



Research article

Some fractional integral type inequalities for differentiable convex functions

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Abstract: In this paper, we propose to establish some fractional parametrized three-point integral inequalities. We start by developing a new integral identity. Based on this identity, we derive many types of integral inequality, including Ostrowski, midpoint, trapeze, Simpson, and Bullen. A number of known results are also derived. The findings' applications are given.

Keywords: fractional parametrized; three-point integral inequalities; Ostrowski; midpoint; trapeze; Simpson; Bullen

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1. Introduction

Integral inequalities are both theoretically fascinating and practically important, as they contribute significantly to the study of the properties of solutions of differential and integral equations, facilitate error estimations, and play a significant role in ensuring the best approximation of computations in numerical integration. In particular, Newton-Cotes inequalities are an effective way to assess the accuracy of integrals in numerical analysis as well as in many other scientific fields, such as engineering, physics, and economics.

Convexity theory has important geometric and algebraic characteristics and properties, as well as a wide range of applications, and is closely and strongly related to inequality. There have been several notable works that have contributed to this area of research. For more details, we advise the interested

readers to [1–3]. We remember that a function $\mathfrak{S} : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex, if

$$\mathfrak{S}(\tilde{t}_o l_1 + (1 - \tilde{t}_o) l_2) \leq \tilde{t}_o \mathfrak{S}(l_1) + (1 - \tilde{t}_o) \mathfrak{S}(l_2)$$

holds for all $l_1, l_2 \in I$ and all $\tilde{t}_o \in [0, 1]$ ([4]).

Fractional calculus is a subfield of mathematical analysis that builds upon standard calculus by integrating derivatives and integrals of non-integer order. It is used to faithfully depict non-local and non-Markovian occurrences in a variety of scientific fields. To improve the modeling of many scientific and technical processes, we have developed a number of fractional integral operators; for Hadamard inequalities, see [5–7], for exploring other types of fractional integrals [8–10].

However, the Riemann-Liouville operators, is the most commonly encountered operators, and are defined as follows:

Definition 1. [11] Assume that $\mathfrak{S} \in L^1[l_1, l_2]$. For order $\alpha > 0$ with $l_1 \geq 0$, the Riemann-Liouville fractional integrals are defined by

$$\begin{aligned} I_{l_1^+}^\alpha \mathfrak{S}(s) &= \frac{1}{\Gamma(\alpha)} \int_{l_1}^s (s - \tilde{t}_o)^{\alpha-1} \mathfrak{S}(\tilde{t}_o) d\tilde{t}_o, \quad s > l_1, \\ I_{l_2^-}^\alpha \mathfrak{S}(s) &= \frac{1}{\Gamma(\alpha)} \int_s^{l_2} (\tilde{t}_o - s)^{\alpha-1} \mathfrak{S}(\tilde{t}_o) d\tilde{t}_o, \quad l_2 > s, \end{aligned}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$, represents the gamma function with $I_{l_1^+}^0 \mathfrak{S}(s) = I_{l_2^-}^0 \mathfrak{S}(s) = \mathfrak{S}(s)$.

This idea has been accepted and approved in many scientific and technical fields due to its wide range of applications. Also, several subfields of mathematics have adopted this principle, including integral inequalities for many classes of functions, the most widespread being convex functions.

Sarikaya and Yildirim [12] gave the following Hermite-Hadamard type inequalities:

$$C\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \frac{2^{\nu-1} \Gamma(\nu+1)}{(\varepsilon_2 - \varepsilon_1)^\nu} \left(I_{\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)^+}^\nu C(\varepsilon_2) + I_{\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)^-}^\nu C(\varepsilon_1) \right) \leq \frac{C(\varepsilon_1) + C(\varepsilon_2)}{2}. \quad (1.1)$$

Nasri et al. [10] created the midpoint-type inequalities using a similar methodology. Bullen and trapezoid type inequalities were established by Saleh et al. [13]. Hai and Wang [4] developed the Simpson-type inequalities.

We direct readers to several current developments and applications of fractional calculus, including [14–16].

Motivated and inspired by the works introduced by Sarikaya [17] and Xi and Qi [18], as well as by the use of Riemann-Liouville integral operators, we first prove a new fractional integral identity. From this identity, we establish several new parameterized fractional integral inequalities for functions whose first derivatives are convex. Numerous known results are derived. Applications of the finding are provided.

2. Main results

The following lemma is required to support our findings.

Lemma 1. Let $\mathfrak{S} : [l_1, l_2] \rightarrow \mathbb{R}$ be a differentiable function on $[l_1, l_2]$ with $l_1 < l_2$, and $\mathfrak{S}' \in L^1 [l_1, l_2]$; then the following equality holds:

$$\begin{aligned} & \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \\ &= \frac{\kappa - l_1}{2} \int_0^1 (\tilde{t}_o^\alpha - \lambda) \mathfrak{S}'((1 - \tilde{t}_o) l_1 + \tilde{t}_o \kappa) d\tilde{t}_o \\ & \quad + \frac{l_2 - \kappa}{2} \int_0^1 (\mu - (1 - \tilde{t}_o)^\alpha) \mathfrak{S}'((1 - \tilde{t}_o) \kappa + \tilde{t}_o l_2) d\tilde{t}_o, \end{aligned} \quad (2.1)$$

where $\lambda, \mu \in [0, 1]$, $\alpha > 0$ and $\kappa \in [l_1, l_2]$.

Proof. The first integral of the right side of (2.1) may be integrated by parts to yield

$$\begin{aligned} & \frac{\kappa - l_1}{2} \int_0^1 (\tilde{t}_o^\alpha - \lambda) \mathfrak{S}'((1 - \tilde{t}_o) l_1 + \tilde{t}_o \kappa) d\tilde{t}_o \\ &= \frac{1}{2} (\tilde{t}_o^\alpha - \lambda) \mathfrak{S}((1 - \tilde{t}_o) l_1 + \tilde{t}_o \kappa) \Big|_0^1 - \frac{\alpha}{2} \int_0^1 \tilde{t}_o^{\alpha-1} \mathfrak{S}((1 - \tilde{t}_o) l_1 + \tilde{t}_o \kappa) d\tilde{t}_o \\ &= \frac{1-\lambda}{2} \mathfrak{S}(\kappa) + \frac{\lambda}{2} \mathfrak{S}(l_1) - \frac{\alpha}{2(\kappa - l_1)^\alpha} \int_{l_1}^{\kappa} (u - l_1)^{\alpha-1} \mathfrak{S}(u) du \\ &= \frac{1-\lambda}{2} \mathfrak{S}(\kappa) + \frac{\lambda}{2} \mathfrak{S}(l_1) - \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1). \end{aligned} \quad (2.2)$$

Similarly, the second integral gives

$$\begin{aligned} & \frac{l_2 - \kappa}{2} \int_0^1 (\mu - (1 - \tilde{t}_o)^\alpha) \mathfrak{S}'((1 - \tilde{t}_o) \kappa + \tilde{t}_o l_2) d\tilde{t}_o \\ &= \frac{1}{2} (\mu - (1 - \tilde{t}_o)^\alpha) \mathfrak{S}((1 - \tilde{t}_o) \kappa + \tilde{t}_o l_2) \Big|_0^1 - \frac{\alpha}{2} \int_0^1 (1 - \tilde{t}_o)^{\alpha-1} \mathfrak{S}((1 - \tilde{t}_o) \kappa + \tilde{t}_o l_2) d\tilde{t}_o \\ &= \frac{\mu}{2} \mathfrak{S}(l_2) + \frac{1-\mu}{2} \mathfrak{S}(\kappa) - \frac{\alpha}{2(l_2 - \kappa)^\alpha} \int_{\kappa}^{l_2} (l_2 - u)^{\alpha-1} \mathfrak{S}(u) du \\ &= \frac{\mu}{2} \mathfrak{S}(l_2) + \frac{1-\mu}{2} \mathfrak{S}(\kappa) - \frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2). \end{aligned} \quad (2.3)$$

Adding (2.2) and (2.3), we get the desired result. $\square$

Theorem 1. Let $\mathfrak{S}$ be as in Lemma 1. If $|\mathfrak{S}'|$ is convex, then we have

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_1)| \\ & \quad + \left(\frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+2)\lambda}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \lambda^{1+\frac{2}{\alpha}} \right) + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+2)\mu}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \mu^{1+\frac{2}{\alpha}} \right) \right) |\mathfrak{S}'(\kappa)| \\ & \quad + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\mu}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \mu^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_2)|, \end{aligned}$$

where $\lambda, \mu \in [0, 1]$, $\alpha > 0$ and $\kappa \in [l_1, l_2]$.

