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Research article

Exact solutions and conservation laws for the time-fractional nonlinear dirac system: A study of classical and nonclassical lie symmetries

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Abstract: Time-fractional Dirac-type systems arise in quantum field theory, plasma physics, and condensed matter systems where fractional calculus captures nonlocal interactions. In this study, we employ classical and nonclassical Lie symmetry methods to analyze the underlying symmetry structure of the system. By deriving infinitesimal generators and performing similarity reductions, we transform the governing fractional partial differential equations (FPDEs) into fractional ordinary differential equations (FODEs). Exact solutions are constructed using the power series method. Furthermore, we establish conservation laws in the fractional setting, ensuring the physical consistency of the system. Our findings offer new insights into the interplay among symmetry, conservation principles, and exact solutions in fractional quantum field models, expanding the analytical toolkit for studying nonlinear relativistic wave equations.

Keywords: classical Lie symmetries; nonclassical lie symmetries; time-fractional nonlinear Dirac system; exact solutions; conservation laws
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1. Introduction

Nonlinear fractional partial differential equations (NLFPDEs) are equations in which the derivatives are of fractional order. Due to their non-local characteristics and specific complexities, these equations have applications in many scientific fields such as physics, engineering, chemistry, and biology. FPDEs are used to model systems that exhibit delay effects or history-dependent behavior. These equations are commonly observed in nonlinear systems and complex phenomena, such as fluid dynamics, disease models, and social dynamics. One of the advantages of using FPDEs is their greater accuracy in describing natural phenomena and the ability to model behaviors that cannot be represented by ordinary differential equations. Specifically, these equations can explain phenomena where the system's memory and history-dependent effects are influential.

Solving NLFPDEs often requires complex numerical and analytical methods because the properties of these equations make traditional solution techniques, which depend on integer-order derivatives, ineffective. As a result, researchers are increasingly seeking methods that can effectively solve these equations and provide a better understanding of the phenomena being studied. In this regard, NLFPDEs have been studied using various methods. One common method is the Laplace transform, which converts fractional equations into algebraic equations, enabling their solution.

For example, the authors of [1] applied the Laplace transform to solve families of fractional differential equations. They extended the classical Frobenius method and derived explicit particular solutions using binomial series expansions. Vatsala and Sambandham [2] developed a Laplace transform method to solve sequential Caputo fractional differential equations. They addressed both initial and boundary value problems, providing solutions in terms of Mittag–Leffler functions. Their approach generalizes classical methods and offers a framework for analyzing fractional systems with memory effects. In [3], the authors investigated Laplace transforms with respect to functions and their applications to fractional differential equations. They established fundamental properties, including an inversion formula, and demonstrate how these transforms can be used to solve fractional equations efficiently.

Additionally, series methods, such as Maclaurin or Taylor series, are used to expand the solution as a series of power functions, especially when the exact solution of the equation is not accessible. Meanwhile, Cang et al. [4] applied the homotopy analysis method to derive series solutions for nonlinear Riccati differential equations of fractional order. Further extending power series techniques, Angstmann and Henry [5] developed a generalized fractional power series method for solving fractional differential equations. This approach refines traditional methods by incorporating the complexities of fractional calculus, offering a versatile tool for obtaining analytical solutions across a wide range of fractional differential equations. Ali, Kalim, and Khan [6] employed the fractional power series method (FPSM) to solve FPDEs. They demonstrated that the FPSM effectively constructs series solutions for a variety of FPDEs, providing a systematic approach to handling the complexities introduced by fractional derivatives. Tashtoush et al. [7] focused on obtaining exact solutions to the space-time conformable fractional (4+1)-dimensional Fokas equation with Kerr law nonlinearity. The authors of [8] applied a fractional nonlinear dispersive model to describe wave propagation in Murnaghan's rods using β-fractional and M-truncated derivatives. They presented exact solutions and phase portraits to analyze the system's dynamic behavior and singularities.

In more complex cases, semi-analytical algorithms and numerical methods are employed to approximate fractional derivatives and solve the equations. For instance, Kheybari et al. [9] presented a novel semi-analytical algorithm designed to solve time-fractional modified anomalous sub-diffusion equations. In [10], the author applied pseudospectral methods based on different fractional derivative operator matrices for solving time-space FPDEs characterized by variable coefficients and governed by the Caputo derivative. In [11], Hashemi, Mirzazadeh, and Baleanu proposed an innovative method for computing approximate solutions to non-homogeneous wave equations featuring generalized fractional derivatives. Javeed et al. [12] analyzed the homotopy perturbation method for solving FPDEs.

Some other numerical approaches for solving FPDEs have been proposed in the literature, such as those in [13–16]. Furthermore, symmetry methods and Lie group analysis are used to obtain exact solutions for FPDEs, particularly in physics and engineering. Lie group analysis is a powerful analytical method used for studying fractional and nonlinear differential equations. In this method, Lie groups and their principles are employed to find analytical solutions to differential equations. Lie groups are particularly useful for nonlinear differential equations because they can identify the symmetries of the equation and, through them, obtain general solutions [17–19]. This method is especially applicable to equations that have spatial or temporal symmetries. In fact, Lie group analysis allows for the modeling of the behavior of complex nonlinear equations using a simpler and more precise symmetry structure, thus enabling the extraction of both specific and general solutions to these equations [20–22].

In the present work, the time-fractional nonlinear Dirac system (TFNLDS), expressed as follows, is investigated:

$$\begin{aligned} \Lambda_1 : \mathcal{D}_t^{\alpha} p &= \frac{1}{2} q_{xx} - p^2 q - q^3, \\ \Lambda_2 : \mathcal{D}_t^{\alpha} q &= -\frac{1}{2} p_{xx} + p q^2 + p^3, \end{aligned} \tag{1.1}$$

where p and q are functions of (t, x), and $\mathcal{D}_t^{\alpha}(.)$ represents the time-fractional Riemann–Liouville (RL) derivative of order α, where $\alpha \in (0, 1)$. By setting $\alpha = 1$, the classical type of the nonlinear Dirac system can be recovered from the system (1.1) [23–25]. In the original Dirac system, as described by Frolov [26] and Schratz et al. [27], the functions $p(x, t)$ and $q(x, t)$ evolve according to the coupled nonlinear equations given in [28]. This system, which describes the motion of relativistic spin-$\frac{1}{2}$ particles in external electromagnetic fields, has significant applications in applied sciences. However, to account for more complex dynamics and long-range interactions, the system can be generalized into a fractional form. In this work, we extend the nonlinear Dirac system by introducing fractional derivatives of order $\alpha \in (0, 1)$ to model the system with non-local effects. The fractional form of the Dirac system provides a more accurate representation of phenomena exhibiting memory effects, such as the self-interaction of nonlinear particles and the influence of long-range forces. By transforming the system into a fractional setting, we can explore new solutions and behaviors that arise from the fractional order, opening up avenues for further research in both theoretical and applied contexts.

This paper investigates the Lie symmetries and conservation laws of the TFNLDS. Section 2 presents the necessary preliminaries and mathematical framework. Section 3 is dedicated to the Lie symmetry analysis of the TFNLDS, identifying both classical symmetries (Section 3.1.1) and nonclassical symmetries (Section 3.1.2). Utilizing these symmetries, exact solutions are derived, and their implications are explored. Section 4 constructs the conservation laws associated with the TFNLDS, providing insights into the fundamental invariants of the system. Finally, Section 5 summarizes the findings and discusses their significance.

2. Preliminaries

Some fundamental definitions and properties of fractional order derivatives are presented in this section. Interested readers are referred to [29–31] for their definitions and properties.

Definition 1. [29] Let $\alpha \in \mathbb{R}_+$. The operator J^α defined by

$$J^\alpha f(t,x) = \int_0^t \frac{(t-w)^{\alpha-1}}{\Gamma(\alpha)} f(w,x)dw,$$

where $\Gamma(\cdot)$ is the gamma function, is called the RL fractional integral operator of order α. When $\alpha = 0$, $J^\alpha = I$ is the identity operator.

Definition 2. [30] Let $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$, the operator $\mathcal{D}^\alpha$ formulated as

$$\mathcal{D}^\alpha = \mathcal{D}^n J^{n-\alpha},$$

which referred to as the RL fractional differential operator of order α. When $\alpha = 0$, $\mathcal{D}^\alpha = I$ is the identity operator. Therefore, the fractional RL derivative of order α of the function $f(t,x)$ is given by

$$\mathcal{D}_t^\alpha f(t,x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial t} \displaystyle\int_0^t (t-w)^{-\alpha} f(w,x)dw, & 0 < \alpha < 1, \\ \\ \dfrac{\partial}{\partial t} f(t,x), & \alpha = 1, \end{cases}$$

and the RL fractional derivative of t^β is represented by

$$\mathcal{D}_t^\alpha t^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\alpha-\beta \notin \mathbb{N}), \beta > -1, \\ 0, & (\alpha - \beta \in \mathbb{N}). \end{cases}$$

It is evident that when $\alpha - \beta \in \mathbb{N}$, the right-hand side is the $\lceil \alpha \rceil$-th derivative of the classical polynomial of degree $\lceil \alpha \rceil - (\alpha - \beta) \in \{0, 1, ..., \lceil \alpha \rceil - 1\}$, where $\lceil \cdot \rceil$ shows the ceiling function.

The fractional integral and the RL fractional derivative possess the following properties:

1) $J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \quad \alpha > 0, \ \beta > -1.$
2) $J^{\alpha_1} J^{\alpha_2} f(t) = J^{\alpha_2} J^{\alpha_1} f(t) = J^{\alpha_1+\alpha_2} f(t), \quad \alpha_1, \alpha_2 \geq 0.$
3) $\mathcal{D}^\alpha J^\alpha f(t) = f(t), \quad \alpha \geq 0.$
4) $\mathcal{D}^\alpha (c_1 f(t) + c_2 g(t)) = c_1 \mathcal{D}^\alpha f(t) + c_2 \mathcal{D}^\alpha g(t), \quad c_1, c_2 \in \mathbb{R}, \ \alpha > 0.$
5) $\mathcal{D}^\alpha [fg](t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \binom{\alpha}{k} (\mathcal{D}^\alpha f)(t) (\mathcal{D}^{\alpha-k} g)(t) + \sum_{k=\lfloor \alpha \rfloor+1}^{\infty} \binom{\alpha}{k} \left(\mathcal{D}^k f\right)(t) (J^{k-\alpha} g)(t), \quad \alpha > 0,$
   where $\lfloor \cdot \rfloor$ shows the flooring function.

