AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 11465–11487.
DOI:10.3934/math.2025522
Received: 12 February 2025
Revised: 06 May 2025
Accepted: 09 May 2025
Published: 20 May 2025

Research article

An accelerated conjugate method for split variational inclusion problems with applications

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Abstract: In this work, split variational inclusion problems were investigated by combining new stepsizes and inertia with conjugate gradient methods in real Hilbert spaces, in which the inertial steps were used to speed up the convergent rate of the methods, and the new stepsizes not only avoided computing the operator norm, but also ensured that the strong convergence of the methods holds without Lipschitz continuity of the monotone operator. Also, the proximal operator was computed less than that in the original method. Further, the split feasibility and split minimization problems were considered. Finally, several examples were used for illustration and comparison.

Keywords: split variational inclusion problem; inertial conjugate gradient method; new stepsize; strong convergence
Mathematics Subject Classification: 35A15, 47J20, 47J25, 49J40

1. Introduction

Let χ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $G:\chi\to 2^{\chi}$ be a set-valued mapping with its domain $\mathbb{D}(G):=\{x\in\chi : G(x)\neq\emptyset\}$. Furthermore, G is named monotone, if

$$\langle \hat{c}-\hat{d}, x-y\rangle \geq 0,\ \forall\ \hat{c}\in Gx,\ \hat{d}\in Gy. \tag{1.1}$$

The graph of G is the subset of $\chi\times\chi$ presented by

$$\mathrm{Graph}(G) := \{(x,\hat{c})\in\mathbb{D}(G)\times Gx\}. \tag{1.2}$$

A monotone mapping G is named *maximal* if the graph is not properly contained in the graph of any other monotone mapping. As is known to us, a monotone mapping G is *maximal*, iff, for all

$(x,\hat{c}) \in \chi \times \chi$,

$$\langle \hat{c} - \hat{d}, x - y \rangle \geq 0, \ \forall \ (y, \hat{d}) \in \text{Graph}(G), \tag{1.3}$$

and the resulted $\hat{c} \in Gx$. Subsequently, the resolvent of a maximally monotone operator $G : \chi \to 2^{\chi}$ with parameter $\kappa > 0$ is presented by

$$J_{\kappa}^{G}(x) = (I + \kappa G)^{-1}(x), \ \forall x \in \chi, \tag{1.4}$$

where I denotes the identity operator on χ.

Based on the context of the maximal monotone operator, the following split variational inclusion problem (SVIP) is investigated in two real Hilbert spaces χ_1 and χ_2:

$$\text{Find } z \in \chi_1 \text{ such that } 0 \in G_1(z) \text{ and } 0 \in G_2(Az), \tag{1.5}$$

where $G_1 : \chi_1 \to 2^{\chi_1}$ and $G_2 : \chi_2 \to 2^{\chi_2}$ are two maximally monotone multi-valued operators and $A : \chi_1 \to \chi_2$ is a bounded linear operator. The solution set of the SVIP is denoted by

$$S = \{z \in \chi_1 : 0 \in G_1(z) \text{ and } 0 \in G_2(Az)\}. \tag{1.6}$$

The SVIP was analyzed by Moudafi [23] following by Censor et al. [11] who presented and considered split variational inequalities. Proximal split feasibility problems [11, 24], split variational inequality problems [2, 3], and split null point problems [4, 30, 37] are some special cases of the SVIP. These studied problems have been applied to some important fields such as intensity-modulated radiation therapy treatment planning [7, 25, 34] and data compression [10, 18, 26]. These applications have been deeply solved by many numerical algorithms, for example proximal algorithms [15, 16, 37, 38], inertial proximal methods [22, 28] and conjugate gradient methods [15, 17, 33, 40] and the references therein. Among them, a classical approach to solve the SVIP is the following Byrne algorithm [8].

$$x_{n+1} = J_{\kappa}^{G_1}\left(x_n - \tilde{\ell} A^*\left(I - J_{\kappa}^{G_2}\right) A x_n\right), \tag{1.7}$$

where $\kappa > 0$, and A^* is the adjoint operator of A. The weak convergence of algorithm (1.7) was proved by using the condition $\tilde{\ell} \in \left(0, \frac{2}{\|A\|^2}\right)$. Its strong convergence is also established by the following iterative scheme:

$$x_{n+1} = \tilde{\alpha}_n x_0 + (1 - \tilde{\alpha}_n) J_{\kappa}^{G_1}\left(x_n - \tilde{\ell} A^*\left(I - J_{\kappa}^{G_2}\right) A x_n\right), \ x_0 \in H_1, \tag{1.8}$$

where $\{\tilde{\alpha}_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \tilde{\alpha}_n = 0$ and $\sum\limits_{n=1}^{\infty} \tilde{\alpha}_n = \infty$. Inspired by his work, Chuang et al. [12] presented the following conjugate method for solving the SVIP and obtained its strong convergence.

$$\begin{cases} \tilde{\Gamma}_n = -A^*\left(I - J_{\kappa_n}^{G_2}\right) A x_n + \tilde{\omega}_n \tilde{\Gamma}_{n-1}, \ y_n = J_{\kappa_n}^{G_1}\left(\left(1 - \tilde{b}_n \tilde{\ell}_n\right) x_n - \tilde{\ell}_n A^*\left(I - J_{\kappa_n}^{G_2}\right) A x_n + \tilde{\nu}_n \tilde{\Gamma}_n\right), \\ x_{n+1} = J_{\kappa_n}^{G_1}\left(x_n - \tilde{\mu}_n L_n\right)), L_n = x_n - y_n - \tilde{\ell}_n\left[A^*\left(I - J_{\kappa_n}^{G_2}\right) A x_n - A^*\left(I - J_{\kappa_n}^{G_2}\right) A y_n\right], \\ \tilde{\mu}_n = \frac{\langle x_n - y_n, L_n \rangle}{\|L_n\|^2}, \end{cases} \tag{1.9}$$

where the stepsize $\tilde{\ell}_n \in \left(0, \ \min\left(\frac{\delta}{\|A\|^2}, \frac{2}{\|A\|^2+2}\right)\right)$ fulfills

$$\tilde{\ell}_n \left\| A^*\left(I - J_{\kappa_n}^{G_2}\right) A x_n - A^*\left(I - J_{\kappa_n}^{G_2}\right) A y_n \right\| \leq \delta \|x_n - y_n\|, 0 < \delta < 1 \tag{1.10}$$

and (a) $\lim_{n\to\infty}\tilde{b}_n = \lim_{n\to\infty}\tilde{\omega}_n = \lim_{n\to\infty}\tilde{\nu}_n = 0$, $\sum_{n=1}^{\infty}\tilde{b}_n = \infty$, $\liminf_{n\to\infty}\tilde{\ell}_n > 0$, and $\liminf_{n\to\infty}\kappa_n > 0$; (b) $\{\tilde{b}_n\} \subset [0,1]$, $\{\tilde{\nu}_n\} \subset [0,1]$, $\{\tilde{\omega}_n\} \subset [0,1]$,$\lim_{n\to\infty}\frac{\tilde{\nu}_n}{\tilde{b}_n} = \tilde{t}$ for some $\tilde{t} \geq 0$, and $\{x_n\}$ is a bounded sequence.

It is noticed that the stepsize $\tilde{\ell}_n$ depends on the operator norm $\|A\|$, which is usually overconservative in practice [13].