Proof. Using the absolute value of both sides of (2.1) and the convexity of $|\mathfrak{S}'|$, we deduce

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa-l_1}{2} \int_0^1 |\tilde{l}_o^\alpha - \lambda| |\mathfrak{S}'((1-\tilde{l}_o)l_1 + \tilde{l}_o\kappa)| d\tilde{l}_o \\ & \quad + \frac{l_2-\kappa}{2} \int_0^1 |\mu - (1-\tilde{l}_o)^\alpha| |\mathfrak{S}'((1-\tilde{l}_o)\kappa + \tilde{l}_ol_2)| d\tilde{l}_o \\ & \leq \frac{\kappa-l_1}{2} \int_0^1 |\tilde{l}_o^\alpha - \lambda| ((1-\tilde{l}_o)|\mathfrak{S}'(l_1)| + \tilde{l}_o|\mathfrak{S}'(\kappa)|) d\tilde{l}_o \\ & \quad + \frac{l_2-\kappa}{2} \int_0^1 |\mu - (1-\tilde{l}_o)^\alpha| ((1-\tilde{l}_o)|\mathfrak{S}'(\kappa)| + \tilde{l}_o|\mathfrak{S}'(l_2)|) d\tilde{l}_o \\ & = \frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_1)| \\ & \quad + \left(\frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+2)\lambda}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \lambda^{1+\frac{2}{\alpha}} \right) + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+2)\mu}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \mu^{1+\frac{2}{\alpha}} \right) \right) |\mathfrak{S}'(\kappa)| \\ & \quad + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\mu}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \mu^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_2)|, \\ & \int_0^1 |\tilde{l}_o^\alpha - \lambda| (1-\tilde{l}_o) d\tilde{l}_o = \frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}}, \end{aligned} \tag{2.4}$$

$$\int_0^1 |\tilde{l}_o^\alpha - \lambda| \tilde{l}_o d\tilde{l}_o = \frac{2-(\alpha+2)\lambda}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \lambda^{1+\frac{2}{\alpha}}, \tag{2.5}$$

$$\int_0^1 |\mu - (1-\tilde{l}_o)^\alpha| (1-\tilde{l}_o) d\tilde{l}_o = \frac{2-(\alpha+2)\mu}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \mu^{1+\frac{2}{\alpha}} \tag{2.6}$$

and

$$\int_0^1 |\mu - (1-\tilde{l}_o)^\alpha| \tilde{l}_o d\tilde{l}_o = \frac{2-(\alpha+1)(\alpha+2)\mu}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \mu^{1+\frac{2}{\alpha}}, \tag{2.7}$$

which concludes the proof. $\square$

Corollary 1. Putting $\alpha = 1$ in Theorem 1, we obtain

$$\begin{aligned} & \left| \frac{\lambda \Xi(l_1) + (2 - \lambda - \mu) \Xi(\kappa) + \mu \Xi(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \Xi(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \Xi(u) du \right) \right| \\ & \leq \frac{\kappa - l_1}{2} \left(\frac{2 - (\alpha + 1)(\alpha + 2)\lambda}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \lambda^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \lambda^{1 + \frac{2}{\alpha}} \right) |\Xi'(l_1)| \\ & \quad + \left(\frac{\kappa - l_1}{2} \left(\frac{2 - (\alpha + 1)(\alpha + 2)\lambda}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \lambda^{1 + \frac{2}{\alpha}} \right) + \frac{l_2 - \kappa}{2} \left(\frac{2 - (\alpha + 2)\mu}{2(\alpha + 2)} + \frac{2\alpha}{2(\alpha + 2)} \mu^{1 + \frac{2}{\alpha}} \right) \right) |\Xi'(\kappa)| \\ & \quad + \frac{l_2 - \kappa}{2} \left(\frac{2 - (\alpha + 1)(\alpha + 2)\mu}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \mu^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \mu^{1 + \frac{2}{\alpha}} \right) |\Xi'(l_2)|. \end{aligned}$$

Corollary 2. In Theorem 1, if we choose $\lambda = \mu = 0$, then we obtain

$$\begin{aligned} & \left| \Xi(\kappa) - \left(\frac{\Gamma(\alpha + 1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \Xi(l_2) + \frac{\Gamma(\alpha + 1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \Xi(l_1) \right) \right| \\ & \leq \frac{\kappa - l_1}{2(\alpha + 1)(\alpha + 2)} |\Xi'(l_1)| + \frac{l_2 - l_1}{2(\alpha + 2)} |\Xi'(\kappa)| + \frac{l_2 - \kappa}{2(\alpha + 1)(\alpha + 2)} |\Xi'(l_2)|. \end{aligned}$$

Moreover, if we take $\alpha = 1$, then we obtain

$$\begin{aligned} & \left| \Xi(\kappa) - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \Xi(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \Xi(u) du \right) \right| \\ & \leq \frac{\kappa - l_1}{12} |\Xi'(l_1)| + \frac{l_2 - l_1}{6} |\Xi'(\kappa)| + \frac{l_2 - \kappa}{12} |\Xi'(l_2)|. \end{aligned}$$

Corollary 3. In Theorem 1, if we take $\lambda = \mu = 1$, then we obtain

$$\begin{aligned} & \left| \frac{\Xi(l_1) + \Xi(l_2)}{2} - \left(\frac{\Gamma(\alpha + 1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \Xi(l_2) + \frac{\Gamma(\alpha + 1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \Xi(l_1) \right) \right| \\ & \leq \frac{\alpha(\alpha + 3)(\kappa - l_1)}{4(\alpha + 1)(\alpha + 2)} |\Xi'(l_1)| + \frac{\alpha(l_2 - l_1)}{4(\alpha + 2)} |\Xi'(\kappa)| + \frac{\alpha(\alpha + 3)(l_2 - \kappa)}{4(\alpha + 1)(\alpha + 2)} |\Xi'(l_2)|. \end{aligned}$$

Moreover, if we take $\alpha = 1$, then we obtain

$$\begin{aligned} & \left| \frac{\Xi(l_1) + \Xi(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \Xi(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \Xi(u) du \right) \right| \\ & \leq \frac{\kappa - l_1}{6} |\Xi'(l_1)| + \frac{l_2 - l_1}{12} |\Xi'(\kappa)| + \frac{l_2 - \kappa}{6} |\Xi'(l_2)|. \end{aligned}$$

Corollary 4. In Theorem 1, if we take $\kappa = \frac{l_1 + l_2}{2}$, then we obtain

$$\begin{aligned} & \left| \frac{\lambda \Xi(l_1) + (2 - \lambda - \mu) \Xi\left(\frac{l_1 + l_2}{2}\right) + \mu \Xi(l_2)}{2} - \frac{2^{\alpha - 1} \Gamma(\alpha + 1)}{(l_2 - l_1)^\alpha} \left(I_{\frac{l_1 + l_2}{2}^+}^\alpha \Xi(l_2) + I_{\frac{l_1 + l_2}{2}^-}^\alpha \Xi(l_1) \right) \right| \\ & \leq \frac{l_2 - l_1}{4} \left(\left(\frac{2 - (\alpha + 1)(\alpha + 2)\lambda}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \lambda^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \lambda^{1 + \frac{2}{\alpha}} \right) |\Xi'(l_1)| \right. \\ & \quad \left. + \left(\frac{4 - (\alpha + 2)(\lambda + \mu)}{2(\alpha + 2)} + \frac{\alpha}{\alpha + 2} \left(\lambda^{1 + \frac{2}{\alpha}} + \mu^{1 + \frac{2}{\alpha}} \right) \right) \left| \Xi'\left(\frac{l_1 + l_2}{2}\right) \right| \right. \\ & \quad \left. + \left(\frac{2 - (\alpha + 1)(\alpha + 2)\mu}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \mu^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \mu^{1 + \frac{2}{\alpha}} \right) |\Xi'(l_2)| \right). \end{aligned}$$