Definition 3. [30] Let $\alpha, \beta > 0$. The two-parameter Mittag–Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)}.$$

Definition 4. [31] The fractional integral operator of Erdélyi–Kober for $f(t)$ is

$$\left(K_{\beta}^{\nu,\alpha} f\right)(t) = \begin{cases} \int_{1}^{\infty} \frac{(w-1)^{\alpha-1}}{\Gamma(\alpha) w^{-(\nu+\alpha)}} f(tw^{\frac{1}{\beta}}) dw, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases}$$

Definition 5. [31] The fractional derivative operator of Erdélyi–Kober for $f(t)$ is

$$\left(\mathcal{P}_{\beta}^{\nu,\alpha} f\right)(t) = \prod_{i=0}^{m-1} \left(\nu + i - \frac{1}{\beta} t \frac{d}{dt}\right) \left(K_{\beta}^{\nu+\alpha, m-\alpha} f\right)(t),$$

$$m = \begin{cases} \lfloor \alpha \rfloor + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}$$

3. Lie symmetries of the TFNLDS

In this section, we examine the Lie group and both the classical and nonclassical symmetries of the main system (1.1). To this end, we present the relevant concepts of Lie group analysis for the system of time-fractional partial differential equations (STFPDEs), which will be used later.

3.1. Lie group method

In this subsection, a general description is provided of how the Lie group method is applied to STFPDEs of order α, where $\alpha \in (0, 1)$, expressed as follows:

$$\begin{aligned} \Xi_1 &: \mathcal{D}_t^{\alpha} p - \sigma_1 (t, x, p, q, p_x, q_x, p_{xx}, q_{xx}, \cdots) = 0, \\ \Xi_2 &: \mathcal{D}_t^{\alpha} q - \sigma_2 (t, x, p, q, p_x, q_x, p_{xx}, q_{xx}, \cdots) = 0. \end{aligned} \tag{3.1}$$

The system (3.1) consists of p and q as dependent variables, while t and x serve as independent variables. Additionally, the subscripts denote integer-order derivatives. Considering that the system (3.1) remains invariant under the following one-parameter Lie group transformations

$$\begin{aligned} \check{x} &= x + \epsilon \zeta_1(t, x, p, q) + O(\epsilon^2), \\ \check{t} &= t + \epsilon \zeta_2(t, x, p, q) + O(\epsilon^2), \\ \check{p} &= u + \epsilon \varpi_1(t, x, p, q) + O(\epsilon^2), \\ \check{q} &= v + \epsilon \varpi_2(t, x, p, q) + O(\epsilon^2), \\ \mathcal{D}_t^{\alpha} \check{p} &= \mathcal{D}_t^{\alpha} p + \epsilon \varpi_1^{\alpha,t}(t, x, p, q) + O(\epsilon^2), \\ \mathcal{D}_t^{\alpha} \check{q} &= \mathcal{D}_t^{\alpha} q + \epsilon \varpi_2^{\alpha,t}(t, x, p, q) + O(\epsilon^2), \\ \frac{\partial^j \check{p}}{\partial \check{x}^j} &= \frac{\partial^j p}{\partial x^j} + \epsilon \varpi_1^{j,x}(t, x, p, q) + O(\epsilon^2), \quad j = 1, 2, 3, \ldots, \\ \frac{\partial^j \check{q}}{\partial \check{x}^j} &= \frac{\partial^j q}{\partial x^j} + \epsilon \varpi_2^{j,x}(t, x, p, q) + O(\epsilon^2), \quad j = 1, 2, 3, \ldots, \end{aligned}$$

where ϵ is the group parameter. Furthermore, the vector field corresponding to these transformations is given as follows:

$$\mathcal{W} = \zeta_1(t,x,p,q)\frac{\partial}{\partial x} + \zeta_2(t,x,p,q)\frac{\partial}{\partial t} + \varpi_1(t,x,p,q)\frac{\partial}{\partial u} + \varpi_2(t,x,p,q)\frac{\partial}{\partial v}.$$

For the system (3.1), the infinitesimal generator admits a symmetry precisely when the following conditions are satisfied:

$$\wp r^{(\alpha,k_1,h_1)}\mathcal{W}(\Xi_1)\Big|_{\Xi_1=0} = 0,$$
$$\wp r^{(\alpha,k_2,h_2)}\mathcal{W}(\Xi_2)\Big|_{\Xi_2=0} = 0,$$

where k_l and h_l for $l = 1,2$, represent the leading orders in the l-th equation in the system (3.1), and it is also important to note that the fractional prolongation operator $\wp r^{(\alpha,k,h)}\mathcal{W}$ is expressed as follows:

$$\begin{aligned}\wp r^{(\alpha,k,h)}\mathcal{W} =& \mathcal{W} + \varpi_1^{\alpha,t}\frac{\partial}{\partial(\mathcal{D}_x^\alpha p)} + \varpi_1^{1,x}\frac{\partial}{\partial p_x} + \ldots + \varpi_1^{k,x}\frac{\partial}{\partial p_{kx}}\\ &+ \varpi_2^{\alpha,t}\frac{\partial}{\partial(\mathcal{D}_x^\alpha q)} + \varpi_2^{1,x}\frac{\partial}{\partial q_x} + \ldots + \varpi_2^{h,x}\frac{\partial}{\partial q_{hx}},\end{aligned}$$

where $\varpi_1^{j,x}$ and $\varpi_2^{j,x}$ represent the extensions of the infinitesimals in the integer-order framework, as follows:

$$\begin{aligned}\varpi_1^{j,x} &= D_x\varpi_1^{j-1,x} - (D_x\zeta_1)\frac{\partial^j p}{\partial x^j} - (D_x\zeta_2)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1}p}{\partial x^{j-1}}\right),\\ \varpi_2^{j,x} &= D_x\varpi_2^{j-1,x} - (D_x\zeta_1)\frac{\partial^j q}{\partial x^j} - (D_x\zeta_2)\frac{\partial}{\partial t}\left(\frac{\partial^{j-1}q}{\partial x^{j-1}}\right), \quad j \in \mathbb{N},\end{aligned}$$

where the symbol D_x signifies the total derivative, i.e.,

$$D_x = \frac{\partial}{\partial x} + p_x\frac{\partial}{\partial p} + p_{xx}\frac{\partial}{\partial p_x} + \cdots + q_x\frac{\partial}{\partial q} + q_{xx}\frac{\partial}{\partial q_x} + \cdots .$$

Therefore, $\varpi_1^{\alpha,t}$ represents the α-order extensions of the infinitesimal operator as

$$\varpi_1^{\alpha,t} = D_t^\alpha\varpi_1 + \zeta_1 D_t^\alpha(p_x) - D_t^\alpha(\zeta_1 p_x) + D_t^\alpha\left(D_t(\zeta_2)p\right) - D_t^{\alpha+1}(\zeta_2 p) + \zeta_2 D_t^{\alpha+1}(p),$$

where the symbol D_t^α denotes the total α-order fractional derivative. Utilizing the generalized Leibnitz formula and the chain rule [29, 32], $\varpi_1^{\alpha,t}$ can be defined as follows:

$$\begin{aligned}\varpi_1^{\alpha,t} =& \frac{\partial^\alpha\varpi_1}{\partial t^\alpha} + \left(\varpi_{1_p} - \alpha D_t(\zeta_2)\right)\frac{\partial^\alpha p}{\partial t^\alpha} - p\frac{\partial^\alpha\varpi_{1_p}}{\partial t^\alpha} + \left(\varpi_{1_q}\frac{\partial^\alpha q}{\partial t^\alpha} - q\frac{\partial^\alpha\varpi_{1_q}}{\partial t^\alpha}\right)\\ &+ \sum_{j=1}^{\infty}\left[\binom{\alpha}{j}\frac{\partial^j\varpi_{1_p}}{\partial t^j} - \binom{\alpha}{j+1}D_t^{j+1}(\zeta_2)\right]D_t^{\alpha-j}(p)\\ &+ \sum_{j=1}^{\infty}\binom{\alpha}{i}\frac{\partial^j\varpi_{1_q}}{\partial t^j}D_t^{\alpha-j}(q) - \sum_{j=1}^{\infty}\binom{\alpha}{j}D_t^j(\zeta_1)D_t^{\alpha-j}(p_x) + \lambda_1,\end{aligned}$$

where

$$\lambda_1 = \sum_{j=2}^{\infty}\sum_{k=2}^{j}\sum_{l=2}^{k}\sum_{m=0}^{l-1}\binom{\alpha}{j}\binom{j}{k}\binom{l}{m}\frac{(-1)^m t^{j-\alpha}}{l!\Gamma(j-\alpha+1)}\left(p^m\frac{\partial^k(p^{l-m})}{\partial t^k}\frac{\partial^{j-k+l}\varpi_1}{\partial t^{j-k}\partial p^l}+q^m\frac{\partial^k(q^{l-m})}{\partial t^k}\frac{\partial^{j-k+l}\varpi_1}{\partial t^{j-k}\partial q^l}\right).$$

Similarly, $\varpi_2^{\alpha,t}$ can be written as follows:

$$\begin{aligned}\varpi_2^{\alpha,t} =&\frac{\partial^\alpha\varpi_2}{\partial t^\alpha}+\Big(\varpi_{2_q}-\alpha D_t(\zeta_2)\Big)\frac{\partial^\alpha q}{\partial t^\alpha}-q\frac{\partial^\alpha\varpi_{2_q}}{\partial t^\alpha}+\Big(\varpi_{2_p}\frac{\partial^\alpha p}{\partial t^\alpha}-p\frac{\partial^\alpha\varpi_{2_p}}{\partial t^\alpha}\Big)\\&+\sum_{j=1}^{\infty}\Big[\binom{\alpha}{j}\frac{\partial^j\varpi_{2_q}}{\partial t^j}-\binom{\alpha}{j+1}D_t^{j+1}(\zeta_2)\Big]D_t^{\alpha-j}(q)+\sum_{j=1}^{\infty}\binom{\alpha}{j}\frac{\partial^j\varpi_{2_u}}{\partial t^j}D_t^{\alpha-j}(p)\\&-\sum_{j=1}^{\infty}\binom{\alpha}{j}D_t^j(\zeta_1)D_t^{\alpha-j}(q_x)+\lambda_2,\end{aligned}$$

where

$$\lambda_2 = \sum_{j=2}^{\infty}\sum_{k=2}^{j}\sum_{l=2}^{k}\sum_{m=0}^{l-1}\binom{\alpha}{j}\binom{j}{k}\binom{l}{m}\frac{(-1)^m t^{j-\alpha}}{l!\Gamma(j-\alpha+1)}\left(p^m\frac{\partial^k(p^{l-m})}{\partial t^k}\frac{\partial^{j-k+l}\varpi_2}{\partial t^{j-k}\partial p^l}+q^m\frac{\partial^k(q^{l-m})}{\partial t^k}\frac{\partial^{j-k+l}\varpi_2}{\partial t^{j-k}\partial q^l}\right).$$

Since ϖ_1 and ϖ_2 depend linearly on the variables p and q, their partial derivatives $\frac{\partial^i\varpi_1}{\partial p^i}$ and $\frac{\partial^i\varpi_2}{\partial q^i}$ vanish for all $i\in\mathbb{N}-\{1\}$. Consequently, it follows that $\lambda_1=0$ and $\lambda_2=0$.