The inertia was adopted by Nesterov [29] to speed up the convergence rate of the following heavy ball method [32]:

$$\begin{aligned} \tilde{\varphi}_n &= x_n + \tilde{\theta}_n(x_n - x_{n-1}) \\ x_{n+1} &= \tilde{\varphi}_n - \tilde{\ell}_n \nabla q_n(\tilde{\varphi}_n), n \geq 1, \end{aligned} \tag{1.11}$$

where $\tilde{\theta}_n \in [0,1)$ and ∇q_n is a gradient of the differential function q_n. The term $\tilde{\theta}_n(x_n - x_{n-1})$ stands for the inertia. Using it, Alvarez and Attouch [1] presented an inertial proximal point algorithm. Recently, some proximal algorithms were accelerated for the SVIP [14, 21, 35]. However, the study of the algorithm (1.9) with inertia has yet to be founded. Also, the proof of the algorithm's convergence needs some strong conditions such as Lipschitz continuity of the monotone operator $A^*\left(I - J_{\kappa_n}^{G_2}\right)A$. The above considerations develop the following question:

Question: Can we make new modifications of algorithm (1.9) such that the stepsize $\tilde{\ell}_n$ is not determined by the norm $\|A\|$ and the strong convergence holds without Lipschitz continuity of the monotone operator $A^*\left(I - J_{\kappa_n}^{G_2}\right)A$?

From this question, our main contributions are the following:

• We combine the inertial technique [5] with algorithm (1.9), which accelerates the convergence rate of algorithm (1.9) and creates a new convergence result;

• We propose a new stepsize which is not determined by the norm $\|A\|$ [15, 20, 36] for reducing the computation of algorithm (1.9), thereby an accelerated convergent algorithm is generated;

• We prove the strong convergence of our methods without Lipschitz continuity of the monotone operator $A^*\left(I - J_{\kappa_n}^{G_2}\right)A$ [36, 39];

• We apply our proposed algorithms to solve split feasibility and split minimization problems, and some numerical examples are provided for illustration.

The paper is constructed as follows. In Section 2, some useful notions and results are stated. The main theorem is analyzed in Section 3, an application is stated in Section 4, numerical experiments are provided in Section 5, and we give a conclusion in the final section.

2. Preliminaries

The symbols $\rightharpoonup$ and $\to$ represent weak and strong convergence, respectively. Let χ be a real Hilbert space and $C \subset \chi$ be a non-empty closed convex set. Let $x \in \chi$ and the orthogonal projection of x onto C bd denoted by $P_C x$, namely,

$$P_C x = \text{argmin}\{\|x - y\| \mid y \in C\}.$$

Definition 2.1. Let $\tilde{h} : \chi \to \chi$ be a mapping, then
(a) $\tilde{h}$ is called nonexpansive if

$$\|\tilde{h}x - \tilde{h}y\| \leq \|x - y\|, \ \forall\ x,\ y \in \chi,$$

(b) $\tilde{h}$ is said to be firmly nonexpansive if

$$\langle \tilde{h}x - \tilde{h}y, x - y\rangle \geq \|\tilde{h}x - \tilde{h}y\|^2, \ \forall\ x,\ y \in \chi;$$

or, equivalently,

$$\|\tilde{h}x - \tilde{h}y\|^2 \leq \|x - y\|^2 - \|(I - \tilde{h})x - (I - \tilde{h})y\|^2,\ \forall\ x, y \in \chi.$$

Lemma 2.1. Let $\forall\ x,\ y,\ \mu,\ \nu \in \chi$, then
(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,
(b) $2\langle x - y, \mu - \nu\rangle = \|x - \nu\|^2 + \|y - \mu\|^2 - \|x - \mu\|^2 - \|y - \nu\|^2$.
Lemma 2.2. (**Demiclosedness principle** [19]) Assume that $\tilde{h} : C \to C$ is a nonexpansive mapping. Then the following implication holds:

$$x_n \rightharpoonup x \in C \text{ and } \lim_{n\to\infty}\|x_n - \tilde{h}x_n\| = 0 \Rightarrow x = \tilde{h}x.$$

To investigate the SVIP, we denote by $G^{-1}(0) = \{x \in \chi : 0 \in Gx\}$, $D(\tilde{h})$ the domain of $\tilde{h}$ and $Fix(\tilde{h})$ the fixed point set of $\tilde{h}$, or equivalently, $Fix(\tilde{h}) = \{x \in \chi : x = \tilde{h}x\}$.
Lemma 2.3. [6] Let $\{\bar{a}_n\}$ be a nonnegative real sequence, and $\exists N > 0$, $\forall n \geq N$, such that $\bar{a}_{n+1} \leq (1 - \bar{\alpha}_n)\bar{a}_n + \bar{\alpha}_n\bar{\iota}_n$, where $\{\bar{\alpha}_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \bar{\alpha}_n = \infty$, and $\{\bar{\iota}_n\}$ is a sequence such that $\limsup_{n\to\infty} \bar{\iota}_n \leq 0$. Then $\lim_{n\to\infty}\bar{a}_n = 0$.
Lemma 2.4. [27, 38] Let $G : \chi \to 2^\chi$ be a set-valued maximal monotone mapping and $\kappa > 0$. The following statements hold:
(a) J_κ^G is a single-valued and firmly nonexpansive mapping;
(b) $D(J_\kappa^G) = \chi$ and $Fix(J_\kappa^G) = \{x \in D(G) : 0 \in Gx\}$;
(c) $\|x - J_\kappa^G x\| \leq \|x - J_{\tilde{\nu}}^G x\|$ for all $0 < \kappa < \tilde{\nu}$ and $x \in \chi$;
(d) $(I - J_\kappa^G)$ is a firmly nonexpansive mapping;
(e) for $G^{-1}(0) \neq \emptyset$, one has

$$\|x - J_\kappa^G x\|^2 + \|J_\kappa^G x - \tilde{x}\|^2 \leq \|x - \tilde{x}\|^2, \forall\ x \in \chi,\ \tilde{x} \in G^{-1}(0),$$

and

$$\langle x - J_\kappa^G x, J_\kappa^G x - \tilde{w}\rangle \geq 0, \forall\ x \in \chi,\ \tilde{w} \in G^{-1}(0).$$

Lemma 2.5. [16] Let χ_1 and χ_2 be two real Hilbert spaces, $A : \chi_1 \to \chi_2$ be an operator, and $\kappa > 0$. Let $G : \chi_2 \to 2^{\chi_2}$ be a set-valued maximal monotone mapping. Define a mapping $\tilde{h} : \chi_1 \to \chi_1$ by $\tilde{h}x = A^*(I - J_\kappa^G)Ax$ for all $x \in \chi_1$. Then, we have the following:
(a) $\|(I - J_\kappa^G)Ax - (I - J_\kappa^G)Ay\|^2 \leq \langle \tilde{h}x - \tilde{h}y, x - y\rangle, \forall\ x,\ y \in \chi_1$;
(b) $\|\tilde{h}x - \tilde{h}y\|^2 \leq \|A\|^2\langle \tilde{h}x - \tilde{h}y, x - y\rangle, \forall\ x,\ y \in \chi_1$.
Lemma 2.6. [28] Let $\{\bar{a}_n\}$ be a nonnegative real sequence such that there exists a subsequence $\{\bar{a}_{n_i}\}$ of $\{\bar{a}_n\}$ such that $\bar{a}_{n_i} < \bar{a}_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$, and the following relations are satisfied by all (sufficiently large) number $k \in \mathbb{N}$,

$$\bar{a}_{m_k} \leq \bar{a}_{m_k+1},\ \bar{a}_k \leq \bar{a}_{m_k+1}.$$

Indeed, $\{m_k\}$ is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $\bar{a}_n < \bar{a}_{n+1}$.
Lemma 2.7. [31] Let $\{\pi_n\}$ be a sequence of nonnegative numbers fulfilling:

$$\pi_{n+1} \leq \phi_n\pi_n + \psi_n,\ \forall n \in \mathbb{N},$$

where $\{\phi_n\}$ and $\{\psi_n\}$ are sequences of nonnegative numbers such that $\{\phi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty}(\phi_n - 1) < \infty$, and $\sum_{n=1}^{\infty} \psi_n < \infty$. Then $\lim_{n\to\infty}\pi_n$ exists.