Corollary 5. Using the convexity of $|\mathfrak{S}'|$ i.e., $\left|\mathfrak{S}'\left(\frac{l_1+l_2}{2}\right)\right| \leq \frac{1}{2}|\mathfrak{S}'(l_1)| + \frac{1}{2}|\mathfrak{S}'(l_2)|$, Corollary 4 becomes

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}\left(\frac{l_1+l_2}{2}\right) + \mu \mathfrak{S}(l_2)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(l_2-l_1)^\alpha} \left(I_{\frac{l_1+l_2}{2}^+}^\alpha \mathfrak{S}(l_2) + I_{\frac{l_1+l_2}{2}^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{l_2-l_1}{4} \left(\left(\frac{4-(3\lambda+\mu)(\alpha+1)}{4(\alpha+1)} + \frac{\alpha}{2(\alpha+2)} \left(\mu^{1+\frac{2}{\alpha}} - \lambda^{1+\frac{2}{\alpha}} \right) + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} \right) |\mathfrak{S}'(l_1)| \right. \\ & \quad \left. + \left(\frac{4-(\lambda+3\mu)(\alpha+1)}{4(\alpha+1)} + \frac{\alpha}{2(\alpha+2)} \left(\lambda^{1+\frac{2}{\alpha}} - \mu^{1+\frac{2}{\alpha}} \right) + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} \right) |\mathfrak{S}'(l_2)| \right). \end{aligned}$$

Corollary 6. In Corollary 4, if we take $\alpha = 1$, then we obtain

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}\left(\frac{l_1+l_2}{2}\right) + \mu \mathfrak{S}(l_2)}{2} - \frac{1}{l_2-l_1} \int_{l_1}^{l_2} \mathfrak{S}(u) du \right| \\ & \leq \frac{l_2-l_1}{4} \left(\frac{1-3\lambda+6\lambda^2-2\lambda^3}{6} |\mathfrak{S}'(l_1)| + \frac{1-3\mu+6\mu^2-2\mu^3}{6} |\mathfrak{S}'(l_2)| \right. \\ & \quad \left. + \frac{4-3(\lambda+\mu)+2(\lambda^3+\mu^3)}{6} \left| \mathfrak{S}'\left(\frac{l_1+l_2}{2}\right) \right| \right). \end{aligned}$$

Corollary 7. Using the convexity of $|\mathfrak{S}'|$, Corollary 6 becomes

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}\left(\frac{l_1+l_2}{2}\right) + \mu \mathfrak{S}(l_2)}{2} - \frac{1}{l_2-l_1} \int_{l_1}^{l_2} \mathfrak{S}(u) du \right| \\ & \leq \frac{l_2-l_1}{4} \left(\frac{6-3(3\lambda+\mu)+12\lambda^2+2(\mu^3-\lambda^3)}{12} |\mathfrak{S}'(l_1)| + \frac{6-3(\lambda+3\mu)+12\mu^2+2(\lambda^3-\mu^3)}{12} |\mathfrak{S}'(l_2)| \right). \end{aligned}$$

Corollary 8. In Corollary 4, if we take $\lambda = \mu$, then we obtain

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + 2(1-\lambda) \mathfrak{S}\left(\frac{l_1+l_2}{2}\right) + \lambda \mathfrak{S}(l_2)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(l_2-l_1)^\alpha} \left(I_{\frac{l_1+l_2}{2}^+}^\alpha \mathfrak{S}(l_2) + I_{\frac{l_1+l_2}{2}^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{l_2-l_1}{4} \left(\left(\frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) (|\mathfrak{S}'(l_1)| + |\mathfrak{S}'(l_2)|) \right. \\ & \quad \left. + \left(\frac{2-(\alpha+2)\lambda}{\alpha+2} + \frac{2\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) \left| \mathfrak{S}'\left(\frac{l_1+l_2}{2}\right) \right| \right). \end{aligned}$$

Remark 1. Corollary 8 recaptures.

- Corollary 3 with $s = 1$ from [13], if we take $\lambda = 0$.
- The second point of Remark 3.6 from [19] if we take $\lambda = \frac{1}{2}$.
- Theorem 2.2 with $l_1 = 1$ (inequality (2.17)) from [20], if we take $\lambda = \frac{1}{3}$.
- The first point of Remark 3.6 from [19], if we take $\lambda = 1$.

Corollary 9. Using the convexity of $|\mathfrak{S}'|$, Corollary 8 gives

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + 2(1-\lambda) \mathfrak{S}\left(\frac{l_1+l_2}{2}\right) + \lambda \mathfrak{S}(l_2)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(l_2-l_1)^\alpha} \left(I_{\frac{l_1+l_2}{2}^+}^\alpha \mathfrak{S}(l_2) + I_{\frac{l_1+l_2}{2}^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{l_2-l_1}{4} \left(\frac{1-(\alpha+1)\lambda}{\alpha+1} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} \right) (|\mathfrak{S}'(l_1)| + |\mathfrak{S}'(l_2)|). \end{aligned}$$

Remark 2. Corollary 9 recaptures.

- Theorem 5 from [21], if we take $\lambda = 0$.
- Corollary 3.3 from [22], if we take $\lambda = \frac{1}{2}$.
- Theorem 2.3 with $l_1 = 1$ (inequality (2.21)) from [20], if we take $\lambda = \frac{1}{3}$.
- Corollary 5.4 from [23], if we take $\lambda = 1$.

Corollary 10. In Corollary 8, if we take $\alpha = 1$, then we get

$$\left| \frac{\lambda \Xi(l_1) + 2(1-\lambda) \Xi\left(\frac{l_1+l_2}{2}\right) + \lambda \Xi(l_2)}{2} - \frac{1}{l_2-l_1} \int_{l_1}^{l_2} \Xi(u) du \right|$$

$$\leq \frac{l_2-l_1}{4} \left(\frac{1-3\lambda+6\lambda^2-2\lambda^3}{6} (|\Xi'(l_1)| + |\Xi'(l_2)|) + \frac{2-3\lambda+2\lambda^3}{3} \left| \Xi'\left(\frac{l_1+l_2}{2}\right) \right| \right).$$

Remark 3. Corollary 10 recaptures.

- Corollary 3 from [24], if we take $\lambda = 0$.
- Corollary 10 from [17], if we take $\lambda = \frac{1}{2}$.
- Corollary 2.4 from [25], if we take $\lambda = \frac{1}{3}$.
- Corollary 2 from [26], if we take $\lambda = 1$.

Corollary 11. Using the convexity of $|\Xi'|$, Corollary 10 gives

$$\left| \frac{\lambda \Xi(l_1) + 2(1-\lambda) \Xi\left(\frac{l_1+l_2}{2}\right) + \lambda \Xi(l_2)}{2} - \frac{1}{l_2-l_1} \int_{l_1}^{l_2} \Xi(u) du \right|$$

$$\leq \frac{l_2-l_1}{8} (1 - 2\lambda + 2\lambda^2) (|\Xi'(l_1)| + |\Xi'(l_2)|).$$

Remark 4. Corollary 11 recaptures.

