f are concave functions, then $(\overline{x},\overline{u})$ is a global minimizer of functional $\mathcal{J}$.*

4. Illustrative examples

We now present two examples in order to illustrate our main result.

Example 4.1. *Consider the following problem:*

$$\mathcal{J}(x,u)=\int_0^1\left(\Big(x(t)-(\sigma(t)-\sigma(0))^2\Big)^2+\left(u(t)-\frac{(\sigma(t)-\sigma(0))^2-\sigma(t)+\sigma(0)}{\ln(\sigma(t)-\sigma(0))}\right)^2\right)dt\to extr,$$

subject to

$${}^{C}\mathrm{D}_{0^+}^{\phi(\gamma),\sigma}x(t)=u(t),\quad t\in[0,1],\ \ and\ \ x(0)=0.$$

The order-weighting function is $\phi:[0,1]\to[0,1]$ defined by

$$\phi(\gamma)=\frac{\Gamma(3-\gamma)}{2}.$$

The Hamiltonian in this case is given by

$$H(t,x,u,\lambda)=\Big(x(t)-(\sigma(t)-\sigma(0))^2\Big)^2+\left(u(t)-\frac{(\sigma(t)-\sigma(0))^2-\sigma(t)+\sigma(0)}{\ln(\sigma(t)-\sigma(0))}\right)^2+\lambda(t)u(t).$$

From (3.3) – (3.5)*, we get*

$$\lambda(t)=-2\left(u(t)-\frac{(\sigma(t)-\sigma(0))^2-\sigma(t)+\sigma(0)}{\ln(\sigma(t)-\sigma(0))}\right),\ t\in[0,1],$$

$$\left(\mathrm{D}_{1^-}^{\phi(\gamma),\sigma}\frac{\lambda(t)}{\sigma'(t)}\right)\sigma'(t) = 2\Big(x(t) - (\sigma(t)-\sigma(0))^2\Big),\ t\in[0,1],$$

and

$$I_{1^-}^{1-\phi(\gamma),\sigma}\frac{\lambda(1)}{\sigma'(1)} = 0.$$

Note that the triple $(\overline{x},\overline{u},\lambda)$ *given by:*

$$\overline{x}(t) = (\sigma(t)-\sigma(0))^2,\quad \overline{u}(t) = \frac{(\sigma(t)-\sigma(0))^2 - \sigma(t) + \sigma(0)}{\ln(\sigma(t)-\sigma(0))},\quad \lambda(t) = 0,\quad t\in[0,1],$$

satisfies the necessary optimality conditions given by the Pontryagin maximum principle. Since the Lagrangian L and $f(t,x(t),u(t)) = u(t)$ *are convex, then* $(\overline{x},\overline{u})$ *is a global minimizer of functional* $\mathcal{J}$.

Example 4.2. *Consider the optimal control problem:*

$$\mathcal{J}(x,u) = \int_0^1 (\sin(x(t)) + \cos(x(t)) + x(t)u(t))\,dt \to extr,$$

subject to the restriction

$${}^{C}\mathrm{D}_{0^+}^{\phi(\gamma),\sigma}x(t) = x(t)u(t),\quad t\in[0,1],\quad with\ \ x(0)=0.$$

The corresponding Hamiltonian is given by

$$H(t,x,u,\lambda) = \sin(x(t)) + \cos(x(t)) + x(t)u(t) + \lambda(t)x(t)u(t).$$

From the stationarity condition (3.3)*, we obtain*

$$x(t) + \lambda(t)x(t) = 0 \Leftrightarrow \lambda(t) = -1,\ t\in[0,1].$$

The remaining necessary conditions, as given by (3.4) *and* (3.5)*, are*

$$\left(\mathrm{D}_{1^-}^{\phi(\gamma),\sigma}\frac{-1}{\sigma'(t)}\right)\sigma'(t) = \cos(x(t)) - \sin(x(t)),\ t\in[0,1],$$

and

$$I_{1^-}^{1-\phi(\gamma),\sigma}\frac{-1}{\sigma'(1)} = 0.$$

5. Conclusions and future work

In this work, we explored necessary and sufficient conditions for optimality in fractional OC problems involving a novel fractional operator. Specifically, we addressed the challenge of determining the optimal processes for a fractional OC problem, where the regularity of the optimal solution is assured. The solution to this question is provided by an extension of the PMP, for which we have established the basic version applicable to fractional OC problems, involving a new generalized fractional derivative with respect to a smooth kernel. Additionally, we presented sufficient conditions of optimality for this class of problems using the newly defined fractional derivative. Future research will aim to extend these theorems to OC problems with bounded control constraints. Additionally, it is crucial to develop numerical methods for determining the optimal pair. One potential approach is to approximate the fractional operators using a finite sum, thereby transforming the fractional problem into a finite-dimensional problem, which can then be more effectively solved numerically.

Author contributions

Fátima Cruz, Ricardo Almeida and Natália Martins: Formal analysis, investigation, methodology, and writing – review & editing, of this manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Professor Ricardo Almeida is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

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