3. Strong convergence

In this section, we introduce an inertial conjugate method with non-Lipschitz stepsizes for split variational inclusion problems and prove its strong convergence without Lipschitz continuity of the monotone operator $A^*(I - J_{\kappa_n}^{G_2})A$. Before offering our method, some important assumptions are the following.

(a1) The solution set of the SVIP is nonempty, i.e., $S \neq \emptyset$.

(a2) Let χ_1 and χ_2 be real Hilbert spaces and $A : \chi_1 \to \chi_2$ be a bounded linear operator with its adjoint A^*. Assume that $G_1 : \chi_1 \to 2^{\chi_1}$ and $G_2 : \chi_2 \to 2^{\chi_2}$ are maximal monotone mappings.

(a3) Let $\{\tilde{\tau}_n\} \subset [0, \tilde{\theta})$ be a positive sequence such that $\tilde{\tau}_n = o(\tilde{b}_n)$, i.e., $\lim_{n\to\infty} \frac{\tilde{\tau}_n}{\tilde{b}_n} = 0$, where $\{\tilde{b}_n\} \subset (0, 1)$ satisfies $\sum_{n=1}^{\infty} \tilde{b}_n = \infty$, $\lim_{n\to\infty} \tilde{b}_n = 0$.

Algorithm 1

Step 0. Given $\tilde{\ell}_1 > 0$, $\Gamma_0 = 0$, $\delta \in (0, \mu) \subset \left(0, \frac{1}{2}\right)$, x_0, $x_1 \in \chi_1$, $\{\tilde{\theta}_n\} \subset [0, \theta) \subset [0, 1)$ and $\{\kappa_n\} \subset (0, \infty)$. The sequence $\{\phi_n\}$ from Lemma 2.7, and $\{\tilde{\nu}_n\} \subset [0, 1]$.

Step 1. Take x_{n-1}, x_n $(n \geq 1)$ and compute

$$\begin{aligned} \tilde{\varphi}_n &= x_n + \tilde{\theta}_n (x_n - x_{n-1}), \\ \tilde{\Gamma}_n &= -A^* \left(I - J_{\kappa_n}^{G_2}\right) A \tilde{\varphi}_n + \tilde{\omega}_n \tilde{\Gamma}_{n-1}, \\ y_n &= J_{\kappa_n}^{G_1} \left((1 - \tilde{b}_n \tilde{\ell}_n) \tilde{\varphi}_n - \tilde{\ell}_n A^* \left(I - J_{\kappa_n}^{G_2}\right) A \tilde{\varphi}_n + \tilde{\nu}_n \tilde{\Gamma}_n\right), \end{aligned} \tag{3.1}$$

where the stepsize $\tilde{\ell}_n$ is computed by

$$\tilde{\ell}_{n+1} = \begin{cases} \min\left\{ \frac{\delta \|\tilde{\varphi}_n - y_n\|^2}{\left\langle A^*\left(I - J_{\kappa_n}^{G_2}\right) A \tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right) A y_n, \tilde{\varphi}_n - y_n \right\rangle}, \phi_n \tilde{\ell}_n \right\}, & \text{if } t_n > 0, \\ \phi_n \tilde{\ell}_n, & \text{otherwise}, \end{cases} \tag{3.2}$$

where $t_n = \left\langle A^* \left(I - J_{\kappa_n}^{G_2}\right) A \tilde{\varphi}_n - A^* \left(I - J_{\kappa_n}^{G_2}\right) A y_n, \tilde{\varphi}_n - y_n \right\rangle$. If $y_n = \tilde{\varphi}_n$, then stop; y_n is a solution of the SVIP. Otherwise, go to Step 2.

Step 2. Compute

$$x_{n+1} = \tilde{\varphi}_n - \tilde{\mu}_n L_n, \tag{3.3}$$

where

$$\begin{aligned} L_n &= \tilde{\varphi}_n - y_n - \tilde{\ell}_n \left[A^* \left(I - J_{\kappa_n}^{G_2}\right) A \tilde{\varphi}_n - A^* \left(I - J_{\kappa_n}^{G_2}\right) A y_n \right], \\ \tilde{\mu}_n &= \frac{\langle \tilde{\varphi}_n - y_n, L_n \rangle}{\|L_n\|^2}. \end{aligned} \tag{3.4}$$

Remark 3.1. In Algorithm 1, the inertial parameter $\tilde{\theta}_n$ is chosen as

$$\bar{\theta}_n = \begin{cases} \min\left\{ \frac{\tilde{\tau}_n}{\|x_n - x_{n-1}\|}, \theta \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}. \end{cases}$$

Lemma 3.1. Let $\{\tilde{\ell}_n\}$ be a sequence developed by Algorithm 1, then we get $\lim_{n\to\infty} \tilde{\ell}_n = \tilde{\ell}$, $\tilde{\ell} \geq \min\left\{ \frac{\delta}{\|A\|^2}, \tilde{\ell}_1 \right\}$

and $\tilde{\ell}_1$ is an initial stepsize.

Proof. In the view of $\left\langle A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n, \tilde{\varphi}_n - y_n\right\rangle > 0$ and $J_{\kappa_n}^{G_2}$ being firmly nonexpansive, we get

$$\begin{aligned}
&\left\langle A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n, \tilde{\varphi}_n - y_n\right\rangle \\
&\qquad = \left\langle \left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - \left(I - J_{\kappa_n}^{G_2}\right)Ay_n, A\tilde{\varphi}_n - Ay_n\right\rangle \\
&\qquad = \|A\tilde{\varphi}_n - Ay_n\|^2 - \left\langle J_{\kappa_n}^{G_2}A\tilde{\varphi}_n - J_{\kappa_n}^{G_2}Ay_n, A\tilde{\varphi}_n - Ay_n\right\rangle \\
&\qquad \le \|A\tilde{\varphi}_n - Ay_n\|^2 - \left\|J_{\kappa_n}^{G_2}A\tilde{\varphi}_n - J_{\kappa_n}^{G_2}Ay_n\right\|^2 \\
&\qquad \le \|A\tilde{\varphi}_n - Ay_n\|^2 \\
&\qquad \le \|A\|^2\|\tilde{\varphi}_n - y_n\|^2,
\end{aligned}$$

which yields that

$$\frac{\delta\|\tilde{\varphi}_n - y_n\|^2}{\left\langle A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n, \tilde{\varphi}_n - y_n\right\rangle} \ge \frac{\delta}{\|A\|^2}.$$

This further deduces that

$$\tilde{\ell}_{n+1} = \min\left\{\frac{\delta\|\tilde{\varphi}_n - y_n\|^2}{\left\langle A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n, \tilde{\varphi}_n - y_n\right\rangle}, \phi_n\tilde{\ell}_n\right\} \ge \min\left\{\frac{\delta}{\|A\|^2}, \tilde{\ell}_n\right\},$$

where $\phi_n \ge 1$. By induction, we know that the sequence $\{\tilde{\ell}_n\}$ has a lower bound $\min\left\{\frac{\delta}{\|A\|^2}, \tilde{\ell}_1\right\}$. Using (3.2), we have

$$\tilde{\ell}_{n+1} \le \phi_n\tilde{\ell}_n.$$

Using Lemma 2.7, it follows that $\lim_{n\to\infty}\tilde{\ell}_n$ exists and we denote $\lim_{n\to\infty}\tilde{\ell}_n = \tilde{\ell}$. Since the sequence $\{\tilde{\ell}_n\}$ has the lower bound $\min\left\{\frac{\delta}{\|A\|^2}, \tilde{\ell}_1\right\}$. Thus, $\tilde{\ell} > 0$. □

Lemma 3.2. Assume that $\{\tilde{\varphi}_n\}$ and $\{y_n\}$ are the two sequences produced by Algorithm 1, $\{\kappa_n\} \subset [\kappa, \infty)$ for some $\kappa > 0$, $\{x_n\}$ is a bounded sequence, and $\lim_{n\to\infty}\tilde{b}_n = \lim_{n\to\infty}\tilde{v}_n = 0$. If $\tilde{\varphi}_{n_k} \rightharpoonup z$ as $k \to \infty$ and $\lim_{n\to\infty}\|y_n - \tilde{\varphi}_n\| = 0$, then $z \in S$.