- Theorem 2.2 from [27], if we take $\lambda = 0$.
- Corollary 3.2 from [28] and Remark 3.2 from [29], if we take $\lambda = \frac{1}{2}$.
- Theorem 5 from [30], if we take $\lambda = \frac{1}{3}$.
- Theorem 2.2 from [31], if we take $\lambda = 1$.

The following special functions are used for the following outcomes.

The incomplete beta function is given by

$$B_a(t, f) = \int_0^a \tilde{t}_o^{-1} (1 - \tilde{t}_o)^{f-1} d\tilde{t}_o, \quad t, f > 0 \text{ and } 0 < a < 1.$$

When we replace a by 1, we obtain the classical beta function.

The hypergeometric function is given by

$${}_2F_1(a, b, c; z) = \frac{1}{B(b, c-b)} \int_0^1 \tilde{t}_o^{b-1} (1 - \tilde{t}_o)^{c-b-1} (1 - z\tilde{t}_o)^{-a} d\tilde{t}_o,$$

where $c > b > 0$, $|z| < 1$ and $B(., .)$ is the beta function.

To establish our next result, we need the following lemma.

Lemma 2. [32] Let $\mathfrak{S} : [l_1, l_2] \rightarrow (0, \infty)$, if $\mathfrak{S}^q$ is convex on $[l_1, l_2]$ for all $q \in (0, 1]$, we can derive two important consequences below.

(i) For $0 < q \leq \frac{1}{2}$, we have

$$\begin{aligned} & \mathfrak{S}((1 - \tilde{t}_o)l_1 + \tilde{t}_ol_2) \\ & \leq q2^{\frac{1}{q}-1} \left((1 - \tilde{t}_o)^{\frac{1}{q}} \mathfrak{S}(l_1) + \tilde{t}_o^{\frac{1}{q}} \mathfrak{S}(l_2) + \left(\frac{2}{q} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (\mathfrak{S}(l_1) \mathfrak{S}(l_2))^{\frac{1}{2}} \right). \end{aligned}$$

(ii) For $\frac{1}{2} < q \leq 1$, we have

$$\begin{aligned} & \mathfrak{S}((1 - \tilde{t}_o)l_1 + \tilde{t}_ol_2) \\ & \leq (1 - \tilde{t}_o)^{\frac{1}{q}} \mathfrak{S}(l_1) + \tilde{t}_o^{\frac{1}{q}} \mathfrak{S}(l_2) + \left(2^{\frac{1}{q}} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (\mathfrak{S}(l_1) \mathfrak{S}(l_2))^{\frac{1}{2}}. \end{aligned}$$

Theorem 2. Let $\mathfrak{S}$ be as in Lemma 1. If $|\mathfrak{S}'|^q$ is convex, then we have:

For $0 < q < \frac{1}{2}$

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{q2^{\frac{1}{q}(\kappa - l_1)}}{4} (\varphi(q, \alpha, \lambda) |\mathfrak{S}'(l_1)| + \psi(q, \alpha, \kappa) |\mathfrak{S}'(\kappa)|) \\ & \quad + \left(\frac{2}{q} - 2 \right) l_2(q, \alpha, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \\ & \quad + \frac{q2^{\frac{1}{q}(l_2 - \kappa)}}{4} (\psi(q, \alpha, \mu) |\mathfrak{S}'(\kappa)| + \varphi(q, \alpha, \mu) |\mathfrak{S}'(l_2)|) \\ & \quad + \left(\frac{2}{q} - 2 \right) l_2(q, \alpha, \mu) (|\mathfrak{S}'(l_2)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}. \end{aligned}$$

For $\frac{1}{2} < q < 1$

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa - l_1}{2} (\varphi(q, \alpha, \lambda) |\mathfrak{S}'(l_1)| + \psi(q, \alpha, \kappa) |\mathfrak{S}'(\kappa)|) \\ & \quad + \left(2^{\frac{1}{q}} - 2 \right) l_2(q, \alpha, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \\ & \quad + \frac{l_2 - \kappa}{2} (\psi(q, \alpha, \mu) |\mathfrak{S}'(\kappa)| + \varphi(q, \alpha, \mu) |\mathfrak{S}'(l_2)|) \\ & \quad + \left(2^{\frac{1}{q}} - 2 \right) l_2(q, \alpha, \mu) (|\mathfrak{S}'(l_2)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}, \end{aligned}$$

where $0 < q < 1$, $\lambda, \mu \in [0, 1]$, $\alpha > 0$ and $\kappa \in [l_1, l_2]$.

Proof. Using the absolute value of both sides of (2.1), we deduce

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa - l_1}{2} \int_0^1 |\tilde{t}_o^\alpha - \lambda| |\mathfrak{S}'((1 - \tilde{t}_o)l_1 + \tilde{t}_o\kappa)| d\tilde{t}_o \\ & \quad + \frac{l_2 - \kappa}{2} \int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| |\mathfrak{S}'((1 - \tilde{t}_o)\kappa + \tilde{t}_ol_2)| d\tilde{t}_o. \end{aligned} \tag{2.8}$$

Assume that $0 < q < \frac{1}{2}$, from the convexity of $|\mathfrak{S}'|^q$ and Lemma 2, we obtain

$$|\mathfrak{S}'((1 - \tilde{t}_o)l_1 + \tilde{t}_o\kappa)| \leq q2^{\frac{1}{q}-1} \left((1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| \right. \\ \left. + \left(\frac{2}{q} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right) \quad (2.9)$$

and

$$|\mathfrak{S}'((1 - \tilde{t}_o)\kappa + \tilde{t}_ol_2)| \leq q2^{\frac{1}{q}-1} \left((1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \\ \left. + \left(\frac{2}{q} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right). \quad (2.10)$$

Using (2.9) and (2.10) in (2.8), we obtain

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{q2^{\frac{1}{q}-1}(\kappa - l_1)}{4} \int_0^1 |\tilde{t}_o^\alpha - \lambda| \left((1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| \right. \\ & \quad \left. + \left(\frac{2}{q} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right) d\tilde{t}_o \\ & \quad + \frac{q2^{\frac{1}{q}-1}(l_2 - \kappa)}{4} \int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| \left((1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \\ & \quad \left. + \left(\frac{2}{q} - 2 \right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_2)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right) d\tilde{t}_o \\ & = \frac{q2^{\frac{1}{q}-1}(\kappa - l_1)}{4} \left(\left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| (1 - \tilde{t}_o)^{\frac{1}{q}} d\tilde{t}_o \right) |\mathfrak{S}'(l_1)| + \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| \tilde{t}_o^{\frac{1}{q}} d\tilde{t}_o \right) |\mathfrak{S}'(\kappa)| \right. \\ & \quad \left. + \left(\frac{2}{q} - 2 \right) \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} d\tilde{t}_o \right) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right) \\ & \quad + \frac{q2^{\frac{1}{q}-1}(l_2 - \kappa)}{4} \left(\left(\int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| (1 - \tilde{t}_o)^{\frac{1}{q}} d\tilde{t}_o \right) |\mathfrak{S}'(\kappa)| \right. \\ & \quad \left. + \left(\int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| \tilde{t}_o^{\frac{1}{q}} d\tilde{t}_o \right) |\mathfrak{S}'(l_2)| \right. \\ & \quad \left. + \left(\frac{2}{q} - 2 \right) \left(\int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} d\tilde{t}_o \right) (|\mathfrak{S}'(l_2)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right) \\ & = \frac{q2^{\frac{1}{q}-1}(\kappa - l_1)}{4} (\varphi(q, \alpha, \lambda) |\mathfrak{S}'(l_1)| + \psi(q, \alpha, z) |\mathfrak{S}'(\kappa)| \\ & \quad + \left(\frac{2}{q} - 2 \right) l_2(q, \alpha, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}) \end{aligned}$$

$$\begin{aligned}
& + \frac{q2^{\frac{1}{q}}(l_2-\kappa)}{4} (\psi(q, \alpha, \mu) |\mathfrak{S}'(\kappa)| + \varphi(q, \alpha, \mu) |\mathfrak{S}'(l_2)| \\
& + \left(\frac{2}{q} - 2\right) l_2(q, \alpha, \mu) (|\mathfrak{S}'(l_2)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}).
\end{aligned} \tag{2.11}$$

