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Research article

Multi-criteria evaluation of tree species for afforestation in arid regions using a hybrid cubic bipolar fuzzy soft rough set framework

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Abstract: Decision rules are effective tools for managing information and characterizing datasets. As a result, they contribute significantly to fuzzy rough-set theory-based decision-making procedures. Rough set theory (RS) is a robust method for analyzing ambiguity in data. Moreover, cubic bipolar fuzzy sets (CBFS), an extension of bipolar fuzzy sets, can discuss both uncertainty and bipolarity in numerous situations. This article presents the robust decision-making approach named cubic bipolar fuzzy soft rough sets (CBFSRSs) by integrating RS and cubic bipolar fuzzy soft sets. This study explores the construction and fundamental characteristics of a novel approach based on CBFSs. We introduce and examine the concept of rough sets based on CBFSs, develop level sets for CBFSs, and highlight their key properties through illustrative examples. Additionally, we propose a decision-making framework based on CBFSRS that is capable of effectively managing uncertain, conflicting, and imprecise information. This approach demonstrates the potential of CBFSs in enhancing decision-making processes in large data environments. To demonstrate the practical benefit of CBFSRSs in decision-making, we provide an example of how CBFSRS standards might be used in decision-making processes to help decision-makers make well-informed and reasoned decisions. The example shows that the proposed strategies are useful and effective by applying them to real-life problems. It proves that they can handle complex, uncertain, and conflicting information in real decision-making situations.

Keywords: CBFSRSs; bipolar fuzzy set; rough set; decision-making
Mathematics Subject Classification: 05C25, 11E04, 20G15

1. Introduction

Multi-criteria decision-making (MCDM) is a promotion approach in which a set of decision-making (DM) problems simultaneously select the optimal choice from a set of viable options based on diverse conditions. However, due to the deficiency of data and incoherent human judgements, this approach produces unclear and imprecise results. The DM frequently focus on addressing the two-sided conflicting aspects of bipolar information. When there is ambiguity and bipolarity, traditional methodologies cannot establish the optimal alternative. The notion of fuzzy set (FS) first introduced by Zadeh [1] to enhance the range of traditional sets from $\{0, 1\} \rightarrow [0, 1]$. The membership function (MF) is the core idea in FS. Pawlak's rough set theory [2, 3] is an key statistical framework that used approximations to handle vagueness and granularity. Pawlak and Skowron established rough set extensions in approximation spaces [4]. Pawlak explored how the proximity of an object relates to its associated information by comparing its lower and upper approximations. This concept has numerous practical applications and, as a result, garnered significant interest from scientists and researchers, spurring further lines of investigation. Dubois and Prade established the concept of fuzzy rough sets and rough fuzzy sets (RFSs) [5]. Molodtsov [6] first proposed the notion of soft sets (SSs) as a new mathematical apparatus for handling uncertainties and imprecisions. The dimensions of the information are critical for analyzing and evaluating data or making decisions. The effectiveness of SSs as a parameterization tool has been demonstrated, addressing many issues associated with previous theories. This advancement has been thoroughly examined due to its broad applicability. An article [7] outlined some fundamental operations involving SSs. Maji et al. [8] applied the concept of SSs to tackle a DM issue. Aktas and Cagman established links between FSs, rough sets, and SSs [9], analyzing the attributes of SSs and presenting their findings. Ali et al. [10] proposed novel approaches to SSs. Feng et al. [11] presented a framework that integrates soft, rough, and FSs. Ali et al. [12] investigated the algebraic framework of SSs associated with new operations, while Ali [13] provided a novel viewpoint on parameter reduction in SSs.

Bipolarity is an important idea in data analysis since it allows for the advancement of strong mathematical models in a various domains. It includes both positive and negative entities, which reflect certainty and falsity, limitations, or impossibilities within the dataset, respectively. Understanding these different types of information is critical to understanding how humans make decisions because judgements often occur along a spectrum with two opposite ends. To declare a decision, such as selecting between competition and contribution, being friendly and hostile, or differentiating between sweet and sour in food, these observations shape our decisions and interactions, demonstrating how human minds are naturally bipolar. The stability of social institutions, business platforms, and organizational dynamics is often dependent on the negotiation and resolution of bipolar similarities. For example, successful leadership blends firmness and empathy, encouraging cooperation while maintaining healthy rivalry.

Numerous investigators have effectively used these models in recent decades, such as SSs and FSs, which are useful equipment for data analysis. However, they might be unable to accurately depict bipolarity. For instance, the lack of a defined aesthetic grade does not imply that a property is "ugly". This small variation demonstrates the limits of conventional methods for handling bipolar information. Zhang [14] established BFSs as a solution to this problem. This method makes it possible to show unclear and contradictory data in a more perfect and nuanced way by adding bipolarity to the FS structure. This is an easy way to integrate complex decision-making processes into mathematical models. For BFSs, it is simpler to process both positive and negative data at the same time, which is useful for various tasks such as sent iment analysis, risk assessment, and decision-making based on several factors. This multi-modal addressing allows for better balance and comprehension of opposing signals. Zhang established the more modified structure of FS as a BFS by increasing its flexibility and precision in dealing with complicated uncertainties. This development leads to improved decision-making models that are better at mimicking human judgements involving ambiguity and conflicting opinions.

Shabir and Gul [15] established the concept of the modified structure of RBSSs as a coupling of rough sets and bipolar SSs. Moreover, their qualities were investigated, and their relevance was established their relevance in Malik et al. [16] demonstrated the practical application of BSSs in diagnosing and treating diseases in the medical field were demonstrated, highlighting their efficacy in real-world healthcare mechanisms. Furthermore, they explored a robust method for defining and measuring the roughness of BSSs, which they subsequently applied to MCGDM. The study [17] indicated how advantageous the technique is in real life for dealing with ambiguity and bipolarity, making it a successful strategy for MCGDM in a range of scenarios. Kousar and Kausar [18] introduced an approach that uses FSR for MCDM based on data collected from a survey using questionnaires. The study investigates client opinions regarding sustainable development in agritourism and aims to identify the essential elements and alternatives required for effective DM in this sector. Shen [19] studied the relationship between economic performance, environmental, social, and governance (ESG) issues, stock returns, and RFSs. Gul [20] created a combined DM method using BF preference δ-covering-BFRSs with the VIKOR technique and demonstrated its effectiveness through a real-world example, along with a detailed assessment of the proposed method with some common decision-makers.

Naz and Shabir [21] made vital improvements to the algebraic structure of BFSS, which pushed the field forward even more. Juni et al. [22] added to this theory with new ideas, such as P-union, R-intersection, and R-union for both internal and external cubic sets. The introduction of the CBFS by Riaz and Tehrim [23, 24] marked a notable breakthrough. This model joined the features of BFSs and interval-valued BFS, offering a flexible approach to handling uncertain and conflicting information. The CBFSS constructed on BFSS enhances the representation of complex issues that previous FS theory extensions had difficulty capturing effectively. Many academics proposed the connection of various FS extensions with rough sets. Zhang and Shu developed the generalized interval-valued FSR and IFSRS models to handle uncertainty and imprecision in DM processes. They also explored key properties of the GIVFRS, such as approximation operators and dependency degrees [25, 26]. Jiang [27] presented a new fundamental cloud model capable of handling decision-making in big groups with several languages. This model adapts to decision-makers' preferences and addresses the ambiguity and unpredictability of their choices. The experiments indicated that this new strategy works effectively and outperforms others. Yang et al. [28] suggested a way to change BFRSs into formats that are easier to understand and use more efficiently. They did this by focusing on changing BFRS models to make them more useful for dealing with uncertainty and bipolarity in data. Malik and Shabir [29] demonstrated the DM application based on rough fuzzy bipolar SSs and explored its theoretical properties. Furthermore, many other decision makers established the DM approach under the influence of RS (see, for instance, [30–32]) also contributes to this composition. Inspired by the definition of level sets in some FS extensions [33–36], we further explore the concept of level sets of CBFSs. Jiang et al. [27] developed a unique rough integrated asymmetric cloud framework for addressing large group DM in a multi-granularity linguistic context. This approach not only depicts decision-makers' choices in a variety of ways, but also can objectively address the uncertainty and unpredictability inherent in large group DM by leveraging priority relevance. Yin et al. [37] investigated a rural integrated green energy system by considering the interests of many stakeholders and developing a complete benefit assessment framework. They used Sugeno-Weber aggregation operations and a multi-attribute decision-making approach to ensure robust and reliable outcomes. The case study confirmed the approach by finding solar energy as the best alternative among the investigated sources.

Motivation and contribution

(1) Zhang extended the FS as a BFS with an MF range of $[-1, 1]$. While a member with an MD of $[-1, 0]$ partially satisfies the implicit counter property, an element with an MD of (0,1] partially satisfies the property, and an element with an MD of 0 is irrelevant to the related property. The more generalized form of the BFS is the CBFS established by [4].

(2) Nanda and Majumdar [38] proposed an approach that combines FSs with rough sets, overcoming the constraints of conventional rough sets in managing uncertainty and ambiguity. Additionally, it establishes a mathematical basis for FSRs by delineating lower and higher approximations inside the framework of FSs.

(3) The DM applications utilizing FSRs and BFSs tackle uncertainty and bipolarity in intricate situations. Both are essential for tasks such as data reduction, feature selection, and medical diagnosis. They assist in managing ambiguous or incomplete information by presenting both the advantages and disadvantages of the data. This renders them suitable for MCDM, risk assessment, and social network analysis, where it is crucial to evaluate opposing elements, such as opportunities versus dangers or collaboration versus discord [25, 26, 29].

(4) While RS-theory and BFSSs have been studied independently, their hybrid combination as CBFSRSs remains under explored. Existing decision-making models often address either uncertainty or bipolarity separately, but CBFSRSs effectively handle both simultaneously, making them suitable for unclear and contradictory data. Despite theoretical advancements in combining rough sets and FSs, practical applications are scarce. Our research work bridges this gap by providing an illustrative example that demonstrates the practicality and scalability of CBFSRSs in real-world decision-making (see Table 1). By addressing these gaps, our research significantly contributes to RS, CFSS, and DM under uncertainty and bipolarity.

Table 1. Comparison of CBFSRS to the existing approaches.

Sets	Researchers	Uper & Lower Approximation	Degree of Membership	Degree of Non-membership	Interval Grading	Interval Grading Positive & Negative
FSs	Zadeh [1]	×	✓	×	×	×
IFSs	Atanassov [39]	×	✓	✓	×	×
IVFS	Zadeh [40]	×	✓	×	✓	×
BFS	Zhang [14]	×	×	×	✓	✓
Cubic set	Juni et al. [22]	×	✓	×	✓	×
Cubic bipolar fuzzy set	Riaz and Tehrim [23]	×	✓	×	✓	✓
FSR	Pawlak, Zdzis [5]	✓	✓	×	×	×
CBFSRs	Proposed Approach	✓	✓	✓	✓	✓

In this paper, we integrate CBFSs and CBFSSs with rough sets (RSs) and present an enhanced CBFSRS model that effectively bridges the gap between theoretical concepts and practical applications. We develop the CBFRS and CBFSRS approaches, resulting from joining CBFSs and CBFSSs with RS. Moreover, we provide comprehensive preliminaries and fundamental characteristics of the proposed approach, laying the groundwork for further exploration and application. Additionally, we build a DM algorithm using CBFSRS, capitalizing on the model's enhanced capacities to address intricate decision-making scenarios more efficiently. This research represents significant progress in the field, delivering a comprehensive framework for dealing with cubic bipolarity and ambiguity in decision-making DM processes. Figure 1 shows the proposed model:

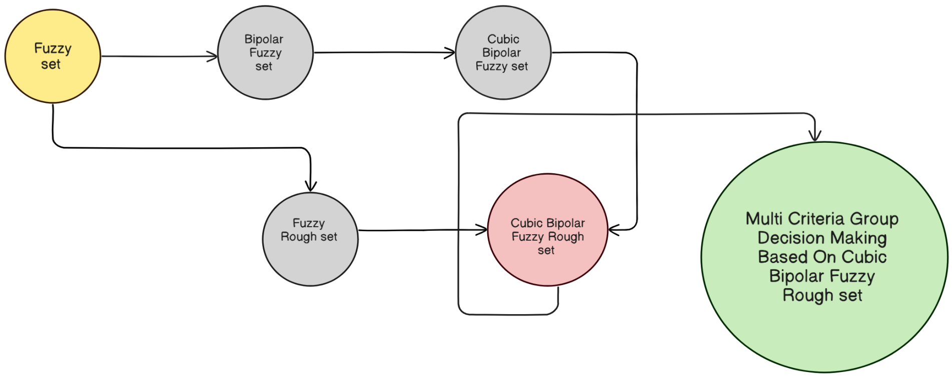

Figure 1. Proposed model.

Section 2 covers important definitions and key ideas about RSs, CBFSs, SSs, crisp soft relations (CSRs), and fuzzy soft relations (FSRs), looking at the basic traits needed to understand the more complex models discussed later. In Section 3, a rough set model is created based on the CBFR, exploring its features and showing how it builds on traditional rough set theory to manage the complexities of cubic bipolar fuzzy relations. In Section 4, a rough set model is created using the CBFR, looking at its features and showing how it builds on traditional rough sets to deal with the complexities of cubic bipolar fuzzy relations. Section 4 focuses on CBFSSs, discussing their key values and explaining how they combine elements of FSs and SSs within a cubic, bipolar framework, highlighting their unique attributes. Section 5 explores the features of a rough set model that uses the cubic bipolar FSR, showing how this model improves the rough set framework, especially in handling fuzzy and soft data with bipolar traits. Additionally, a definition of level sets for CBFSs is provided, and some of its most important aspects are analyzed. Section 6 gives a clear example to show a general way to do decision-making using CBSRSs, highlighting the main parts of the suggested decision-making strategy and showing how it works and is effective in real-life situations. Section 7 wraps up the paper with a detailed summary of the work, reviewing the key findings and contributions, discussing the impact of the research, and suggesting areas for future study and use of the developed models and algorithms.

2. Preliminaries

This section discusses some basic review regarding to RSs, CBFSs, SSs, crisp soft relation (CSR), and FSR.