3.1.1. Classical symmetries

Based on the proposed Lie group method, the invariance condition of the system (3.1) is that the following relations hold:

$$\begin{cases}\wp r^{(\alpha,k_1,h_1)}\mathcal{W}\Big(\mathcal{D}_t^\alpha p-\sigma_1\left(t,x,p,q,p_x,q_x,p_{xx},q_{xx},\cdots\right)\Big)\Big|_{System(3.1)}=0,\\[2ex]\wp r^{(\alpha,k_2,h_2)}\mathcal{W}\Big(\mathcal{D}_t^\alpha q-\sigma_2\left(t,x,p,q,p_x,q_x,p_{xx},q_{xx},\cdots\right)\Big)\Big|_{System(3.1)}=0.\end{cases}\tag{3.2}$$

Thus, on the basis of the relation (3.2) and system (1.1), we have

$$\begin{cases}\wp r^{(\alpha,2,2)}\mathcal{W}\Big(\mathcal{D}_t^\alpha p-\frac{1}{2}q_{xx}+p^2q+q^3\Big)\Big|_{System(1.1)}=0,\\[2ex]\wp r^{(\alpha,2,2)}\mathcal{W}\Big(\mathcal{D}_t^\alpha q+\frac{1}{2}p_{xx}-pq^2-p^3\Big)\Big|_{System(1.1)}=0.\end{cases}$$

Therefore, the following determining equations are obtained:

$$
\begin{aligned}
&3p^2\varpi_1 + q^2\varpi_1 - p^3\varpi_{2_q} - pq^2\varpi_{2_q} + \alpha p^3\zeta_{2_t} + 2pq\varpi_2 - \frac{\partial^\alpha \varpi_2}{\partial t^\alpha} + q\frac{\partial^\alpha \varpi_{2_q}}{\partial t^\alpha} + \alpha pq^2\zeta_{2_t} - \frac{1}{2}\varpi_{1_{xx}} = 0,\\
&q^3\varpi_{1_p} - p^2\varpi_2 - 3q^2\varpi_2 + p^2q\varpi_{1_p}\alpha q^3\zeta_{2_t} - 2pq\varpi_1 - \alpha p^2 q\zeta_{2_t} - \frac{\partial^\alpha \varpi_1}{\partial t^\alpha} + p\frac{\partial^\alpha \varpi_{1_p}}{\partial t^\alpha} + \frac{1}{2}\varpi_{2_{xx}} = 0,\\
&\alpha(\alpha^2 - 2\alpha + 1)\zeta_{2_{ttt}} - 3\alpha(\alpha - 1)\varpi_{2_{ttq}} = 0,\\
&\alpha(\alpha^2 - 2\alpha + 1)\zeta_{2_{ttt}} - 3\alpha(\alpha - 1)\varpi_{1_{ttp}} = 0,\\
&\binom{\alpha}{j}\frac{\partial^j \varpi_{1_p}}{\partial t^j} - \binom{\alpha}{j+1}D_t^{j+1}(\zeta_2) = 0,\\
&\binom{\alpha}{j}\frac{\partial^j \varpi_{2_q}}{\partial t^j} - \binom{\alpha}{j+1}D_t^{j+1}(\zeta_2) = 0,\\
&\frac{1}{2}\varpi_{1_p} - \frac{1}{2}\alpha\zeta_{2_t} + \zeta_{1_x} - \frac{1}{2}\varpi_{2_q} = 0,\\
&\frac{1}{2}\varpi_{2_q} - \frac{1}{2}\alpha\zeta_{2_t} + \zeta_{1_x} - \frac{1}{2}\varpi_{1_p} = 0,\\
&\varpi_{1_q} = \varpi_{2_p} = \varpi_{1_{pp}} = \varpi_{2_{qq}} = 0,\\
&\alpha(1-\alpha)\zeta_{2_{tt}} + 2\alpha\varpi_{1_{tp}} = 0,\\
&\alpha(1-\alpha)\zeta_{2_{tt}} + 2\alpha\varpi_{2_{tq}} = 0,\\
&\zeta_{2_x} = \zeta_{2_p} = \zeta_{2_q} = 0,\\
&\zeta_{1_p} = \zeta_{1_q} = \zeta_{1_t} = 0,\\
&\varpi_{1_{xp}} - \frac{1}{2}\zeta_{1_{xx}} = 0,\\
&\frac{1}{2}\zeta_{1_{xx}} - \varpi_{2_{xq}} = 0,\\
&\frac{\partial^j \varpi_{1_p}}{\partial t^j} = 0,\\
&\frac{\partial^j \varpi_{2_q}}{\partial t^j} = 0,\\
&D_t^j(\zeta_1) = 0.
\end{aligned}
$$

The vector fields are derived on the basis of the determining equations as follows:

$$
\begin{aligned}
\mathcal{W}_1 &= \frac{\partial}{\partial x},\\
\mathcal{W}_2 &= 4t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + \alpha p\frac{\partial}{\partial p} + \alpha q\frac{\partial}{\partial q},\\
\mathcal{W}_3 &= p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q},\\
\mathcal{W}_4 &= h(t,x)\frac{\partial}{\partial q},\\
\mathcal{W}_5 &= k(t,x)\frac{\partial}{\partial p}.
\end{aligned}
$$

Considering the vector field $\mathcal{W}_1 = \dfrac{\partial}{\partial x}$, the corresponding characteristic equation is

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dp}{0} = \frac{dq}{0}.$$

By analyzing the given equation, one can derive the corresponding invariant solutions related to the associated vector field, expressed as $p(t, x) = \omega(t)$ and $q(t, x) = \vartheta(t)$. By inserting the derived functions into the main equation, the resulting system takes the following form:

$$\begin{cases} \mathcal{D}_t^{\alpha}\omega(t) = -\vartheta(t)\omega^2(t) - \vartheta^3(t), \\ \\ \mathcal{D}_t^{\alpha}\vartheta(t) = \omega(t)\vartheta^2(t) + \omega^3(t). \end{cases} \tag{3.3}$$

If we suppose that $\omega(t) = i\vartheta(t)$, the system (3.3) can be written as

$$\begin{cases} i\mathcal{D}_t^{\alpha}\vartheta(t) = 0, \\ \\ \mathcal{D}_t^{\alpha}\vartheta(t) = 0, \end{cases}$$

and the exact solutions of the given system, dependent on time, can be formulated as follows:

$$\begin{cases} \omega(t) = ic_1 t^{\alpha-1}, \\ \\ \vartheta(t) = c_2 t^{\alpha-1}, \end{cases}$$

where c_1 and c_2 are constants. Therefore the final solutions of the main system are $p(t, x) = ic_1 t^{\alpha-1}$ and $q(t, x) = c_2 t^{\alpha-1}$.

The characteristic equation for the vector field $\mathcal{W}_2$ is given by

$$\frac{dt}{4t} = \frac{dx}{2\alpha x} = \frac{dp}{\alpha p} = \frac{dz}{\alpha q}.$$

The invariant solutions corresponding to the vector field $\mathcal{W}_2$ are

$$\begin{aligned} &p(t, x) = t^{\frac{\alpha}{4}} f(\varepsilon), \\ &q(t, x) = t^{\frac{\alpha}{4}} g(\varepsilon), \qquad \varepsilon = x t^{-\frac{\alpha}{2}}. \end{aligned} \tag{3.4}$$

Theorem 1. By applying the solutions (3.4), the system (1.1) simplifies to the following system of FODEs:

$$\begin{cases} \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} f\right)(\varepsilon) - \frac{1}{2}g''(\varepsilon) = 0, \\ \\ \left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} g\right)(\varepsilon) + \frac{1}{2}f''(\varepsilon) = 0, \\ \\ g(\varepsilon)f^2(\varepsilon) + g^3(\varepsilon) = 0, \\ \\ f(\varepsilon)g^2(\varepsilon) + f^3(\varepsilon) = 0. \end{cases} \tag{3.5}$$

Proof. Let $n \in \mathbb{N}$ and $\alpha \in (n-1, n)$. Taking the α-order temporal RL derivative of Eq (3.4) yields

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-w)^{n-\alpha-1} w^{\frac{\alpha}{4}} f(xw^{-\frac{\alpha}{2}})dw\right]. \tag{3.6}$$

By using the change in the variable $\nu = \dfrac{t}{w}$, the relation (3.6) can be written as follows:

$$\begin{aligned}
\frac{\partial^\alpha p}{\partial t^\alpha} =& \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\frac{3\alpha}{4}}}{\Gamma(n-\alpha)}\int_1^\infty (\nu-1)^{n-\alpha-1}\nu^{\frac{3\alpha}{4}-n-1} f(xt^{-\frac{\alpha}{2}}\nu^{\frac{\alpha}{2}})d\nu\right],\\
\frac{\partial^\alpha p}{\partial t^\alpha} =& \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{4}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha} f\right)(\varepsilon)\right].
\end{aligned}$$

