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Research article

On the oscillation of fourth-order neutral differential equations with multiple delays

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Abstract: This work focuses on the canonical scenario and examines the oscillatory and asymptotic features of fourth-order differential equations with numerous delays and mixed neutral terms. The Riccati methodology is employed as a useful mathematical tool to simplify the theoretical analysis and derive stringent conditions that rule out the existence of positive solutions satisfying the examined equation. By systematically combining these conditions, precise criteria ensuring the oscillation of all solutions are obtained. These findings contribute qualitatively to the scientific literature by advancing the theoretical understanding of the oscillatory behavior of such equations. Furthermore, to highlight the practical importance of the established results, two applied examples are provided to demonstrate the effectiveness of the derived criteria in handling relevant mathematical models.

Keywords: oscillation properties; neutral differential equations; Riccati method; multiple delays; canonical case
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

A key area of mathematics, differential equations (DEs) define the relationships between variables and the rates at which they change. As such, they are a vital tool for comprehending intricate biological, engineering, and physical models. Equations with complex nonlinear characteristics and higher orders are exciting in applied sciences and mathematical analysis. Due to their capacity to characterize intricate oscillatory behavior, fourth-order differential equations are especially significant in a wide range of scientific and engineering applications. Because they offer a comprehensive theoretical foundation for comprehending and managing intricate dynamic events, this class of equations continues to draw researchers (see [1, 2]).

Neutral differential equations (NDEs) play an important role in the mathematical modeling of systems that depend on the values of present, past, or future variables. These equations are characterized by the existence of the highest-order derivative of the unknown function at the present time s and also at a later or future time $\mathrm{s} - \tau$. In addition to their theoretical importance, these equations have wide applications, such as the analysis of networks with lossless transmission lines, which are used in high-speed computers to connect switching circuits [3, 4]. Therefore, the study of these equations is essential in many engineering and physical fields.

This paper examines fourth-order NDEs, which are denoted by the following form:

$$\left(r(\mathrm{s})\left[\mathcal{G}'''(\mathrm{s})\right]^{\alpha}\right)' + \sum_{i=1}^{m} \kappa_i(\mathrm{s})\chi^{\beta}(\sigma_i(\mathrm{s})) = 0, \ \mathrm{s} \geq \mathrm{s}_0, \tag{1.1}$$

where

$$\mathcal{G}(\mathrm{s}) := \chi(\mathrm{s}) + \mathfrak{a}_1(\mathrm{s})\chi^{\gamma}(\varsigma(\mathrm{s})) - \mathfrak{a}_2(\mathrm{s})\chi^{\delta}(\varsigma(\mathrm{s})). \tag{1.2}$$

This study operates under the following hypotheses:

($\mathcal{A}_1$) α, β, γ, and δ are quotients of positive odd integers, with $\gamma < 1$ and $\delta > 1$;
($\mathcal{A}_2$) $\mathfrak{a}_1, \mathfrak{a}_2, \kappa_i \in C([\mathrm{s}_0, \infty), [0, \infty))$, $i = 1, 2, ..., m$;
($\mathcal{A}_3$) $\varsigma, \sigma_i \in C^1([\mathrm{s}_0, \infty), \mathbb{R})$ satisfies $\sigma_i(\mathrm{s}) \leq \mathrm{s}$, $\varsigma(\mathrm{s}) \leq \mathrm{s}$, $\sigma_i'(\mathrm{s}) > 0$ and $\lim_{\mathrm{s}\to\infty} \varsigma(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} \sigma_i(\mathrm{s}) = \infty$, $i = 1, 2, ..., m$;
($\mathcal{A}_4$) $r \in C^1([\mathrm{s}_0, \infty), (0, \infty))$ satisfies

$$\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{r^{1/\alpha}(\hbar)} \mathrm{d}\hbar \to \infty \text{ as } \mathrm{s} \to \infty. \tag{1.3}$$

A function $\chi \in C([\mathrm{s}_\chi, \infty), \mathbb{R})$, $\mathrm{s}_\chi \geqslant \mathrm{s}_0$, is said to be a solution of (1.1) which has the property $r(\mathrm{s})\left[\mathcal{G}'''(\mathrm{s})\right]^{\alpha} \in C^1[\mathrm{s}_\chi, \infty)$, and it satisfies the Eq (1.1) for all $\mathrm{s} \in [\mathrm{s}_\chi, \infty)$. We consider only those solutions χ of (1.1) that exist on some half-line $[\mathrm{s}_\chi, \infty)$ and satisfy the condition

$$\sup\{|\chi(\mathrm{s})| : \mathrm{s} \geqslant \mathrm{S}\} > 0, \text{ for all } \mathrm{S} \geq \mathrm{s}_\chi.$$

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

The area of studying oscillatory processes in differential equations has a long history and is well-established, thanks to the work of early pioneers like Alexander Lyapunov and Henri Poincaré. A particular family of differential equations has attracted the interest of several scholars due to its oscillating nature (see [5–7]).

Numerous scholars have devoted their attention to comprehending the oscillatory nature of neutral differential equations in various orders. In the second order, using sophisticated methods for examining nonlinear neutral effects, Baculíková and Džurina [8], and Aldiaij et al. [9] presented significant results on the oscillation of delay equations. For the oscillation of equations with infinite neutral coefficients, Chatzarakis et al. [10] gave exact conditions at the third order. Using novel methodologies, Grace et al. [11] demonstrated sophisticated findings on the oscillation of delay equations at the fourth order. Graef et al. [12], Xing et al. [13], Alnafisah et al. [14], and Alqahtan et al. [15] expanded on the knowledge of the oscillation of delay neutral equations for higher orders.

Special instances of these equations have been the subject of the majority of research. The most well-known researchers in this area are Jadlowska et al. [16], who examined the linear delay equations

$$\chi'''(\mathrm{s}) + \kappa(\mathrm{s})\chi(\sigma(\mathrm{s})) = 0.$$

Using methods to enhance the special qualities of non-oscillatory solutions, they were able to offer accurate oscillation criteria for a linear differential equation with a delay.

Zhang et al. [17] studied the oscillatory properties of the quasi-linear delay differential equation

$$(r(\mathrm{s})\,[\chi'''(\mathrm{s})]^{\alpha})' + \kappa(\mathrm{s})\chi^{\alpha}(\sigma(\mathrm{s})) = 0,$$

Their analyses focused on developing precise oscillation criteria through rigorous analytical and comparison techniques. These studies contributed significantly to the theory by clarifying how the delay and nonlinear structure affect the qualitative properties of solutions.

Subsequently, Kusano et al. [18] and Kamo et al. [19] considered a more general nonlinear form, namely

$$(r(\mathrm{s})\,[\chi'''(\mathrm{s})]^{\alpha})' + \kappa(\mathrm{s})\chi^{\beta}(\sigma(\mathrm{s})) = 0,$$

allowing for distinct exponents in the delayed term. Their work provided broader oscillation criteria under more flexible structural assumptions, relying on refined comparison techniques and inequalities to ensure that all solutions oscillate under specific parameter regimes.

Bazighifan and Cesarano [20] and Alatwi et al. [21] created specific criteria to assure oscillation in neutral nonlinear equations of the following form:

$$(r(\mathrm{s})\,[(\chi(\mathrm{s}) + \mathfrak{a}_1(\mathrm{s})\chi(\varsigma(\mathrm{s})))''']^{\alpha})' + \kappa(\mathrm{s})\chi^{\beta}(\sigma(\mathrm{s})) = 0.$$

Masood et al. [22] studied nonlinear differential equations with a sublinear neutral term, given by the following form:

$$\left(r(\mathrm{s})\,[(\chi(\mathrm{s}) + \mathfrak{a}_1(\mathrm{s})\chi^{\gamma}(\varsigma(\mathrm{s})))''']^{\alpha}\right)' + \kappa(\mathrm{s})\chi^{\beta}(\sigma(\mathrm{s})) = 0.$$

Based on prior research, the current study broadens the scope to include fourth-order neutral differential equations with mixed neutral terms, concentrating on the oscillatory behavior of all solutions. This inclusion of mixed neutral terms represents a significant extension of previous models. The complexity introduced by the higher-order nature of the equations, combined with neutral terms and multiple delays, reflects real-world dynamics found in applications such as engineering, biological systems, and control theory. Therefore, studying such equations is not only of theoretical interest but also of practical relevance. This research improves our present understanding and allows for the incorporation of new criteria that ensure the oscillations of these equations under various conditions.