Definition 2.1 ([2, 3]). *An approximation space (AS) is an element of the type* $(\mho, \varpi)$*, where* ϖ *is an equivalence relation (ER) on* $\mho$ *and* $\mho$ *is a non-void finite discourse. assuming any* ϱ *subset of* $\mho$*. If* ϱ

can be defined as the latter, it is known as definable; if not, it is an RS. ϱ can be described as a union of equivalence classes induced by ϖ. If is an RS, then ϱ can be roughly represented by two defined sets:

$$\underline{\varpi}(\varrho) = \{\mho \in \mho : [\mho]_{\varpi} \subseteq \varrho\},$$
$$\overline{\varpi}(\varrho) = \{\mho \in \mho : [\mho]_{\varpi} \cap \varrho \neq \phi\}.$$

They are known as the lower and upper-approximations of ϱ. In the order mentioned, the pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ is named an RS of ϱ, where

$$[\mho]_{\varpi} = \{y \in \mho; (\mho, y) \in \varpi\},$$

and the rough set RS $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ leads to the description of ϱ in light of current understanding, i.e., the $\mho$ distribution. Additionally, the positive, negative, and boundary regions of ϱ are

$$\mathrm{Pos}_{\varpi}(\varrho) = \underline{\varpi}(\varrho), \mathrm{Neg}_{\varpi}(\varrho) = [\overline{\varpi}(\varrho)]^c, \mathrm{Bnd}_{\varpi}(\varrho) = \overline{\varpi}(\varrho) - \underline{\varpi}(\varrho).$$

Definition 2.2 ([23, 24])**.** *Let $\mho$ be the initial set discourse, Then, the object of the form:*

$$\varrho = \left\{\left\langle \mho, I_{\varrho}(\mho), B_{\varrho}(\mho)\right\rangle, \mho \in \mho\right\}$$

is called a CBFS on $\mho$, where I_{ϱ} is an IVBFS, B_{ϱ} is a BFS on $\mho$ and CBFS can organize as:

$$\varrho = \left\{\left\langle \mho, \left[\Gamma^{+}_{L\varrho}(\mho), \Gamma^{+}_{u\varrho}(\mho)\right], \left[\Gamma^{-}_{L\varrho}(\mho), \Gamma^{-}_{u\varrho}(\mho)\right], \left(\Gamma^{+}_{\varrho}(\mho), \Gamma^{-}_{\varrho}(\mho)\right)\right\rangle : \mho \in \mho\right\}$$

where $\left[\Gamma^{+}_{L\varrho}(\mho), \Gamma^{+}_{u\varrho}(\mho)\right] \in \mathrm{int}([0,1])$ and $\left[\Gamma^{-}_{L\varrho}(\mho), \Gamma^{-}_{u\varrho}(\mho)\right] \in \mathrm{int}([-1,0])$ are expressed as the IV positive and negative MD, respectively, $\Gamma^{+}_{\varrho}(\mho) \in [0,1]$, $\Gamma^{-}_{\varrho}(\mho) \in [-1,0]$ indicate the single-valued positive and negative MD orderly of an element $\mho \in \mho$, where $\mathrm{int}([-1,0])$ and $\mathrm{int}([0,1])$ denote to the set of all closed subdivisions in $[-1,0]$, $[0,1]$, respectively. CBFS $(\mho)$ is a representation of the set of all CBF subsets of $\mho$.

Definition 2.3 ([24])**.** *Let*

$$\varrho = \left\{\left\langle \mho, \left[\Gamma^{+}_{L\varrho}(\mho), \Gamma^{+}_{u\varrho}(\mho)\right], \left[\Gamma^{-}_{L\varrho}(\mho), \Gamma^{-}_{u\varrho}(\mho)\right], \left(\Gamma^{+}_{\varrho}(\mho), \Gamma^{-}_{\varrho}(\mho)\right)\right\rangle, \mho \in \mho\right\}$$

and

$$\varkappa = \{\langle \mho, [\Gamma^{+}_{L\varkappa}(\mho), \Gamma^{+}_{u\varkappa}(\mho)], [\Gamma^{-}_{L\varkappa}(\mho), \Gamma^{-}_{u\varkappa}(\mho)], (\Gamma^{+}_{\varkappa}(\mho), \Gamma^{-}_{\varkappa}(\mho))\rangle, \mho \in \mho\}$$

be two CBFSs on $\mho$. Then, some P-order operation on these CBFSs are given as follows:

(i)

$$\begin{aligned}\varrho \cup_P \varkappa = &\left\{\left\langle \mho, \left[\max\left\{\Gamma^{+}_{L\varrho}(\mho), \Gamma^{+}_{L\varkappa}(\mho)\right\}, \max\left\{\Gamma^{+}_{u\varrho}(\mho), \Gamma^{+}_{u\varkappa}(\mho)\right\}\right],\right.\right.\\ &\left[\min\left\{\Gamma^{-}_{L\varrho}(\mho), \Gamma^{-}_{L\varkappa}(\mho)\right\}, \min\left\{\Gamma^{-}_{u\varrho}(\mho), \Gamma^{-}_{u\varkappa}(\mho)\right\}\right],\\ &\left.\left.\left(\max\left\{\Gamma^{+}_{\varrho}(\mho), \Gamma^{+}_{\varkappa}(\mho)\right\}, \min\left\{\Gamma^{-}_{\varrho}(\mho), \Gamma^{-}_{\varkappa}(\mho)\right\}\right)\right\rangle, \mho \in \mho\right\};\end{aligned} \tag{2.1}$$

(ii)

$$\begin{aligned}\varrho \cap_P \varkappa = &\left\{\left\langle \mho, \left[\min\left\{\Gamma^{+}_{L\varrho}(\mho), \Gamma^{+}_{L\varkappa}(\mho)\right\}, \min\left\{\Gamma^{+}_{u\varrho}(\mho), \Gamma^{+}_{u\varkappa}(\mho)\right\}\right],\right.\right.\\ &\left[\max\left\{\Gamma^{-}_{L\varrho}(\mho), \Gamma^{-}_{L\varkappa}(\mho)\right\}, \max\left\{\Gamma^{-}_{u\varrho}(\mho), \Gamma^{-}_{u\varkappa}(\mho)\right\}\right],\\ &\left.\left.\left(\min\left\{\Gamma^{+}_{\varrho}(\mho), \Gamma^{+}_{\varkappa}(\mho)\right\}, \max\left\{\Gamma^{-}_{\varrho}(\mho), \Gamma^{-}_{\varkappa}(\mho)\right\}\right)\right\rangle, \mho \in \mho\right\};\end{aligned} \tag{2.2}$$

(iii)

$$\varrho \oplus_P \varkappa = \Big\{\Big\langle \mho, \Big[\Gamma^+_{L\varrho}(\mho) + \Gamma^+_{L\varkappa}(\mho) - \Gamma^+_{L\varrho}(\mho)\Gamma^+_{u\varkappa}(\mho), \Gamma^+_{u\varrho}(\mho) + \Gamma^+_{u\varkappa}(\mho) - \Gamma^+_{u\varrho}(\mho)\Gamma^+_{u\varkappa}\Big], \Big[-\Big(\Gamma^-_{L\varrho}(\mho)\Gamma^-_{L\varkappa}(\mho)\Big), -\Big(\Gamma^-_{u\varrho}(\mho)\Gamma^-_{u\varkappa}(\mho)\Big)\Big], \Big(\Gamma^+_{\varrho}(\mho) + \Gamma^+_{\varkappa}(\mho) - \Gamma^+_{\varrho}(\mho)\Gamma^+_{\varkappa}(\mho), -\Big(\Gamma^-_{\varrho}(\mho)\Gamma^-_{\varkappa}(\mho)\Big)\Big)\Big\rangle, \mho \in \mho\Big\}; \tag{2.3}$$

(iv) $\varrho \subseteq_P \varkappa$ *if* $\Big[\Gamma^+_{L\varrho}(\mho), \Gamma^+_{u\varrho}(\mho)\Big] \leqslant \Big[\Gamma^+_{L\varkappa}(\mho), \Gamma^+_{u\varkappa}(\mho)\Big]$ *and*

$$\Big[\Gamma^-_{L\varrho}(\mho), \Gamma^-_{u\varrho}(\mho)\Big] \geqslant [\Gamma^+_{L\varkappa}(\mho), \Gamma^+_{u\varkappa}(\mho)], \ \ \Gamma^+_{\varrho}(\mho) \leqslant \Gamma^+_{\varkappa}(\mho) \text{ and } \Gamma^-_{\varrho}(\mho) \geqslant \Gamma^-_{\varkappa}(\mho).$$

Definition 2.4 ([24]). *Let*

$$\varrho = \Big\{\Big\langle \mho, \Big[\Gamma^+_{L\varrho}(\mho), \Gamma^+_{u\varrho}(\mho)\Big], \Big[\Gamma^-_{L\varrho}(\mho), \Gamma^-_{u\varrho}(\mho)\Big], \Big(\Gamma^+_{\varrho}(\mho), \Gamma^-_{\varrho}(\mho)\Big)\Big\rangle, \mho \in \mho\Big\}$$

and

$$\varkappa = \{\langle \mho, [\Gamma^+_{L\varkappa}(\mho), \Gamma^+_{u\varkappa}(\mho)], [\Gamma^-_{L\varkappa}(\mho), \Gamma^-_{u\varkappa}(\mho)], (\Gamma^+_{\varkappa}(\mho), \Gamma^-_{\varkappa}(\mho))\rangle, \mho \in \mho\}$$

be two CBFSs on $\mho$. *Consequently, we possess the following attributes:*

(i)

$$\varrho \cup_R \varkappa = \Big\{\Big\langle \mho, \Big[\max\Big\{\Gamma^+_{L\varrho}(\mho), \Gamma^+_{L\varkappa}(\mho)\Big\}, \max\Big\{\Gamma^+_{u\varrho}(\mho), \Gamma^+_{u\varkappa}(\mho)\Big\}\Big], \Big[\min\Big\{\Gamma^-_{L\varrho}(\mho), \Gamma^-_{L\varkappa}(\mho)\Big\}, \min\Big\{\Gamma^-_{u\varrho}(\mho), \Gamma^-_{u\varkappa}(\mho)\Big\}\Big], \Big(\min\Big\{\Gamma^+_{\varrho}(\mho), \Gamma^+_{\varkappa}(\mho)\Big\}, \max\Big\{\Gamma^-_{\varrho}(\mho), \Gamma^-_{\varkappa}(\mho)\Big\}\Big)\Big\rangle, \mho \in \mho\Big\};$$

(ii)

$$\varrho \cap_R \varkappa = \Big\{\Big\langle \mho, \Big[\min\Big\{\Gamma^+_{L\varrho}(\mho), \Gamma^+_{L\varkappa}(\mho)\Big\}, \min\Big\{\Gamma^+_{u\varrho}(\mho), \Gamma^+_{u\varkappa}(\mho)\Big\}\Big], \Big[\max\Big\{\Gamma^-_{L\varrho}(\mho), \Gamma^-_{L\varkappa}(\mho)\Big\}, \max\Big\{\Gamma^-_{u\varrho}(\mho), \Gamma^-_{u\varkappa}(\mho)\Big\}\Big], \Big(\max\Big\{\Gamma^+_{\varrho}(\mho), \Gamma^+_{\varkappa}(\mho)\Big\}, \min\Big\{\Gamma^-_{\varrho}(\mho), \Gamma^-_{\varkappa}(\mho)\Big\}\Big)\Big\rangle, \mho \in \mho\Big\};$$

(iii)

$$\varrho \oplus_R \varkappa = \Big\{\Big\langle \mho, \Big[\Gamma^+_{L\varrho}(\mho) + \Gamma^+_{L\varkappa}(\mho) - \Gamma^+_{L\varrho}(\mho)\Gamma^+_{u\varkappa}(\mho), \Gamma^+_{u\varrho}(\mho) + \Gamma^+_{u\varkappa}(\mho) - \Gamma^+_{u\varrho}(\mho)\Gamma^+_{u\varkappa}\Big], \Big[-\Big(\Gamma^-_{L\varrho}(\mho)\Gamma^-_{L\varkappa}(\mho)\Big), -\Big(\Gamma^-_{u\varrho}(\mho)\Gamma^-_{u\varkappa}(\mho)\Big)\Big], \Big(\Gamma^+_{\varrho}(\mho)\Gamma^+_{\varkappa}(\mho) - \Big(-\Gamma^+_{\varrho}(\mho) - \Gamma^+_{\varkappa}(\mho), -\Big(\Gamma^-_{\varrho}(\mho)\Gamma^-_{\varkappa}(\mho)\Big)\Big)\Big)\Big\rangle, \mho \in \mho\Big\};$$

(iv) $\varrho \subseteq_R \varkappa$ *if* $\Big[\Gamma^+_{L\varrho}(\mho), \Gamma^+_{u\varrho}(\mho)\Big] \leqslant \Big[\Gamma^+_{L\varkappa}(\mho), \Gamma^+_{u\varkappa}(\mho)\Big]$ *and*

$$\Big[\Gamma^-_{L\varrho}(\mho), \Gamma^-_{u\varrho}(\mho)\Big] \geqslant [\Gamma^-_{L\varkappa}(\mho), \Gamma^-_{u\varkappa}(\mho)], \ \ \Gamma^+_{\varrho}(\mho) \geqslant \Gamma^+_{\varkappa}(\mho) \text{ and } \Gamma^-_{\varrho}(\mho) \leqslant \Gamma^-_{\varkappa}(\mho).$$

Definition 2.5 ([6]). *Let* $\aleph$ *be the parameter set defined for the discourse constituents* $\mho$. *A pair* $(f, \aleph)$, *where* f *is a transforming given by* $f : \aleph \to P(\Psi)$, *where* $P(\Psi)$ *is the set of all subsets of* Ψ, *which is an SS over* Ψ. *According to this definition, an SS over* Ψ *is an ordered list of subsets of the cosmos* Ψ. $f(\ell)$ *is defined as the set of* ℓ*-approximate elements of the SS* $(f, \aleph)$ *for any* $\ell \in \aleph$.

Definition 2.6 ([41]). *Let Ψ be the universe of discourse, and let $(f, \aleph)$ an SS on Ψ. Then a subset of $\Psi \times \aleph$ is known as a CSR, which defined as*

$$\varpi = \{\langle(\psi, \ell), \mu_{\varpi}(\psi, \ell)\rangle, (\psi, \ell) \in \Psi \times \aleph\}$$

where $\mu_{\varpi} : \Psi \times \aleph \to \{0, 1\}$ such that:

$$\mu_{\varpi}(\psi, \ell) = \begin{cases} 1, & \text{if } (\psi, \ell) \in \varpi, \\ 0, & \text{if } (\psi, \ell) \notin \varpi. \end{cases}$$

Definition 2.7 ([42]). *Let $(f, \aleph)$ be a fuzzy SS over Ψ. Then, a fuzzy subset of $\Psi \times \aleph$ is known as a FSR from Ψ to $\aleph$ is expressly described by:*

$$\varpi = \{\langle(\psi, \ell), \mu_{\varpi}(\psi, \ell)\rangle, (\psi, \ell) \in \Psi \times \aleph\}$$

where $\mu_{\varpi} : \Psi \times \aleph \to [0, 1]$, $\mu_{\varpi}(\psi, \ell) = \mu_{f(\ell)}(\psi)$.