Proof. Clearly, $y_{n_k} \rightharpoonup z$. Since $z \in S$, then $z \in G_1^{-1}(0)$ and $Az \in G_2^{-1}(0)$. Using Lemma 2.4(e), we attain that

$$\left\langle y_{n_k} - z, \tilde{\varphi}_{n_k} - y_{n_k} - \tilde{\ell}_{n_k}A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)A\tilde{\varphi}_{n_k}\right\rangle \ge \left\langle \tilde{b}_{n_k}\tilde{\ell}_{n_k}\tilde{\varphi}_{n_k} - \tilde{v}_{n_k}\Gamma_{n_k}, y_{n_k} - z\right\rangle. \tag{3.5}$$

Also, we obtain $J_{\kappa_{n_k}}^{G_2}Az = Az$. From Lemma 2.5(a), it follows that

$$\begin{aligned}
\left\langle y_{n_k} - z, A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k}\right\rangle &= \left\langle y_{n_k} - z, A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k} - A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)Az\right\rangle \\
&= \left\langle Ay_{n_k} - Az, \left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k} - \left(I - J_{\kappa_{n_k}}^{G_2}\right)Az\right\rangle \\
&\ge \left\|\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k}\right\|^2.
\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we derive

$$\begin{aligned}
\left\langle y_{n_k} - z, d\left(\tilde{\varphi}_{n_k}, \tilde{\ell}_{n_k}\right)\right\rangle &= \left\langle y_{n_k} - z, \tilde{\varphi}_{n_k} - y_{n_k} - \tilde{\ell}_{n_k}\left(A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_{n_k} - A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k}\right)\right\rangle \\
&= \left\langle y_{n_k} - z, \tilde{\varphi}_{n_k} - y_{n_k} - \tilde{\ell}_{n_k}A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)A\tilde{\varphi}_{n_k}\right\rangle \\
&\quad + \tilde{\ell}_{n_k}\left\langle y_{n_k} - z, A^*\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k}\right\rangle \\
&\ge \tilde{\ell}_{n_k}\left\|\left(I - J_{\kappa_{n_k}}^{G_2}\right)Ay_{n_k}\right\|^2 + \left\langle \tilde{b}_{n_k}\tilde{\ell}_{n_k}\tilde{\varphi}_{n_k} - \tilde{v}_{n_k}\Gamma_{n_k}, y_{n_k} - z\right\rangle.
\end{aligned}$$

After arrangement, and by Lemma 2.5(b), we have

$$\begin{aligned}\left\|\left(I-J^{G_2}_{\kappa_{n_k}}\right)Ay_{n_k}\right\|^2 &\le \tfrac{1}{\tilde{\ell}_{n_k}}\left(\left\langle y_{n_k}-z, d\left(\tilde{\varphi}_{n_k},\tilde{\ell}_{n_k}\right)\right\rangle-\left\langle\tilde{b}_{n_k}\tilde{\ell}_{n_k}\tilde{\varphi}_{n_k}-\tilde{\nu}_{n_k}\tilde{\Gamma}_{n_k}, y_{n_k}-z\right\rangle\right)\\ &\le \tfrac{1}{\tilde{\ell}_{n_k}}\left(\left\langle y_{n_k}-z, d(\tilde{\varphi}_{n_k},\tilde{\ell}_{n_k})\right\rangle+\left|\left\langle\tilde{b}_{n_k}\tilde{\ell}_{n_k}\tilde{\varphi}_{n_k}-\tilde{\nu}_{n_k}\tilde{\Gamma}_{n_k}, y_{n_k}-z\right\rangle\right|\right)\\ &\le \tfrac{1}{\tilde{\ell}_{n_k}}\left(\left\|y_{n_k}-z\right\|\left\|d(\tilde{\varphi}_{n_k},\tilde{\ell}_{n_k})\right\|+\left(\tilde{b}_{n_k}\tilde{\ell}_{n_k}\|\tilde{\varphi}_{n_k}\|+\tilde{\nu}_{n_k}\|\tilde{\Gamma}_{n_k}\|\right)\|y_{n_k}-z\|\right)\\ &\le \tfrac{1+L\tilde{\ell}_{n_k}}{\tilde{\ell}}\|y_{n_k}-z\|\|\tilde{\varphi}_{n_k}-y_{n_k}\|\\ &+\left(\tilde{b}_{n_k}\left\|\tilde{\varphi}_{n_k}\right\|+\tfrac{\tilde{\nu}_{n_k}}{\tilde{\ell}}\left\|\tilde{\Gamma}_{n_k}\right\|\right)\|y_{n_k}-z\|,\end{aligned}\tag{3.7}$$

where $\tilde{\ell}=\min\left\{\frac{\delta}{\|A\|^2},\tilde{\ell}_1\right\}$ and

$$\begin{aligned}\left\|d(\tilde{\varphi}_{n_k},\tilde{\ell}_{n_k})\right\| &= \left\|\tilde{\varphi}_{n_k}-y_{n_k}-\tilde{\ell}_{n_k}\left[A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}-A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)Ay_{n_k}\right]\right\|\\ &\le \left\|\tilde{\varphi}_{n_k}-y_{n_k}\right\|+\tilde{\ell}_{n_k}\left\|A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}-A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)Ay_{n_k}\right\|\\ &\le \left(1+\tilde{\ell}_{n_k}\|A\|^2\right)\left\|\tilde{\varphi}_{n_k}-y_{n_k}\right\|.\end{aligned}$$

Similar to the proof of the boundness of $\left\{\tilde{\Gamma}_{n_k}\right\}$ in [12], $\{x_{n_k}\}$ bounds, then $\{\tilde{\varphi}_{n_k}\}$, $\left\{A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}\right\}$ bounds, and by the induction method, then $\{\Gamma_{n_k}\}$ is also bounded. Note that $\lim\limits_{k\to\infty}\tilde{b}_{n_k}=\lim\limits_{k\to\infty}\tilde{\nu}_{n_k}=0$ and $\lim\limits_{k\to\infty}\left\|\tilde{\varphi}_{n_k}-y_{n_k}\right\|=0$. These, together with the boundness of $\{y_{n_k}\}$, further reveal that

$$\lim_{k\to\infty}\left\|\left(I-J^{G_2}_{\kappa_{n_k}}\right)Ay_{n_k}\right\|=0.\tag{3.8}$$

Now, we have

$$\begin{aligned}\left\|A\tilde{\varphi}_{n_k}-J^{G_2}_{\kappa_{n_k}}A\tilde{\varphi}_{n_k}\right\| &\le \left\|A\tilde{\varphi}_{n_k}-J^{G_2}_{\kappa_{n_k}}A\tilde{\varphi}_{n_k}-Ay_{n_k}+J^{G_2}_{\kappa_{n_k}}Ay_{n_k}\right\|+\left\|(I-J^{G_2}_{\kappa_{n_k}})Ay_{n_k}\right\|\\ &\le 2\|A\|\left\|\tilde{\varphi}_{n_k}-y_{n_k}\right\|+\left\|\left(I-J^{G_2}_{\kappa_{n_k}}\right)Ay_{n_k}\right\|.\end{aligned}$$

Combining with (3.8), we obtain

$$\lim_{k\to\infty}\left\|\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}\right\|=0.$$