2. Preliminary results

Initially, we offer numerous helpful lemmas about the monotonic characteristics of the non-oscillatory solutions to the examined equations. In what follows, we let

$$\sigma(\mathrm{s}) := \min\{\sigma_i(\mathrm{s}) : i = 1, 2, ..., m\},$$

$$\kappa(\mathrm{s}) := c^\beta \sum_{i=1}^{m} \kappa_i(\mathrm{s}),$$

$$g_1(\mathrm{s}) := (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \mathfrak{a}_1^{\frac{1}{1-\gamma}}(\mathrm{s})\, \mathfrak{a}^{\frac{\gamma}{\gamma-1}}(\mathrm{s}),$$

and

$$g_2(\mathrm{s}) := (\delta-1)\delta^{\frac{\delta}{1-\delta}} \mathfrak{a}_2^{\frac{1}{1-\delta}}(\mathrm{s})\, \mathfrak{a}^{\frac{\delta}{\delta-1}}(\mathrm{s}).$$

Definition 2.1. *[23] A function $\chi(\mathrm{s})$ is said to be eventually positive (or eventually negative) if there exist $\mathrm{s}_2 \geq \mathrm{s}_1 \geq \mathrm{s}_0$ such that $\chi(\mathrm{s})$ is a solution on the interval $[\mathrm{s}_2, \infty)$, and satisfies $\chi(\mathrm{s}) > 0$ (or $\chi(\mathrm{s}) < 0$) for all $\mathrm{s} \geq \mathrm{s}_2$.*

Lemma 2.1. *[24] Suppose that $y \in C^n([\mathrm{s}_0, \infty), \mathbb{R}^+)$, $y^{(n)}(\mathrm{s})$ is of fixed sign and not identically zero on $[\mathrm{s}_0, \infty)$ and that there exists $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $y^{(n-1)}(\mathrm{s})\, y^{(n)}(\mathrm{s}) \leq 0$ for all $\mathrm{s}_1 \geq \mathrm{s}_0$. If $\lim_{\mathrm{s}\to\infty} y(\mathrm{s}) \neq 0$, then, for every $\delta \in (0,1)$, there exists $\mathrm{s}_\epsilon \in [\mathrm{s}_1, \infty)$ such that*

$$y(\mathrm{s}) \geq \frac{\epsilon}{(n-1)!}\mathrm{s}^{n-1}\left|y^{(n-1)}(\mathrm{s})\right|,$$

for $\mathrm{s} \in [\mathrm{s}_\epsilon, \infty)$.

Lemma 2.2. *[25] If A and B are nonnegative, then*

$$A^\lambda + (\lambda-1)B^\lambda - \lambda A B^{\lambda-1} \geq 0 \quad \text{for } \lambda > 1, \tag{2.1}$$

$$A^\lambda - (1-\lambda)B^\lambda - \lambda A B^{\lambda-1} \leq 0 \quad \text{for } 0 < \lambda < 1, \tag{2.2}$$

where the equality holds if and only if $A = B$.

Lemma 2.3. *[26] Let α be a ratio of two odd positive integers; A and B are constants. Then*

$$Bu - Au^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, \; A > 0. \tag{2.3}$$

Lemma 2.4. *[27] Let $y \in \mathbf{C}^n([\mathrm{s}_0, \infty), (0, \infty))$, $y^{(i)}(\mathrm{s}) > 0$ for $i = 1, 2, ..., n$, and $y^{(n+1)}(\mathrm{s}) \leq 0$, eventually. Then, eventually,*

$$\frac{y(\mathrm{s})}{y'(\mathrm{s})} \geq \frac{\epsilon \mathrm{s}}{n},$$

for every $\epsilon \in (0,1)$.

Lemma 2.5. *[22] Assume that $\chi(\mathrm{s})$ is an eventually positive solution of (1.1). Then, for sufficiently large s, $\mathcal{G}(\mathrm{s})$ satisfies one of the following cases:*

$$(\mathrm{C}_1) \; : \; \mathcal{G}(\mathrm{s}) > 0,\ \mathcal{G}'(\mathrm{s}) > 0,\ \mathcal{G}''(\mathrm{s}) > 0,\ \mathcal{G}'''(\mathrm{s}) > 0,\ \left(r(\mathrm{s})\left(\mathcal{G}'''(\mathrm{s})\right)^\alpha\right)' < 0,$$

$$(\mathrm{C}_2) \; : \; \mathcal{G}(\mathrm{s}) > 0,\ \mathcal{G}'(\mathrm{s}) > 0,\ \mathcal{G}''(\mathrm{s}) < 0,\ \mathcal{G}'''(\mathrm{s}) > 0,$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

Lemma 2.6. *Let χ be an eventually positive solution of (1.1), and assume $\mathfrak{a} \in C([s_0,\infty),(0,\infty))$ such that $\mathfrak{a}_2(s) \neq 0$ is bounded and*

$$\lim_{s\to\infty}[g_1(s)+g_2(s)]=0. \tag{2.4}$$

Then:
(i) $\chi(s) \geq c\mathcal{G}(s)$ *for some* $c \in (0,1)$;
(ii) $(r(s)[\mathcal{G}'''(s)]^{\alpha})' + \kappa(s)\mathcal{G}^{\beta}(\sigma(s)) \leq 0$,
for sufficiently large s

Proof. From the definition of $\mathcal{G}$, it is obvious that

$$\mathcal{G}(s)=\chi(s)+\left[\mathfrak{a}(s)\chi(\varsigma(s))-\mathfrak{a}_2(s)\chi^{\delta}(\varsigma(s))\right]+\left[\mathfrak{a}_1(s)\chi^{\gamma}(\varsigma(s))-\mathfrak{a}(s)\chi(\varsigma(s))\right],$$

or

$$\chi(s)=\mathcal{G}(s)-\left[\mathfrak{a}(s)\chi(\varsigma(s))-\mathfrak{a}_2(s)\chi^{\delta}(\varsigma(s))\right]-\left[\mathfrak{a}_1(s)\chi^{\gamma}(\varsigma(s))-\mathfrak{a}(s)\chi(\varsigma(s))\right]. \tag{2.5}$$

If we apply the inequality (2.1) with $\lambda=\delta>1$, $A=\mathfrak{a}_2^{1/\delta}(s)\chi(\varsigma(s))$, and $B=\left(\frac{1}{\delta}\mathfrak{a}(s)\mathfrak{a}_2^{-1/\delta}(s)\right)^{\frac{1}{\delta-1}}$, we get

$$\mathfrak{a}(s)\chi(\varsigma(s))-\mathfrak{a}_2(s)\chi^{\delta}(\varsigma(s))\leq(\delta-1)\,\delta^{\frac{\delta}{1-\delta}}\mathfrak{a}_2^{\frac{1}{1-\delta}}(s)\,\mathfrak{a}^{\frac{\delta}{\delta-1}}(s)=g_2(s). \tag{2.6}$$

Similarly, if we apply (2.2) with $\lambda=\gamma<1$, $A=\mathfrak{a}_1^{1/\gamma}(s)\chi(\varsigma(s))$, and $B=\left(\frac{1}{\gamma}\mathfrak{a}(s)\mathfrak{a}_1^{-1/\gamma}(s)\right)^{\frac{1}{\gamma-1}}$, we get

$$\mathfrak{a}_1(s)\chi^{\gamma}(\varsigma(s))-\mathfrak{a}(s)\chi(\varsigma(s))\leq(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\mathfrak{a}_1^{\frac{1}{1-\gamma}}(s)\,\mathfrak{a}^{\frac{\gamma}{\gamma-1}}(s)=g_1(s). \tag{2.7}$$