Now, assume that $\frac{1}{2} < q < 1$; from the convexity of $|\mathfrak{S}'|^q$ and Lemma 2, we have

$$\begin{aligned}
& |\mathfrak{S}'((1 - \tilde{t}_o)l_1 + \tilde{t}_o\kappa)| \\
& \leq (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
& |\mathfrak{S}'((1 - \tilde{t}_o)\kappa + \tilde{t}_ol_2)| \\
& \leq (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}}.
\end{aligned} \tag{2.13}$$

Using (2.12) and (2.13) in (2.8), we obtain

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\
& \leq \frac{\kappa-l_1}{2} \int_0^1 \left| \tilde{t}_o^\alpha - \lambda \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right. \\
& \quad \left. + \frac{l_2-\kappa}{2} \int_0^1 \left| \mu - (1 - \tilde{t}_o)^\alpha \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right. \\
& = \frac{\kappa-l_1}{2} \left(\left(\int_0^1 \left| \tilde{t}_o^\alpha - \lambda \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| \right. \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right) |\mathfrak{S}'(l_1)| \\
& \quad \left. + \left(\int_0^1 \left| \tilde{t}_o^\alpha - \lambda \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right) |\mathfrak{S}'(\kappa)| \\
& \quad \left. + \left(2^{\frac{1}{q}} - 2\right) \left(\int_0^1 \left| \tilde{t}_o^\alpha - \lambda \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(l_1)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| \right. \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \\
& \quad \left. + \frac{l_2-\kappa}{2} \left(\left(\int_0^1 \left| \mu - (1 - \tilde{t}_o)^\alpha \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \right. \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right) |\mathfrak{S}'(l_2)| \\
& \quad \left. + \left(2^{\frac{1}{q}} - 2\right) \left(\int_0^1 \left| \mu - (1 - \tilde{t}_o)^\alpha \left| (1 - \tilde{t}_o)^{\frac{1}{q}} |\mathfrak{S}'(\kappa)| + \tilde{t}_o^{\frac{1}{q}} |\mathfrak{S}'(l_2)| \right. \right. \right. \right. \\
& \quad \left. \left. + \left(2^{\frac{1}{q}} - 2\right) \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right| d\tilde{t}_o \right) (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right) \\
& = \frac{\kappa-l_1}{2} (\varphi(q, \alpha, \lambda) |\mathfrak{S}'(l_1)| + \psi(q, \alpha, \lambda) |\mathfrak{S}'(\kappa)| \\
& \quad + \left(2^{\frac{1}{q}} - 2\right) l_2(q, \alpha, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}}) \\
& \quad + \frac{l_2-\kappa}{2} (\psi(q, \alpha, \mu) |\mathfrak{S}'(\kappa)| + \varphi(q, \alpha, \mu) |\mathfrak{S}'(l_2)|
\end{aligned}$$

$$+ \left(2^{\frac{1}{q}} - 2\right) l_2(q, \alpha, \mu) (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}}, \quad (2.14)$$

where we have used

$$\begin{aligned} \varphi(q, \alpha, z) &= \int_0^1 |\tilde{t}_o^\alpha - z| (1 - \tilde{t}_o)^{\frac{1}{q}} d\tilde{t}_o \\ &= \frac{qz}{1+q} - \frac{2qz}{1+q} \left(1 - z^{\frac{1}{\alpha}}\right)^{\frac{1}{q}+1} + B_{1-z^{\frac{1}{\alpha}}} \left(\frac{1+q}{q}, \alpha + 1\right) - B_{z^{\frac{1}{\alpha}}} \left(\alpha + 1, \frac{1+q}{q}\right), \\ \psi(q, \alpha, z) &= \int_0^1 |\tilde{t}_o^\alpha - z| \tilde{t}_o^{\frac{1}{q}} d\tilde{t}_o = \frac{(1-z)q(q+1)-z\alpha q^2}{(1+q)(\alpha q+q+1)} + \frac{2\alpha q^2}{(1+q)(\alpha q+q+1)} z^{\frac{\alpha q+1+q}{\alpha q}} \end{aligned}$$

and

$$\begin{aligned} l_2(q, \alpha, z) &= \int_0^1 |\tilde{t}_o^\alpha - \lambda| \tilde{t}_o^{\frac{1}{2q}} (1 - \tilde{t}_o)^{\frac{1}{2q}} d\tilde{t}_o \\ &= z \left(B_{z^{\frac{1}{\alpha}}} \left(\frac{1+2q}{2q}, \frac{1+2q}{2q}\right) - B_{1-z^{\frac{1}{\alpha}}} \left(\frac{1+2q}{2q}, \frac{1+2q}{2q}\right) \right) \\ &\quad - \left(B_{z^{\frac{1}{\alpha}}} \left(\frac{2q(\alpha+1)+1}{2q}, \frac{1+2q}{2q}\right) - B_{1-z^{\frac{1}{\alpha}}} \left(\frac{2q(\alpha+1)+1}{2q}, \frac{1+2q}{2q}\right) \right). \end{aligned}$$

The desired result follows from (2.11) and (2.14). $\square$

Corollary 12. Putting $\alpha = 1$ in Theorem 2, we obtain

For $0 < q < \frac{1}{2}$

$$\begin{aligned} &\left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \mathfrak{S}(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \mathfrak{S}(u) du \right) \right| \\ &\leq \frac{q 2^{\frac{1}{q}} (\kappa - l_1)}{4} \left(\frac{q\lambda + q^2(2\lambda - 1) + 2q^2(1 - \lambda)^{\frac{1+2q}{q}}}{(1+q)(1+2q)} |\mathfrak{S}'(l_1)| + \frac{(1 - \lambda)q(q+1) - \lambda q^2 + 2q^2 \lambda^{\frac{2q+1}{q}}}{(1+q)(2q+1)} |\mathfrak{S}'(\kappa)| \right) \\ &\quad + \left(\frac{2}{q} - 2 \right) \omega(q, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \\ &\quad + \frac{q 2^{\frac{1}{q}} (l_2 - \kappa)}{4} \left(\frac{(1 - \mu)q(q+1) - \mu q^2 + 2q^2 \mu^{\frac{2q+1}{q}}}{(1+q)(2q+1)} |\mathfrak{S}'(\kappa)| + \frac{q\mu + q^2(2\mu - 1) + 2q^2(1 - \mu)^{\frac{1+2q}{q}}}{(1+q)(1+2q)} |\mathfrak{S}'(l_2)| \right) \\ &\quad + \left(\frac{2}{q} - 2 \right) \omega(q, \mu) (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}}. \end{aligned}$$