Furthermore, if we let $\varepsilon = xt^{-\frac{\alpha}{2}}$, $0 < \rho < \infty$, then

$$t\frac{\partial}{\partial t}\rho(\varepsilon) = t\frac{\partial \varepsilon}{\partial t}\frac{d\rho(\varepsilon)}{d\varepsilon} = -\frac{\alpha}{2}\varepsilon\frac{d\rho(\varepsilon)}{d\varepsilon}.$$

Therefore

$$\begin{aligned}
\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{4}}\left(K_{\frac{2}{\alpha}}^{1+\frac{\alpha}{4},n-\alpha} f\right)(\varepsilon)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{3\alpha}{4}}\left(K_{\frac{2}{\alpha}}^{1+\frac{\alpha}{4},n-\alpha} f\right)(\varepsilon)\right)\right]\\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{3\alpha}{4}-1}\left(n-\frac{3\alpha}{4}-\frac{\alpha}{2}\varepsilon\frac{d}{d\varepsilon}\right)\left(K_{\frac{2}{\alpha}}^{1+\frac{\alpha}{4},n-\alpha} f\right)(\varepsilon)\right]\\
&\vdots\\
&= t^{-\frac{3\alpha}{4}}\prod_{j=0}^{n-1}\left(1-\frac{3\alpha}{4}+j-\frac{\alpha}{2}\varepsilon\frac{d}{d\varepsilon}\right)\left(K_{\frac{2}{\alpha}}^{1+\frac{\alpha}{4},n-\alpha} f\right)(\varepsilon)\\
&= t^{-\frac{3\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} f\right)(\varepsilon).
\end{aligned}$$

Thus

$$\frac{\partial^\alpha p}{\partial t^\alpha} = t^{-\frac{3\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} f\right)(\varepsilon).$$

Similarly

$$\frac{\partial^\alpha q}{\partial t^\alpha} = t^{-\frac{3\alpha}{4}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} f\right)(\varepsilon),$$

and thus the proof is completed. □

Applying a power series approach to obtain exact solutions

To find an exact solution to the system (1.1), let $g(\varepsilon) = if(\varepsilon)$. Consequently, it is enough to solve the following equation:

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{4},\alpha} f\right)(\varepsilon) - \frac{i}{2}f''(\varepsilon) = 0. \tag{3.7}$$

By employing a power series approach, we assume that $f(\varepsilon)$ can be expanded as follows:

$$f(\varepsilon) = \sum_{j=0}^{\infty} a_j \varepsilon^j, \qquad \left(\mathcal{P}_{\beta}^{\tau,\alpha} f\right)(\varepsilon) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\tau - \frac{k}{\beta} + 1)}{\Gamma(\tau - \frac{k}{\beta} + 1 - \alpha)} \varepsilon^k, \tag{3.8}$$

and

$$\begin{aligned} f'(\varepsilon) &= \sum_{j=1}^{\infty} j a_j \varepsilon^{j-1}, \\ f''(\varepsilon) &= \sum_{j=2}^{\infty} j(j-1) a_j \varepsilon^{j-2}. \end{aligned} \tag{3.9}$$

Substituting Eqs (3.8) and (3.9) into Eq (3.7) yields the following equation:

$$\sum_{j=0}^{\infty} a_j \frac{\Gamma(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2})}{\Gamma(2 - \frac{7\alpha}{4} - \frac{\alpha j}{2})} \varepsilon^j - \frac{i}{2} \sum_{j=0}^{\infty} (j+1)(j+2) b_{j+2} \varepsilon^j = 0. \tag{3.10}$$

If we compare the coefficients in Eq (3.10), for $j = 0$, we have

$$a_2 = \frac{2\Gamma(2 - \frac{3\alpha}{4})}{i\Gamma(2 - \frac{7\alpha}{4})} a_0,$$

and for $j \geqslant 1$, we have

$$a_{j+2} = \frac{2a_j \Gamma\left(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2}\right)}{i(j+1)(j+2)\Gamma\left(2 - \frac{7\alpha}{4} + \frac{\alpha j}{2}\right)}.$$

Therefore, by inserting the obtained coefficients into the series (3.8) we have

$$f(\varepsilon) = a_0 + a_1 \varepsilon - i \left(\frac{2\Gamma(2 - \frac{3\alpha}{4})}{\Gamma(2 - \frac{7\alpha}{4})} a_0 \varepsilon^2 - \sum_{j=1}^{\infty} \frac{2b_j \Gamma\left(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2}\right)}{(j+1)(j+2)\Gamma\left(2 - \frac{7\alpha}{4} + \frac{\alpha j}{2}\right)} \varepsilon^j \right). \tag{3.11}$$

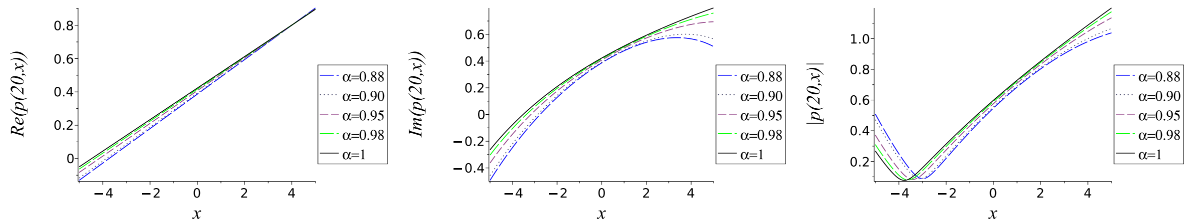
Thus

$$g(\varepsilon) = i(a_0 + a_1 \varepsilon) + \left(\frac{2\Gamma(2 - \frac{3\alpha}{4})}{\Gamma(2 - \frac{7\alpha}{4})} a_0 \varepsilon^2 - \sum_{j=1}^{\infty} \frac{2b_j \Gamma\left(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2}\right)}{(j+1)(j+2)\Gamma\left(2 - \frac{7\alpha}{4} + \frac{\alpha j}{2}\right)} \varepsilon^j \right). \tag{3.12}$$

Through the application of (3.11), (3.12), and (3.4), the following exact solutions are obtained for the governing system (1.1):

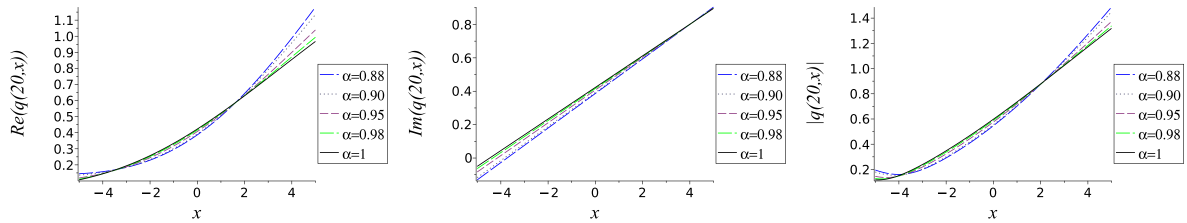
$$\begin{aligned} p(t,x) &= t^{\frac{\alpha}{4}} \left(a_0 + a_1 x t^{-\frac{\alpha}{2}} - i \sum_{j=0}^{\infty} \frac{2a_j \Gamma\left(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2}\right)}{(j+1)(j+2)\Gamma\left(2 - \frac{7\alpha}{4} + \frac{\alpha j}{2}\right)} (x t^{-\frac{\alpha}{2}})^j \right), \\ q(t,x) &= t^{\frac{\alpha}{4}} \left(i(a_0 + a_1 x t^{-\frac{\alpha}{2}}) + \sum_{j=0}^{\infty} \frac{2a_j \Gamma\left(2 - \frac{3\alpha}{4} - \frac{\alpha j}{2}\right)}{(j+1)(j+2)\Gamma\left(2 - \frac{7\alpha}{4} + \frac{\alpha j}{2}\right)} (x t^{-\frac{\alpha}{2}})^j \right). \end{aligned} \tag{3.13}$$

To illustrate that the plots are consistent with the solutions in Eq (3.13), the series are truncated at $N = 30$. Figures 1 and 2 present the real and imaginary parts, as well as the absolute values of $p(t, x)$ and $q(t, x)$ for $a_0 = a_1 = 0.2$ and different fractional orders α at $t = 20$.



(a) **(b)** **(c)**

Figure 1. The plots derived from the classical vector field $\mathcal{W}_2$ show (a) the real part, (b) the imaginary part, and (c) the absolute value of $p(t, x)$ for various values of α at $t = 20$. These plots indicate that the obtained solutions consistently exhibit convergent behavior.



(a) **(b)** **(c)**

Figure 2. The plots derived from the classical vector field $\mathcal{W}_2$ show (a) the real part, (b) the imaginary part, and (c) the absolute value of $q(t, x)$ for various values of α at $t = 20$. These plots demonstrate that the obtained solutions exhibit convergence behavior.