By substituting (2.6) and (2.7) into (2.5), we obtain

$$\chi(s)\geq\mathcal{G}(s)-g_1(s)-g_2(s)=\left(1-\frac{g_1(s)+g_2(s)}{\mathcal{G}(s)}\right)\mathcal{G}(s). \tag{2.8}$$

Since $\mathcal{G}'(s)>0$, we find $\mathcal{G}(s)\geq c_0$ for some $c_0>0$. Therefore, (2.8) leads to

$$\chi(s)\geq\left(1-\frac{g_1(s)+g_2(s)}{c_0}\right)\mathcal{G}(s).$$

In light of (2.4), we can identify a constant $c\in(0,1)$ such that

$$\chi(s)\geq c\mathcal{G}(s). \tag{2.9}$$

By substituting (2.9) into (1.1), we obtain

$$(r(s)[\mathcal{G}'''(s)]^{\alpha})'=-\sum_{i=1}^{m}\kappa_i(s)\chi^{\beta}(\sigma_i(s))\leq-c^{\beta}\sum_{i=1}^{m}\kappa_i(s)\mathcal{G}^{\beta}(\sigma_i(s)).$$

Since $\mathcal{G}'(s)>0$, and $\sigma(s)\leq\sigma_i(s)$ for all $i=1,2,...,m$, then

$$(r(s)[\mathcal{G}'''(s)]^{\alpha})'\leq-c^{\beta}\mathcal{G}^{\beta}(\sigma(s))\sum_{i=1}^{m}\kappa_i(s)=-\kappa(s)\,\mathcal{G}^{\beta}(\sigma(s)).$$

This completes the proof. □

3. Main results

This section introduces additional conditions that ensure the oscillatory behavior of solutions to Eq (1.1) using the Riccati technique with varied substitutions. These requirements are based on a rigorous study that takes into consideration the equation's unique structure, eventually focusing on positive solutions. We assume that the functional inequalities hold for all large enough s, which simplifies the proof without losing generality.

3.1. *The scenario* (C_1)

This subsection discusses the criteria that exclude the existence of positive solutions to Eq (1.1) within (C_1).

Lemma 3.1. *Let $\beta \geq \alpha$. If there is a nondecreasing function $\mu \in C^1([s_0, \infty), (0, \infty))$ such that*

$$\limsup_{s \to \infty} \int_{s_0}^{s} \left(L\mu(\hbar)\kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{3\beta/\epsilon} - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(\hbar)(\mu'(\hbar))^{\alpha+1}}{(\epsilon_1 \hbar^2 \mu(\hbar))^\alpha} \right) d\hbar = \infty, \tag{3.1}$$

holds for every $\epsilon, \epsilon_1 \in (0,1)$, $L > 0$, then $C_1 = \emptyset$.

Proof. Let $\chi \in C_1$. Then, there exists a $s_1 \geq s_0$, such that $\chi(s) > 0$, $\chi(\varsigma(s)) > 0$, and $\chi(\sigma_i(s)) > 0$ for $s \geq s_1 \geq s_0$ and $i = 1, 2, ..., m$. Let us now definet

$$w(s) := \mu(s) \frac{r(s)(\mathcal{G}'''(s))^\alpha}{\mathcal{G}^\alpha(s)} > 0. \tag{3.2}$$

So

$$w'(s) = \mu'(s) \frac{r(s)(\mathcal{G}'''(s))^\alpha}{\mathcal{G}^\alpha(s)} + \mu(s) \frac{(r(s)(\mathcal{G}'''(s))^\alpha)'}{\mathcal{G}^\alpha(s)} - \alpha\mu(s) \frac{r(s)(\mathcal{G}'''(s))^\alpha \mathcal{G}'(s)}{\mathcal{G}^{\alpha+1}(s)}. \tag{3.3}$$

Using Lemma 2.6 (ii), (3.2), and (3.3), we deduce that

$$w'(s) \leq -\mu(s)\kappa(s) \frac{\mathcal{G}^\beta(\sigma(s))}{\mathcal{G}^\alpha(s)} + \frac{\mu'(s)}{\mu(s)} w(s) - \alpha \frac{\mathcal{G}'(s)}{\mathcal{G}(s)} w(s). \tag{3.4}$$

From Lemma 2.4, we have that

$$\mathcal{G}(s) \geq \frac{\epsilon}{3} s \mathcal{G}'(s),$$

and hence,

$$\frac{\mathcal{G}(\sigma(s))}{\mathcal{G}(s)} \geq \left(\frac{\sigma(s)}{s} \right)^{3/\epsilon}. \tag{3.5}$$

It follows from Lemma 2.1 that

$$\mathcal{G}'(s) \geq \frac{\epsilon_1}{2} s^2 \mathcal{G}'''(s), \tag{3.6}$$

for all $\epsilon_1 \in (0,1)$ and every sufficiently large s. Thus, by (3.4)-(3.6), we have

$$w'(s) \leq -\mu(s)\kappa(s) \frac{\mathcal{G}^\beta(\sigma(s))}{\mathcal{G}^\alpha(s)} + \frac{\mu'(s)}{\mu(s)} w(s) - \frac{\epsilon_1 \alpha}{2} s^2 \frac{\mathcal{G}'''(s)}{\mathcal{G}(s)} w(s)$$

$$= \quad -\mu(\mathrm{s})\,\kappa(\mathrm{s})\mathcal{G}^{\beta-\alpha}(\mathrm{s})\,\frac{\mathcal{G}^{\beta}(\sigma(\mathrm{s}))}{\mathcal{G}^{\beta}(\mathrm{s})} + \frac{\mu'(\mathrm{s})}{\mu(\mathrm{s})}w(\mathrm{s}) - \frac{\epsilon_1\alpha\mathrm{s}^2}{2\,(r(\mathrm{s})\,\mu(\mathrm{s}))^{1/\alpha}}\frac{\mu^{1/\alpha}(\mathrm{s})\,r^{1/\alpha}(\mathrm{s})\,\mathcal{G}'''(\mathrm{s})}{\mathcal{G}(\mathrm{s})}w(\mathrm{s})\,.$$

By using (3.2) and (3.5), we obtain

$$w'(\mathrm{s}) \le -\mu(\mathrm{s})\,\kappa(\mathrm{s})\mathcal{G}^{\beta-\alpha}(\mathrm{s})\left(\frac{\sigma(\mathrm{s})}{\mathrm{s}}\right)^{3\beta/\epsilon} + \frac{\mu'(\mathrm{s})}{\mu(\mathrm{s})}w(\mathrm{s}) - \frac{\epsilon_1\alpha\mathrm{s}^2}{2\,(r(\mathrm{s})\,\mu(\mathrm{s}))^{1/\alpha}}w^{(1+\alpha)/\alpha}(\mathrm{s})\,. \tag{3.7}$$

Since $\mathcal{G}'(\mathrm{s}) > 0$, and $\beta \ge \alpha$, then there exist a $\mathrm{s}_1 \ge \mathrm{s}_0$ and a constant $L > 0$ such that

$$\mathcal{G}^{\beta-\alpha}(\mathrm{s}) > L. \tag{3.8}$$

Thus, the inequality (3.7) gives

$$w'(\mathrm{s}) \le -L\mu(\mathrm{s})\,\kappa(\mathrm{s})\left(\frac{\sigma(\mathrm{s})}{\mathrm{s}}\right)^{3\beta/\epsilon} + \frac{\mu'(\mathrm{s})}{\mu(\mathrm{s})}w(\mathrm{s}) - \frac{\epsilon_1\alpha\mathrm{s}^2}{2\,(r(\mathrm{s})\,\mu(\mathrm{s}))^{1/\alpha}}w^{(1+\alpha)/\alpha}(\mathrm{s})\,. \tag{3.9}$$