3. Rough set model based on CBFSs

In this study, and for the remaining sections of this study, we shall use the *R*-order, *R*-union, *R*-intersection, and *R*-ring sum.

3.1. Rough CBFSs

Definition 3.1. *Let $\Psi \neq \emptyset$ and for the finite discourse for any an random crisp relation (CR) ϖ on Ψ, a set-valued function (SVF) $\varpi^* : \Psi \to P(\Psi)$ is represented as*

$$\varpi^*(\psi) = \{y \in \Psi/(\psi, y) \in \varpi\}, \psi \in \Psi$$

if $\varpi(\psi) \neq \emptyset$ for all $\psi \in \Psi$, then ϖ is said to be serial in this context. The pair (Ψ, ϖ) is referred to as a crisp approximation space (CAS).

For any $\varrho \subseteq \Psi$, the lower and upper approximation of ϱ with respect to (Ψ, ϖ) presented by $\underline{\varpi}(\varrho)$ and $\overline{\varpi}(\varrho)$, respectively, are as follows:

$$\underline{\varpi}(\varrho) = \{\langle\psi \in \Psi, \varpi^*(\psi) \subseteq \varrho\rangle\},$$
$$\overline{\varpi}(\varrho) = \{\langle\psi \in \Psi, \varpi^*(\psi) \cap \varrho \neq \emptyset\rangle\}.$$

The pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$, known as an RS and $\underline{\varpi}, \overline{\varpi} : P(\Psi) \to P(\Psi)$ are pointed to as the lower and upper crisp rough approximation tools, respectively. Moreover, if $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$ then ϱ is known as a definable set.

Definition 3.2. *Let (Ψ, ϖ) be a CAS and let $\varrho \in$ CBFS(Ψ). Then the lower and upper approximation of ϱ with respect to (Ψ, ϖ), denoted by $\underline{\varpi}(\varrho)$ and $\underline{\varpi}(\varrho)$, are given by*

$$\underline{\varpi}(\varrho) = \left\{\left\langle\psi, I^+_{\underline{\varpi}(\varrho)}(\psi), I^-_{\underline{\varpi}(\varrho)}(\psi), \left(B^+_{\underline{\varpi}(\varrho)}(\psi), B^-_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$
$$\overline{\varpi}(\varrho) = \left\{\left\langle\psi, I^+_{\overline{\varpi}(\varrho)}(\psi), I^-_{\overline{\varpi}(\varrho)}(\psi), \left(B^+_{\overline{\varpi}(\varrho)}(\psi), B^-_{\overline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$

or it can be written with more explanation as follows:

$$\underline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^{+}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\underline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^{-}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\underline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^{+}_{\underline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$

$$\overline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^{+}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\overline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^{-}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\overline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^{+}_{\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{\overline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$

where

$$\left[\Gamma^{+}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{y\in\varpi^*(\psi)} \left[\Gamma^{+}_{L\varrho}(y), \Gamma^{+}_{u\varrho}(y)\right],$$

$$\left[\Gamma^{-}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{y\in\varpi^*(\psi)} \left[\Gamma^{-}_{L\varrho}(y), \Gamma^{-}_{u\varrho}(y)\right],$$

$$\Gamma^{+}_{\underline{\varpi}(\varrho)}(\psi) = \bigvee_{y\in\varpi^*(\psi)} \Gamma^{+}_{\varrho}(y), \quad \textit{and} \quad \Gamma^{-}_{\underline{\varpi}(\varrho)}(\psi) = \bigwedge_{y\in\varpi^*(\psi)} \Gamma^{-}_{\varrho}(y).$$

And, for $\overline{\varpi}(\varrho)$, we obtain

$$\left[\Gamma^{+}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{y\in\varpi^*(\psi)} \left[\Gamma^{+}_{L\varrho}(y), \Gamma^{+}_{u\varrho}(y)\right],$$

$$\left[\Gamma^{-}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{y\in\varpi^*(\psi)} \left[\Gamma^{-}_{L\varrho}(y), \Gamma^{-}_{u\varrho}(y)\right],$$

$$\Gamma^{+}_{\overline{\varpi}(\varrho)}(\psi) = \bigwedge_{y\in\varpi^*(\psi)} \Gamma^{+}_{\varrho}(y), \quad \textit{and} \quad \Gamma^{-}_{\overline{\varpi}(\varrho)}(\psi) = \bigvee_{y\in\varpi^*(\psi)} \Gamma^{-}_{\varrho}(y).$$

In relation to (Ψ, ϖ), the pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ is pointed to as the rough CBFS of ϱ. Moreover, ϱ is narrated to be defined if $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$.

Remark 3.3. *Let $A, B \in IVFS(\Psi)$, where*

$$A = [A_L, A_u], \quad B = [B_L, B_u].$$

Then we have the following:

(1) $A \wedge B(\psi) = [\min(A_L(\psi), B_L(\psi)), \min(A_u(\psi), B_u(\psi))]$,
(2) $A \vee B(\psi) = [\max(A_L(\psi), B_L(\psi)), \max(A_u(\psi), B_u(\psi))]$.

In the following illustration, we find the lower and upper approximations $\underline{\varpi}(\varrho)$ and $\overline{\varpi}(\varrho)$ of ϱ with respect to (Ψ, ϖ) defined on CBFS.

Example 3.4. *Let $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ be a universe set. We define a CR ϖ on $\Psi \times \Psi$ as follows:*

$$\varpi = \{(\psi_1, \psi_2), (\psi_1, \psi_3), (\psi_2, \psi_1), (\psi_2, \psi_4), (\psi_3, \psi_1), (\psi_3, \psi_2), (\psi_3, \psi_3), (\psi_3, \psi_4), (\psi_4, \psi_1), (\psi_4, \psi_2)\}.$$

So, we obtain the SVF ϖ^*, as follows:

$$\varpi^*(\psi_1) = \{\psi_2, \psi_3\}, \ \varpi^*(\psi_2) = \{\psi_1, \psi_4\}, \ \varpi^*(\psi_3) = \{\psi_1, \psi_2, \psi_3, \psi_4\} = \Psi, \ \varpi^*(\psi_4) = \{\psi_1, \psi_2\}.$$

Now, we consider $\varrho \in CBFS(\Psi)$, as follows:

$$\varrho = \{\langle\psi_1, [0.25, 0.5], [-0.5, 0], (0.5, -0.5)\rangle, \langle\psi_2, [0, 0.25], [-0.75, -0.25], (0.3, -0.3)\rangle,$$
$$\langle\psi_3, [0.5, 0.75], [-0.75, 0], (0.6, -0.6)\rangle, \langle\psi_4, [0.3, 0.6], [-0.6, -0.3], (0.45, -0.5)\rangle\}.$$

With regard to Definition 3.2, we can obtain the lower and upper approximation $\underline{\varpi}(\varrho)$ and $\overline{\varpi}(\varrho)$ of ϱ with respect to (Ψ, ϖ) as follows:

$$\underline{\varpi}(\varrho) = \{\langle\psi_1, [0, 0.25], [-0.75, 0], (0.6, -0.6)\rangle, \langle\psi_2, [0.25, 0.5], [-0.5, 0], (0.5, -0.5)\rangle, \\ \langle\psi_3, [0, 0.25], [-0.5, 0], (-0.6, -0.6)\rangle, \langle\psi_4, [0, 0.25], [-0.5, 0], (0.5, -0.5)\rangle\},$$

$$\overline{\varpi}(\varrho) = \{\langle\psi_1, [0.5, 0.75], [-0.75, -0.25], (0.3, -0.3)\rangle, \langle\psi_2, [0.3, 0.6], [-0.6, -0.3], (0.45, -0.5)\rangle, \\ \langle\psi_3, [0.5, 0.75], [-0.75, -0.3], (0.3, -0.3)\rangle, \langle\psi_4, [0.25, 0.5], [-0.75, -0.25](0.3, -0.3)\rangle\}.$$

The pair $\left(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)\right)$ is called a rough CBFS.

The following are some key characteristics related to the approximation space under the CBFS environment. These results establish the foundation for the definition of lower and upper approximations, boundary regions, and decision rules in this hybrid environment. By integrating the cubic structure and BF logic in the SS, the rough CBFS approximation space provides a more expressive and flexible approach to modeling uncertainty and dual-opinion information, which is particularly useful in complex DM scenarios.

Theorem 3.5. *Let (Ψ, ϖ) be a CAS. For any $\varrho, \varkappa \in CBFS(\Psi)$, the lower and upper rough cubic bipolar, approximation operator $\underline{\varpi}(\varrho)$, and $\overline{\varpi}(\varrho)$, respectively, hold the these characteristics:*

(1) $\underline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varrho), \quad \forall \varrho \in CBFS(\Psi)$,
(2) $\varrho \subseteq \varkappa \Rightarrow \underline{\varpi}(\varrho) \subseteq \underline{\varpi}(\varkappa)$ *and* $\overline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varkappa)$,
(3) $\underline{\varpi}(\varrho^c) = (\overline{\varpi}(\varrho))^c, \quad \overline{\varpi}(\varrho^c) = \left(\underline{\varpi}(\varrho)\right)^c$,
(4) $\underline{\varpi}(\varrho \cap \varkappa) = \underline{\varpi}(\varrho) \cap \underline{\varpi}(\varkappa), \quad \overline{\varpi}(\varrho \cup \varkappa) = \overline{\varpi}(\varrho) \cup \overline{\varpi}(\varkappa)$,
(5) $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa), \quad \underline{\varpi}(\varrho \cup \varkappa) \supseteq \underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa)$.

Proof. (1) The proof of (1) is straightforward.
(2) Let $\varrho \subseteq \varkappa$. Then,

$$\begin{aligned}
\underline{\varpi}(\varrho) &= \left\{\left\langle\psi, I^+_{\underline{\varpi}(\varrho)}(\psi), I^-_{\underline{\varpi}(\varrho)}(\psi), \left(B^+_{\underline{\varpi}(\varrho)}(\psi), B^-_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\} \\
&= \left\{\left\langle\psi, \bigwedge_{y \in \varpi^*(\psi)} I^+_\varrho(y), \bigvee_{y \in \varpi^*(\psi)} I^-_\varrho(y), \left(\bigvee_{y \in \varpi^*(\psi)} B^+_\varrho(y), \bigwedge_{y \in \varpi^*(\psi)} B^-_\varrho(y)\right)\right\rangle, \psi \in \Psi\right\} \\
&\leq \left\{\left\langle\psi, \bigwedge_{y \in \varpi^*(\psi)} I^+_\varkappa(y), \bigvee_{y \in \varpi^*(\psi)} I^-_\varkappa(y), \left(\bigvee_{y \in \varpi^*(\psi)} B^+_\varkappa(y), \bigwedge_{y \in \varpi^*(\psi)} B^-_\varkappa(y)\right)\right\rangle, \psi \in \Psi\right\} \\
&= \underline{\varpi}(\varkappa).
\end{aligned}$$

Similarly, we can obtain that $\overline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varkappa)$.
(3) We need to show that, for $\psi \in \Psi$,

$$\underline{\left(I^+_\varrho\right)^c(\psi)} = \left(\overline{\left(I^+_\varrho\right)(\psi)}\right)^c, \quad \underline{\left(I^-_\varrho\right)^c(\psi)} = \left(\overline{\left(I^-_\varrho\right)(\psi)}\right)^c,$$
$$\underline{\left(B^+_\varrho\right)^c(\psi)} = \left(\overline{\left(B^+_\varrho\right)(\psi)}\right)^c \quad \text{and} \quad \underline{\left(B^-_\varrho\right)^c(\psi)} = \left(\overline{\left(B^-_\varrho\right)(\psi)}\right)^c$$

to prove $\underline{\varpi}(\varrho^c) = [\overline{\varpi}(\varrho)]^c$. Now, we consider

$$\begin{aligned}\underline{\left(I_\varrho^+\right)^c(\psi)} &= \bigwedge_{y\in\varpi^*(\psi)}\left[1-\left[\Gamma_{L\varrho}^+(y),\Gamma_{u\varrho}^+(y)\right]\right] = 1-\bigvee_{y\in\varpi^*(\psi)}\left[\Gamma_{L\varrho}^+(y),\Gamma_{u\varrho}^+(y)\right]\\ &= 1-\overline{\left(I_\varrho^+\right)(\psi)} = \left(\overline{\left(I_\varrho^+\right)(\psi)}\right)^c,\\ \underline{\left(I_\varrho^-\right)^c(\psi)} &= \bigvee_{y\in\varpi^*(\psi)}\left[-1-\left[\Gamma_{L\varrho}^-(y),\Gamma_{u\varrho}^-(y)\right]\right] = -1-\bigwedge_{y\in\varpi^*(\psi)}\left[\Gamma_{L\varrho}^-(y),\Gamma_{u\varrho}^-(y)\right]\\ &= -1-\overline{\left(I_\varrho^-\right)(\psi)} = -1-\left(\overline{\left(I_\varrho^-\right)(\psi)}\right)^c.\end{aligned}$$

Also,

$$\underline{\left(B_\varrho^+\right)^c(\psi)} = \bigvee_{y\in\varpi^*(\psi)}\left[1-B_\varrho^+(y)\right] = 1-\bigwedge_{y\in\varpi^*(\psi)}B_\varrho^+(y) = 1-\overline{\left(B_\varrho^+\right)(\psi)} = \left(\overline{\left(B_\varrho^+\right)(\psi)}\right)^c,$$

and, finally,

$$\underline{\left(B_\varrho^-\right)^c(\psi)} = \bigwedge_{y\in\varpi^*(\psi)}\left[-1-B_\varrho^-(y)\right] = -1-\bigvee_{\varpi^*(\psi)}B_\varrho^-(y) = -1-\overline{\left(B_\varrho^-\right)(\psi)} = \left(\overline{\left(B_\varrho^-\right)(\psi)}\right)^c.$$

The proof of $\overline{\varpi}(\varrho^c) = \left(\underline{\varpi}(\varrho)\right)^c$ is by similar steps.