For $\frac{1}{2} < q < 1$

$$\begin{aligned} &\left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \mathfrak{S}(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \mathfrak{S}(u) du \right) \right| \\ &\leq \frac{\kappa - l_1}{2} \left(\frac{q\lambda + q^2(2\lambda - 1) + 2q^2(1 - \lambda)^{\frac{1+2q}{q}}}{(1+q)(1+2q)} |\mathfrak{S}'(l_1)| + \frac{(1 - \lambda)q(q+1) - \lambda q^2 + 2q^2 \lambda^{\frac{2q+1}{q}}}{(1+q)(2q+1)} |\mathfrak{S}'(\kappa)| \right) \end{aligned}$$

$$\begin{aligned}
& + \left(2^{\frac{1}{q}} - 2\right) \omega(q, \lambda) (|\mathfrak{S}'(l_1)| |\mathfrak{S}'(\kappa)|)^{\frac{1}{2}} \\
& + \frac{l_2 - \kappa}{2} \left(\frac{(1-\mu)q(q+1) - \mu q^2 + 2q^2 \mu^{\frac{2q+1}{q}}}{(1+q)(2q+1)} |\mathfrak{S}'(\kappa)| + \frac{q\mu + q^2(2\mu-1) + 2q^2(1-\mu)^{\frac{1+2q}{q}}}{(1+q)(1+2q)} |\mathfrak{S}'(l_2)| \right. \\
& \left. + \left(2^{\frac{1}{q}} - 2\right) \omega(q, \mu) (|\mathfrak{S}'(\kappa)| |\mathfrak{S}'(l_2)|)^{\frac{1}{2}} \right),
\end{aligned}$$

where

$$\begin{aligned}
\omega(q, z) = & z \left(B_z \left(\frac{1+2q}{2q}, \frac{1+2q}{2q} \right) - B_{1-z} \left(\frac{1+2q}{2q}, \frac{1+2q}{2q} \right) \right) \\
& - \left(B_z \left(\frac{2q(\alpha+1)+1}{2q}, \frac{1+2q}{2q} \right) - B_{1-z} \left(\frac{2q(\alpha+1)+1}{2q}, \frac{1+2q}{2q} \right) \right).
\end{aligned}$$

Theorem 3. Let $\mathfrak{S}$ be as in Lemma 1. If $|\mathfrak{S}'|^q$ is convex then we have

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\
\leq & \frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_1)| \\
& + \left(\frac{\kappa-l_1}{2} \left(\frac{2-(\alpha+2)\lambda}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \lambda^{1+\frac{2}{\alpha}} \right) + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+2)\mu}{2(\alpha+2)} + \frac{2\alpha}{2(\alpha+2)} \mu^{1+\frac{2}{\alpha}} \right) \right) |\mathfrak{S}'(\kappa)| \\
& + \frac{l_2-\kappa}{2} \left(\frac{2-(\alpha+1)(\alpha+2)\mu}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \mu^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_2)|,
\end{aligned}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $B(.,.)$ and ${}_2F_1(.,.,.;.)$ are beta and hypergeometric functions, respectively.

Proof. Using the absolute value of both sides of (2.1), then applying Hölder's inequality and convexity of $|\mathfrak{S}'|^q$, it yields

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\
\leq & \frac{\kappa-l_1}{2} \left(\int_0^1 |\tilde{t}_\circ^\alpha - \lambda|^p d\tilde{t}_\circ \right)^{\frac{1}{p}} \left(\int_0^1 |\mathfrak{S}'((1-\tilde{t}_\circ)l_1 + \tilde{t}_\circ\kappa)|^q d\tilde{t}_\circ \right)^{\frac{1}{q}} \\
& + \frac{l_2-\kappa}{2} \left(\int_0^1 |\mu - (1-\tilde{t}_\circ)^\alpha|^p d\tilde{t}_\circ \right)^{\frac{1}{p}} \left(\int_0^1 |\mathfrak{S}'((1-\tilde{t}_\circ)\kappa + \tilde{t}_\circ l_2)|^q d\tilde{t}_\circ \right)^{\frac{1}{q}} \\
\leq & \frac{\kappa-l_1}{2} \left(\int_0^1 |\tilde{t}_\circ^\alpha - \lambda|^p d\tilde{t}_\circ \right)^{\frac{1}{p}} \left(\int_0^1 ((1-\tilde{t}_\circ)|\mathfrak{S}'(l_1)|^q + \tilde{t}_\circ|\mathfrak{S}'(\kappa)|^q) d\tilde{t}_\circ \right)^{\frac{1}{q}} \\
& + \frac{l_2-\kappa}{2} \left(\int_0^1 |\mu - (1-\tilde{t}_\circ)^\alpha|^p d\tilde{t}_\circ \right)^{\frac{1}{p}} \left(\int_0^1 ((1-\tilde{t}_\circ)|\mathfrak{S}'(\kappa)|^q + \tilde{t}_\circ|\mathfrak{S}'(l_2)|^q) d\tilde{t}_\circ \right)^{\frac{1}{q}} \\
= & \frac{\kappa-l_1}{2} \left(\frac{\lambda^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha} + \frac{(1-\lambda)^{p+1} ({}_2F_1(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\lambda)))}{\alpha(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathfrak{S}'(l_1)|^q + |\mathfrak{S}'(\kappa)|^q}{2} \right)^{\frac{1}{q}} \\
& + \frac{l_2-\kappa}{2} \left(\frac{\mu^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha} + \frac{(1-\mu)^{p+1} ({}_2F_1(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\mu)))}{\alpha(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathfrak{S}'(\kappa)|^q + |\mathfrak{S}'(l_2)|^q}{2} \right)^{\frac{1}{q}},
\end{aligned}$$

where we have utilized

$$\int_0^1 |\tilde{t}_o^\alpha - z|^p d\tilde{t}_o = \frac{z^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha} + \frac{(1-z)^{p+1} ({}_2F_1(\frac{\alpha-1}{\alpha}, 1, p+2; 1-z))}{\alpha(p+1)}. \quad (2.15)$$

The proof is finished. $\square$

Corollary 13. Putting $\alpha = 1$ in Theorem 3, we obtain

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{1}{2(\kappa-l_1)} \int_{l_1}^{\kappa} \mathfrak{S}(u) du + \frac{1}{2(l_2-\kappa)} \int_{\kappa}^{l_2} \mathfrak{S}(u) du \right) \right| \\ & \leq \frac{\kappa-l_1}{2} \left(\frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathfrak{S}'(l_1)|^q + |\mathfrak{S}'(\kappa)|^q}{2} \right)^{\frac{1}{q}} \\ & \quad + \frac{l_2-\kappa}{2} \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{\alpha(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathfrak{S}'(\kappa)|^q + |\mathfrak{S}'(l_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 4. Let $\mathfrak{S}$ be as in Lemma 1. If $|\mathfrak{S}'|^q$ is convex, where $q \geq 1$, then we have

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa-l_1}{2} \left(\frac{1-(\alpha+1)\lambda+2\alpha\lambda^{1+\frac{1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\left(\frac{2-(\alpha+1)(\alpha+2)\lambda}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \lambda^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \lambda^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_1)|^q \right. \\ & \quad \left. + \frac{2-(\alpha+2)\lambda+2\alpha\lambda^{1+\frac{2}{\alpha}}}{2(\alpha+2)} |\mathfrak{S}'(\kappa)|^q \right)^{\frac{1}{q}} \\ & \quad + \frac{l_2-\kappa}{2} \left(\frac{1-(\alpha+1)\mu+2\alpha\mu^{1+\frac{1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{2-(\alpha+2)\mu+2\alpha\mu^{1+\frac{2}{\alpha}}}{2(\alpha+2)} |\mathfrak{S}'(\kappa)|^q \right. \\ & \quad \left. + \left(\frac{2-(\alpha+1)(\alpha+2)\mu}{2(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1} \mu^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2} \mu^{1+\frac{2}{\alpha}} \right) |\mathfrak{S}'(l_2)|^q \right)^{\frac{1}{q}}, \end{aligned}$$

where $\lambda, \mu \in [0, 1]$, $\alpha > 0$ and $\kappa \in [\varepsilon_1, l_2]$.