Using Lemma 2.3, where we define $B = \mu'(\mathrm{s})/\mu(\mathrm{s})$, $A = \epsilon_1\alpha\mathrm{s}^2/2\,(r(\mathrm{s})\,\mu(\mathrm{s}))^{1/\alpha}$, and $u(\mathrm{s}) = w(\mathrm{s})$, we derive

$$w'(\mathrm{s}) \le -L\mu(\mathrm{s})\,\kappa(\mathrm{s})\left(\frac{\sigma(\mathrm{s})}{\mathrm{s}}\right)^{3\beta/\epsilon} + \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\mathrm{s})\,(\mu'(\mathrm{s}))^{\alpha+1}}{(\epsilon_1\mathrm{s}^2\mu(\mathrm{s}))^{\alpha}}. \tag{3.10}$$

Integrating (3.10) from $\mathrm{s}_2 \ge \mathrm{s}_1$ to s, one arrives at

$$\int_{\mathrm{s}_2}^{\mathrm{s}}\left(L\mu(\hbar)\,\kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{3\beta/\epsilon} - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\hbar)\,(\mu'(\hbar))^{\alpha+1}}{(\epsilon_1\hbar^2\mu(\hbar))^{\alpha}}\right)\mathrm{d}\hbar \le w(\mathrm{s}_2)\,,$$

this contradicts (3.11) as $\mathrm{s} \to \infty$. This completes the proof. □

Lemma 3.2. *Let $\beta \ge \alpha$. If there is a nondecreasing functions $\mu_1 \in C^1([\mathrm{s}_0,\infty),(0,\infty))$ such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left(\mu_1(\hbar)\,\kappa(\hbar) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(\hbar))\left[\mu_1'(\hbar)\right]^{\alpha+1}}{(L_1\epsilon_1\sigma^2(\hbar)\,\sigma'(\hbar)\,\mu_1(\hbar))^{\alpha}}\right)\mathrm{d}\hbar = \infty, \tag{3.11}$$

holds for every $\epsilon_1 \in (0,1)$, $L_1 > 0$, then $\mathrm{C}_1 = \varnothing$.

Proof. Let $\chi(\mathrm{s}) \in \mathrm{C}_1$. Then there exists a $\mathrm{s}_1 \ge \mathrm{s}_0$, such that $\chi(\mathrm{s}) > 0$, $\chi(\varsigma(\mathrm{s})) > 0$, and $\chi(\sigma_i(\mathrm{s})) > 0$ for $\mathrm{s} \ge \mathrm{s}_1 \ge \mathrm{s}_0$ and $i = 1, 2, ..., m$. Now, define a function $w_1(\mathrm{s})$ by

$$w_1(\mathrm{s}) := \mu_1(\mathrm{s})\,\frac{r(\mathrm{s})\,(\mathcal{G}'''(\mathrm{s}))^{\alpha}}{\mathcal{G}^{\beta}(\sigma(\mathrm{s}))} > 0,\ \ \mathrm{s} \ge \mathrm{s}_1. \tag{3.12}$$

Thus

$$\begin{aligned} w_1'(\mathrm{s}) \quad = \quad & \mu_1'(\mathrm{s})\,\frac{r(\mathrm{s})\,(\mathcal{G}'''(\mathrm{s}))^{\alpha}}{\mathcal{G}^{\beta}(\sigma(\mathrm{s}))} + \mu_1(\mathrm{s})\,\frac{(r(\mathrm{s})\,(\mathcal{G}'''(\mathrm{s}))^{\alpha})'}{\mathcal{G}^{\beta}(\sigma(\mathrm{s}))} \\ & -\beta\mu_1(\mathrm{s})\,\sigma'(\mathrm{s})\,\frac{r(\mathrm{s})\,(\mathcal{G}'''(\mathrm{s}))^{\alpha}\,\mathcal{G}'(\sigma(\mathrm{s}))}{\mathcal{G}^{\beta+1}(\sigma(\mathrm{s}))}. \end{aligned} \tag{3.13}$$

We see from Lemma 2.6 (ii), (3.12), and (3.13) that

$$w_1'(s) \le \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \mu_1(s)\kappa(s) - \beta\sigma'(s)\frac{\mathcal{G}'(\sigma(s))}{\mathcal{G}(\sigma(s))} w_1(s). \tag{3.14}$$

Using (3.6), we obtain

$$\mathcal{G}'(\sigma(s)) \ge \frac{\epsilon_1}{2}\sigma^2(s)\mathcal{G}'''(\sigma(s)), \tag{3.15}$$

for any $\epsilon_1 \in (0,1)$ and sufficiently large s. When we replace (3.15) with (3.14), we see that

$$\begin{aligned} w_1'(s) &\le \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \mu_1(s)\kappa(s) - \frac{\epsilon_1}{2}\beta\sigma^2(s)\sigma'(s)\frac{\mathcal{G}'''(\sigma(s))}{\mathcal{G}(\sigma(s))} w_1(s) \\ &= -\mu_1(s)\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1}{2}\beta\frac{\sigma^2(s)\sigma'(s)}{r^{1/\alpha}(\sigma(s))}\frac{r^{1/\alpha}(\sigma(s))\mathcal{G}'''(\sigma(s))}{\mathcal{G}(\sigma(s))} w_1(s). \end{aligned}$$

Since $(r(s)(\mathcal{G}'''(s))^{\alpha})' < 0$, then

$$r^{1/\alpha}(s)\mathcal{G}'''(s) \le r^{1/\alpha}(\sigma(s))\mathcal{G}'''(\sigma(s)).$$

Then

$$\begin{aligned} w_1'(s) &\le -\mu_1(s)\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\beta}{2}\frac{\sigma^2(s)\sigma'(s)}{r^{1/\alpha}(\sigma(s))}\frac{r^{1/\alpha}(s)\mathcal{G}'''(s)}{\mathcal{G}(\sigma(s))} w_1(s) \\ &= -\mu_1(s)\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\beta}{2}\frac{\sigma^2(s)\sigma'(s)}{[\mu_1(s)r(\sigma(s))]^{1/\alpha}}[\mathcal{G}(\sigma(s))]^{\frac{\beta-\alpha}{\alpha}} w_1^{\frac{\alpha+1}{\alpha}}(s). \end{aligned} \tag{3.16}$$

Since $\beta \ge \alpha$ and $\mathcal{G}' > 0$, there are constants $L_1 > 0$ and $s_2 \ge s_1$ such that

$$\mathcal{G}^{\frac{\beta-\alpha}{\alpha}}(\sigma(s)) \ge L_1, \ s \ge s_2. \tag{3.17}$$

Thus, the inequality (3.16) gives

$$w_1'(s) \le -\mu_1(s)\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\alpha L_1}{2}\frac{\sigma^2(s)\sigma'(s)}{[\mu_1(s)r(\sigma(s))]^{1/\alpha}} w_1^{\frac{\alpha+1}{\alpha}}(s). \tag{3.18}$$

Using Lemma 2.3, where we define $B = \mu_1'(s)/\mu_1(s)$, $A = \epsilon_1\alpha L_1\sigma^2(s)\sigma'(s)/2[\mu_1(s)r(\sigma(s))]^{1/\alpha}$, and $u(s) = w_1(s)$, we derive

$$w_1'(s) \le -\mu_1(s)\kappa(s) + \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(s))\left[\mu_1'(s)\right]^{\alpha+1}}{(L_1\epsilon_1\sigma^2(s)\sigma'(s)\mu_1(s))^{\alpha}}. \tag{3.19}$$

Integrating (3.19) from $s_3 \ge s_2$ to s, one arrives at

$$\int_{s_3}^{s}\left(\mu_1(\hbar)\kappa(\hbar) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(\hbar))\left[\mu_1'(\hbar)\right]^{\alpha+1}}{(L_1\epsilon_1\sigma^2(\hbar)\sigma'(\hbar)\mu_1(\hbar))^{\alpha}}\right)d\hbar \le w_1(s_3),$$

this contradicts (3.11) as $s \to \infty$.