(4)

$$\begin{aligned}\underline{\varpi}(\varrho\cap\varkappa) &= \left\{\left\langle\psi, I^+_{\underline{\varpi}((\varrho\cap\varkappa))}(\psi), I^-_{\underline{\varpi}((\varrho\cap\varkappa))}(\psi), \left(B^+_{\underline{\varpi}((\varrho\cap\varkappa))}(\psi), B^-_{\underline{\varpi}((\varrho\cap\varkappa))}(\psi)\right)\right\rangle, \psi\in\Psi\right\}\\ &= \Bigg\{\Bigg\langle\psi, \bigwedge_{y\in\varpi^*(\psi)} I^+_{(\varrho\cap\varkappa)}(y), \bigvee_{y\in\varpi^*(\psi)} I^-_{(\varrho\cap\varkappa)}(y),\\ &\qquad\left(\bigvee_{y\in\varpi^*(\psi)} B^+_{(\varrho\cap\varkappa)}(y), \bigwedge_{y\in\varpi^*(\psi)} B^-_{(\varrho\cap\varkappa)}(y)\right)\Bigg\rangle, \psi\in\Psi\Bigg\}\\ &= \Bigg\{\Bigg\langle\psi, \bigwedge_{y\in\varpi^*(\psi)}\left[I^+_{(\varrho)}(y)\wedge I^+_{(\varkappa)}(y)\right], \bigvee_{y\in\varpi^*(\psi)}\left[I^-_{(\varrho)}(y)\vee I^-_{(\varkappa)}(y)\right],\\ &\qquad\left(\bigvee_{y\in\varpi^*(\psi)}\left[B^+_{(\varrho)}(y)\vee B^+_{(\varkappa)}(y)\right], \bigwedge_{y\in\varpi^*(\psi)}\left[B^-_{(\varrho)}(y)\wedge B^-_{(\varkappa)}(y)\right]\right)\Bigg\rangle, \psi\in\Psi\Bigg\}\\ &= \Bigg\{\Bigg\langle\psi, \left[\bigwedge_{y\in\varpi^*(\psi)} I^+_{(\varrho)}(y)\wedge\bigwedge_{y\in\varpi^*(\psi)} I^+_{(\varkappa)}(y)\right], \left[\bigvee_{y\in\varpi^*(\psi)} I^-_{(\varrho)}(y)\vee\bigvee_{y\in\varpi^*(\psi)} I^-_{(\varkappa)}(y)\right],\\ &\qquad\Bigg(\left[\bigvee_{y\in\varpi^*(\psi)} B^+_{(\varrho)}(y)\vee\bigvee_{y\in\varpi^*(\psi)} B^+_{(\varkappa)}(y)\right],\\ &\qquad\left[\bigwedge_{y\in\varpi^*(\psi)} B^-_{(\varrho)}(y)\wedge\bigwedge_{y\in\varpi^*(\psi)} B^-_{(\varkappa)}(y)\right]\Bigg)\Bigg\rangle, \psi\in\Psi\Bigg\}\\ &= \Big\{\Big\langle\psi, \left[I^+_{\underline{\varpi}(\varrho)}(y)\wedge I^+_{\underline{\varpi}(\varkappa)}(y)\right], \left[I^-_{\underline{\varpi}(\varrho)}(y)\vee I^-_{\underline{\varpi}(\varkappa)}(y)\right],\end{aligned}$$

$$\left(\left[B^{+}_{\underline{\varpi}(\varrho)}(y) \vee B^{+}_{\underline{\varpi}(\varkappa)}(y)\right], \left[B^{-}_{\underline{\varpi}(\varrho)}(y) \wedge B^{-}_{\underline{\varpi}(\varkappa)}(y)\right]\right)\right\rangle, \psi \in \Psi\right\}$$
$$= \underline{\varpi}(\varrho) \cap \underline{\varpi}(\varkappa).$$

In the same way, we can obtain $\overline{\varpi}(\varrho \cup \varkappa) = \overline{\varpi}(\varrho) \cup \overline{\varpi}(\varkappa)$.

(5) Therefore, $\varrho \subseteq \varrho \cup \varkappa$ and $\varkappa \subseteq \varrho \cup \varkappa$. According to part (1), it implies that $\underline{\varpi}(\varrho) \subseteq \underline{\varpi}(\varrho \cup \varkappa)$ and $\underline{\varpi}(\varkappa) \subseteq \underline{\varpi}(\varrho \cup \varkappa)$. This means that $\underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa) \subseteq \underline{\varpi}(\varrho \cup \varkappa)$. Similar, we can prove $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa)$. □

Theorem 3.6. *Let (Ψ, ϖ) be a CAS, if ϖ is a reflexive binary relation. Then the lower and upper rough CB approximation operator $\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)$, respectively, satisfy the following property:*

$$\underline{\varpi}(\varrho) \subseteq \varrho \subseteq \overline{\varpi}(\varrho), \quad \textit{for all} \quad \varrho \in \text{CBFS}(\Psi).$$

Proof.

$$\begin{aligned}
\underline{\varpi}(\varrho) &= \left\{\left\langle \psi, I^{+}_{\underline{\varpi}(\varrho)}(\psi), I^{-}_{\underline{\varpi}(\varrho)}(\psi), \left(B^{+}_{\underline{\varpi}(\varrho)}(\psi), B^{-}_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\} \\
&= \left\{\left\langle \psi, \bigwedge_{y \in \varpi^{*}(\psi)} I^{+}_{\varrho}(y), \bigvee_{y \in \varpi^{*}(\psi)} I^{-}_{\varrho}(y), \left(\bigvee_{y \in \varpi^{*}(\psi)} B^{+}_{\varrho}(y), \bigwedge_{y \in \varpi^{*}(\psi)} B^{-}_{\varrho}(y)\right)\right\rangle, \psi \in \Psi\right\} \\
&\subseteq \left\{\left\langle \psi, I^{+}_{\varrho}(\psi), I^{-}_{\varrho}(\psi), \left(B^{+}_{\varrho}(\psi), B^{-}_{\varrho}(\psi)\right)\right\rangle, \psi \in \Psi\right\} \\
&\subseteq \left\{\left\langle \psi, \bigwedge_{y \in \varpi^{*}(\psi)} I^{+}_{\varrho}(y), \bigvee_{y \in \varpi^{*}(\psi)} I^{-}_{\varrho}(y), \left(\bigvee_{y \in \varpi^{*}(\psi)} B^{+}_{\varrho}(y), \bigwedge_{y \in \varpi^{*}(\psi)} B^{-}_{\varrho}(y)\right)\right\rangle, \psi \in \Psi\right\} \\
&= \overline{\varpi}(\varrho).
\end{aligned}$$

This implies $\underline{\varpi}(\varrho) \subseteq \varrho \subseteq \overline{\varpi}(\varrho)$. □

3.2. CBF rough sets

Definition 3.7. *A CBF relation ϖ over Ψ can defined as:*

$$\varpi = \{\langle(\psi, y), I^{+}_{\varpi}(\psi, y), I^{-}_{\varpi}(\psi, y), (B^{+}_{\varpi}(\psi, y), B^{-}_{\varpi}(\psi, y))\rangle, (\psi, y) \in \Psi \times \Psi\}$$

where $I^{+}_{\varpi} : \Psi \times \Psi \to \text{int}([0,1])$, $I^{-}_{\varpi} : \Psi \times \Psi \to \text{int}([-1,0])$, $B^{+}_{\varpi} : \Psi \times \Psi \to [0,1]$, and $B^{-}_{\varpi} : \Psi \times \Psi \to [-1,0]$ are mappings for CBFR ϖ over $\Psi, I^{+}_{\varpi}(\psi, y), B^{+}_{\varpi}(\psi, y)$ the positive MD interval and single valued, respectively, which displays an object's level of delight (ψ, y) to the property related to ϖ, and it is negative MD interval and single valued respectively $I^{-}_{\varpi}(\psi, y), B^{-}_{\varpi}(\psi, y)$ represent the agreed degree to some implicit-counter-property related with ϖ.

Definition 3.8. *Let Ψ be a set of discourse, and let ϖ be a CBFR over Ψ, then the pair (Ψ, ϖ) is called the cubic bipolar approximation space (CBAS). For any $\varrho \in$ CBFS(Ψ). The lower and upper approximation of ϱ with respect to (Ψ, ϖ) denoted by $\underline{\varpi}(\varrho)$ and $\overline{\varpi}(\varrho)$ are two CBFS(Ψ) defined as follows, respectively:*

$$\underline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^{+}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\underline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^{-}_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\underline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^{+}_{\underline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$
$$\overline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^{+}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\overline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^{-}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\overline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^{+}_{\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{\overline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$

where

$$\left[\Gamma^+_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{y\in\Psi} \left\{(1 - [\Gamma^+_{L\varpi}(\psi, y), \Gamma^+_{u\varpi}(\psi, y)]) \vee \left[\Gamma^+_{L\varrho}(y), \Gamma^+_{u\varrho}(y)\right]\right\},$$

$$\left[\Gamma^-_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{y\in\Psi} \left\{([\Gamma^-_{L\varpi}(\psi, y), \Gamma^-_{u\varpi}(\psi, y)]) \wedge \left[\Gamma^-_{L\varrho}(y), \Gamma^-_{u\varrho}(y)\right]\right\},$$

$$\Gamma^+_{\underline{\varpi}(\varrho)}(\psi) = \bigvee_{y\in\Psi} \left\{\Gamma^+_{\varpi}(\psi, y) \wedge \Gamma^+_{\varrho}(y)\right\},$$

$$\Gamma^-_{\underline{\varpi}(\varrho)}(\psi) = \bigwedge_{y\in\Psi} \left\{-1 - \Gamma^-_{\varpi}(\psi, y) \vee \Gamma^-_{\varrho}(y)\right\}.$$

And, for $\overline{\varpi}(\varrho)$, we obtain

$$\left[\Gamma^+_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{y\in\Psi} \left\{([\Gamma^+_{L\varpi}(\psi, y), \Gamma^+_{u\varpi}(\psi, y)]) \wedge \left[\Gamma^+_{L\varrho}(y), \Gamma^+_{u\varrho}(y)\right]\right\},$$

$$\left[\Gamma^-_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{y\in\Psi} \left\{(-1 - [\Gamma^-_{L\varpi}(\psi, y), \Gamma^-_{u\varpi}(\psi, y)]) \vee \left[\Gamma^-_{L\varrho}(y), \Gamma^-_{u\varrho}(y)\right]\right\},$$

$$\Gamma^+_{\overline{\varpi}(\varrho)}(\psi) = \bigwedge_{y\in\Psi} \left\{1 - \Gamma^+_{\varpi}(\psi, y) \vee \Gamma^+_{\varrho}(y)\right\},$$

$$\Gamma^-_{\overline{\varpi}(\varrho)}(\psi) = \bigvee_{y\in\Psi} \left\{\Gamma^-_{\varpi}(\psi, y) \wedge \Gamma^-_{\varrho}(y)\right\}.$$

The pair $\left(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)\right)$ is referred to as a CBFRS of ϱ with respect to (Ψ, ϖ) and $\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)$ respectively known as lower and upper CB rough approximation operators. Additionally, if $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$, then ϱ is deemed defined.

Example 3.9. *Assume that $\Psi = \{\psi_1, \psi_2, \psi_3\}$ a finite universe and let ϖ be a CBFR over Ψ. Their representation is shown in Table 2.*

Table 2. The CBFR over Ψ.

	ψ_1	ψ_2	ψ_3
ψ_1	$\langle[0, 0.5], [-0.7, 0], (0.6, -0.8)\rangle$	$\langle[0.2, 0.8], [-0.7, -0.4], (0.7, -0.3)\rangle$	$\langle[0.15, 0.8], [-0.3, 0], (0.45, -0.5)\rangle$
ψ_2	$\langle[0.1, 0.3], [-0.4, 0], (0.9, -0.8)\rangle$	$\langle[0.3, 1], [-0.8, 0], (0.6, -0.8)\rangle$	$\langle[0.2, 0.7], [-0.5, 0], (0.6, -0.8)\rangle$
ψ_3	$\langle[0.2, 0.7], [-0.15, 0], (0.75, -0.9)\rangle$	$\langle[0.3, 1], [-0.7, -0.1], (0.7, -0.8)\rangle$	$\langle[0.1, 0.8], [-0.7, -0.2], (0.6, -0.2)\rangle$

Now, consider a $\varrho \in$ CBFS(Ψ) as follows:

$$\varrho = \{\langle\psi_1, [0, 0.8], [-0.4, 0], (0, 7, -0.2), \langle\psi_2, [0.1, 0.4], [-0.8, -0.1], (0.4, -0.7)\rangle, \langle\psi_3, [0.3, 0.7], [-0.85, -0.1], (0.3, -0.9)\rangle\}.$$

The lower and upper CB rough approximations of ϱ with respect to (Ψ, ϖ) are obtained by employing Definition 3.8 as follows:

$$\underline{\varpi}(\varrho) = \{\langle\psi_1, [0.3, 0.8], [-0.7, 0], (0.6, -0.2)\rangle, \langle\psi_2, [0.1, 0.7], [-0.7, -0.1], (-0.7, -0.2)\rangle, \langle\psi_3, [0.2, 0.8], [-0.4, 0], (0.45, -0.8)\rangle\},$$

$$\overline{\varpi}(\varrho) = \{\langle\psi_1, [0.2, 0.7], [-0.85, -0.1], (0.3, -0.8)\rangle, \langle\psi_2, [0.1, 0.7], [-0.85, -0.1], (0.3, -0.3)\rangle, \langle\psi_3, [0.1, 0.8], [-0.8, -0.1], (0.4, -0.51)\rangle\}.$$

The pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ *is knows as a CBFRS.*

Theorem 3.10. *Let* (Ψ, ϖ) *be a CBAS for any* $\varrho, \varkappa \in$ CBFS(Ψ)*, the lower and upper CB rough approximations operators* $\underline{\varpi}(\varrho)$ *and* $\overline{\varpi}(\varrho)$*, respectively. Verify the given conditions:*

(1) $\varrho \subseteq \varkappa \Rightarrow \underline{\varpi}(\varrho) \subseteq \underline{\varpi}(\varkappa)$ *and* $\overline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varkappa)$*,*
(2) $\underline{\varpi}(\varrho^c) = (\overline{\varpi}(\varrho))^c$ *and* $\overline{\varpi}(\varrho^c) = \left(\underline{\varpi}(\varrho)\right)^c$*,*
(3) $\underline{\varpi}(\varrho \cap \varkappa) = \underline{\varpi}(\varrho) \cap \underline{\varpi}(\varkappa)$ *and* $\overline{\varpi}(\varrho \cup \varkappa) = \overline{\varpi}(\varrho) \cup \overline{\varpi}(\varkappa)$*,*
(4) $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa)$ *and* $\underline{\varpi}(\varrho \cup \varkappa) \supseteq \underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa)$*.*

Proof. (1) This is clearly demonstrable by Definition 3.8.