By Lemma 2.4(c), we get

$$\lim_{k\to\infty}\|(I-J^{G_2}_{\kappa})A\tilde{\varphi}_{n_k}\|\le\lim_{k\to\infty}\|(I-J^{G_2}_{\kappa_{n_k}})A\tilde{\varphi}_{n_k}\|=0.\tag{3.9}$$

Using Lemma 2.4(a), we derive

$$\begin{aligned}&\left\|y_{n_k}-J^{G_1}_{\kappa_{n_k}}\tilde{\varphi}_{n_k}\right\|\\ &= \left\|J^{G_1}_{\kappa_{n_k}}\left[\left(1-\tilde{b}_{n_k}\tilde{\ell}_{n_k}\right)\tilde{\varphi}_{n_k}-\tilde{\ell}_{n_k}A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}+\tilde{\nu}_{n_k}\tilde{\Gamma}_{n_k}\right]-J^{G_1}_{\kappa_{n_k}}\tilde{\varphi}_{n_k}\right\|\\ &\le \left\|\left(1-\tilde{b}_{n_k}\tilde{\ell}_{n_k}\right)\tilde{\varphi}_{n_k}-\tilde{\ell}_{n_k}A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}-\tilde{\varphi}_{n_k}\right\|+\tilde{\nu}_{n_k}\left\|\tilde{\Gamma}_{n_k}\right\|\\ &\le \tilde{b}_{n_k}\tilde{\ell}_{n_k}\left\|\tilde{\varphi}_{n_k}\right\|+\tilde{\ell}_{n_k}\left\|A^*\left(I-J^{G_2}_{\kappa_{n_k}}\right)Aw_{n_k}\right\|+\tilde{\nu}_{n_k}\|d_{n_k}\|\\ &\le \tilde{b}_{n_k}\tilde{\ell}_{n_k}\left\|\tilde{\varphi}_{n_k}\|+\tilde{\ell}_{n_k}\right\|A^*\|\cdot\left\|\left(I-J^{G_2}_{\kappa_{n_k}}\right)A\tilde{\varphi}_{n_k}\right\|+\tilde{\nu}_{n_k}\left\|\tilde{\Gamma}_{n_k}\right\|.\end{aligned}$$

Since $\lim_{k\to\infty} \tilde{b}_{n_k} = \lim_{k\to\infty} \tilde{\nu}_{n_k} = 0$ and $\lim_{k\to\infty} \tilde{\ell}_{n_k} = \tilde{\ell}$, then we obtain

$$\lim_{k\to\infty} \left\| y_{n_k} - J^{G_2}_{\kappa_{n_k}} \tilde{\varphi}_{n_k} \right\| = 0.$$

This further shows that

$$\begin{aligned}\left\| \tilde{\varphi}_{n_k} - J^{G_1}_{\kappa_{n_k}} \tilde{\varphi}_{n_k} \right\| &= \left\| \tilde{\varphi}_{n_k} - y_{n_k} + y_{n_k} - J^{G_1}_{\kappa_{n_k}} y_{n_k} \right\| \\ &\leq \left\| \tilde{\varphi}_{n_k} - y_{n_k} \right\| + \| y_{n_k} - J^{G_2}_{\kappa_{n_k}} \tilde{\varphi}_{n_k} \| \\ &\to 0 \ \ as \ \ k \to \infty.\end{aligned}$$

From Lemma 2.4(c), it follows that

$$\lim_{k\to\infty} \left\| \tilde{\varphi}_{n_k} - J^{G_1}_{\kappa} \tilde{\varphi}_{n_k} \right\| \leq \lim_{k\to\infty} \left\| \left(I - J^{G_2}_{\kappa_{n_k}} \right) \tilde{\varphi}_{n_k} \right\| = 0. \tag{3.10}$$

Since $\tilde{\varphi}_{n_k} \rightharpoonup z$, and A is a bounded and linear operator, we get $A\tilde{\varphi}_{n_k} \rightharpoonup Az$. By (3.10), Lemma 2.2 and Lemma 2.4(a), we have $z \in Fix(J^{G_1}_{\kappa})$. By (3.9), Lemma 2.2, and Lemma 2.4(a), we get $Az \in Fix\left(J^{G_2}_{\kappa}\right)$. Hence, we conclude that $z \in S$. □

Now, our main theorem is established without Lipschitz continuity of $A^*\left(I - J^{G_2}_{\kappa_n}\right)A$ in this paper.

Theorem 3.1. Assume that assumptions $(a1)-(a3)$ hold and the sequence $\{\kappa_n\} \subset [\kappa, \infty)$ for some $\kappa > 0$. For the sequence $\{x_n\}$ in Algorithm 1, we further suppose that:

(a) $\lim_{n\to\infty} \tilde{\omega}_n = \lim_{n\to\infty} \tilde{\nu}_n = 0$.

(b) $\lim_{n\to\infty} \frac{\tilde{\nu}_n}{\tilde{b}_n} = \tilde{t}$ for some $\tilde{t} > 0$, and $\{x_n\}$ is a bounded sequence.

Then $\lim_{n\to\infty} x_n = z$, where $z = P_S o0$.

Proof. Our proof is divided into six parts.

Part 1. By (3.2), we have

$$\begin{aligned}&\langle \tilde{\varphi}_n - y_n, L_n \rangle \\ &\quad = \langle \tilde{\varphi}_n - y_n, \tilde{\varphi}_n - y_n - \tilde{\ell}_n \left[A^*\left(I - J^{G_2}_{\kappa_n}\right)A\tilde{\varphi}_n - A^*\left(I - J^{G_2}_{\kappa_n}\right)Ay_n \right] \rangle \\ &\quad = \langle \tilde{\varphi}_n - y_n, \tilde{\varphi}_n - y_n \rangle - \left\langle \tilde{\varphi}_n - y_n, \tilde{\ell}_n \left[A^*\left(I - J^{G_2}_{\kappa_n}\right)A\tilde{\varphi}_n - A^*\left(I - J^{G_2}_{\kappa_n}\right)Ay_n \right] \right\rangle \\ &\quad = \|\tilde{\varphi}_n - y_n\|^2 - \frac{\tilde{\ell}_n}{\tilde{\ell}_{n+1}} \tilde{\ell}_{n+1} \langle \tilde{\varphi}_n - y_n, A^*\left(I - J^{G_2}_{\kappa_n}\right)A\tilde{\varphi}_n - A^*\left(I - J^{G_2}_{\kappa_n}\right)Ay_n \rangle \\ &\quad \geq \|\tilde{\varphi}_n - y_n\|^2 - \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}} \|\tilde{\varphi}_n - y_n\|^2 \\ &\quad = \left(1 - \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\right) \|\tilde{\varphi}_n - y_n\|^2.\end{aligned}$$

Since

$$\lim_{n\to\infty} \left(1 - \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\right) = 1 - \delta > 1 - \mu,$$

where $\delta \in (0, \mu) \subset \left(0, \frac{1}{2}\right)$, then $\exists N \geq 0$ such that $\forall n \geq N$, $1 - \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}} > 0$. Hence, $\forall n \geq N$,

$$\langle \tilde{\varphi}_n - y_n, L_n \rangle \geq (1 - \mu)\|\tilde{\varphi}_n - y_n\|^2. \tag{3.11}$$