Thus, the proof is finished. □

Lemma 3.3. *Let* $0 < \beta < \alpha$. *If there is a nondecreasing function* $\mu_1 \in C^1([s_0, \infty), (0, \infty))$ *such that*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \left(\mu_1(\hbar)\,\kappa(\hbar) - \frac{2^{\beta}}{(\beta+1)^{\beta+1}} \frac{r(\hbar)\left[\mu_1'(\hbar)\right]^{\beta+1}}{\left(L_2\epsilon_1\mu_1(\hbar)\,\sigma^2(\hbar)\,\sigma'(\hbar)\right)^{\beta}} \right) d\hbar = \infty, \tag{3.20}$$

holds for every $\epsilon_1 \in (0,1)$, $L_2 > 0$, *then* $C_1 = \varnothing$.

Proof. Let $\chi(s) \in C_1$. Then there exists a $s_1 \geq s_0$, such that $\chi(s) > 0$, $\chi(\varsigma(s)) > 0$, and $\chi(\sigma_i(s)) > 0$ for $s \geq s_1 \geq s_0$ and $i = 1, 2, ..., m$. As in the proof of (3.11) in Lemma 3.2. The function $w_1(s)$ is defined in (3.12), then (3.13) holds. By using Lemma 2.6 (ii), (3.12), and (3.13), we conclude that

$$w_1'(s) \leq -\mu_1(s)\,\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \beta\sigma'(s)\,\frac{\mathcal{G}'(\sigma(s))}{\mathcal{G}(\sigma(s))} w_1(s).$$

By using (3.15), we see that

$$\begin{aligned} w_1'(s) &\leq -\mu_1(s)\,\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\beta}{2}\sigma^2(s)\,\sigma'(s)\left[\mathcal{G}'''(s)\right]^{\frac{\beta-\alpha}{\beta}} \frac{\left[\mathcal{G}'''(s)\right]^{\alpha/\beta}}{\mathcal{G}(\sigma(s))} w_1(s) \\ &= -\mu_1(s)\,\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\beta}{2} \frac{\sigma^2(s)\,\sigma'(s)}{\left(\mu_1(s)\,r(s)\right)^{1/\beta}} \left[\mathcal{G}'''(s)\right]^{\frac{\beta-\alpha}{\beta}} w_1^{\frac{\beta+1}{\beta}}(s). \end{aligned} \tag{3.21}$$

Note that $0 < \beta < \alpha$ and (C_1) hold. Since $r'(s) \geq 0$, we deduce that $\mathcal{G}^{(4)}(s) \leq 0$; this readily infers that $\mathcal{G}'''(s)$ is nonincreasing. Then there are $L_2 > 0$ and $s_2 \geq s_1$ such that

$$\left[\mathcal{G}'''(s)\right]^{\frac{\beta-\alpha}{\beta}} \geq L_2, \; s \geq s_2. \tag{3.22}$$

From (3.21) and (3.22), it follows that

$$w_1'(s) \leq -\mu_1(s)\,\kappa(s) + \frac{\mu_1'(s)}{\mu_1(s)} w_1(s) - \frac{\epsilon_1\beta L_2}{2} \frac{\sigma^2(s)\,\sigma'(s)}{\left(\mu_1(s)\,r(s)\right)^{1/\beta}} w_1^{\frac{\beta+1}{\beta}}(s). \tag{3.23}$$

By applying Lemma 2.3, where $B = \mu_1'(s)/\mu_1(s)$, $r = \epsilon_1\beta L_2\sigma^2(s)\,\sigma'(s)/2\left[\mu_1(s)\,r(s)\right]^{1/\beta}$, and $u(s) = w_1(s)$, we drive

$$w_1'(s) \leq -\mu_1(s)\,\kappa(s) + \frac{2^{\beta}}{(\beta+1)^{\beta+1}} \frac{r(s)\left[\mu_1'(s)\right]^{\beta+1}}{\left(L_2\epsilon_1\mu_1(s)\,\sigma^2(s)\,\sigma'(s)\right)^{\beta}}. \tag{3.24}$$

Integrating (3.24) throughout the interval $[s_3, s]$ allows us to conclude

$$\int_{s_2}^{s} \left(\mu_1(\hbar)\,\kappa(\hbar) - \frac{2^{\beta}}{(\beta+1)^{\beta+1}} \frac{r(\hbar)\left[\mu_1'(\hbar)\right]^{\beta+1}}{\left(L_2\epsilon_1\mu_1(\hbar)\,\sigma^2(\hbar)\,\sigma'(\hbar)\right)^{\beta}} \right) d\hbar \leq w_1(s_2),$$

This contradicts (3.20) when $s \to \infty$.

Thus, the proof is finished. □

3.2. *The scenario* (C_2)

This subsection discusses the criteria that exclude the existence of positive solutions to Eq (1.1) within (C_2).

Lemma 3.4. *Let $\beta \geq \alpha$. If there is a nondecreasing function $\mu_2 \in C^1([s_0, \infty), (0, \infty))$ such that*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \left(L_3^{\frac{\beta-\alpha}{\alpha}} \mu_2(v) \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du - \frac{\left[\mu_2'(v)\right]^2}{4\mu_2(v)} \right) dv = \infty, \tag{3.25}$$

holds for every $\epsilon_2 \in (0,1)$, $L_3 > 0$, then $C_2 = \emptyset$.

Proof. Let $\chi(s) \in C_2$. Then, there exists a $s_1 \geq s_0$, such that $\chi(s) > 0$, $\chi(\varsigma(s)) > 0$, and $\chi(\sigma_i(s)) > 0$ for $s \geq s_1 \geq s_0$ and $i = 1, 2, ..., m$. Integrating (ii) from s to ∞ and applying the fact that $(r(\mathcal{G}''')^\alpha)' \leq 0$, we deduce

$$r(s)\left[\mathcal{G}'''(s)\right]^\alpha \geq \int_s^\infty \kappa(\hbar)\mathcal{G}^\beta(\sigma(\hbar))d\hbar. \tag{3.26}$$

As $\mathcal{G}(s) > 0$, $\mathcal{G}'(s) > 0$, and $\mathcal{G}''(s) < 0$, Lemma 2.4 implies that

$$\mathcal{G}(s) \geq \epsilon_2 s\mathcal{G}'(s), \text{ forall } \epsilon_2 \in (0,1). \tag{3.27}$$

Integrating (3.27) from $\sigma(s)$ to s, we obtain

$$\frac{\mathcal{G}(\sigma(s))}{\mathcal{G}(s)} \geq \left(\frac{\sigma(s)}{s}\right)^{1/\epsilon_2}.$$

Therefore, (3.26) becomes

$$r(s)\left[\mathcal{G}'''(s)\right]^\alpha \geq \int_s^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} \mathcal{G}^\beta(\hbar)\, d\hbar.$$

Since $\mathcal{G}'(s) > 0$, then

$$r(s)\left[\mathcal{G}'''(s)\right]^\alpha \geq \mathcal{G}^\beta(s) \int_s^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar,$$

or equivalently

$$\mathcal{G}'''(s) \geq \mathcal{G}^{\beta/\alpha}(s)\left(\frac{1}{r(s)}\int_s^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha}.$$

Integrating this inequality from s to ∞, we have

$$\mathcal{G}''(s) \leq -\mathcal{G}^{\beta/\alpha}(s)\int_s^\infty \left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du. \tag{3.28}$$

Now, define

$$F(s) := \mu_2(s)\frac{\mathcal{G}'(s)}{\mathcal{G}(s)}. \tag{3.29}$$

Then, $F(s) \geq 0$ for $s \geq s_1 \geq s_0$ and

$$\begin{aligned} F'(s) &= \mu_2'(s)\frac{\mathcal{G}'(s)}{\mathcal{G}(s)} + \mu_2(s)\frac{\mathcal{G}''(s)}{\mathcal{G}(s)} - \mu_2(s)\frac{(\mathcal{G}'(s))^2}{\mathcal{G}^2(s)} \\ &= \mu_2(s)\frac{\mathcal{G}''(s)}{\mathcal{G}(s)} + \frac{\mu_2'(s)}{\mu_2(s)}F(s) - \frac{1}{\mu_2(s)}F^2(s). \end{aligned}$$

Hence, by (3.28), we obtain

$$\begin{aligned} F'(s) &\leq -\mu_2(s)\mathcal{G}^{\beta/\alpha-1}(s)\int_s^\infty\left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du \\ &\quad + \frac{\mu_2'(s)}{\mu_2(s)}F(s) - \frac{1}{\mu_2(s)}F^2(s). \end{aligned} \tag{3.30}$$