For the vector field $\mathcal{W}_1 + \mathcal{W}_3$, the following invariant solutions are obtained:

$$p(t, x) = e^x f(t), \quad q(t, x) = e^x g(t). \tag{3.14}$$

Utilizing the transformation (3.14) to the system (1.1), the following fractional and integer order ODE system is obtained:

$$\begin{cases} \mathcal{D}_t^\alpha f(t) - \frac{1}{2} g(t) = 0, \\ \mathcal{D}_t^\alpha g(t) + \frac{1}{2} f(t) = 0, \\ g(t) f^2(t) + g^3(t) = 0, \\ f(t) g^2(t) + f^3(t) = 0. \end{cases} \tag{3.15}$$

To obtain an exact solution of the system (3.15), let $g(t) = i f(t)$. Thus, it is sufficient to determine the solution of the following equation:

$$\mathcal{D}_t^\alpha f(t) - \frac{i}{2} f(t) = 0. \tag{3.16}$$

The solutions to Eq (3.16) are obtained using the fractional Laplace transforms (FLTs) [33–35] as follows:

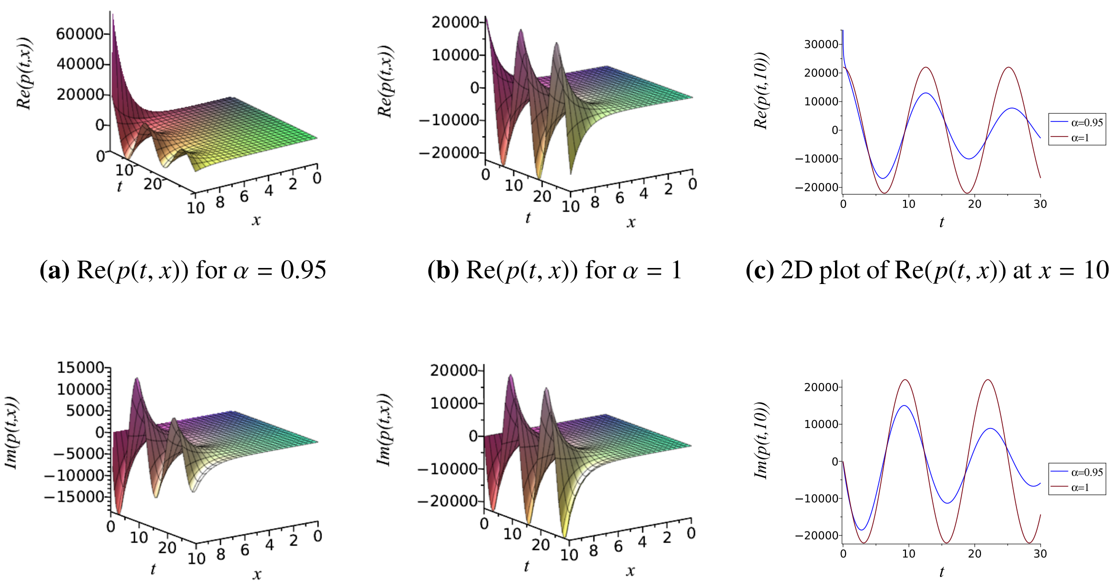
$$f(t) = \lambda t^{2\alpha-2} E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^{\alpha}\right),$$
$$g(t) = i\lambda t^{2\alpha-2} E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^{\alpha}\right), \tag{3.17}$$

where $\lambda = \frac{k}{\Gamma(1-\alpha)}$ and k is a constant. Substituting the relations (3.17) into the system (3.14) yields the exact solutions of the system (1.1) as follows:

$$p(t,x) = \lambda e^{x} t^{2\alpha-2} E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^{\alpha}\right),$$
$$q(t,x) = i\lambda e^{x} t^{2\alpha-2} E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^{\alpha}\right).$$

Due to definition of the Mittage–Leffler function, we have

$$p(t,x) = \lambda e^{x} t^{2\alpha-2}\left(\sum_{j=0}^{\infty}\frac{(-1)^{j}(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)} + i\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right),$$
$$q(t,x) = \lambda e^{x} t^{2\alpha-2}\left(i\sum_{j=0}^{\infty}\frac{(-1)^{j}(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)} - \sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right).$$

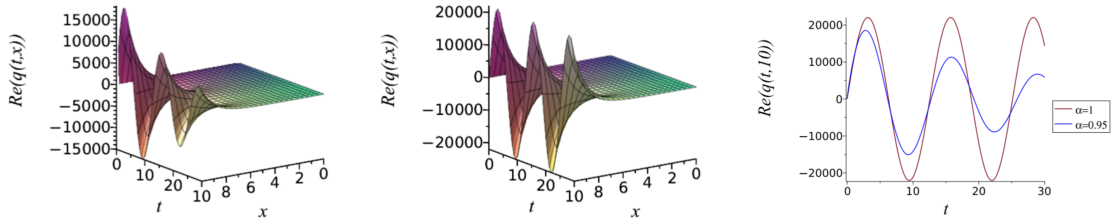


(a) Re($p(t,x)$) for $\alpha = 0.95$ (b) Re($p(t,x)$) for $\alpha = 1$ (c) 2D plot of Re($p(t,x)$) at $x = 10$

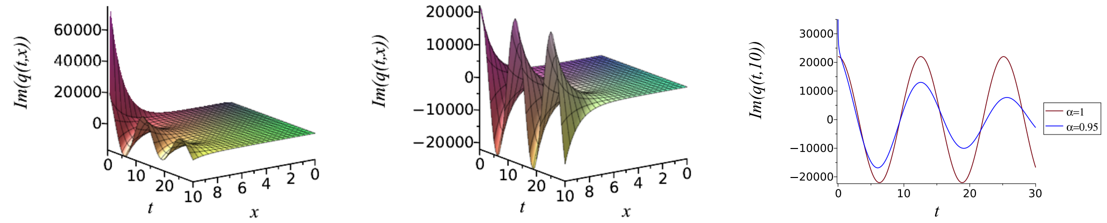
(d) Im($p(t,x)$) for $\alpha = 0.95$ (e) Im($p(t,x)$) for $\alpha = 1$ (f) 2D plot of Im($p(t,x)$) at $x = 10$

Figure 3. Subfigures $\{(a),(d)\}$ and $\{(b),(e)\}$ display the real and imaginary parts of $p(t,x)$, respectively, for $\alpha = 0.95$ and the classical case $\alpha = 1$, with $\lambda = 1$. Subfigures (c) and (f) present 2D plots of the real and imaginary parts at $x = 10$ for both values of α. These plots reveal that as α approaches 1, the amplitude of the oscillations increases noticeably.

In Figures 3 and 4, the three-dimensional (3D) and two-dimensional (2D) plots of the real and imaginary parts of the solutions $p(t,x)$ and $q(t,x)$, obtained from the classical vector field $\mathcal{W}_1 + \mathcal{W}_3$, are presented for two different values of α.



(a) Re($q(t,x)$) for $\alpha = 0.95$ **(b)** Re($q(t,x)$) for $\alpha = 1$ **(c)** 2D plot of Re($p(t,x)$) at $x = 10$



(d) Im($q(t,x)$) for $\alpha = 0.95$ **(e)** Im($q(t,x)$) for $\alpha = 1$ **(f)** 2D plot of Im($q(t,x)$) at $x = 10$

Figure 4. Subfigures $\{(a),(d)\}$ and $\{(b),(e)\}$ represent the real and imaginary parts of $q(t,x)$ for $\alpha = 0.95$ and the classical case $\alpha = 1$, with $\lambda = 1$. The 2D plots in (c) and (f), evaluated at $x = 10$, illustrate the effect of α on the oscillatory behavior, showing that as α approaches 1, the amplitude of the oscillations increases.

3.1.2. Nonclassical symmetries

To obtain new exact solutions for the heat equation, Bluman and Cole proposed a nonclassical reduction method [36]. The essence of this approach lies in incorporating a fixed surface condition, meaning that applying this condition along with the primary determining equations leads to a system of nonlinear determining equations for infinitesimals. In the analysis of the nonclassical scenario within Lie's symmetry theory, beyond the condition $\wp r^{(\alpha,k,h)}\mathcal{W} = 0$, the invariant surface conditions must also be satisfied. In the nonclassical symmetry method, by applying the invariant surface condition along with the governing differential equations, a nonlinear system of partial differential equations is obtained. This system yields the infinitesimals that characterize the nonclassical symmetries of the original problem. Unlike the classical Lie symmetry method, which requires the invariance of the entire differential equation under prolonged vector fields, the nonclassical approach imposes a more restrictive criterion by requiring invariance on a solution manifold defined by both the differential equation and the invariant surface condition.

As a result, the number of determining equations in the nonclassical framework is generally fewer than those in the classical method. This reduction in the determining system, however, comes at the cost of increased complexity due to its nonlinearity. Despite this, the nonclassical symmetry method is capable of uncovering a wider class of symmetry reductions and exact solutions that are not accessible through classical methods. Consequently, the overall solution set in the nonclassical case is typically more extensive, providing deeper insights into the structure and integrability of nonlinear differential equations.

In this work, we aim to apply this method to our system to derive new exact solutions. Consider the following invariant surface conditions:

$$\Omega_1 \equiv \zeta_1(t,x,p,q)p_x + \zeta_2(t,x,p,q)p_t - \varpi_1 = 0,$$
$$\Omega_2 \equiv \zeta_1(t,x,p,q)q_x + \zeta_2(t,x,p,q)q_t - \varpi_2 = 0.$$

Assume that, $\zeta_1 = 1$ and $\zeta_2 = 0$. Thus, the surface conditions are as follows:

$$p_x = \varpi_1, \quad q_x = \varpi_2.$$

Therefore

$$p_{xx} = \varpi_{1_x} + \varpi_1\varpi_{1_p} + \varpi_2\varpi_{1_q}, \qquad q_{xx} = \varpi_{2_x} + \varpi_1\varpi_{2_p} + \varpi_2\varpi_{2_q}.$$

By substituting these relations into the system (1.1), we obtain $\varpi_2 = p$ and $\varpi_1 = -q$. Consequently, we derive the vector field $\mathcal{W}_6 = \frac{\partial}{\partial x} - q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}$, whose corresponding invariant solutions for this case are given as follows:

$$p(t,x) = f(t)\cos(x) + g(t)\sin(x), \qquad q(t,x) = -g(t)\cos(x) + f(t)\sin(x). \tag{3.18}$$

Moreover, by applying (3.18), the reduced system takes the following form:

$$\begin{cases} \mathcal{D}_t^\alpha f(t) - \frac{1}{2}g(t) - f^2(t)g(t) - g^3(t) = 0, \\ \\ \mathcal{D}_t^\alpha g(t) + \frac{1}{2}f(t) + g^2(t)g(t) + f^3(t) = 0. \end{cases} \tag{3.19}$$

To compute a set of solutions for the system (3.19), it suffices to assume $g(t) = if(t)$ and solve the following equation:

$$\mathcal{D}_t^\alpha f(t) - \frac{i}{2}f(t) = 0. \tag{3.20}$$

The solution to Eq (3.20) can be obtained using the FLTs as follows:

$$f(t) = \lambda t^{2\alpha-2}E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^\alpha\right), \tag{3.21}$$