On the other hand,

$$\begin{aligned}\|L_n\|^2 &= \|\tilde{\varphi}_n - y_n - \tilde{\ell}_n[A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n - A^*(I - J_{\kappa_n}^{G_2})Ay_n]\|^2 \\ &= \|\tilde{\varphi}_n - y_n\|^2 + \tfrac{\tilde{\ell}_n^2}{\tilde{\ell}_{n+1}^2}\tilde{\ell}_{n+1}^2\|A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n - A^*(I - J_{\kappa_n}^{G_2})Ay_n\|^2 \\ &\quad -\tfrac{2\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\tilde{\ell}_{n+1}\left\langle \tilde{\varphi}_n - y_n, A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n\right\rangle \\ &\le \|\tilde{\varphi}_n - y_n\|^2 + \tfrac{\delta^2\tilde{\ell}_n^2}{\tilde{\ell}_{n+1}^2}\|\tilde{\varphi}_n - y_n\|^2 \\ &\quad +\tfrac{2\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\tilde{\ell}_{n+1}|\langle\tilde{\varphi}_n - y_n, A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n - A^*(I - J_{\kappa_n}^{G_2})Ay_n\rangle| \\ &\le \|\tilde{\varphi}_n - y_n\|^2 + \tfrac{\delta^2\tilde{\ell}_n^2}{\tilde{\ell}_{n+1}^2}\|\tilde{\varphi}_n - y_n\|^2 + \tfrac{2\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\|\tilde{\varphi}_n - y_n\|^2 \\ &= \left(1 + \tfrac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\right)^2\|\tilde{\varphi}_n - y_n\|^2.\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(1 + \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\right) = 1 + \delta < 1 + \mu,$$

where $\delta \in (0,\mu) \subset \left(0, \frac{1}{2}\right)$, then $\exists N \ge 0$ such that $\forall n \ge N$, $1 + \frac{\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}} < 1 + \mu$. So, $\forall n \ge N$,

$$\|L_n\|^2 \le (1 + \mu)^2\|\tilde{\varphi}_n - y_n\|^2. \tag{3.12}$$

Combining (3.11) and (3.12), we have that $\forall n \ge N$,

$$\tilde{\mu}_n = \frac{\langle\tilde{\varphi}_n - y_n, L_n\rangle}{\|L_n\|^2} \ge \frac{1 - \mu}{(1 + \mu)^2}. \tag{3.13}$$

Next, we get

$$\begin{aligned}\|L_n\|^2 &= \left\|\tilde{\varphi}_n - y_n + \tilde{\ell}_n\left[A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right]\right\|^2 \\ &= \|\tilde{\varphi}_n - y_n\|^2 + \tilde{\ell}_n^2\left\|A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\|^2 \\ &\quad +\tfrac{2\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\tilde{\ell}_{n+1}\left\langle\tilde{\varphi}_n - y_n, A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\rangle \\ &\ge \|\tilde{\varphi}_n - y_n\|^2 + \tilde{\ell}_n^2\left\|A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\|^2 \\ &\quad -\tfrac{2\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\tilde{\ell}_{n+1}\left|\left\langle\tilde{\varphi}_n - y_n, A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\rangle\right| \\ &\ge \|\tilde{\varphi}_n - y_n\|^2 + \tilde{\ell}_n^2\left\|A^*\left(I - J_{\kappa_n}^{G_2}\right)Ay_n - A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\|^2 \\ &\quad -\tfrac{2\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\|\tilde{\varphi}_n - y_n\|^2 \\ &\ge \left(1 - \tfrac{2\delta\tilde{\ell}_n}{\tilde{\ell}_{n+1}}\right)\|\tilde{\varphi}_n - y_n\|^2.\end{aligned}$$

Similar to the analysis of (3.11), we get that $\forall n \ge N$,

$$\|L_n\|^2 \ge (1 - 2\mu)\|\tilde{\varphi}_n - y_n\|^2,\ \delta \in (0,\mu) \subset \left(0, \frac{1}{2}\right).$$

This implies that

$$\tilde{\mu}_n^2 \le \left(\frac{\|\tilde{\varphi}_n - y_n\| \cdot \|L_n\|}{\|L_n\|^2}\right)^2 \le \frac{\|\tilde{\varphi}_n - y_n\|^2}{(1 - 2\mu)\|\tilde{\varphi}_n - y_n\|^2} = \frac{1}{1 - 2\mu}. \tag{3.14}$$

By (3.2), (3.3), and (3.13), we obtain $\forall n \geq N$,

$$\begin{aligned}\langle \tilde{\varphi}_n - y_n, L_n \rangle = \tilde{\mu}_n \|L_n\|^2 &= \tfrac{1}{\tilde{\mu}_n}\|\tilde{\mu}_n L_n\|^2 \\ &= \tfrac{1}{\tilde{\mu}_n}\|x_{n+1} - \tilde{\varphi}_n\|^2 \\ &\leq \tfrac{(1+\mu)^2}{(1-\mu)}\|x_{n+1} - \tilde{\varphi}_n\|^2.\end{aligned}$$

This, along with (3.11), verifies that $\forall n \geq N$,

$$\|\tilde{\varphi}_n - y_n\|^2 \leq \frac{1}{1-\mu}\langle \tilde{\varphi}_n - y_n, L_n \rangle \leq \frac{(1+\mu)^2}{(1-\mu)^2}\|x_{n+1} - \tilde{\varphi}_n\|^2.$$

Part 2. Since $z \in S$, then $z \in G_1^{-1}(0)$, which along with Lemma 2.4(e) and the definition of y_n implies that

$$\left\langle y_n - z, \tilde{\varphi}_n - y_n - \tilde{\ell}_n A^* \left(I - J_{\kappa_n}^{G_2}\right) A\tilde{\varphi}_n \right\rangle \geq \left\langle b_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle. \tag{3.15}$$

Furthermore, we have $Az \in G_2^{-1}(0)$ and $Az \in Fix\left(J_{\kappa}^{G_2}\right)$. We obtain directly that $J_{\kappa_n}^{G_2}Az = Az$. By Lemma 2.5(a), we obtain

$$\left\langle y_n - z, \tilde{\ell}_n \left(A^* \left(I - J_{\kappa_n}^{G_2}\right) Ay_n - A^* \left(I - J_{\kappa_n}^{G_2}\right) Az\right)\right\rangle \geq 0,$$

which means that

$$\left\langle y_n - z, \tilde{\ell}_n A^*(I - J_{\kappa_n}^{G_2})Ay_n \right\rangle \geq 0. \tag{3.16}$$

By the direct sum of (3.15) and (3.16), one has

$$\langle y_n - z, L_n \rangle \geq \left\langle \tilde{b}_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle.$$

This shows that

$$\begin{aligned}\langle \tilde{\varphi}_n - z, L_n \rangle = \langle \tilde{\varphi}_n - y_n + y_n - z, L_n \rangle &= \langle \tilde{\varphi}_n - y_n, L_n \rangle + \langle y_n - z, L_n \rangle \\ &\geq \langle \tilde{\varphi}_n - y_n, L_n \rangle + \left\langle \tilde{b}_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle.\end{aligned}$$

Combining with ***Part 1*** and Lemma 2.1(b), we obtain that $\forall n \geq N$,

$$\begin{aligned}\|x_{n+1} - z\|^2 = \|\tilde{\varphi}_n - \tilde{\mu}_n L_n - z\|^2 &= \|\tilde{\varphi}_n - z\|^2 - 2\tilde{\mu}_n \langle \tilde{\varphi}_n - z, L_n \rangle + \tilde{\mu}_n^2 \|L_n\|^2 \\ &\leq \|\tilde{\varphi}_n - z\|^2 - 2\tilde{\mu}_n \left(\left\langle \tilde{\varphi}_n - y_n, L_n \right\rangle + \langle \tilde{b}_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle\Big) \\ &\quad + \tilde{\mu}_n^2 \|L_n\|^2 \\ &= \|\tilde{\varphi}_n - z\|^2 - \tilde{\mu}_n^2 \|L_n\|^2 - 2\tilde{\mu}_n \left\langle \tilde{b}_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle \\ &= \|\tilde{\varphi}_n - z\|^2 - \|x_{n+1} - \tilde{\varphi}_n\|^2 - 2\tilde{\mu}_n \left\langle \tilde{b}_n \tilde{\ell}_n \tilde{\varphi}_n - \tilde{\nu}_n \tilde{\Gamma}_n, y_n - z \right\rangle \\ &\leq \left(1 - \tilde{\mu}_n \tilde{b}_n \tilde{\ell}_n\right) \left\|\tilde{\varphi}_n - z\right\|^2 - \left(\tfrac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n \tilde{b}_n \tilde{\ell}_n\right) \left\| y_n - \tilde{\varphi}_n\right\|^2 \\ &\quad + \tilde{\mu}_n \tilde{b}_n \tilde{\ell}_n (\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n \tilde{\nu}_n \left\langle \tilde{\Gamma}_n, y_n - z \right\rangle.\end{aligned} \tag{3.17}$$