Because $\mathcal{G}'(s) > 0$ and $\beta \geq \alpha$, there are constants $L_3 > 0$ and $s_2 \geq s_1$ such that

$$\mathcal{G}^{\beta/\alpha-1}(s) \geq L_3^{\beta/\alpha-1} \text{ (If } \alpha = \beta, \text{ then } L_3 = 1). \tag{3.31}$$

Substituting (3.31) into (3.30), we have

$$\begin{aligned} F'(s) &\leq -L_3^{\beta/\alpha-1}\mu_2(s)\int_s^\infty\left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du \\ &\quad + \frac{\mu_2'(s)}{\mu_2(s)}F(s) - \frac{1}{\mu_2(s)}F^2(s). \end{aligned} \tag{3.32}$$

Using Lemma 2.3 with $B = \mu_2'(s)/\mu_2(s)$, $A = 1/\mu_2(s)$, and $u(s) = F(s)$, we obtain

$$\frac{\mu_2'(s)}{\mu_2(s)}F(s) - \frac{1}{\mu_2(s)}F^2(s) \leq \frac{\left[\mu_2'(s)\right]^2}{4\mu_2(s)}.$$

Consequently, (3.32) leads to

$$F'(s) \leq -L_3^{\beta/\alpha-1}\mu_2(s)\int_s^\infty\left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du + \frac{\mu_2^2(s)}{4\mu_2(s)}.$$

When we integrate this inequality between s_2 to s, we obtain

$$\int_{s_2}^{s}\left(L_3^{\beta/\alpha-1}\mu_2(v)\int_v^\infty\left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du - \frac{\left[\mu_2'(v)\right]^2}{4\mu_2(v)}\right)dv \leq F(s_2),$$

which contradicts (3.25) as $s \to \infty$. This completes the proof. □

The following corollary is obtained by setting $\mu_2(s) = 1$ in Lemma 3.4.

Corollary 3.1. *Let $\beta \geq \alpha$. If*

$$\limsup_{s\to\infty}\int_{s_0}^{s}\int_v^\infty\left(\frac{1}{r(u)}\int_u^\infty \kappa(\hbar)\left(\frac{\sigma(\hbar)}{\hbar}\right)^{\beta/\epsilon_2} d\hbar\right)^{1/\alpha} du\,dv = \infty, \tag{3.33}$$

holds for every $\epsilon_2 \in (0,1)$, then $C_2 = \varnothing$.

Lemma 3.5. *Let $0 < \beta < \alpha$. If there are nondecreasing functions $\mu_2 \in C^1([s_0, \infty), (0, \infty))$ such thatand*

$$\limsup_{s\to\infty} \int_{s_0}^{s} \left[L_4^{\frac{\beta-\alpha}{\alpha}} \mu_2(v)(v - s_0)^{\frac{\beta-\alpha}{\alpha}} \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du + \frac{\left(\mu_2'(v)\right)^2}{4\mu_2(v)} \right] dv = \infty, \tag{3.34}$$

holds for every $\epsilon_2 \in (0, 1)$, $L_4 > 0$, then $C_2 = \emptyset$.

Proof. Let $\chi(s) \in C_2$. Then there exists a $s_1 \geq s_0$, such that $\chi(s) > 0$, $\chi(\varsigma(s)) > 0$, and $\chi(\sigma_i(s)) > 0$ for $s \geq s_1 \geq s_0$ and $i = 1, 2, ..., m$. As demonstrated in the proof of (3.25) in Lemma 3.4. The function $F(s)$ is defined in (3.29), and so (3.30) holds, which can be written as follows:

$$\begin{aligned} F'(s) \quad \leq \quad & -\mu_2(s) \mathcal{G}^{\beta/\alpha-1}(s) \int_s^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du \\ & + \frac{\mu_2'(s)}{\mu_2(s)} F(s) - \frac{1}{\mu_2(s)} F^2(s). \end{aligned} \tag{3.35}$$

Since $\mathcal{G}'' < 0$, it follows that $\mathcal{G}'$ is decreasing. Consequently, we have

$$\mathcal{G}(s) = \int_{s_1}^{s} \mathcal{G}'(\hbar) d\hbar \leq \mathcal{G}'(s_1)(s - s_1) = L_4(s - s_1), \; L_4 := \mathcal{G}'(s_1) > 0. \tag{3.36}$$

Since $0 < \beta < \alpha$, we obtain $0 < \beta/\alpha < 1$, which, together with (3.36), leads to

$$\mathcal{G}^{\beta/\alpha-1}(s) \geq L_4^{\beta/\alpha-1}(s - s_1)^{\beta/\alpha-1}. \tag{3.37}$$

Hence, the inequality (3.35) becomes

$$\begin{aligned} F'(s) \quad \leq \quad & -L_4^{\beta/\alpha-1} \mu_2(s)(s - s_1)^{\beta/\alpha-1} \int_s^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du \\ & + \frac{\mu_2'(s)}{\mu_2(s)} F(s) - \frac{1}{\mu_2(s)} F^2(s). \end{aligned} \tag{3.38}$$

Using Lemma 2.3 with $B = \mu_2'(s)/\mu_2(s)$, $A = 1/\mu_2(s)$, and $u(s) = F(s)$, we can deduce that

$$\frac{\mu_2'(s)}{\mu_2(s)} F(s) - \frac{1}{\mu_2(s)} F^2(s) \leq \frac{\left(\mu_2'(s)\right)^2}{4\mu_2(s)}.$$

Consequently, (3.38) leads to

$$F'(s) \leq -L_4^{\frac{\beta-\alpha}{\alpha}} \mu_2(s)(s - s_1)^{\frac{\beta-\alpha}{\alpha}} \int_s^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du + \frac{\left(\mu_2'(s)\right)^2}{4\mu_2(s)}.$$

When we integrate this inequality between s_2 to s, we obtain

$$\int_{s_2}^{s} \left[L_4^{\frac{\beta-\alpha}{\alpha}} \mu_2(v)(v - s_1)^{\frac{\beta-\alpha}{\alpha}} \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} d\hbar \right)^{1/\alpha} du + \frac{\left(\mu_2'(v)\right)^2}{4\mu_2(v)} \right] dv \leq F(s_2),$$

which contradicts (3.34) as $s \to \infty$. Thus, we have completed the proof. □

The following corollary is obtained by setting $\mu_2(\mathrm{s}) = 1$ in Lemma 3.5.

Corollary 3.2. *Let* $0 < \beta < \alpha$. *If*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} (v - \mathrm{s}_0)^{\frac{\beta-\alpha}{\alpha}} \int_v^\infty \left(\frac{1}{r(u)} \int_u^\infty \kappa(\hbar) \left(\frac{\sigma(\hbar)}{\hbar} \right)^{\beta/\epsilon_2} \mathrm{d}\hbar \right)^{1/\alpha} \mathrm{d}u\mathrm{d}v = \infty, \tag{3.39}$$

hold for some $\epsilon_2 \in (0,1)$. *Then (1.1) is oscillatory.*

4. Oscillation theorem

In this part, we develop oscillatory criteria for the studied equation by combining the findings from the preceding sections.