(2) For $\underline{\varpi}(\varrho^c) = [\overline{\varpi}(\varrho)]^c$, we must demonstrate that for every $\psi \in \Psi$,

$$\underline{\left(I_\varrho^+\right)^c(\psi)} = \left(\overline{\left(I_\varrho^+\right)(\psi)}\right)^c, \quad \underline{\left(I_\varrho^-\right)^c(\psi)} = \left(\overline{\left(I_\varrho^-\right)(\psi)}\right)^c,$$

$$\underline{\left(B_\varrho^+\right)^c(\psi)} = \left(\overline{\left(B_\varrho^+\right)(\psi)}\right)^c \text{ and } \underline{\left(B_\varrho^-\right)^c(\psi)} = \left(\overline{\left(B_\varrho^-\right)(\psi)}\right)^c.$$

Now, consider

$$\begin{aligned}
\underline{\left(I_\varrho^+\right)^c(\psi)} &= \bigwedge_{y\in\Psi}\left\{(1 - [\Gamma^+_{L\varpi}(\psi, y), \Gamma^+_{u\varpi}(\psi, y)]) \vee \left(1 - \left[\Gamma^+_{L\varrho}(y), \Gamma^+_{u\varrho}(y)\right]\right)\right\} \\
&= 1 - \bigvee_{y\in\Psi}\left\{[\Gamma^+_{L\varpi}(\psi, y), \Gamma^+_{u\varpi}(\psi, y)] \wedge \left[\Gamma^+_{L\varrho}(y), \Gamma^+_{u\varrho}(y)\right]\right\} \\
&= 1 - \overline{\left(I_\varrho^+\right)(\psi)} \\
&= \left[\overline{\left(I_\varrho^+\right)(\psi)}\right]^c.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{\left(I_\varrho^-\right)^c(\psi)} &= \bigvee_{y\in\Psi}\left\{[\Gamma^-_{L\varpi}(\psi, y), \Gamma^-_{u\varpi}(\psi, y)] \wedge \left(-1 - \left[\Gamma^-_{L\varrho}(y), \Gamma^-_{u\varrho}(y)\right]\right)\right\} \\
&= -1 - \bigwedge_{y\in\Psi}\left\{(-1 - [\Gamma^-_{L\varpi}(\psi, y), \Gamma^-_{u\varpi}(\psi, y)]) \vee \left[\Gamma^-_{L\varrho}(y), \Gamma^-_{u\varrho}(y)\right]\right\} \\
&= -1 - \overline{\left(I_\varrho^-\right)(\psi)} \\
&= \left[\overline{\left(I_\varrho^-\right)(\psi)}\right]^c.
\end{aligned}$$

Also,

$$\begin{aligned}
\underline{\left(B_\varrho^+\right)^c(\psi)} &= \bigvee_{y\in\Psi}\left\{\Gamma^+_{\varpi}(\psi, y) \wedge \left(1 - \Gamma^+_{\varrho}(y)\right)\right\} \\
&= 1 - \bigwedge_{y\in\Psi}\left\{(1 - \Gamma^+_{\varpi}(\psi, y)) \vee \Gamma^+_{\varrho}(y)\right\} \\
&= 1 - \overline{\left(B_\varrho^+\right)(\psi)} \\
&= \left[\overline{\left(B_\varrho^+\right)(\psi)}\right]^c.
\end{aligned}$$

Finally,

$$\underline{\left(B_{\varrho}^{-}\right)^{c}(\psi)} = \bigwedge_{y\in\Psi}\left\{(-1-\Gamma_{\varpi}^{-}(\psi,y))\vee\left(-1-\Gamma_{\varrho}^{-}(y)\right)\right\}$$
$$= -1-\bigvee_{y\in\Psi}\left\{\Gamma_{\varpi}^{-}(\psi,y)\wedge\Gamma_{\varrho}^{-}(y)\right\}$$
$$= 1-\overline{\left(B_{\varrho}^{-}\right)(\psi)}$$
$$= \left[\overline{\left(B_{\varrho}^{-}\right)(\psi)}\right]^{c}.$$

Similar procedures are used in $\overline{\varpi}\left(\varrho^{c}\right) = \left[\underline{\varpi}(\varrho)\right]^{c}$.

(3)

$$\underline{\varpi}(\varrho\cap\varkappa) = \left\{\psi, \bigwedge_{y\in\Psi}\left\{(1-[\Gamma_{L\varpi}^{+}(\psi,y),\Gamma_{u\varpi}^{+}(\psi,y)])\vee\left[\Gamma_{L(\varrho\cap\varkappa)}^{+}(y),\Gamma_{u(\varrho\cap\varkappa)}^{+}(y)\right]\right\},\right.$$
$$\bigvee_{y\in\psi}\left\{[\Gamma_{L\varpi}^{-}(\psi,y),\Gamma_{u\varpi}^{-}(\psi,y)]\wedge\left[\Gamma_{L(\varrho\cap\varkappa)}^{-}(y),\Gamma_{u(\varrho\cap\varkappa)}^{-}(y)\right]\right\},$$
$$\left.\bigvee_{y\in\Psi}\left\{\Gamma_{\varpi}^{+}(\psi,y)\wedge\Gamma_{(\varrho\cap\varkappa)}^{+}(y)\right\}, \bigwedge_{y\in\Psi}\left\{(-1-\Gamma_{\varpi}^{-}(\psi,y))\vee\Gamma_{(\varrho\cap\varkappa)}^{-}(y)\right\}, \psi\in\Psi\right\}$$
$$= \left\{\psi, \bigwedge_{y\in\Psi}\left\{(1-[\Gamma_{L\varpi}^{+}(\psi,y),\Gamma_{u\varpi}^{+}(\psi,y)])\vee\left[\Gamma_{L\varrho}^{+}(y)\wedge\Gamma_{L\varkappa}^{+}(y),\Gamma_{u\varrho}^{+}(y)\wedge\Gamma_{u\varkappa}^{+}(y)\right]\right\},\right.$$
$$\bigvee_{y\in\psi}\left\{[\Gamma_{L\varpi}^{-}(\psi,y),\Gamma_{u\varpi}^{-}(\psi,y)]\wedge\left[\Gamma_{L\varrho}^{-}(y)\vee\Gamma_{L\varkappa}^{-}(y),\Gamma_{u\varrho}^{-}(y)\vee\Gamma_{u\varkappa}^{-}(y)\right]\right\},$$
$$\bigvee_{y\in\Psi}\left\{\Gamma_{\varpi}^{+}(\psi,y)\wedge\left(\Gamma_{(\varrho)}^{+}(y)\vee\Gamma_{(\varkappa)}^{+}(y)\right)\right\}, \bigwedge_{y\in\Psi}\{(1-\Gamma_{\varpi}^{-}(\psi,y))\vee$$
$$\left.\left(\Gamma_{\varrho}^{-}(y)\wedge\Gamma_{\varkappa}^{-}(y)\right)\right\}, \psi\in\Psi\right\}$$
$$= \left\{\psi, \bigwedge_{y\in\psi}\left\{(1-[\Gamma_{L\varpi}^{+}(\psi,y),\Gamma_{u\varpi}^{+}(\psi,y)])\vee\left[\Gamma_{L\varrho}^{+}(y),\Gamma_{u\varrho}^{+}(y)\right]\wedge\right.\right.$$
$$\bigwedge_{y\in\Psi}\{(1-[\Gamma_{L\varpi}^{+}(\psi,y),\Gamma_{u\varpi}^{+}(\psi,y])\vee[\Gamma_{L\varkappa}^{+}(y),\Gamma_{u\varkappa}^{+}(y)]\},$$
$$\bigvee_{y\in\Psi}\left\{[\Gamma_{L\varpi}^{-}(\psi,y),\Gamma_{u\varpi}^{-}(\psi,y)]\wedge\left[\Gamma_{L\varrho}^{-}(y),\Gamma_{u\varrho}^{-}(y)\right]\right\}\vee\bigvee_{y\in\Psi}\{[\Gamma_{L\varpi}^{-}(\psi,y),\Gamma_{u\varpi}^{-}(\psi,y)]\wedge$$
$$[\Gamma_{L\varkappa}^{-}(y),\Gamma_{u\varkappa}^{-}(y)]\}, \bigvee_{y\in\psi}\left\{\Gamma_{\varpi}^{+}(\psi,y)\wedge\Gamma_{\varrho}^{+}(y)\right\}\vee\bigvee_{y\in\Psi}\{\Gamma_{\varpi}^{+}(\psi,y)\wedge\Gamma_{\varkappa}^{+}(y)\},$$
$$\left.\bigwedge_{y\in\psi}\left\{(-1-\Gamma_{\varpi}^{-}(\psi,y))\vee\Gamma_{\varrho}^{-}(y)\right\}\wedge\bigwedge_{y\in\psi}\left\{(-1-\Gamma_{\varpi}^{-}(\psi,y))\vee\Gamma_{\varkappa(y)}^{-}\right\}, \psi\in\Psi\right)$$
$$=\underline{\varpi}(\varrho)\cap\underline{\varpi}(\varrho).$$

Likewise, we can determine that $\overline{\varpi}(\varrho\cup\varkappa) = \overline{\varpi}(\varrho)\cup\overline{\varpi}(\varkappa)$.

(4) Since $\varrho \cap \varkappa \subseteq \varrho$ and $\varrho \cap \varkappa \subseteq \varkappa$ so, regarding part (1), it implies that $\overline{\varpi}(\varrho \cap \varkappa) \subset \overline{\varpi}(\varrho)$ and $\overline{\varpi}(\varrho \cap \varkappa) \subset \overline{\varpi}(\varkappa)$. This means that $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa)$.

In a similar vein, we can prove $\underline{\varpi}(\varrho \cup \varkappa) \supseteq \underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa)$. □

4. Cubic bipolar fuzzy SSs

The notion of CBFSS is an extension of BFSS that was introduced in [21]. The following are some fundamental findings about the approximation space under the CBFSS paradigm. These findings provide the foundation for creating lower and higher approximations, border regions, and decision rules in this hybrid context.

Definition 4.1. *CBSS over Ψ is a pair $(f, \aleph)$, where f is a mapping provided by*

$$f : \aleph \longrightarrow \mathrm{CBFS}(\Psi).$$

Clearly, a CBFSS is a parameterized set of cubic bipolar subset of Ψ for $\ell \in \aleph$, $f(\ell)$ is regarded as the set of ℓ-approximate elements of the CBSS $(f, \aleph)$.

Example 4.2. *Let $\Psi = \{\psi_1, \psi_2, \psi_3\}$ be the set of three schools being examined and $\aleph = \{\ell_1 =$ costly, $\ell_2 =$ beautiful building, $\ell_3 =$ Excellent education$\}$ be the set of parameters. Then the CBFSS is described by*

$$\begin{aligned} f(\ell_1) = \{ & \langle\psi_1, [0.3, 1], [-0.25, 0], (0.8, -0.45)\rangle, \langle\psi_2, [0, 0.32], [-0.82, -0.3], (0.73, -0.15)\rangle, \\ & \langle\psi_3, [0, 0.8], [-0.9, -0.7], (1, -0.2)\rangle\}, \end{aligned}$$

$$\begin{aligned} f(\ell_2) = \{ & \langle\psi_1, [0, 0.5], [-0.23, -0.15], (0.18, -0.31)\rangle, \langle\psi_2, [0.51, 1], [-0.91, 0], (0.73, -0.83)\rangle, \\ & \langle\psi_3, [0.15, 0.9], [-0.81, 0], (0.91, -0.52)\rangle\}, \\ f(\ell_3) = \{ & \langle\psi_1, [0, 0.54], [-0.12, 0], (0.33, -0.15)\rangle, \langle\psi_3, [0.12, 0.95], [-0.72, 0], (0.17, -0.8)\rangle\}. \end{aligned}$$

Definition 4.3. *Let $(f_1, \aleph_1), (f_2, \aleph_2) \in \mathrm{CBFSS}(\Psi)$, $(f_1, \aleph_1)$ be a CBFSS subset of $(f_2, \aleph_2)$, if $\aleph_1 \subseteq \aleph_2$ and $f_1(\ell) \subseteq f_2(\ell)$ for all $\ell \in \aleph_1$.*

Definition 4.4. *A CBFSS $(f, \aleph)$ is named as a null CBFSS and indicated by the empty set ϕ if for all $\ell \in \aleph$, $f(\ell) = \emptyset$, and it is called an absolute CBFSS if for all $\ell \in \aleph$, $f(\ell) = \mathrm{CBFSS}(\Psi)$.*

Definition 4.5. *The complement of a CBFSS $(f, \aleph)$ is indicated by $(f, \aleph)^c$ and defined by $(f, \aleph)^c = (f^c, \aleph)$, where $f^c : \aleph \to \mathrm{CBFS}(\Psi)$ is provided by mapping*

$$f^c(\ell) = (f(\ell))^c \quad \text{for all} \quad \forall \in \aleph.$$

Remark 4.6. *Evidently, $((f, \ell)^c)^c = (f, \ell)$.*

Definition 4.7. *The intersection of two CBFSSs $(f_1, \aleph_1)$ and $(f_2, \aleph_2)$ is a $\mathrm{CBFSS}(g, L)$ where $L = \aleph_1 \cap \aleph_2 \neq \emptyset$ and $g : L \to \mathrm{CBFS}(\Psi)$ is presented by $g(\ell) = f_1(\ell) \cap f_2(\ell)\ \forall \ell \in L$.*

Definition 4.8. *The union of two CBFSSs $(f_1, \aleph_1)$ and $(f_2, \aleph_2)$ through a universe Ψ is a $\mathrm{CBFSS}(g, L)$ where $L = \aleph_1 \cup \aleph_2$ and $g : L \to \mathrm{CBFS}(\Psi)$ is given by*

$$\left.\begin{array}{ll} g(\ell) = f_1(\ell), & \text{if } \ell \in \aleph_1 \backslash \aleph_2, \\ g(\ell) = f_2(\ell), & \text{if } \ell \in \aleph_2 \backslash \aleph_1, \\ g(\ell) = f_1(\ell) \cup f_2(\ell), & \text{if } \ell \in \aleph_1 \cap \aleph_2. \end{array}\right\}$$

Theorem 4.9. *Let* $(f_1, \aleph_1)$ *and* $(f_2, \aleph_2)$ *be two CBFSSs. Then*

(1) $(f_1, \aleph_1) \cup (f_1, \aleph_1) = (f_1, \aleph_1)$,
(2) $(f_1, \aleph_1) \cap (f_1, \aleph_1) = (f_1, \aleph_1)$,
(3) $(f_1, \aleph_1) \cap \varnothing = \varnothing$, *where* $\varnothing$ *is a null CBFSS,*
(4) $(f_1, \aleph_1) \cup \varnothing = (f_1, \aleph_1)$,
(5) $(f_1, \aleph_1) \cup \Psi = \Psi$,
(6) $(f_1, \aleph_1) \cap \Psi = (f_1, \aleph_1)$,
(7) $(f_1, \aleph_1) \cap (f_2, \aleph_2) = (f_2, \aleph_2) \cap (f_1, \aleph_1)$,
(8) $(f_1, \aleph_1) \cup (f_2, \aleph_2) = (f_2, \aleph_2) \cup (f_1, \aleph_1)$.