Proof. Using the absolute value of both sides of (2.1), then applying power mean inequality, and convexity of $|\mathfrak{S}'|^q$, it yields

$$\begin{aligned} & \left| \frac{\lambda \mathfrak{S}(l_1) + (2-\lambda-\mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha+1)}{2(l_2-\kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha+1)}{2(\kappa-l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\ & \leq \frac{\kappa-l_1}{2} \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| d\tilde{t}_o \right)^{1-\frac{1}{q}} \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| |\mathfrak{S}'((1-\tilde{t}_o)l_1 + \tilde{t}_o\kappa)|^q d\tilde{t}_o \right)^{\frac{1}{q}} \\ & \quad + \frac{l_2-\kappa}{2} \left(\int_0^1 |\mu - (1-\tilde{t}_o)^\alpha| d\tilde{t}_o \right)^{1-\frac{1}{q}} \left(\int_0^1 |\mu - (1-\tilde{t}_o)^\alpha| |\mathfrak{S}'((1-\tilde{t}_o)\kappa + \tilde{t}_ol_2)|^q d\tilde{t}_o \right)^{\frac{1}{q}} \\ & \leq \frac{\kappa-l_1}{2} \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| d\tilde{t}_o \right)^{1-\frac{1}{q}} \left(\int_0^1 |\tilde{t}_o^\alpha - \lambda| (|(1-\tilde{t}_o)| |\mathfrak{S}'(l_1)|^q + \tilde{t}_o |\mathfrak{S}'(\kappa)|^q) d\tilde{t}_o \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{l_2 - \kappa}{2} \left(\int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| d\tilde{t}_o \right)^{1 - \frac{1}{q}} \\
& \times \left(\int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| ((1 - \tilde{t}_o) |\mathfrak{S}'(\kappa)|^q + \tilde{t}_o |\mathfrak{S}'(l_2)|^q) d\tilde{t}_o \right)^{\frac{1}{q}} \\
& = \frac{\kappa - l_1}{2} \left(\frac{1 - (\alpha + 1)\lambda + 2\alpha\lambda^{1 + \frac{1}{\alpha}}}{\alpha + 1} \right)^{1 - \frac{1}{q}} \\
& \times \left(\left(\frac{2 - (\alpha + 1)(\alpha + 2)\lambda}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \lambda^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \lambda^{1 + \frac{2}{\alpha}} \right) |\mathfrak{S}'(l_1)|^q \right. \\
& \left. + \frac{2 - (\alpha + 2)\lambda + 2\alpha\lambda^{1 + \frac{2}{\alpha}}}{2(\alpha + 2)} |\mathfrak{S}'(\kappa)|^q \right)^{\frac{1}{q}} \\
& + \frac{l_2 - \kappa}{2} \left(\frac{1 - (\alpha + 1)\mu + 2\alpha\mu^{1 + \frac{1}{\alpha}}}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left(\frac{2 - (\alpha + 2)\mu + 2\alpha\mu^{1 + \frac{2}{\alpha}}}{2(\alpha + 2)} |\mathfrak{S}'(\kappa)|^q \right. \\
& \left. + \left(\frac{2 - (\alpha + 1)(\alpha + 2)\mu}{2(\alpha + 1)(\alpha + 2)} + \frac{2\alpha}{\alpha + 1} \mu^{1 + \frac{1}{\alpha}} - \frac{\alpha}{\alpha + 2} \mu^{1 + \frac{2}{\alpha}} \right) |\mathfrak{S}'(l_2)|^q \right)^{\frac{1}{q}},
\end{aligned}$$

where (2.4)–(2.7) have been utilized. The proof is finished. $\square$

Corollary 14. Putting $\alpha = 1$ in Theorem 4, we obtain

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \mathfrak{S}(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \mathfrak{S}(u) du \right) \right| \\
& \leq \frac{\kappa - l_1}{2} \left(\frac{1 - 2\lambda + 2\lambda^2}{2} \right)^{1 - \frac{1}{q}} \left(\frac{1 - 3\lambda + 6\lambda^2 - 2\lambda^3}{6} |\mathfrak{S}'(l_1)|^q + \frac{2 - 3\lambda + 2\lambda^3}{6} |\mathfrak{S}'(\kappa)|^q \right)^{\frac{1}{q}} \\
& + \frac{l_2 - \kappa}{2} \left(\frac{1 - 2\mu + 2\mu^2}{2} \right)^{1 - \frac{1}{q}} \left(\frac{2 - 3\mu + 2\mu^3}{6} |\mathfrak{S}'(\kappa)|^q + \frac{1 - 3\mu + 6\mu^2 - 2\mu^3}{6} |\mathfrak{S}'(l_2)|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 5. Let $\mathfrak{S}$ be as in Lemma 1. If $|\mathfrak{S}'|^q$ is convex, then we have

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha + 1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha + 1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\
& \leq \frac{\kappa - l_1}{2} \left(\frac{\lambda^{p + \frac{1}{\alpha}} B(\frac{1}{\alpha}, p + 1)}{\alpha^p} + \frac{(1 - \lambda)^{p + 1} \left({}_2F_1\left(\frac{\alpha - 1}{\alpha}, 1, p + 2; (1 - \lambda)\right) \right)}{\alpha(p + 1)^p} + \frac{|\mathfrak{S}'(l_1)|^q + |\mathfrak{S}'(\kappa)|^q}{2q} \right) \\
& + \frac{l_2 - \kappa}{2} \left(\frac{\mu^{p + \frac{1}{\alpha}} B(\frac{1}{\alpha}, p + 1)}{\alpha^p} + \frac{(1 - \mu)^{p + 1} \left({}_2F_1\left(\frac{\alpha - 1}{\alpha}, 1, p + 2; (1 - \mu)\right) \right)}{\alpha(p + 1)^p} + \frac{|\mathfrak{S}'(\kappa)|^q + |\mathfrak{S}'(l_2)|^q}{2q} \right),
\end{aligned}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda, \mu \in [0, 1]$, $\alpha > 0$ and $\kappa \in [l_1, l_2]$.

Proof. Using the absolute value of both sides of (2.1), then applying Young's inequality and convexity of $|\mathfrak{S}'|^q$, we have

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{\Gamma(\alpha + 1)}{2(l_2 - \kappa)^\alpha} I_{\kappa^+}^\alpha \mathfrak{S}(l_2) + \frac{\Gamma(\alpha + 1)}{2(\kappa - l_1)^\alpha} I_{\kappa^-}^\alpha \mathfrak{S}(l_1) \right) \right| \\
& \leq \frac{\kappa - l_1}{2} \int_0^1 |\tilde{t}_o^\alpha - \lambda| |\mathfrak{S}'((1 - \tilde{t}_o)l_1 + \tilde{t}_o\kappa)| d\tilde{t}_o
\end{aligned}$$

$$\begin{aligned}
& + \frac{l_2 - \kappa}{2} \int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha| |\mathfrak{S}'((1 - \tilde{t}_o)\kappa + \tilde{t}_o l_2)| d\tilde{t}_o \\
& \leq \frac{\kappa - l_1}{2} \left(\frac{1}{p} \int_0^1 |\tilde{t}_o^\alpha - \lambda|^p d\tilde{t}_o + \frac{1}{q} \int_0^1 |\mathfrak{S}'((1 - \tilde{t}_o)l_1 + \mathfrak{I}\kappa)|^q d\tilde{t}_o \right) \\
& \quad + \frac{l_2 - \kappa}{2} \left(\frac{1}{p} \int_0^1 |\mu - (1 - \tilde{t}_o)^\alpha|^p d\tilde{t}_o + \frac{1}{q} \int_0^1 |\mathfrak{S}'((1 - \tilde{t}_o)\kappa + \tilde{t}_o l_2)|^q d\tilde{t}_o \right) \\
& \leq \frac{\kappa - l_1}{2} \left(\frac{1}{p} \int_0^1 |\tilde{t}_o^\alpha - \lambda|^p d\tilde{t}_o + \frac{1}{q} \int_0^1 ((1 - \tilde{t}_o) |\mathfrak{S}'(l_1)|^q + \tilde{t}_o |\mathfrak{S}'(\kappa)|^q) d\tilde{t}_o \right) \\
& \quad + \frac{l_2 - \kappa}{2} \left(\frac{1}{p} \int_0^1 |\mu - \tilde{t}_o^\alpha|^p d\tilde{t}_o + \frac{1}{q} \int_0^1 ((1 - \tilde{t}_o) |\mathfrak{S}'(\kappa)|^q + \tilde{t}_o |\mathfrak{S}'(l_2)|^q) d\tilde{t}_o \right) \\
& = \frac{\kappa - l_1}{2} \left(\frac{\lambda^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha p} + \frac{(1-\lambda)^{p+1} \left({}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\lambda)\right) \right)}{\alpha(p+1)p} + \frac{|\mathfrak{S}'(l_1)|^q + |\mathfrak{S}'(\kappa)|^q}{2q} \right) \\
& \quad + \frac{l_2 - \kappa}{2} \left(\frac{\mu^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha p} + \frac{(1-\mu)^{p+1} \left({}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\mu)\right) \right)}{\alpha(p+1)p} + \frac{|\mathfrak{S}'(\kappa)|^q + |\mathfrak{S}'(l_2)|^q}{2q} \right).
\end{aligned}$$