Theorem 4.1. *If condition (2.4) is satisfied, then Eq (1.1) will exhibit oscillatory behavior provided that at least one of the following conditions is met:*
(1) *Both conditions (3.1) and (3.25);*
(2) *Both conditions (3.11) and (3.25);*
(3) *Both conditions (3.1) and (3.33);*
(4) *Both conditions (3.11) and (3.33);*
(5) *Both conditions (3.20) and (3.34);*
(6) *Both conditions (3.20) and (3.39).*

Proof. We prove the first case (1) and apply the same method to the rest of the cases. Let us assume that χ is eventually a positive solution to the Eq (1.1). According to Lemma 2.5, there are two possible cases of behavior: (C_1) and (C_2). Using Lemmas 3.1 and 3.2, we find that the conditions (3.1) and (3.25) exclude the existence of solutions that satisfy both of the mentioned cases, which leads to a contradiction with the initial assumption, and therefore, the solutions cannot be positive eventually, which proves the oscillation. This approach can be generalized to the other cases by the same method. Thus, the proof is complete. □

Example 4.1. *Consider the NDE given by:*

$$\left(\mathrm{s}^{-1} \left(\chi(\mathrm{s}) + \frac{1}{\mathrm{s}} \chi^{1/5}\left(\frac{1}{2}\mathrm{s}\right) - \chi^5\left(\frac{1}{2}\mathrm{s}\right) \right)''' \right)' + \frac{\kappa_0}{\mathrm{s}^5} \left(\chi\left(\frac{1}{2}\mathrm{s}\right) + \chi\left(\frac{1}{3}\mathrm{s}\right) + \chi\left(\frac{1}{4}\mathrm{s}\right) \right) = 0, \tag{4.1}$$

for $\mathrm{s} \geq 1$ *and* $\kappa_0 > 0$. *Here we have*

$$\begin{aligned} \alpha &= \beta = 1,\ \gamma = \frac{1}{5},\ \delta = 5,\ m = 3,\ \varsigma(\mathrm{s}) = \frac{1}{2}\mathrm{s},\ \sigma(\mathrm{s}) = \frac{1}{4}\mathrm{s}, \\ r(\mathrm{s}) &= \mathrm{s},\ \mathfrak{a}(\mathrm{s}) = \mathfrak{a}_1(\mathrm{s}) = \frac{1}{\mathrm{s}},\ \textit{and}\ \mathfrak{a}_2(\mathrm{s}) = 1. \end{aligned}$$

Now, we calculate

$$\int_{\mathrm{s}_0}^\infty \frac{1}{r^{1/\alpha}(\hbar)} \mathrm{d}\hbar = \int_1^\infty \hbar \mathrm{d}\hbar = \infty,$$

$$\kappa(\mathrm{s}) = c^\beta \sum_{i=1}^m \kappa_i(\mathrm{s}) = c \sum_{i=1}^3 \frac{\kappa_0}{\mathrm{s}^5} = \frac{3c\kappa_0}{\mathrm{s}^5},$$

$$g_1(s) := (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\mathfrak{a}_1^{\frac{1}{1-\gamma}}(s)\,\mathfrak{a}^{\frac{\gamma}{\gamma-1}}(s) = \frac{4}{5^{5/4}}\frac{1}{s}.$$

and

$$g_2(s) := (\delta-1)\delta^{\frac{\delta}{1-\delta}}\mathfrak{a}_2^{\frac{1}{1-\delta}}(s)\,\mathfrak{a}^{\frac{\delta}{\delta-1}}(s) = \frac{4}{5^{5/4}}\frac{1}{s^{5/4}}.$$

Therefore

$$\lim_{s\to\infty}[g_1(s)+g_2(s)] = \lim_{s\to\infty}\left[\frac{4}{5^{5/4}}\frac{1}{s}+\frac{4}{5^{5/4}}\frac{1}{s^{5/4}}\right] = 0,$$

which means that condition (2.4) is satisfied.

Now, consider the test function $\mu(s) = s^4$. *Applying condition (3.1), we obtain:*

$$\begin{aligned}
&\limsup_{s\to\infty}\int_1^s\left(\hbar^4\frac{3c\kappa_0}{\hbar^5}\left(\frac{\frac{1}{4}\hbar}{\hbar}\right)^{3/\epsilon}-\frac{2}{2^2}\frac{\hbar^{-1}\left(4\hbar^3\right)^2}{\epsilon_1\hbar^2\hbar^4}\right)d\hbar\\
= &\limsup_{s\to\infty}\int_1^s\left(3c\kappa_0\left(\frac{1}{4}\right)^{3/\epsilon}-\frac{8}{\epsilon_1}\right)\frac{1}{\hbar}d\hbar\\
= &\left(3c\kappa_0\left(\frac{1}{4}\right)^{3/\epsilon}-\frac{8}{\epsilon_1}\right)\limsup_{s\to\infty}\ln s = \infty,
\end{aligned}$$

which is satisfied provided that

$$\kappa_0 > \frac{8}{3c\epsilon_1(2)^{6/\epsilon}}. \tag{4.2}$$

Similarly, consider the test function $\mu_1(s) = s^4$. *Applying condition (3.11), we derive:*

$$\begin{aligned}
\limsup_{s\to\infty}\int_1^s\left(\hbar^4\frac{3c\kappa_0}{\hbar^5}-\frac{2}{2^2}\frac{4\hbar^{-1}\left[4\hbar^3\right]^2}{L_1\epsilon_1\frac{1}{16}\hbar^2\frac{1}{4}\hbar^4}\right)d\hbar &= \limsup_{s\to\infty}\int_1^s\left(3c\kappa_0-\frac{2^{11}}{L_1\epsilon_1}\right)\frac{1}{\hbar}d\hbar\\
&= \left(3c\kappa_0-\frac{2^{11}}{L_1\epsilon_1}\right)\limsup_{s\to\infty}\ln s = \infty,
\end{aligned}$$

which holds provided that

$$\kappa_0 > \frac{2^{11}}{3cL_1\epsilon_1}. \tag{4.3}$$

Now, let us consider the test function $\mu_2(s) = s$. *Applying condition (3.25), we obtain:*

$$\begin{aligned}
&\limsup_{s\to\infty}\int_1^s\left(v\int_v^\infty\left(u\int_u^\infty\frac{3c\kappa_0}{\hbar^5}\left(\frac{\hbar}{4\hbar}\right)^{1/\epsilon_2}d\hbar\right)du-\frac{1}{4v}\right)dv\\
= &\limsup_{s\to\infty}\int_1^s\left(v\int_v^\infty\frac{3c\kappa_0}{2^{2/\epsilon_2}}\frac{1}{4u^3}du-\frac{1}{4v}\right)dv\\
= &\limsup_{s\to\infty}\int_1^s\left(\frac{3c\kappa_0}{2^{2/\epsilon_2}}\frac{1}{8}-\frac{1}{4}\right)\frac{1}{v}dv\\
= &\left(\frac{3c\kappa_0}{2^{2/\epsilon_2}}\frac{1}{8}-\frac{1}{4}\right)\limsup_{s\to\infty}\ln s = \infty,
\end{aligned}$$

which holds if

$$\kappa_0 > \frac{2^{1+2/\epsilon_2}}{3c}. \tag{4.4}$$

Hence, in accordance with conditions (1)−(4) *of Theorem 4.1, the satisfaction of (4.2)–(4.4) ensures that Eq (4.1) exhibits oscillatory behavior.*

Example 4.2. *Consider the NDE given by*

$$\left(\mathrm{s}^{-1/3}\left(\left(\chi(\mathrm{s}) + \frac{1}{\mathrm{s}}\chi^{1/3}(0.5\mathrm{s}) - \chi^3(0.5\mathrm{s})\right)'''\right)^{1/3}\right)' + \sum_{i=1}^{m}\frac{\kappa_0}{\mathrm{s}^{7/3}}\chi^{1/3}(\sigma_i \mathrm{s}) = 0,\ \mathrm{s} \geq 1, \tag{4.5}$$

for $\mathrm{s} \geq 1$, $\kappa_0 > 0$ *and* , $\sigma_i \in (0,1)$, $i = 1,2,...,m$. *Clearly,* $\alpha = \beta = 1/3$, $\gamma = \frac{1}{3}$, $\delta = 3$, $\varsigma(\mathrm{s}) = 0.5\mathrm{s}$, $\sigma(\mathrm{s}) = \sigma\mathrm{s}$, *where* $\sigma = \min\{\sigma_i, i = 1,2,...,m\}$. *Additionally,* $r(\mathrm{s}) = \mathrm{s}^{-1/3}$, $\mathfrak{a}(\mathrm{s}) = \mathfrak{a}_1(\mathrm{s}) = 1/\mathrm{s}$, *and* $\mathfrak{a}_2(\mathrm{s}) = 1$. *We first evaluate the integral:*

$$\int_{\mathrm{s}_0}^{\infty}\frac{1}{r^{1/\alpha}(\hbar)}\mathrm{d}\hbar = \int_{1}^{\infty}\frac{1}{\left(\hbar^{-1/3}\right)^3}\mathrm{d}\hbar = \int_{1}^{\infty}\hbar \mathrm{d}\hbar = \infty,$$

and

$$\kappa(\mathrm{s}) = c^{\beta}\sum_{i=1}^{m}\kappa_i(\mathrm{s}) = c^{1/3}\sum_{i=1}^{m}\frac{\kappa_0}{\mathrm{s}^{7/3}} = \frac{mc^{1/3}k_0}{\mathrm{s}^{7/3}}.$$