5. Rough set model based on CBFSSs

5.1. Rough CBFSSs

Motivated by the notion of RCBFS in Section 3, we will present the idea of rough CBFSSs by merging the RCBFS and the CSR from Ψ to $\aleph$, and explore the characteristics of rough CBS approximation operators.

Definition 5.1. *Suppose that* Ψ *is a discourse set and let* $\aleph$ *be a set of parameters. For any CSR* ϖ *over* $\Psi \times \aleph$, *an SVF* $\varpi^* : \Psi \to P(\aleph)$ *is given by*

$$\varpi^*(\psi) = \{\ell \in \aleph : (\psi, \ell) \in \varpi, \psi \in \Psi\}.$$

If for every ψ *in* Ψ, $\varpi^*(\psi) \neq \varphi$, *then* ϖ *is said to be serial and it is known that the triple* $(\Psi, \aleph, \varpi)$ *is a CSAS.* $\underline{\varpi}(\varrho)$ *and* $\overline{\varpi}(\varrho)$, *which stand for the lower and upper soft approximation of* ϱ *with regard to* $(\Psi, \aleph, \varpi)$ *for any* $\varrho \subseteq \aleph$, *are defined as follows:*

$$\underline{\varpi}(\varrho) = \{\langle \psi \in \Psi, \varpi^*(\psi) \subseteq \varrho\},$$
$$\overline{\varpi}(\varrho) = \{\langle \psi \in \Psi, \varpi^*(\psi) \cap \varrho \neq \varnothing\}.$$

It is argued that the pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ *is a crisp soft RS, and* $\underline{\varpi}, \overline{\varpi} : P(\aleph) \to P(\Psi)$ *are the lower and upper crisp soft rough approximation operators, respectively. Moreover,* ϱ *is referred to as a explodable if* $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$.

Definition 5.2. *Let* $(\Psi, \aleph, \varpi)$ *be a CSAS. For any* $\varrho \in \mathrm{CBFS}(\aleph)$ *the lower and upper soft approximation of* ϱ *with respect to* $(\Psi, \aleph, \varpi)$, *respectively, presented by* $\underline{\varpi}(\varrho)$, *and* $\overline{\varpi}(\varrho)$, *are described as:*

$$\underline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^+_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\underline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^-_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\underline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^+_{\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{\underline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$
$$\overline{\varpi}(\varrho) = \left\{\left\langle \psi, \left[\Gamma^+_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\overline{\varpi}(\varrho)}(\psi)\right], \left[\Gamma^-_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\overline{\varpi}(\varrho)}(\psi)\right], \left(\Gamma^+_{\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{\overline{\varpi}(\varrho)}(\psi)\right)\right\rangle, \psi \in \Psi\right\},$$

where

$$\left[\Gamma^+_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{\ell \in \varpi^*(\psi)} \left[\Gamma^+_{L\varrho}(\ell), \Gamma^+_{u\varrho}(\ell)\right],$$
$$\left[\Gamma^-_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\underline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{\ell \in \varpi^*(\psi)} \left[\Gamma^-_{L\varrho}(\ell), \Gamma^-_{u\varrho}(\ell)\right],$$
$$\Gamma^+_{\underline{\varpi}(\varrho)}(\psi) = \bigvee_{\ell \in \varpi^*(\psi)} \Gamma^+_{\varrho}(\ell), \quad \textit{and} \quad \Gamma^-_{\underline{\varpi}(\varrho)}(\psi) = \bigwedge_{\ell \in \varpi^*(\psi)} \Gamma^-_{\varrho}(\ell).$$

Likewise, for $\overline{\varpi}(\varrho)$ *we get*

$$\left[\Gamma^{+}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{+}_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigvee_{\ell\in\varpi^{*}(\psi)} \left[\Gamma^{+}_{L\varrho}(\ell), \Gamma^{+}_{u\varrho}(\ell)\right],$$
$$\left[\Gamma^{-}_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^{-}_{u\overline{\varpi}(\varrho)}(\psi)\right] = \bigwedge_{\ell\in\varpi^{*}(\psi)} \left[\Gamma^{-}_{L\varrho}(\ell), \Gamma^{-}_{u\varrho}(\ell)\right],$$
$$\Gamma^{+}_{\overline{\varpi}(\varrho)}(\psi) = \bigwedge_{\ell\in\varpi^{*}(\psi)} \Gamma^{+}_{\varrho}(\ell), \quad and \quad \Gamma^{-}_{\overline{\varpi}(\varrho)}(\psi) = \bigvee_{\ell\in\varpi^{*}(\psi)} \Gamma^{-}_{\varrho}(\ell).$$

The pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ *is referred to as a rough CBFSS of* ϱ *with respect to* $(\Psi, \aleph, \varpi)$*. Additionally, if* $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$*, then* ϱ *is called definable.*

Example 5.3. *Let* $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ *be a universe and* $\aleph = \{\ell_1, \ell_2, \ell_3\}$ *be the collection of specifications. The SS* $(f, \aleph)$ *is represented in Table 3.*

Table 3. The SS $(f, \aleph)$.

$(f, \aleph)$	ψ_1	ψ_2	ψ_3	ψ_4
ℓ_1	1	0	1	1
ℓ_2	0	1	1	0
ℓ_3	1	1	1	1

Then, a CSR ϖ *over* $-\Psi \times \aleph$ *is given by:*

$$\varpi = \{(\psi_1, \ell_1), (\psi_1, \ell_3), (\psi_2, \ell_2), (\psi_2, \ell_3), (\psi_3, \ell_1), (\psi_1, \ell_2), (\psi_3, \ell_3), (\psi_4, \ell_1), (\psi_4, \ell_3)\}.$$

Consider $\varrho \in$ CBFS($\aleph$) as follows:

$$\varrho = \{\langle \ell_1, [0.12, 0.9], [-0.1, -0.1], (0.7, -0.2)\rangle, \langle \ell_2, [0, 0.8], [-0.7, -0.3], (0.15, -0.3)\rangle, \langle \ell_3, [0.1, 0.4], [-0.5, -0.2], (0.8, -0.6)\rangle\}.$$

The lower and upper soft approximation of ϱ with respect to $(\Psi, \aleph, \varpi)$ are obtained by applying Definition 5.2 as follows:

$$\underline{\varpi}(\varrho) = \{\langle \psi_1, [0.1, 0.4], [-0.5, -0.1], (0.8, -0.6)\rangle, \langle \psi_2, [0, 0.4], [-0.5, -0.2](0.8, -0.6)\rangle, \langle \psi_3, [0, 0.4], [-0.5, -0.1], (0.8, -0.6)\rangle, \langle \psi_4, [0.1, 0.4], [-0.5, -0.1], (0.8, -0.6)\rangle\},$$

$$\overline{\varpi}(\varrho) = \{\langle \psi_1, [0.12, 0.9], [-0.9, -0.2], (0.7, -0.2)\rangle, \langle \psi_2, [0.1, 0.8], [-0.7, -0.8], (0.15, -0.3)\rangle, \langle \psi_3, [0.12, 0.9], [-0.9, -0.3], (0.15, -0.2)\rangle \langle \psi_4, [0.12, 0.9], [-0.9, -0.2], (0.7, -0.2)\rangle\}.$$

The pair $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ is known as a soft rough CBFS of ϱ.

Theorem 5.4. *Let* $(\Psi, \aleph, \varpi)$ *CSAS. For any* $\varrho, \varkappa \in$ CBFS($\aleph$)*, the lower and upper crisp soft Cubic bipolar approximation operators* $\underline{\varpi}(\varrho)$ $\overline{\varpi}(\varrho)$*, respectively. Verify the following conditions:*

(1) $\underline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varrho)$, $\forall \varrho \in$ CBFS(Ψ),
(2) $\varrho \subseteq \varkappa \Rightarrow \underline{\varpi}(\varrho) \subseteq \underline{\varpi}(\varkappa)$ *and* $\overline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varkappa)$,
(3) $\underline{\varpi}(\varrho^c) = (\overline{\varpi}(\varrho))^c$, $\overline{\varpi}(\varrho^c) = (\underline{\varpi}(\varrho))^c$,

(4) $\underline{\varpi}(\varrho \cap \varkappa) = \underline{\varpi}(\varrho) \cap \underline{\varpi}(\varkappa)$, $\overline{\varpi}(\varrho \cup \varkappa) = \overline{\varpi}(\varrho) \cup \overline{\varpi}(\varkappa)$,
(5) $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa)$, $\underline{\varpi}(\varrho \cup \varkappa) \supset \underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa)$.

Proof. The proof of Theorem 3.5 follows. □

Theorem 5.5. *Let* $(\Psi, \aleph, \varpi)$ *be a CSAS, if* ϖ *is a reflexive relation. Then the lower and upper soft rough CB approximation* $\underline{\varpi}(\varrho)$ *and* $\overline{\varpi}(\varrho)$ *respectively, have the characteristics:*

$$\underline{\varpi}(\varrho) \subseteq \varrho \subseteq \overline{\varpi}(\varrho), \quad \forall \varrho \in \mathrm{CBFS}(\Psi).$$

Proof. This is line with the proof of Theorem 3.6. □

5.2. CBFS rough sets

Definition 5.6. *A CB soft relation* ϖ *over* $\Psi \times \aleph$*, is an object having the form:*

$$\varpi = \left\{ \left\langle (\psi, \ell), I^+_{\varpi}(\psi, \ell), I^-_{\varpi}(\psi, \ell), (B^+_{\varpi}(\psi, \ell), B^-_{\varpi}(\psi, \ell)) \right\rangle, (\psi, \ell) \in \Psi \times \aleph \right\}.$$

Clearly, $\varpi \in \mathrm{CBFS}(\Psi \times \aleph)$ *which shows that satisfaction unit interval degree of an elemet* (ψ, ℓ) *to the set of attribute equivalent to* ϖ *and to counter-property associated with* ϖ*.*

Definition 5.7. *Let* Ψ *be a discourse set, and let* $\aleph$ *be a set of parameters, for any* $\varpi \in \mathrm{CBSR}$ *over* $\Psi \times \aleph$*, then the triple* $(\Psi, \aleph, \varpi)$ *is called a cubic bipolar soft approximation space (CBSAS). For any* $\varrho \in \mathrm{CBFS}(\aleph)$*, the lower and upper approximation of* ϱ *with respect to* $(\Psi, \aleph, \varpi)$*is a* $\mathrm{CBFS}(\Psi)$*, defined as follows:*

$$\underline{\varpi}(\varrho) = \left\{ \left\langle \psi, \left[\Gamma^+_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\underline{\varpi}(\varrho)}(\psi) \right], \left[\Gamma^-_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\underline{\varpi}(\varrho)}(\psi) \right], \left(\Gamma^+_{\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{\underline{\varpi}(\varrho)}(\psi) \right) \right\rangle, \psi \in \Psi \right\},$$
$$\overline{\varpi}(\varrho) = \left\{ \left\langle \psi, \left[\Gamma^+_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\overline{\varpi}(\varrho)}(\psi) \right], \left[\Gamma^-_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\overline{\varpi}(\varrho)}(\psi) \right], \left(\Gamma^+_{\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{\overline{\varpi}(\varrho)}(\psi) \right) \right\rangle, \psi \in \Psi \right\},$$

where

$$\left[\Gamma^+_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\underline{\varpi}(\varrho)}(\psi) \right] = \bigwedge_{\ell \in \aleph} \left\{ (1 - [\Gamma^+_{L\varpi}(\psi, \ell), \Gamma^+_{u\varpi}(\psi, \ell)]) \vee \left[\Gamma^+_{L\varrho}(\ell), \Gamma^+_{u\varrho}(\ell) \right] \right\},$$
$$\left[\Gamma^-_{L\underline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\underline{\varpi}(\varrho)}(\psi) \right] = \bigvee_{\ell \in \aleph} \left\{ ([\Gamma^-_{L\varpi}(\psi, \ell), \Gamma^-_{u\varpi}(\psi, \ell)]) \wedge \left[\Gamma^-_{L\varrho}(\ell), \Gamma^-_{u\varrho}(\ell) \right] \right\},$$
$$\Gamma^+_{\underline{\varpi}(\varrho)}(\psi) = \bigvee_{\ell \in \aleph} \left\{ \Gamma^+_{\varpi}(\psi, \ell) \wedge \Gamma^+_{\varrho}(\ell) \right\} \text{ and } \Gamma^-_{\underline{\varpi}(\varrho)}(\psi) = \bigwedge_{\ell \in \aleph} \left\{ -1 - \Gamma^-_{\varpi}(\psi, \ell) \vee \Gamma^-_{\varrho}(\ell) \right\}.$$

And, for $\overline{\varpi}(\varrho)$*, we obtain*

$$\left[\Gamma^+_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^+_{u\overline{\varpi}(\varrho)}(\psi) \right] = \bigvee_{\ell \in \aleph} \left\{ ([\Gamma^+_{L\varpi}(\psi, \ell), \Gamma^+_{u\varpi}(\psi, \ell)]) \wedge \left[\Gamma^+_{L\varrho}(\ell), \Gamma^+_{u\varrho}(\ell) \right] \right\},$$
$$\left[\Gamma^-_{L\overline{\varpi}(\varrho)}(\psi), \Gamma^-_{u\overline{\varpi}(\varrho)}(\psi) \right] = \bigwedge_{\ell \in \aleph} \left\{ (-1 - [\Gamma^-_{L\varpi}(\psi, \ell), \Gamma^-_{u\varpi}(\psi, \ell)]) \vee \left[\Gamma^-_{L\varrho}(\ell), \Gamma^-_{u\varrho}(\ell) \right] \right\},$$
$$\Gamma^+_{\overline{\varpi}(\varrho)}(\psi) = \bigwedge_{\ell \in \aleph} \left\{ 1 - \Gamma^+_{\varpi}(\psi, \ell) \vee \Gamma^+_{\varrho}(\ell) \right\},$$
$$\Gamma^-_{\overline{\varpi}(\varrho)}(\psi) = \bigvee_{\ell \in \aleph} \left\{ \Gamma^-_{\varpi}(\psi, \ell) \wedge \Gamma^-_{\varrho}(\ell) \right\}.$$

The $(\underline{\varpi}(\varrho), \overline{\varpi}(\varrho))$ *is pointed to as a CBFSRS of* ϱ *with respect to* $(\Psi, \aleph, \varpi)$*, and* $\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)$ *are named as lower and upper CB soft rough approximation tools. Moreover, if* $\underline{\varpi}(\varrho) = \overline{\varpi}(\varrho)$*, then* ϱ *is known as definable.*

Theorem 5.8. *Let $(\Psi, \aleph, \varpi)$ be a CBSAS, for any $\varrho, \varkappa$ belongs to* CBFS(Ψ) *then lower and upper cubic bipolar soft approximation operators denotes as $\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)$ respectively. Verify the following properties:*

(1) $\varrho \subseteq \varkappa \Rightarrow \underline{\varpi}(\varrho) \subseteq \underline{\varpi}(\varkappa)$ and $\overline{\varpi}(\varrho) \subseteq \overline{\varpi}(\varkappa)$,
(2) $\underline{\varpi}(\varrho^c) = (\overline{\varpi}(\varrho))^c$ and $\overline{\varpi}(\varrho^c) = \left(\underline{\varpi}(\varrho)\right)^c$,
(3) $\underline{\varpi}(\varrho \cap \varkappa) = \underline{\varpi}(\varrho) \cap \underline{\varpi}(\varkappa)$ and $\overline{\varpi}(\varrho \cup \varkappa) = \overline{\varpi}(\varrho) \cup \overline{\varpi}(\varkappa)$,
(4) $\overline{\varpi}(\varrho \cap \varkappa) \subseteq \overline{\varpi}(\varrho) \cap \overline{\varpi}(\varkappa)$ and $\underline{\varpi}(\varrho \cup \varkappa) \supseteq \underline{\varpi}(\varrho) \cup \underline{\varpi}(\varkappa)$.