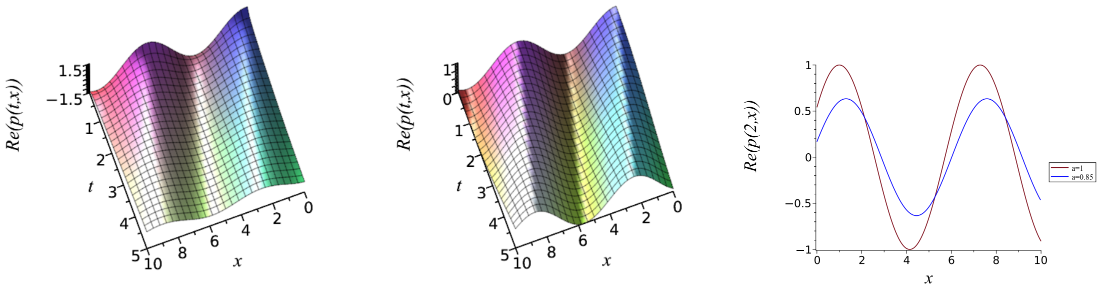
where $\lambda = \frac{k}{\Gamma(1-\alpha)}$ and k is a constant. Substituting Eq (3.21) to Eq (3.18), the exact solutions of system (1.1) are derived as follows:

$$p(t,x) = \lambda t^{2\alpha-2}\Big(E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^\alpha\right)\cos(x) + iE_{\alpha,2\alpha-1}\left(\frac{i}{2}t^\alpha\right)\sin(x)\Big),$$
$$q(t,x) = \lambda t^{2\alpha-2}\Big(E_{\alpha,2\alpha-1}\left(\frac{i}{2}t^\alpha\right)\sin(x) - iE_{\alpha,2\alpha-1}\left(\frac{i}{2}t^\alpha\right)\cos(x)\Big).$$

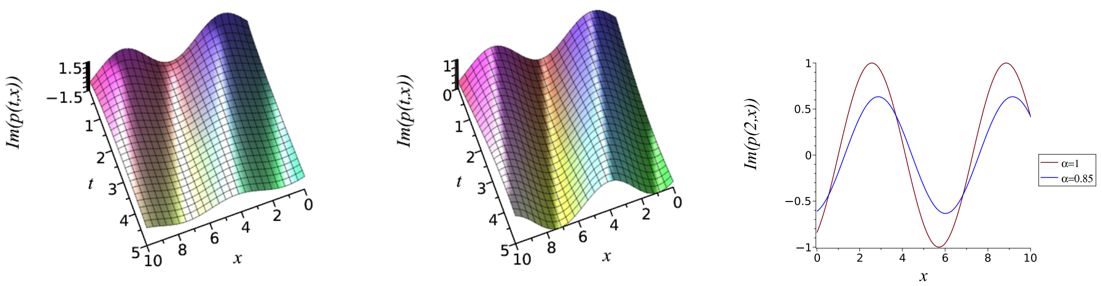
Hence,

$$p(t,x)=\lambda t^{2\alpha-2}\left[\cos(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)}\right)-\sin(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right)\right]$$
$$+\,i\lambda t^{2\alpha-2}\left[\sin(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)}\right)+\cos(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right)\right],$$
$$q(t,x)=\lambda t^{2\alpha-2}\left[\sin(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)}\right)+\cos(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right)\right]$$
$$-\,i\lambda t^{2\alpha-2}\left[\cos(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{i}{2}t^{\alpha})^{2j}}{\Gamma(2j\alpha+2\alpha-1)}\right)-\sin(x)\left(\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(\frac{i}{2}t^{\alpha})^{2j+1}}{\Gamma(2j\alpha+3\alpha-1)}\right)\right].$$

Figures 5 and 6 provide a detailed visualization of the real and imaginary components of the solutions $p(t,x)$ and $q(t,x)$, computed for two distinct values of α. These plots illustrate the spatiotemporal behavior of the solutions across the specified domain, with a focus on the effects of varying α on the oscillatory characteristics of the solutions. The representations in both 3D and 2D formats help highlight the differences in the behavior of the solutions as α changes, offering a clear insight into the dynamics of the system.

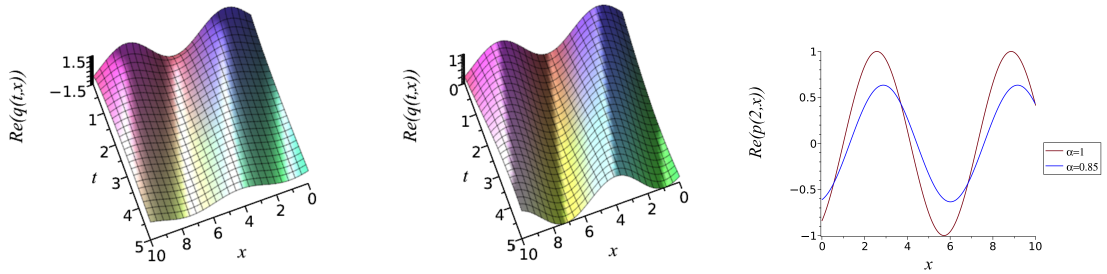


(a) Re($p(t,x)$) for $\alpha = 0.85$ **(b)** Re($p(t,x)$) for $\alpha = 1$ **(c)** 2D plot of Re($p(t,x)$) at $t = 2$

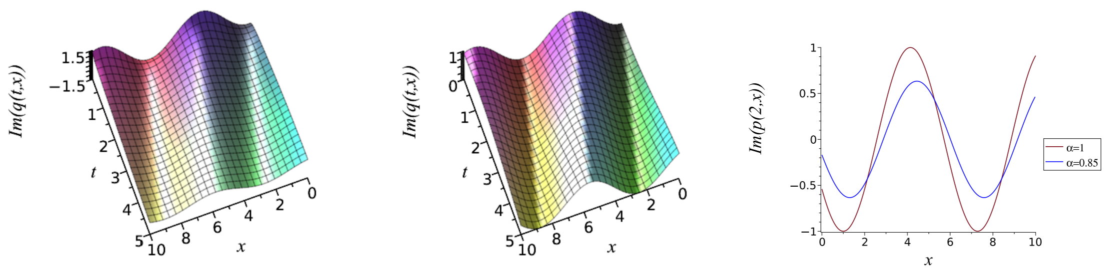


(d) Im($p(t,x)$) for $\alpha = 0.85$ **(e)** Im($p(t,x)$) for $\alpha = 1$ **(f)** 2D plot of Im($p(t,x)$) at $t = 2$

Figure 5. The solution $p(t,x)$, derived from the nonclassical vector field $\mathcal{W}_6$, is displayed for $\alpha = 0.85$ and $\alpha = 1$ with $\lambda = 1$. Subfigures (a), (b), (d), and (e) present 3D views of the real and imaginary parts over space and time. The corresponding 2D profiles at $t = 2$ are shown in (c) and (f). As observed in the plots, increasing the value of α significantly amplifies the oscillatory behavior of the solution.



(a) Re($q(t, x)$) for $\alpha = 0.85$ **(b)** Re($q(t, x)$) for $\alpha = 1$ **(c)** 2D plot of Re($q(t, x)$) at $t = 2$



(d) Im($q(t, x)$) for $\alpha = 0.85$ **(e)** Im($q(t, x)$) for $\alpha = 1$ **(f)** 2D plot of Im($q(t, x)$) at $t = 2$

Figure 6. Real and imaginary parts of the solution $q(t, x)$, associated with the nonclassical vector field $\mathcal{W}_6$, are visualized for $\alpha = 0.85$ and $\alpha = 1$ with $\lambda = 1$. The 3D plots in subfigures (a), (b), (d), and (e) show the real and imaginary parts over space and time, while the 2D plots in (c) and (f), taken at $t = 2$, illustrate the effect of increasing α on the oscillations' amplitude.

In the context of nonclassical symmetries, we consider $\zeta_2 \neq 0$ and $\zeta_1 = 1$, and assume that $\zeta_{2_p} = \zeta_{2_q} = 0$. Under these conditions, the corresponding surface constraints are given as follows:

$$p_x = \varpi_1 - \zeta_2 p_t,$$
$$q_x = \varpi_2 - \zeta_2 q_t.$$

In this case, we obtain

$$\mathcal{W}_7 = \frac{\partial}{\partial x} + c\frac{\partial}{\partial t}.$$

According to the vector field $\mathcal{W}_7$, the following invariant solutions can be derived:

$$\begin{cases} p(t, x) = f(\varepsilon), \\ q(t, x) = g(\varepsilon), \end{cases} \qquad \varepsilon = t - cx,$$

and these variables reduce the system (1.1) to the following system of FODEs:

$$\begin{cases} \mathcal{D}_t^\alpha f(\varepsilon) - \frac{1}{2}c^2 g''(\varepsilon) + f^2(\varepsilon)g(\varepsilon) + g^3(\varepsilon) = 0, \\ \\ \mathcal{D}_t^\alpha g(\varepsilon) + \frac{1}{2}c^2 f''(\varepsilon) - g^2(\varepsilon)f(\varepsilon) - f^3(\varepsilon) = 0. \end{cases}$$

Let $g(\varepsilon) = if(\varepsilon)$. It is enough to solve the following equation:

$$\mathcal{D}_t^\alpha f(\varepsilon) - \frac{i}{2}c^2 f''(\varepsilon) = 0. \tag{3.22}$$

For Eq (3.22), by using the FLTs, we have

$$f(\varepsilon) = \frac{2\lambda}{c^2 i}\varepsilon^\alpha E_{2-\alpha,\alpha+1}\left(\frac{2}{c^2 i}\varepsilon^{2-\alpha}\right) + \lambda_1 E_{2-\alpha,1}\left(\frac{2}{c^2 i}\varepsilon^{2-\alpha}\right) + \lambda_2 \varepsilon E_{2-\alpha,2}\left(\frac{2}{c^2 i}\varepsilon^{2-\alpha}\right),$$

where $\lambda = \dfrac{\lambda_1}{\Gamma(1-\alpha)}$, λ_1, andλ_2 are constants. Thus, the exact solution is given by

$$\begin{aligned} p(t,x) = {} & \frac{2\lambda}{c^2 i}(t-cx)^\alpha E_{2-\alpha,\alpha+1}\left(\frac{2}{c^2 i}(t-cx)^{2-\alpha}\right) + \lambda_1 E_{2-\alpha,1}\left(\frac{2}{c^2 i}(t-cx)^{2-\alpha}\right) \\ & + \lambda_2 (t-cx) E_{2-\alpha,2}\left(\frac{2}{c^2 i}(t-cx)^{2-\alpha}\right), \end{aligned}$$