Since $\lim_{n\to\infty} \tilde{b}_n = 0$, $\lim_{n\to\infty} \tilde{\ell}_n = \tilde{\ell}$, and $\forall n \geq N$, $\{\tilde{\mu}_n\}$ is bounded, and then $\exists N_1 \in \mathbb{N}$ $(N_1 \geq N)$ such that $\forall n \geq N_1$, $0 < \tilde{\mu}_n \tilde{b}_n \tilde{\ell}_n < \frac{(1-\mu)^2}{(1+\mu)^2} < 1$, and $0 < \tilde{b}_n \tilde{\ell}_n < 1 - \mu$ comes from (3.13). So, we obtain ***Part 2.***

Part 3. The sequence $\{y_n\}$ is bounded.
By (3.1) and Lemma 3.1, we arrive at

$$\begin{aligned}\|y_n - z\| \le &\|J_{\kappa_n}^{G_1}\left((1-\tilde{b}_n\tilde{\ell}_n)\tilde{\varphi}_n - \tilde{\ell}_n A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n + \tilde{\nu}_n\tilde{\Gamma}_n\right)\\ &-J_{\kappa_n}^{G_1}((1-\tilde{b}_n\tilde{\ell}_n)z - \tilde{\ell}_n A^*(I - J_{\kappa_n}^{G_2})Az)\|\\ &+\left\|J_{\kappa_n}^{G_1}\left(\left(1-\tilde{b}_n\tilde{\ell}_n\right)z - \tilde{\ell}_n A^*\left(I-J_{\kappa_n}^{G_2}\right)Az\right) - J_{\kappa_n}^{G_1}\left(z - \tilde{\ell}_n A^*\left(I - J_{\kappa_n}^{G_2}\right)Az\right)\right\|\\ \le &(1-\tilde{b}_n\tilde{\ell}_n)\|\tilde{\varphi}_n - z\| + \tilde{\nu}_n\left\|\tilde{\Gamma}_n\right\| + \tilde{b}_n\tilde{\ell}_n\|z\| + \tilde{\ell}_n\left\|A^*\left(I - J_{\kappa_n}^{G_2}\right)A\tilde{\varphi}_n\right\|.\end{aligned}$$

Similar to the analysis of the boundness of $\{\tilde{\Gamma}_n\}$ in [12], since $\{x_n\}$ is bounded, then $\{\tilde{\varphi}_n\}$, $\{A^*(I-J_{\kappa_n}^{G_2})A\tilde{\varphi}_n\}$ bounds, by induction, and then $\{\tilde{\Gamma}_n\}$ bounds and the resulted sequence $\{y_n\}$ is bounded.
Part 4. By ***Part 2,*** then $\forall n \ge N_1$,

$$\begin{aligned}\|x_{n+1} - z\|^2 \le &(1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\left\|\tilde{\varphi}_n - z\|^2 - \left(\tfrac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\right\| y_n - \tilde{\varphi}_n\|^2\\ &+ \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle.\end{aligned} \tag{3.18}$$

Using the definition of $\tilde{\varphi}_n$, we get

$$\begin{aligned}\|\tilde{\varphi}_n - z\| &= \|x_n + \tilde{\theta}_n(x_n - x_{n-1}) - z\|\\ &\le \|x_n - z\| + \tilde{\theta}_n\|x_n - x_{n-1}\|\\ &= \|x_n - z\| + \tilde{b}_n\cdot\tfrac{\tilde{\theta}_n}{\tilde{b}_n}\|x_n - x_{n-1}\|.\end{aligned} \tag{3.19}$$

Note that $\tilde{\theta}_n\|x_n - x_{n-1}\| \le \tilde{\tau}_n$ for all $n \ge 1$, which, together with $\lim\limits_{n\to\infty}\frac{\tilde{\tau}_n}{\tilde{b}_n} = 0$, yields that

$$\lim_{n\to\infty}\frac{\tilde{\theta}_n}{\tilde{b}_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty}\frac{\tilde{\tau}_n}{\tilde{b}_n} = 0.$$

Then, there exists a constant $M_1 > 0$ such that

$$\frac{\tilde{\theta}_n}{\tilde{b}_n}\|x_n - x_{n-1}\| \le M_1.$$

Combining with (3.19), we have that $\forall n \ge N_1$,

$$\|\tilde{\varphi}_n - z\| \le \|x_n - z\| + \tilde{b}_n M_1. \tag{3.20}$$

Substituting (3.20) into (3.18), then $\exists M_2 > 0$ such that $\forall n \ge N_1$,

$$\begin{aligned}\|x_{n+1} - z\|^2 \le &\left(1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\left(\|x_n - z\| + \tilde{b}_n M_1\right)^2 - \left(\tfrac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2\\ &+ \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle\\ = &(1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\|x_n - z\|^2 + (1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)(\tilde{b}_n M_1)^2 + 2(1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\tilde{b}_n M_1\|x_n - z\|\\ &-\left(\tfrac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2\\ &+ \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle\\ \le &\left(1-\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|x_n - z\|^2 - \left(\tfrac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2\\ &+ \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle + \tilde{b}_n M_2.\end{aligned}$$

That is, $\forall\ n \geq N_1$,

$$\left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2 \leq (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle + \tilde{b}_nM_2.$$

Part 5. Recall the definition of $\tilde{\varphi}_n$, and we have

$$\begin{aligned}\|\tilde{\varphi}_n - z\|^2 &= \|x_n + \tilde{\theta}_n(x_n - x_{n-1}) - z\|^2 \\ &= \|x_n - z\|^2 + \tilde{\theta}_n^2\|x_n - x_{n-1}\|^2 + 2\tilde{\theta}_n\langle x_n - z, x_n - x_{n-1}\rangle \\ &\leq \|x_n - z\|^2 + \tilde{\theta}_n^2\|x_n - x_{n-1}\|^2 + 2\tilde{\theta}_n\|x_n - z\|\|x_n - x_{n-1}\|.\end{aligned} \tag{3.21}$$

Let $M = \sup\limits_{n\geq N_1}\{\theta\|x_n - x_{n-1}\|, 2\|x_n - z\|\}$. Substituting (3.21) into (3.18), we obtain $\forall n \geq N_1$,

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\|\tilde{\varphi}_n - z\|^2 - \left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2 \\ &\quad + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle \\ &\leq (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)(\|x_n - z\|^2 + \tilde{\theta}_n^2\|x_n - x_{n-1}\|^2 + 2\tilde{\theta}_n\|x_n - z\|\|x_n - x_{n-1}\|) \\ &\quad - \left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2 \\ &\quad + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle \\ &= (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\|x_n - z\|^2 + (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)(\tilde{\theta}_n\|x_n - x_{n-1}\|(2\|x_n - z\| + \tilde{\theta}_n\|x_n - x_{n-1}\|)) \\ &\quad - \left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2 \\ &\quad + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle \\ &= \left(1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|x_n - z\|^2 + \left(1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\cdot 2M\tilde{\theta}_n\|x_n - x_{n-1}\| \\ &\quad - \left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right)\|y_n - \tilde{\varphi}_n\|^2 \\ &\quad + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n(\|z\|^2 - \|y_n\|^2) + 2\tilde{\mu}_n\tilde{\nu}_n\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle.\end{aligned}$$

This, together with (3.17) and Lemma 2.1(a), implies that $\forall n \geq N_1$,

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad\leq (1 - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n)\|x_n - z\|^2 + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n \cdot 2(\langle -z, y_n - x_n\rangle + \langle -z, x_n - z\rangle) \\ &\quad + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n \cdot \frac{2}{\tilde{\ell}_n}\frac{\tilde{\nu}_n}{\tilde{b}_n}\langle d_n, y_n - z\rangle + \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n \cdot \frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_n} \cdot \frac{\tilde{\theta}_n}{\tilde{b}_n}\|x_n - x_{n-1}\|.\end{aligned}$$

Part 6. $\{\|x_n - z\|^2\}$ converges to zero.