Moreover,

$$g_1(\mathrm{s}) = \frac{2}{3^{3/2}}\frac{1}{\mathrm{s}} \text{ and } g_2(\mathrm{s}) = \frac{2}{3^{3/2}}\frac{1}{\mathrm{s}^{3/2}}.$$

Hence,

$$\lim_{\mathrm{s}\to\infty}[g_1(\mathrm{s}) + g_2(\mathrm{s})] = \lim_{\mathrm{s}\to\infty}\left[\frac{2}{3^{3/2}}\frac{1}{\mathrm{s}} + \frac{2}{3^{3/2}}\frac{1}{\mathrm{s}^{3/2}}\right] = 0,$$

which implies that condition (2.4) is satisfied.

Now, consider the test function $\mu(\mathrm{s}) = \mathrm{s}^{4/3}$. *Applying condition (3.1), we obtain:*

$$\begin{aligned}
&\limsup_{\mathrm{s}\to\infty}\int_1^{\mathrm{s}}\left(L\hbar^{4/3}\frac{mc^{1/3}k_0}{\hbar^{7/3}}\left(\frac{\sigma\hbar}{\hbar}\right)^{1/\epsilon} - \frac{2^{1/3}}{\left(\frac{4}{3}\right)^{4/3}}\frac{\hbar^{-1/3}\left(\frac{4}{3}\hbar^{1/3}\right)^{4/3}}{\left(\epsilon_1\hbar^2\hbar^{4/3}\right)^{1/3}}\right)\mathrm{d}\hbar \\
= &\limsup_{\mathrm{s}\to\infty}\int_1^{\mathrm{s}}\left(Lmc^{1/3}k_0\sigma^{1/\epsilon} - \frac{2^{1/3}}{\epsilon_1^{1/3}}\right)\frac{1}{\hbar}\mathrm{d}\hbar \\
= &\left(Lmc^{1/3}k_0\sigma^{1/\epsilon} - \frac{2^{1/3}}{\epsilon_1^{1/3}}\right)\limsup_{\mathrm{s}\to\infty}\ln \mathrm{s} = \infty,
\end{aligned}$$

which holds provided that

$$k_0 > \frac{1}{Lm\sigma^{1/\epsilon}}\left(\frac{2}{\epsilon_1 c}\right)^{1/3}. \tag{4.6}$$

Similarly, consider the test function $\mu_1(\mathrm{s}) = \mathrm{s}^{4/3}$. *Applying condition (3.11), we derive:*

$$\limsup_{\mathrm{s}\to\infty}\int_1^{\mathrm{s}}\left(\hbar^{4/3}\frac{mc^{1/3}k_0}{\hbar^{7/3}} - \frac{2^{1/3}}{\left(\frac{4}{3}\right)^{4/3}}\frac{\sigma^{-1/3}\hbar^{-1/3}\left[\frac{4}{3}\hbar^{1/3}\right]^{4/3}}{\left(L_1\epsilon_1\sigma^{-2}\hbar^2\sigma'\hbar^{4/3}\right)^{1/3}}\right)\mathrm{d}\hbar$$

$$
\begin{aligned}
&= \limsup_{s\to\infty} \int_{s_0}^{s} \left(mc^{1/3} k_0 - \frac{2^{1/3}}{(L_1 \epsilon_1 \sigma^4)^{1/3}} \right) \frac{1}{\hbar} \mathrm{d}\hbar \\
&= \left(mc^{1/3} k_0 - \frac{2^{1/3}}{(L_1 \epsilon_1 \sigma^4)^{1/3}} \right) \limsup_{s\to\infty} \ln s = \infty,
\end{aligned}
$$

which holds if

$$
k_0 > \frac{1}{m} \left(\frac{2}{L_1 c \epsilon_1 \sigma^4} \right)^{1/3}. \tag{4.7}
$$

Now, take $\mu_2(s) = s$. *Applying condition (3.25), we obtain:*

$$
\begin{aligned}
&\limsup_{s\to\infty} \int_{1}^{s} \left(v \int_{v}^{\infty} \left(u^{1/3} \int_{u}^{\infty} \frac{mc^{1/3} k_0}{\hbar^{7/3}} \left(\frac{\sigma \hbar}{\hbar} \right)^{1/3\epsilon_2} \mathrm{d}\hbar \right)^3 \mathrm{d}u - \frac{1}{4v} \right) \mathrm{d}v \\
&= \limsup_{s\to\infty} \int_{1}^{s} \left(v \int_{v}^{\infty} \left(u^{1/3} \frac{3mc^{1/3} k_0 \sigma^{1/3\epsilon_2}}{4} \frac{1}{u^{4/3}} \right)^3 \mathrm{d}u - \frac{1}{4v} \right) \mathrm{d}v \\
&= \limsup_{s\to\infty} \int_{1}^{s} \left(v \int_{v}^{\infty} \left(\frac{3mc^{1/3} k_0 \sigma^{1/3\epsilon_2}}{4} \right)^3 \frac{1}{u^3} \mathrm{d}u - \frac{1}{4v} \right) \mathrm{d}v \\
&= \limsup_{s\to\infty} \int_{1}^{s} \left(\frac{1}{2} \left(\frac{3mc^{1/3} k_0 \sigma^{1/3\epsilon_2}}{4} \right)^3 - \frac{1}{4} \right) \frac{1}{v} \mathrm{d}v \\
&= \left(\frac{1}{2} \left(\frac{3mc^{1/3} k_0 \sigma^{1/3\epsilon_2}}{4} \right)^3 - \frac{1}{4} \right) \limsup_{s\to\infty} \ln s = \infty,
\end{aligned}
$$

which is satisfied provided that

$$
k_0 > \frac{2^{5/3}}{3} \frac{1}{mc^{1/3} \sigma^{1/3\epsilon_2}}. \tag{4.8}
$$

Therefore, in accordance with conditions (1) *and* (2) *of Theorem 4.1, if inequalities (4.6)–(4.8) are satisfied, then Eq (4.1) exhibits oscillatory behavior.*

5. Conclusions

This study constitutes a pivotal step in exploring the oscillatory behavior of solutions associated with fourth-order differential equations, which contain mixed neutral terms and multiple delays. Using the Riccati methodology, we were able to derive precise criteria that guarantee the oscillation of solutions for this class of equations. Our results are not only a natural extension of previous research but also contribute to laying new foundations for a deeper and more comprehensive understanding. The research horizon opened by this study goes beyond the fourth order, as the results can be generalized in the future to include differential equations of higher order, where $n \geq 4$, adding a new mathematical dimension to the study. Moreover, applying the proposed methodology to the study of uncanonical cases provides a promising framework for expanding the theoretical understanding of these equations. Another interesting aspect is the exploration of equations containing the matching function of the form $\mathcal{G}(s) = \chi(s) + \mathfrak{a}_1(s)\chi^{\gamma}(\varsigma_1(s)) + \mathfrak{a}_2(s)\chi^{\delta}(\varsigma_2(s))$, as this approach opens up new horizons for research creativity and application development. Thus, this study paves the way for more in-depth analyses aimed at enhancing theoretical understanding while laying the foundations for expanding practical applications in different fields.

Author contributions

S.A., F.M., and O.B.: Methodology, investigation. S.A. and F.M.: Writing—original draft preparation. S.A., F.M., and O.B.: Writing—review and editing. O.B.: Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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