Proof. Similar to proof of Theorem 3.10. □

Definition 5.9. *Assume that ϱ be a* CBFS(Ψ) *and let*

$$\gamma = \langle [A, B], [C, D], \alpha, \beta \rangle \in \text{int}([0, 1] \times \text{int}([-1, 0]) \times [0, 1) \times (-1, 0].$$

Then, we describes the γ-cut level set of ϱ (with order R) as a classical set:

$$\varrho^{\alpha,[A,B]}_{\beta,[C,D]} = \left\{\psi \in \Psi, I^+_\varrho(\psi) \succeq [A, B], I^-_\varrho(\psi) \preceq [C, D], B^+_\varrho(\psi) \leqslant \alpha, B^-_\varrho(\psi) \geqslant \beta\right\}.$$

"$\succeq$" and "$\preceq$" are any order of intervals (reflexive, symmetric and transitive) (see [33]).

Additionally, $\varrho^{[A,B]} = \left\{\psi \in \Psi / I^+_\varrho(\psi) \succeq [A, B]\right\}$ and $\varrho^{[A,B]^+} = \left\{\psi \in \Psi \mid I^+_\varrho(\psi) \succ [A, B]\right\}$ are known as the $[A, B]$- level-cut set and the strong $[A, B]$-level cut set of positive interval MD generated by ϱ, and $\varrho_{[C,D]} = \left\{\psi \in \Psi / I^-_\varrho(\psi) \preceq [C, D]\right\}$ and $\varrho_{[C,D]^+} = \{\psi \in \Psi / I^-_\varrho(\psi) \prec [C, D]\}$ are respectively, defined to as the $[C, D]$-strong level cut set and strong $[C, D]$-level cut set of negative interval MD.

Likewise, we can obtain the level cut and level cut strong of positive and negative single MD.

Example 5.10. *Consider a CBF set to demonstrate the definition.*

$$\begin{aligned}\varrho = \{&\langle \psi_1, [0, 0.7], [-0.2, 0], (0.88, -0.13) \rangle, \langle \psi_2, [0.2, 0.91], [-0.23, -0.11], (0.19, -0.32) \rangle, \\ &\langle \psi_3, [0.5, 1], [-0.8, -0.2], (0.98, -0.7) \rangle, \langle \psi_4, [0.1, 0.8], [-0.65, -0.1], (0.68, -0.3) \rangle, \\ &\langle \psi_5, [0, 1], [-0.4, -0.2], (0.3, -0.2) \rangle\}.\end{aligned}$$

For $[A, B] = [0.2, 0.8]$, $[C, D] = [-0.12, -0.05]$, $\alpha = 0.99$ and $\beta = -0.81$, the γ-cut level set of ϱ for R order and lattice order for the intervals is given as

$$\varrho^{\alpha,[A,B]}_{B,[C,D]} = \{\psi_2, \psi_3\}.$$

Definition 5.11. *Let ϖ be a CBFSR, from Ψ to the parameters set $\aleph$, denoted as*

$$\begin{aligned}\varpi^{[A,B]} &= \{(\psi, \ell) / I^+_\varpi(\psi, \ell) \succeq [A, B]\}, \\ \varpi^{[A,B]}(\psi) &= \{\ell \in \aleph / I^+_\varpi(\psi, \ell) \succeq [A, B]\}, \\ \varpi^{[A,B]^+} &= \{(\psi, \ell) / I^+_\varpi(\psi, \ell) \succ [A, B]\}, \\ \varpi^{[A,B]^+}(\psi) &= \{\ell \in \aleph / I^+_\varpi(\psi, \ell) \succ [A, B]\}.\end{aligned}$$

In a similar way, we can define $\varpi^\alpha, \varpi^\alpha(\psi), \varpi^{\alpha^+}$, and $\varpi^{\alpha^+}(\psi)$, as well as $\varpi_\beta, \varpi_\beta(\psi), \varpi_{\beta^+}$, and $\varpi_{\beta^+}(\psi)$. Moreover, $\varpi_{[C,D]}, \varpi_{[C,D]}(\psi), \varpi_{[C,D]^+}$ and $\varpi_{[C,D]^+}(\psi)$. Then, $\varpi^{[A,B]}$, $\varpi^{[A,B]^+}$, ϖ^α, ϖ^{α^+}, ϖ_β, ϖ_{β^+}, $\varpi_{[C,D]}$, $\varpi_{[C,D]^+}$ are CSR on $\Psi \times \aleph$.

The crisp soft rough approximation operator can be used to describe the positive interval MD of the upper CB soft rough approximation operator, according to Theorem 5.12.

Theorem 5.12. *Let $(\Psi, \aleph, \varpi)$ be a CBSAS, and $\varrho \in \mathrm{CBFS}(\aleph)$. Then, the positive interval MD of upper CB soft rough approximation operator can be represented, $\forall \psi \in \Psi$: as follow*

(1)

$$
\begin{aligned}
I^{+}_{\overline{\varpi}(\varrho)}(\psi) &= \bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]}\right)(\psi)\right\} \\
&= \bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]^+}\right)(\psi)\right\} \\
&= \bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]^+}}\left(\varrho^{[A,B]}\right)(\psi)\right\} \\
&= \bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]^+}}\left(\varrho^{[A,B]^+}\right)(\psi)\right\}
\end{aligned}
$$

and, moreover, for any $[A,B] \in \mathrm{int}[0,1]$,

(2)

$$
[\overline{\varpi}(\varrho)]^{[A,B]^+} \subseteq \overline{\varpi^{[A,B]^+}}\left(\varrho^{[A,B]^+}\right) \subseteq \overline{\varpi^{[A,B]^+}}\left(\varrho^{[A,B]}\right) \subseteq \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]}\right) \subseteq [\overline{\varpi}(\varrho)]^{[A,B]}.
$$

Proof. For any $\psi \in \Psi$, we obtain

(1)

$$
\begin{aligned}
&\bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]}\right)(\psi)\right\} \\
&= \sup\left\{[A,B] \in \mathrm{int}[0,1]/\psi \in \overline{\varpi^{[A,B)}}\left(\varrho^{[A,B]}\right)\right\} \\
&= \sup\left\{[A,B] \in \mathrm{int}[0,1]/\varpi^{[A,B]}(\psi) \cap \varrho^{[A,B]} \neq \varnothing\right\} \\
&= \sup\left\{[A,B] \in \mathrm{int}[0,1]/\exists \ell \in \aleph\left[\ell \in \varpi^{[A,B]}(\psi), \ell \in \varrho^{[A,B]}\right]\right\} \\
&= \sup\left\{[A,B] \in \mathrm{int}[0,1]/I^{+}_{\varpi}(\psi,\ell) \succeq_{\mathrm{Lo}} [A,B], I^{+}_{\varrho}(\ell) \succeq_{\mathrm{Lo}} [A,B]\right\} \\
&= \bigvee_{\ell\in\aleph}\left\{I^{+}_{\varpi}(\psi,\ell) \wedge I^{+}_{\varrho}(\ell)\right\} = I^{+}_{\overline{\varpi}(\varrho)}(\psi).
\end{aligned}
$$

(2) It is not difficult to confirm that

$$
\overline{\varpi^{[A,B]^+}}(\varrho^{[A,B]^+}) \subseteq \overline{\varpi^{[A,B]^+}}(\varrho^{[A,B]}) \subseteq \overline{\varpi^{[A,B]}}(\varrho^{[A,B]})
$$

and we have to demonstrate that

$$
[\overline{\varpi}(\varrho)]^{[A,B]^+} \subseteq \overline{\varpi^{[A,B]^+}}(\varrho^{[A,B]^+}) \text{ and } \overline{\varpi^{[A,B]}}(\varrho^{[A,B]}) \subseteq [\overline{\varpi}(\varrho)]^{[A,B]}.
$$

In fact, using lattice order $\forall \psi \in [\overline{\varpi}(\varrho)]^{[A,B]^+}$, we have $I^{+}_{\overline{\varpi}(\varrho)}(\psi) \succ_{\mathrm{Lo}} [A,B]$. According to Definition 5.7, $\bigvee_{\ell\in\aleph}\{I^{+}_{\varpi}(\psi,\ell) \wedge I^{+}_{\varrho}(\ell)\} \succ_{\mathrm{Lo}} [A,B]$ holds. Then $\exists \ell_0 \in \aleph$, such that $I^{+}_{\varpi}(\psi,\ell_0) \wedge I^{+}_{\varrho}(\delta_0) \succ_{\mathrm{Lo}} [A,B]$; that is, $I^{+}_{\varpi}(\psi,\delta_0) \succ_{\mathrm{Lo}} [A,B]$ and $I^{+}_{\varrho}(\delta_0) \succ_{\mathrm{Lo}} [A,B]$. Thus, $\delta_0 \in \varpi^{[A,B]^+}(\psi)$ and $\delta_0 \in \varrho^{[A,B]^+}$. This means, $\varpi^{[A,B]^+}(\psi) \cap \varrho^{[A,B]^+} \neq \varnothing$, by Definition 5.1, we have $\psi \in \overline{\varpi^{[A,B]^+}}(\varrho^{[A,B]^+})$. Hence, $[\overline{\varpi}(\varrho)]^{[A,B]^+} \subseteq \varpi^{[A,B]^+}(\varrho^{[A,B]^+})$, on the other hand, for any $\psi \in \varpi^{[A,B]}(\varrho^{[A,B]})$, we have $\overline{\varpi^{[A,B]}}(\varrho^{[A,B]})(\psi) = [1,1]$. Since

$$
I^{+}_{\overline{\varpi}(\varrho)}(\psi) = \bigvee_{[A,B]\in \mathrm{int}[0,1]} \left\{[A,B] \wedge \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]}\right)(\psi)\right\} \geqslant [A,B] \wedge \overline{\varpi^{[A,B]}}\left(\varrho^{[A,B]}\right)(\psi) = [A,B],
$$

we obtain $\psi \in [\overline{\varpi}(\varrho)]^{[A,B]}$ and hence $\overline{\varpi^{[A,B]}}(\varrho^{[A,B]}) \subseteq [\overline{\varpi}(\varrho)]^{[A,B]}$. □

6. Application of CBFSRSs in decision making

This portion explores the creation of the CBFSRS sets-based DM technique employing the lower and upper CB soft approximation operators. We study a DM problem with alternatives and criteria, i.e., various persons have diverse opinions about the features of the same parameter with respect to a given choice assessment problem. Thus, we can design an optimal normal decision element ϱ that satisfies the decision maker's multiple needs depending on assumptions.

To rank the CBF numbers in the DM process, we determine the cosine similarity measure (CSM) for cubic bipolar elements based on the concept given in [43].

Definition 6.1. *Let ϱ be a CBFS, over Ψ and ϱ^* be the ideal CBFS, with the CSM is defined as*

$$\delta(\varrho,\varrho^*) = \frac{\begin{pmatrix} \Gamma^+_{L\varrho}(\psi)\Gamma^+_{L\varrho^*}(\psi) + \Gamma^+_{u\varrho}(\psi)\Gamma^+_{u\varrho^*}(\psi) + \Gamma^-_{L\varrho}(\psi)\Gamma^-_{L\varrho^*}(\psi) + \Gamma^-_{u\varrho}(\psi)\Gamma^-_{u\varrho^*}(\psi) \\ +\Gamma^+_{\varrho}(\psi)\Gamma^+_{\varrho^*}(\psi) + \Gamma^-_{\varrho}(\psi)\Gamma^-_{\varrho^*}(\psi) \end{pmatrix}}{\sqrt{\begin{array}{l} (\Gamma^+_{L\varrho}(\psi))^2 + (\Gamma^+_{u\varrho}(\psi))^2 \\ +(\Gamma^-_{L\varrho}(\psi))^2 + (\Gamma^-_{u\varrho}(\psi))^2 \\ +(\Gamma^+_{\varrho}(\psi))^2 + (\Gamma^-_{\varrho}(\psi))^2 \end{array}} \times \sqrt{\begin{array}{l} (\Gamma^+_{L\varrho^*}(\psi))^2 + (\Gamma^+_{u\varrho^*}(\psi))^2 \\ +(\Gamma^-_{L\varrho^*}(\psi))^2 + (\Gamma^-_{u\varrho^*}(\psi))^2 \\ +(\Gamma^+_{\varrho^*}(\psi))^2 + (\Gamma^-_{\varrho^*}(\psi))^2 \end{array}}}.$$

The best option in the choice set has been determined by applying the ideal point concept. While there is no such thing as an ideal choice, this idea offers a useful theoretical framework for weighing your options. We can evaluate the quality of an alternative by computing the CSM (closeness similarity measure) between each CB item and the ideal CBFS. Better alternatives are those that are closer to the ideal alternative, as shown by a greater CSM value, $\delta(\varrho,\varrho^*)$.

$$\begin{aligned} \varrho^* &= \left\{\left\langle \psi, I^+_{\varrho^*}(\psi), I^-_{\varrho^*}(\psi), \left(B^+_{\varrho^*}(\psi), B^-_{\varrho^*}(\psi)\right)\right\rangle, \psi \in \Psi\right\} \\ &= \{\langle [1,1], [-1,-1], (0,0)\rangle\}. \end{aligned}$$

The following steps present the DM method's algorithm:

Step 1: Determine the discourse set Ψ.
Step 2: Decide the set of attributes $\aleph$.
Step 3: Build the CB soft relation based on the feedback of decision makers.
Step 4: Provide $\varrho \in \text{CBFS}(\aleph)$ as the testing set.
Step 5: Calculate the lower and upper CB soft approximation operators $\underline{\varpi}(\varrho), \overline{\varpi}(\varrho)$.
Step 6: Obtain $(\underline{\varpi}(\varrho) \oplus_R \overline{\varpi}(\varrho))$.
Step 7: Compute the cosine similarity measure $\delta(\varrho,\varrho^*)$.
Step 8: The most suitable option is to use the greatest value of the cosine similarity metric.