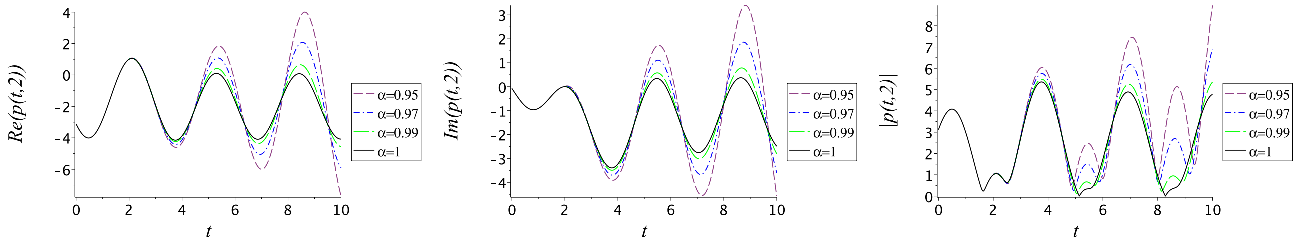
and

$$\begin{aligned} q(t,x) = {} & \frac{2\lambda}{c^2}(t-cx)^\alpha E_{2-\alpha,\alpha+1}\left(\frac{2}{c^2}(t-cx)^{2-\alpha}\right) + i\lambda_1 E_{2-\alpha,1}\left(\frac{2}{c^2 i}(t-cx)^{2-\alpha}\right) \\ & + i\lambda_2 (t-cx) E_{2-\alpha,2}\left(\frac{2}{c^2 i}(t-cx)^{2-\alpha}\right). \end{aligned}$$

By separating the real and imaginary parts of $p(t,x)$ and $p(t,x)$, the following exact solutions are obtained:

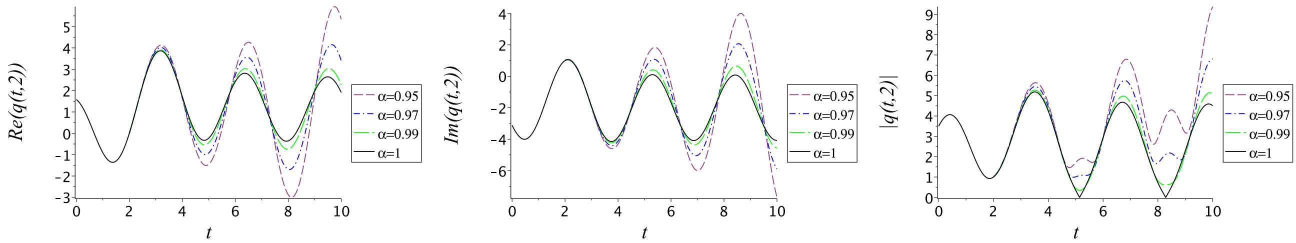
$$\begin{aligned} p(t,x) = {} & \frac{2\lambda}{c^2}(t-cx)^\alpha \left(\sum_{j=0}^{\infty} \frac{(-1)^{j+1}(2(t-cx)^{2-\alpha})^{2j+1}}{c^{4j+2}\Gamma(4j-2j\alpha+3)}\right) + (\lambda_1 + \lambda_2(t-cx))\left(\sum_{j=0}^{\infty} \frac{(-1)^{j}(2(t-cx)^{2-\alpha})^{2j}}{c^{4j}\Gamma(4j-2j\alpha+\alpha+1)}\right) \\ & + i\left[\frac{2\lambda}{c^2}(t-cx)^\alpha \left(\sum_{j=0}^{\infty} \frac{(-1)^{j}(2(t-cx)^{2-\alpha})^{2j}}{c^{4j}\Gamma(4j-2j\alpha+\alpha+1)}\right) + (\lambda_1 + \lambda_2(t-cx))\left(\sum_{j=0}^{\infty} \frac{(-1)^{j+1}(2(t-cx)^{2-\alpha})^{2j+1}}{c^{4j+2}\Gamma(4j-2j\alpha+3)}\right)\right], \\ q(t,x) = {} & \frac{2\lambda}{c^2}(t-cx)^\alpha \left(\sum_{j=0}^{\infty} \frac{(-1)^{j}(2(t-cx)^{2-\alpha})^{2j}}{c^{4j}\Gamma(4j-2j\alpha+\alpha+1)}\right) - (\lambda_1 + \lambda_2(t-cx))\left(\sum_{j=0}^{\infty} \frac{(-1)^{j+1}(2(t-cx)^{2-\alpha})^{2j+1}}{c^{4j+2}\Gamma(4j-2j\alpha+3)}\right) \\ & + i\left[\frac{2\lambda}{c^2}(t-cx)^\alpha \left(\sum_{j=0}^{\infty} \frac{(-1)^{j+1}(2(t-cx)^{2-\alpha})^{2j+1}}{c^{4j+2}\Gamma(4j-2j\alpha+3)}\right) + (\lambda_1 + \lambda_2(t-cx))\left(\sum_{j=0}^{\infty} \frac{(-1)^{j}(2(t-cx)^{2-\alpha})^{2j}}{c^{4j}\Gamma(4j-2j\alpha+\alpha+1)}\right)\right]. \end{aligned} \tag{3.23}$$

To illustrate that the plots are consistent with the solutions (3.23), the series are truncated at $N = 30$. Figures 7 and 8 present the real and imaginary parts, as well as the absolute values of $p(t,x)$ and $q(t,x)$, for $c = \lambda = \lambda_1 = \lambda_2 = 1$ and different fractional orders α at $x = 2$.



(a) (b) (c)

Figure 7. The plots, derived from the classical vector field $\mathcal{W}_7$, show (a) the real part, (b) the imaginary part, and (c) the absolute value of $p(t, x)$ for various values of α at $x = 2$. These plots demonstrate that the obtained solutions exhibit convergence behavior.



(a) (b) (c)

Figure 8. The plots, derived from the classical vector field $\mathcal{W}_7$, show (a) the real part, (b) the imaginary part, and (c) the absolute value of $q(t, x)$ for various values of α at $x = 2$. These plots indicate that the obtained solutions consistently exhibit convergent behavior.

4. Conservation laws of the TFNLDS

The conservation laws (CLs) of the system (1.1) are derived in this section through Ibragimov's method [37], a well-established approach for obtaining CLs in time-fractional nonlinear differential systems. This methodology has been extensively utilized in the study of time-fractional equations [38–43]. The aim is to identify CLs corresponding to both the classical and nonclassical vector fields. A CL for the system (1.1) is expressed as follows:

$$D_x(\mathcal{V}^x) + D_t(\mathcal{V}^t)|_{(3.1)} = 0,$$

where $\mathcal{V}^t$ represents the time flow and $\mathcal{V}^x$ represents the space flow. As observed by Ibragimov [37], the formal Lagrangian of the system (1.1) can be expressed as follows:

$$\mathcal{H} = \mu_1(t, x)\Big(\mathcal{D}_t^\alpha p - \frac{1}{2}q_{xx} + p^2q + q^3\Big) + \mu_2(t, x)\Big(\mathcal{D}_t^\alpha q + \frac{1}{2}p_{xx} - pq^2 - p^3\Big), \tag{4.1}$$

where the variables $\mu_1(t,x)$ and $\mu_2(t,x)$ are treated as dependent, and the corresponding adjoint equations for the formal Lagrangian operator (4.1) are derived as follows:

$$\begin{cases} \mathcal{M}_1^* \equiv \dfrac{\delta \mathcal{H}}{\delta p} = 0, \\ \\ \mathcal{M}_2^* \equiv \dfrac{\delta \mathcal{H}}{\delta q} = 0, \end{cases} \tag{4.2}$$

where $\dfrac{\delta}{\delta p}$ and $\dfrac{\delta}{\delta q}$ represent the Euler–Lagrange operators, which are defined as follows:

$$\frac{\delta}{\delta p} = \frac{\partial}{\partial p} + (\mathcal{D}_t^\alpha)^* \frac{\partial}{\partial(\mathcal{D}_t^\alpha p)} + \sum_{k\geq 1}(-1)^k D_x ... D_x \frac{\partial}{\partial p_{kx}},$$

and

$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} + (\mathcal{D}_t^\alpha)^* \frac{\partial}{\partial(\mathcal{D}_t^\alpha q)} + \sum_{k\geq 1}(-1)^k D_x ... D_x \frac{\partial}{\partial q_{kx}},$$

where the adjoint operator of $\mathcal{D}_t^\alpha$ is denoted by $(\mathcal{D}_t^\alpha)^*$. By taking the RL fractional differential operators into account, the following expression is obtained:

$$\begin{aligned} &(\mathcal{D}_t^\alpha)^* = (-1)^m J_T^{m-\alpha}(\partial_t^n) = (\mathcal{D}_T^\alpha)_t^C, \\ &J_T^{m-\alpha} s(t,x) = \int_t^T \frac{s(\tau,x)(\tau - t)^{m-(1+\alpha)}}{\Gamma(m-\alpha)} d\tau, \qquad m = [\alpha] + 1, \end{aligned}$$

where the operator $(\mathcal{D}_T^\alpha)_t^C$ represents the right-sided Caputo fractional derivative. By replacing Eq (4.1) in Eq (4.2), the adjoint equations for the system (1.1) are obtained:

$$\begin{cases} \mathcal{M}_1^* = \mu_1 (2pq) - \mu_2 \left(q^2 + 3p^2\right) + (\mathcal{D}_t^\alpha)^* \mu_1 + \frac{1}{2}\mu_{2_{xx}}, \\ \mathcal{M}_2^* = \mu_1 \left(p^2 + 3q^2\right) - \mu_2 (2pq) + (\mathcal{D}_t^\alpha)^* \mu_2 - \frac{1}{2}\mu_{2_{xx}}. \end{cases} \tag{4.3}$$