The proof is finished. $\square$

Corollary 15. Putting $\alpha = 1$ in Theorem 5, we obtain

$$\begin{aligned}
& \left| \frac{\lambda \mathfrak{S}(l_1) + (2 - \lambda - \mu) \mathfrak{S}(\kappa) + \mu \mathfrak{S}(l_2)}{2} - \left(\frac{1}{2(\kappa - l_1)} \int_{l_1}^{\kappa} \mathfrak{S}(u) du + \frac{1}{2(l_2 - \kappa)} \int_{\kappa}^{l_2} \mathfrak{S}(u) du \right) \right| \\
& \leq \frac{\kappa - l_1}{2} \left(\frac{\lambda^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha p} + \frac{(1-\lambda)^{p+1} \left({}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\lambda)\right) \right)}{\alpha(p+1)p} + \frac{|\mathfrak{S}'(l_1)|^q + |\mathfrak{S}'(\kappa)|^q}{2q} \right) \\
& \quad + \frac{l_2 - \kappa}{2} \left(\frac{\mu^{p+\frac{1}{\alpha}} B(\frac{1}{\alpha}, p+1)}{\alpha p} + \frac{(1-\mu)^{p+1} \left({}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; (1-\mu)\right) \right)}{\alpha(p+1)p} + \frac{|\mathfrak{S}'(\kappa)|^q + |\mathfrak{S}'(l_2)|^q}{2q} \right).
\end{aligned}$$

3. Applications

We'll look at the means for arbitrary real values l_1 and l_2 .

The formula for the arithmetic mean is $A(l_1, l_2) = \frac{l_1 + l_2}{2}$.

The formula for the weighted arithmetic mean is $A(p, q, l_1, l_2) = \frac{pl_1 + ql_2}{p+q}$.

The geometric mean is equal to $G(l_1, l_2) = \sqrt{l_1 l_2}$.

The harmonic mean is equal to $H(l_1, l_2) = \frac{2}{\frac{1}{l_2} + \frac{1}{l_1}} = \frac{2l_1 l_2}{l_1 + l_2}$.

$L(l_1, l_2) = \frac{l_2 - l_1}{\ln l_2 - \ln l_1}$, $l_1, l_2 > 0$ and $l_1 \neq l_2$, is the logarithmic mean.

$L_p(l_1, l_2) = \left(\frac{l_2 - l_1}{(p+1)(l_2 - l_1)} \right)^{\frac{1}{p}}$, $l_1, l_2 > 0$, $l_1 \neq l_2$ and $p \in \mathbb{R} \setminus \{-1, 0\}$, is the p -logarithmic mean.

Proposition 1. For real numbers l_1, l_2, c such that $0 < l_1 < c < l_2$, we have

$$\begin{aligned} & \left| 2A(l_1^3, l_2^3) - (L_3^3(l_1, c) + L_3^3(c, l_2)) \right| \\ & \leq \frac{l_2 - l_1}{2} (4A(l_2^2, l_1^2) + 2G^2(l_1, l_2) - c^2). \end{aligned}$$

Proof. Applying second inequality of Corollary 3 to the function $\mathfrak{S}(u) = u^3$, yields the statement. $\square$

Proposition 2. For real numbers l_1, l_2 such that $0 < l_1 < l_2$, we have

$$\begin{aligned} & \left| A(1, 3, l_2, A(H(l_1, l_2), l_1)) - G^2(l_1, l_2) L^{-1}(l_1, l_2) \right| \\ & \leq \frac{l_2 - l_1}{1536l_2l_1} (91l_2^2 + 125l_1^2). \end{aligned}$$

Proof. The application of Corollary 7 with $\lambda = \frac{1}{2}$ and $\mu = \frac{3}{4}$ on the interval $[\frac{1}{l_2}, \frac{1}{l_1}]$, gives

$$\begin{aligned} & \left| \frac{\frac{1}{2}\mathfrak{S}\left(\frac{1}{l_2}\right) + \frac{3}{4}\mathfrak{S}\left(\frac{l_1+l_2}{2l_1l_2}\right) + \frac{3}{4}\mathfrak{S}\left(\frac{1}{l_1}\right)}{2} - \frac{l_1l_2}{l_2-l_1} \int_{\frac{1}{l_2}}^{\frac{1}{l_1}} \mathfrak{S}(u) du \right| \\ & \leq \frac{l_2 - l_1}{1536l_1l_2} \left(91 \left| \mathfrak{S}'\left(\frac{1}{l_2}\right) \right| + 125 \left| \mathfrak{S}'\left(\frac{1}{l_1}\right) \right| \right). \end{aligned}$$

Applying the above inequality to the function $\mathfrak{S}(u) = \frac{1}{u}$ the derivative $\mathfrak{S}'(u) = -\frac{1}{u^2}$. So, we obtain $\left| \mathfrak{S}'\left(\frac{1}{l_1}\right) \right| = l_1^2$, $\left| \mathfrak{S}'\left(\frac{1}{l_2}\right) \right| = l_2^2$, and

$$\begin{aligned} \frac{l_1l_2}{l_2-l_1} \int_{\frac{1}{l_2}}^{\frac{1}{l_1}} \mathfrak{S}(u) du &= \frac{l_1l_2}{l_2-l_1} \left(\ln \frac{1}{l_1} - \ln \frac{1}{l_2} \right) \\ &= \frac{l_1l_2}{l_2-l_1} (\ln l_2 - \ln l_1) \\ &= \left(\sqrt{l_1l_2} \right)^2 \frac{\ln l_2 - \ln l_1}{l_2 - l_1} \\ &= G^2(l_1, l_2) L^{-1}(l_1, l_2). \end{aligned}$$

We also note

$$\begin{aligned} \frac{\frac{1}{2}\mathfrak{S}\left(\frac{1}{l_2}\right) + \frac{3}{4}\mathfrak{S}\left(\frac{l_1+l_2}{2l_1l_2}\right) + \frac{3}{4}\mathfrak{S}\left(\frac{1}{l_1}\right)}{2} &= \frac{\frac{1}{2}l_2 + \frac{3}{4}\frac{2l_1l_2}{l_1+l_2} + \frac{3}{4}l_1}{2} \\ &= \frac{1}{4}l_2 + \frac{3}{4} \frac{\left(\frac{2l_1l_2}{l_1+l_2} + l_1\right)}{2} \\ &= \frac{1}{4}l_2 + \frac{3}{4} \frac{(H(l_1, l_2) + l_1)}{2} \\ &= \frac{1}{4}l_2 + \frac{3}{4}A(H(l_1, l_2), l_1) \\ &= A(1, 3, l_2, A(H(l_1, l_2), l_1)). \end{aligned}$$

$\square$

4. Conclusions

In this study, we have established a new integral identity. We have derived several fractional parametrized Newton-Cotes type inequalities for differentiable convex first derivatives. We have discussed some case that can be derived from the finding. We have provided some applications to inequalities involving means. It is our hope that this paper's findings will inspire more study in this area.

Author contributions

Rabah Debbar, Abdelkader Moumen, Hamid Boulares, Badreddine Meftah: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft preparation, Writing-review and editing; Mohamed Bouye: Writing-original draft preparation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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