Figure 2 shows the algorithm for the DM based on CBFSRSs.

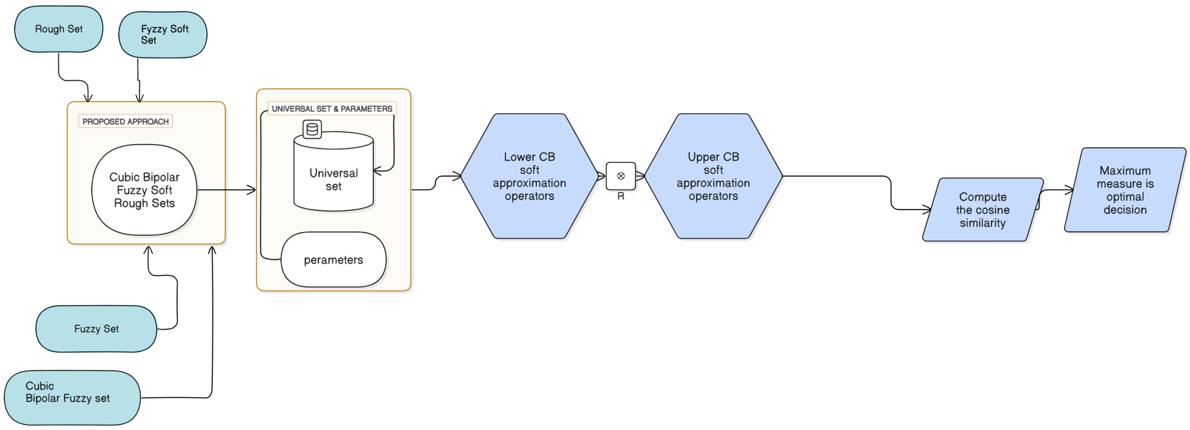

Figure 2. Algorithm for decision making based on CBFSRSs.

Note: The DM and data analysis frequently use the cosine similarity measure to assess how similar two objects are, regardless of their magnitude.

Example 6.2. *Saudi Arabia has aggressively pursued a sustainable future since Vision 2030. Since its launch in 2021, the Green Saudi Initiative has worked to improve environmental protection, accelerate energy transition, and establish sustainability programs to achieve its comprehensive goals of compensating for and reducing carbon emissions, increasing afforestation and land reclamation, and protecting terrestrial and marine areas in the Kingdom. Green Saudi Initiative is crucial to its globalization. A new strategy that engages society is helping the Kingdom create a more sustainable future. Through carefully studied afforestation initiatives throughout the Kingdom, the Green Saudi Initiative aims to increase vegetation cover and combat desertification by planting 10 billion trees, equivalent to rehabilitating 40 million hectares of degraded land. This society-wide program will restore essential environmental functions, reduce dust and sand storms, and more.*

In this example, we will compare several types of different trees found in the center of the Kingdom of Saudi Arabia, specifically in the Qassim region, and determine the most suitable for the environment of the region in relation to several factors with varying degrees of availability. Based on the nature of the region. Assume that the set of trees types is $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$*, for Plumeria alba, grape tree, fig tree, Lavender tree, palm and tree, respectively, and let* $\aleph = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ *be the collection of assessment criteria, where* $\aleph = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ = *{Sunlight, Irrigation water, Temperature, Soil fertility}. Table 4 displays the knowledge base for the selection procedure along with CBFS data.*

Let us now assume that decision makers supply the ideal normal decision object ϱ*, which can be described as follows in terms of cubic bipolar information:*

$$\varrho = \{\langle \ell_1, [0.1, 0.8], [-0.3, -0.1], (0.7, -0.25)\rangle, \langle \ell_2, [0, 0.9], [-0.2, 0], (0.9, -0.15)\rangle, \\ \langle \ell_3, [0, 0.7], [-0.8, -0.2], (0.6, -0.2)\rangle, \langle \ell_4, [0.3, 0.85], [-0.75, -0.4], (0.7, -0.45)\rangle\}.$$

We now compute the lower and upper estimation of the CBSRSs of ϱ *with respect to* $(\Psi, \aleph, \varpi)$*, respectively,*

$$\begin{aligned}\underline{\varpi}(\varrho) = \{&\langle\psi_1, [0.1, 0.8], [-0.3, 0], (0.7, -0.45)\rangle, \langle\psi_2, [0.2, 0.85], [-0.2, 0], (0.7, -0.45)\rangle, \\ &\langle\psi_3, [0.3, 0.9], [-0.3, 0], (0.7, -0.45)\rangle, \langle\psi_4, [0.2, 0.7], [-0.3, 0], (0.7, -0.45)\rangle, \\ &\langle\psi_5, [0.2, 0.8], [-0.4, -0.1], (0.85, -0.25)\rangle\}, \\ \overline{\varpi}(\varrho) = \{&\langle\psi_1, [0.1, 0.8], [-0.8, -0.4], (0.6, -0.2)\rangle, \langle\psi_2, [0.2, 0.8], [-0.8, -0.4], (0.6, -0.2)\rangle, \\ &\langle\psi_3, [0, 0.85], [-0.8, -0.4], (0.7, -0.15)\rangle, \langle\psi_4, [0.1, 0.7], [-0.8, -0.4], (0.7, -0.15)\rangle, \\ &\langle\psi_5, [0.2, 0.8], [-0.8, -0.4], (0.6, -0.3)\rangle\}.\end{aligned}$$

Next, using Definition 2.4, we calculate ($\underline{\varpi}(\varrho) \oplus_R \overline{\varpi}(\varrho)$). As mentioned below,

$$\begin{aligned}\underline{\varpi}(\varrho) \oplus_R \overline{\varpi}(\varrho) = \{&\langle\psi_1, [0.19, 0.96], [-0.24, 0], (0.42, -0.56)\rangle, \\ &\langle\psi_2, [0.36, 0.97], [-0.16, 0], (0.42, -0.56)\rangle, \\ &\langle\psi_3, [0.3, 0.985], [-0.24, 0], (0.49, -0.5325)\rangle, \\ &\langle\psi_4, [0.28, 0.91], [-0.24, 0], (0.49, -0.5325)\rangle, \\ &\langle\psi_5, [0.36, 0.96], [-0.32, -0.04], (0.51, -0.475)\rangle\}.\end{aligned}$$

Next, we compute the CSM $\delta(\varrho, \varrho^*)$ between the actual and the ideal CB numbers using Definition 6.1 as follows:

$$\begin{aligned}\delta(\varrho(\psi_1), \varrho^*) &= 0.56647, \\ \delta(\varrho(\psi_2), \varrho^*) &= 0.59155, \\ \delta(\varrho(\psi_3), \varrho^*) &= 0.59515, \\ \delta(\varrho(\psi_4), \varrho^*) &= 0.58619, \\ \delta(\varrho(\psi_5), \varrho^*) &= 0.65575.\end{aligned}$$

So, the exact decision is to choose ψ_5. Thus, it follows that the decision-makers ought to choose the palm tree as the best option for agriculture and investment in the Qassim region.

Table 4. Cubic bipolar fuzzy soft relation ϖ.

ϖ	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5
ℓ_1	$\langle[0.2, 0.9], [-0.2, 0], (0.8, -0.35)\rangle$	$\langle[0, 0.45], [-0.5, -0.1], (0.6, -0.4)\rangle$	$\langle[0, 0.3], [-0.4, 0], (0.5, -0.25)\rangle$	$\langle[0.1, 0.6], [-0.7, -0.3], (0.7, -0.3)\rangle$	$\langle[0.2, 0.4], [-0.4, 0], (0.5, -0.3)\rangle$
ℓ_2	$\langle[0.1, 0.3], [-0.4, 0], (0.4, -0.2)\rangle$	$\langle[0.2, 0.8], [-0.2, 0], (0.35, -0.45)\rangle$	$\langle[0.12, 0.7], [-0.3, 0], (0.65, -0.15)\rangle$	$\langle[0, 0.2], [-0.3, 0], (0.3, -0.1)\rangle$	$\langle[0.3, 0.7], [-0.5, -0.1], (0.85, -0.6)\rangle$
ℓ_3	$\langle[0, 0.8], [-0.1, 0], (0.7, -0.2)\rangle$	$\langle[0, 0.5], [-0.6, 0], (0.4, -0.2)\rangle$	$\langle[0, 0.1], [-0.6, -0.2], (0.3, -0.6)\rangle$	$\langle[0.3, 0.8], [-0.5, 0], (0.3, -0.2)\rangle$	$\langle[0, 0.8], [-0.3, 0], (0.6, -0.4)\rangle$
ℓ_4	$\langle[0, 0.2], [-0.3, -0.1], (0.15, -0.3)\rangle$	$\langle[0.2, 0.7], [-0.3, 0], (0.8, -0.2)\rangle$	$\langle[0, 1], [-0.2, 0], (1, -0.1)\rangle$	$\langle[0, 0.7], [-0.2, 0], (0.8, -0.3)\rangle$	$\langle[0.2, 0.8], [-0.12, 0], (0.75, -0.81)\rangle$

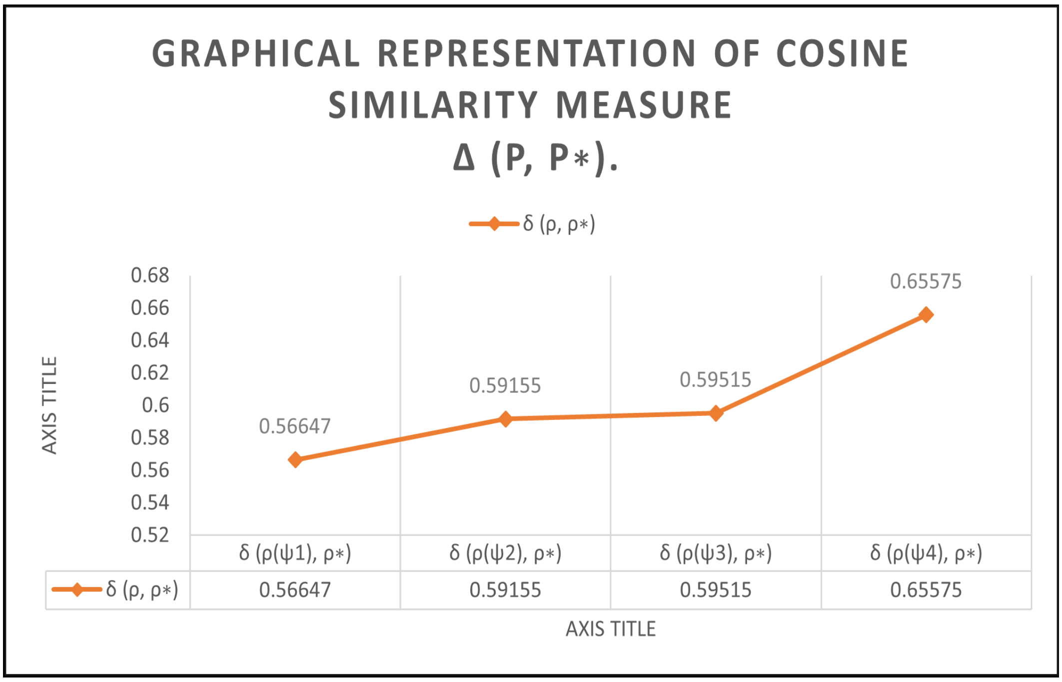

Figure 3. Cosine similarity measure.

7. Conclusions

The CBFSRSs, coupled with three models, such as CBFS, RF, and FSS, successfully handled the previous failure of assessment judgments in complex decision-making situations. Our models give a more complex look at data by employing cubic bipolarity and fuzziness in the rough set framework. This makes decisions that are more accurate and reliable. The findings of our investigations under the CBFRSs and CBFSRSs were quite noteworthy. These findings not only enhance the theoretical grasping of these models, but also highlight their practical use. To be more specific, we showed that these models can handle differences and uncertainty in data very well. This makes them perfect for use in AI, machine learning, and data mining. The decision-making algorithm of this research work integrates the advantages of CBFRSs and CBFSRSs in order to simultaneously evaluate several rational arguments. This comprehensive approach proved that decisions are practical and optimal, considering all facets of the considering problem. The algorithm is a very efficient apparatus for handling real issues because of its ability to comprehend and integrate complex information.

To check the performance, strength, and adaptability of our approach, we used the suggested algorithm in a practical heuristic application and determined our final findings. The implementation demonstrated the models working to adjust to various situations and achieve better outcomes in comparison to existing approaches. The triumph of this application highlights the capacity of CBFRSs and CBFSRSs to fundamentally transform our approach to decision-making in many fields. The FSRs have made great strides forward thanks to the frameworks presented in this research. They provide encompassing answers to the problems with current methods by picking up on both polarity and ambiguity. Future research might concentrate on expanding CBFSRSs to higher-order FSs to handle layered uncertainty and combine machine learning with predictive modeling to obtain robust decision-making. Furthermore, we are intent on investigating real-world applications in healthcare to detect diseases, in agriculture and finance for fraud detection, and combining them with AI techniques, as well as building software equipment, which would improve the usability and uptake of CBFSRSs in various disciplines.

Author contributions

D. S. Aldhuhayyan: Methodology, Conceptualization, Investigating and composing the original manuscript. K. Alsager: Writing assessment and modification. Both authors have reviewed and given their approval to the published version of the article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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