Referring to [37], the system (1.1) admits the following CL:

$$D_x \mathcal{V}_i^x + D_t \mathcal{V}_i^t = 0, \tag{4.4}$$

where the specified equations represent the conserved vectors $\mathcal{V}_i = (\mathcal{V}_i^t, \mathcal{V}_i^x)$

$$\begin{aligned} \mathcal{V}_i^x &= \left({}^p W_i \frac{\delta \mathcal{H}}{\delta p_x} + \sum_{j\geq 1} D_x ... D_x ({}^p W_i) \frac{\partial \mathcal{H}}{\partial p_{(j+1)x}}\right) + \left({}^q W_i \frac{\delta \mathcal{H}}{\delta q_x} + \sum_{j\geq 1} D_x ... D_x ({}^q W_i) \frac{\partial \mathcal{H}}{\partial q_{(j+1)x}}\right), \\ \mathcal{V}_i^t &= \sum_{j=0}^{n-1} (-1)^j \left[\mathcal{D}_t^{\alpha-1-j}({}^p W_i) D_t^j \left(\frac{\partial \mathcal{H}}{\partial(\mathcal{D}_t^\alpha p)}\right) + \mathcal{D}_t^{\alpha-1-j}({}^q W_i) D_t^j \left(\frac{\partial \mathcal{H}}{\partial(\mathcal{D}_t^\alpha q)}\right)\right] \\ &\quad - (-1)^n \left[\mathcal{J}\left({}^p W_i, D_t^n (\frac{\partial \mathcal{H}}{\partial(\mathcal{D}_t^\alpha p)})\right) + \mathcal{J}\left({}^q W_i, D_t^n (\frac{\partial \mathcal{H}}{\partial(\mathcal{D}_t^\alpha q)})\right)\right], \qquad n = [\alpha] + 1, \end{aligned} \tag{4.5}$$

where ${}^{p}W_i = \varpi_{1_i} - \zeta_{1_i} p_x - \zeta_{2_i} p_t$, ${}^{q}W_i = \varpi_{2_i} - \zeta_{1_i} q_x - \zeta_{2_i} q_t$, and the notation $\mathcal{J}$ denotes the following integral:

$$\mathcal{J}(h, s) = \int_0^t \int_t^T \frac{h(\eta, x)s(\sigma, x)(\sigma - \eta)^{n-(\alpha+1)}}{\Gamma(n - \alpha)} d\sigma d\eta. \tag{4.6}$$

If we have the following relation for the time-fractional nonlinear Eq (4.1), then we can say that the system (1.1) is self-adjoint

$$\mathcal{M}^* \equiv \frac{\delta \mathcal{H}}{\delta p} = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2,$$
$$\mathcal{M}^* \equiv \frac{\delta \mathcal{H}}{\delta p} = \lambda_3 \Lambda_1 + \lambda_4 \Lambda_2,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are unknown and are to be determined. Thus, we can write the nonlinear self adjoint condition as follows:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \qquad \mu_1(x, t, p) = A, \mu_2(x, t, q) = B, \qquad A, B \in \mathbb{R}.$$

Hence, if we suppose that $A = B = 1$, then

$$\mathcal{H} = \mathcal{D}_t^{\alpha} p + \mathcal{D}_t^{\alpha} q + \frac{1}{2}(p_{xx} - q_{xx}) + p^2 q + q^3 - pq^2 - p^3. \tag{4.7}$$

Drawing on the previous analysis and the generators of both classical and nonclassical Lie symmetries, the conserved vectors (CVs) for the TFNLDS are derived as follows. Initially, for the classical vector fields, the CVs are computed as follows.

CVs for ($\mathcal{W}_1$): In this case, $\mathcal{W}_1$ is associated with the Lie characteristic functions given by

$${}^{p}W_1 = -p_x, \qquad {}^{q}W_1 = -q_x. \tag{4.8}$$

Using these functions, the CVs corresponding to $\mathcal{W}_1$ are determined as outlined below:

$$\mathcal{V}_1^x = {}^{p}W_1\Big(-D_x \frac{\partial \mathcal{H}}{\partial p_{xx}}\Big) + D_x({}^{p}W_1)\frac{\partial \mathcal{H}}{\partial p_{xx}} + {}^{q}W_1\Big(-D_x \frac{\partial \mathcal{H}}{\partial q_{xx}}\Big) + D_x({}^{q}W_1)\frac{\partial \mathcal{H}}{\partial q_{xx}},$$
$$\mathcal{V}_1^t = -J^{1-\alpha}({}^{p}W_1) - J^{1-\alpha}({}^{q}W_1),$$

and

$$\mathcal{V}_1^x = \frac{1}{2}(q_{xx} - p_{xx}),$$
$$\mathcal{V}_1^t = -J^{1-\alpha} p_x - J^{1-\alpha} q_x.$$

CVs for ($\mathcal{W}_2$): For the generator $\mathcal{W}_2 = 4t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + \alpha p\frac{\partial}{\partial p} + \alpha q\frac{\partial}{\partial q}$, the following Lie characteristic functions are concluded:

$${}^{p}W_2 = \alpha p - 2\alpha x p_x - 4t p_t, \qquad {}^{q}W_2 = \alpha q - 2\alpha x q_x - 4t q_t, \tag{4.9}$$

then

$$\mathcal{V}_2^x = \frac{1}{2}\left(-\alpha p_x - 2\alpha x p_{xx} - 4t p_{tx}\right) - \frac{1}{2}\left(-\alpha q_x - 2\alpha x q_{xx} - 4t q_{tx}\right),$$
$$\mathcal{V}_2^t = -J^{1-\alpha}(\alpha p - 2\alpha x p_x - 4t p_t) - J^{1-\alpha}(\alpha q - 2\alpha x q_x - 4t q_t).$$

CVs for ($\mathcal{W}' = \mathcal{W}_1 + \mathcal{W}_3$): For the generator $\mathcal{W}' = \frac{\partial}{\partial x} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$, we have

$$^{p}W' = p - p_x, \qquad ^{q}W' = q - q_x, \tag{4.10}$$

and thus

$$\mathcal{V}_3^x = \frac{1}{2}\left(p_x - p_{xx}\right) - \frac{1}{2}\left(q_x - q_{xx}\right),$$
$$\mathcal{V}_3^t = -J^{1-\alpha}(p - p_x) - J^{1-\alpha}(q - q_x).$$

Now, the investigation of the nonclassical vector fields proceeds as follows.

CVs for ($\mathcal{W}_6$): In the case where $\zeta_1 = 1$ and $\zeta_2 = 0$, the vector field is expressed as $\mathcal{W}_6 = \frac{\partial}{\partial x} - q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}$. From this, the following Lie characteristic functions are derived:

$$^{p}W_6 = -q - p_x, \qquad ^{q}W_6 = p - q_x. \tag{4.11}$$

Therefore

$$\mathcal{V}_6^x = \frac{1}{2}(-q_x - p_{xx} - p_x + q_{xx}),$$
$$\mathcal{V}_6^t = -J^{1-\alpha}(-q - p_x) - J^{1-\alpha}(p - q_x).$$

CVs for ($\mathcal{W}_7$): In this case, $\zeta_1 = 1$ and $\zeta_2 \neq 0$, and the vector field is expressed as $\mathcal{W}_7 = \frac{\partial}{\partial x} + c\frac{\partial}{\partial t}$. From this, the following Lie characteristic functions are derived:

$$^{p}W_7 = -p_x - c p_t, \qquad ^{q}W_7 = -q_x - c q_t, \tag{4.12}$$

and

$$\mathcal{V}_7^x = \frac{1}{2}(q_{xx} - p_{xx} + c q_{tx} - c p_{tx}),$$
$$\mathcal{V}_7^t = -J^{1-\alpha}(-p_x - c p_t) - J^{1-\alpha}(-q_x - c q_t).$$

5. Conclusions

In this paper, we studied the classical and nonclassical Lie symmetry methods and CLs of the TFNLDS. This study represents the first exploration of exact solutions for the TFNLDS equation incorporating the RL fractional derivative. The significance of this investigation lies in its contribution to the analytical study of nonlinear fractional systems, which are essential for modeling memory-dependent and nonlocal physical phenomena. By applying Lie group analysis, we determined the system's symmetries and employed both classical and nonclassical Lie symmetry methods to derive similarity reductions, transforming the original equation into reduced forms on the basis of the obtained vector fields and corresponding invariant solutions. This methodological framework allowed for systematic reductions and the construction of exact forms, providing new insights into the structure of such fractional systems.

Additionally, we constructed exact solutions using various approaches, including the power series method for the derived generators. The use of different solution strategies emphasized the robustness and flexibility of our analytical treatment.

Our analysis, supported by Figures 1, 2, 7, and 8, demonstrated that the obtained solutions exhibited convergence behavior in both classical and nonclassical cases. These results underscore the reliability of the derived solutions and confirm the effectiveness of the symmetry-based reductions.

Furthermore, on the basis of the Lie symmetry generators, we systematically constructed CLs for the corresponding classical and nonclassical vector fields of the TFNLDS. These conservation laws reflect the underlying invariance properties and provide a deeper physical understanding of the model.

These results highlighted the effectiveness of the proposed approach in finding analytical solutions to a broad class of FPDEs, making it a promising tool for further studies in fractional systems. Therefore, this work not only offers exact solutions for a specific fractional model but also establishes a general pathway for tackling nonlinear fractional PDEs through symmetry and conservation law techniques. Extending this approach to higher-dimensional systems or using other fractional derivative operators (such as Caputo or Hadamard) presents an important challenge for future work. In addition, future research could explore the applicability of this method to coupled fractional equations and systems with more complex boundary conditions, as well as its use in modeling real-world phenomena in fields such as viscoelasticity, fluid dynamics, and anomalous diffusion.

Author contributions

Farzaneh Alizadeh: Conceptualization, writing original draft, methodology, investigation, formal analysis, writing review and editing; Kamyar Hosseini: Writing original draft, methodology, formal analysis; Samad Kheybari: Formal analysis, supervision, writing review and editing.
All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Kamyar Hosseini is a Guest Editor of special issue "Emerging Trends in Algebra, Geometry, and Topology of Soliton Systems" for AIMS Mathematics. Kamyar Hosseini was not involved in the editorial review and the decision to publish this article.

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