Case I There exists $N_2 \in \mathbb{N}$ $(N_2 \geq N_1)$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$, $\forall n \geq N_2$.
Obviously, $\lim\limits_{n\to\infty}\|x_n - z\|$ exists. By ***Part 3***, ***Part 4***,$\lim\limits_{n\to\infty}\tilde{b}_n = 0, \lim\limits_{n\to\infty}\tilde{\nu}_n = 0$, $\lim\limits_{n\to\infty}\tilde{\ell}_n = \tilde{\ell}$, the boundness of $\{\tilde{\mu}_n\}$, and $\lim\limits_{n\to\infty}\left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n\right) = \frac{(1-\mu)^2}{(1+\mu)^2} > 0$, we have

$$\lim_{n\to\infty}\|y_n - \tilde{\varphi}_n\| = 0. \tag{3.22}$$

By (3.22), we have that $\forall n \geq N_1$,

$$\begin{aligned}\|x_{n+1} - \tilde{\varphi}_n\| &= \tilde{\mu}_n\|L_n\| \\ &\leq \tilde{\mu}_n(\|\tilde{\varphi}_n - y_n\| + \tilde{\ell}_n\|A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n - A^*(I - J_{\kappa_n}^{G_2})Ay_n\|) \\ &\leq \tilde{\mu}_n(\|\tilde{\varphi}_n - y_n\| + \tilde{\ell}_n\|A\|^2\|\tilde{\varphi}_n - y_n\|) \\ &\leq \frac{1}{\sqrt{1-2\mu}}(\|\tilde{\varphi}_n - y_n\| + \tilde{\ell}_n\|A\|^2\|\tilde{\varphi}_n - y_n\|) \\ &\to 0\ \ as\ \ n \to \infty.\end{aligned} \tag{3.23}$$

Note that $\lim_{n\to\infty}\tilde{b}_n = 0$, and then

$$\|x_n - \tilde{\varphi}_n\| = \tilde{\theta}_n\|x_n - x_{n-1}\| = \tilde{b}_n\frac{\tilde{\theta}_n}{\tilde{b}_n}\|x_n - x_{n-1}\| \to 0. \tag{3.24}$$

This, together with (3.23), directly shows that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - \tilde{\varphi}_n\| + \|x_n - \tilde{\varphi}_n\| \to 0. \tag{3.25}$$

In view of the sequence $\{x_n\}$ being bounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to a point $z_0 \in H$ such that

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle = \lim_{k\to\infty}\langle -z, x_{n_k} - z\rangle = \langle -z, z_0 - z\rangle.$$

From (3.24), we obtain

$$\tilde{\varphi}_{n_k} \rightharpoonup z_0. \tag{3.26}$$

From (3.24), (3.26), and Lemma 3.2, we have $z_0 \in S$. By the definition of $z = P_S \circ 0$, then

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle = \lim_{k\to\infty}\langle -z, x_{n_k} - z\rangle = \langle -z, z_0 - z\rangle \leq 0. \tag{3.27}$$

Using the definition of $\tilde{\Gamma}_n$, (3.24), $\lim_{n\to\infty}\tilde{\omega}_n = 0$, Lemma 2.5(a), the boundness of $\left\{A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n\right\}$, and $\left\{\tilde{\Gamma}_n\right\}$, we show

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle\tilde{\Gamma}_n, x_n - z\right\rangle\\
&= \limsup_{n\to\infty}\left\langle -A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n + \tilde{\omega}_n\tilde{\Gamma}_{n-1}, x_n - z\right\rangle\\
&= \limsup_{n\to\infty}\left(\left\langle -A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n, x_n - z\right\rangle + \left\langle\tilde{\omega}_n\tilde{\Gamma}_{n-1}, x_n - z\right\rangle\right)\\
&\leq \limsup_{n\to\infty}\left\langle -A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n, x_n - z\right\rangle + \limsup_{n\to\infty}\left\langle\tilde{\omega}_n\tilde{\Gamma}_{n-1}, x_n - z\right\rangle\\
&\leq \limsup_{n\to\infty}\left\langle -A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n, \tilde{\varphi}_n - z\right\rangle + \limsup_{n\to\infty}\left\langle -A^*(I - J_{\kappa_n}^{G_2})A\tilde{\varphi}_n, x_n - \tilde{\varphi}_n\right\rangle\\
&+ \limsup_{n\to\infty}\tilde{\omega}_n\left\langle\tilde{\Gamma}_{n-1}, x_n - z\right\rangle\\
&\leq 0.
\end{aligned} \tag{3.28}$$

Employing (3.22), (3.24), (3.28), and the boundness of $\{\tilde{\Gamma}_n\}$, we attain

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle\\
&= \limsup_{n\to\infty}\left(\left\langle\tilde{\Gamma}_n, y_n - \tilde{\varphi}_n\right\rangle + \left\langle\tilde{\Gamma}_n, \tilde{\varphi}_n - x_n\right\rangle + \left\langle\tilde{\Gamma}_n, x_n - z\right\rangle\right)\\
&\leq \limsup_{n\to\infty}\left\langle\tilde{\Gamma}_n, y_n - \tilde{\varphi}_n\right\rangle + \limsup_{n\to\infty}\left\langle\tilde{\Gamma}_n, \tilde{\varphi}_n - x_n\right\rangle + \limsup_{n\to\infty}\left\langle\tilde{\Gamma}_n, x_n - z\right\rangle\\
&\leq 0.
\end{aligned} \tag{3.29}$$

From (3.22) and (3.24), it follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle -z, y_n - x_n\rangle\\&\quad= \limsup_{n\to\infty}(\langle -z, y_n - \tilde{\varphi}_n\rangle + \langle -z, \tilde{\varphi}_n - x_n\rangle)\\&\quad\le \limsup_{n\to\infty}\langle -z, y_n - \tilde{\varphi}_n\rangle + \limsup_{n\to\infty}\langle -z, \tilde{\varphi}_n - x_n\rangle\\&\quad\le 0.\end{aligned} \tag{3.30}$$

Note that $\lim_{n\to\infty}\frac{\tilde{v}_n}{\tilde{b}_n} = \tilde{t} > 0$ and $\lim_{n\to\infty}\frac{1}{\tilde{\ell}_n} = \frac{1}{\tilde{\ell}} > 0$, and these together with (3.27), (3.29), and (3.30) imply that

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\frac{2}{\tilde{\ell}_n}\frac{\tilde{v}_n}{\tilde{b}_n}\left\langle\tilde{\Gamma}_n, y_n - z\right\rangle + \frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_n}\cdot\frac{\theta_n}{\tilde{b}_n}\|x_n - x_{n-1}\|\\&\quad + 2\left(\langle -z, y_n - x_n\rangle + \langle -z, x_n - z\rangle\right)\Big)\\&\quad\le 0.\end{aligned}$$

Employing $\lim_{n\to\infty}\tilde{\mu}_n\tilde{b}_n\tilde{\ell}_n = 0$ and Lemma 2.3, we finally get that $\lim_{n\to\infty}\|x_n - z\| = 0$.

Case II Assume that if there is some subsequence $\{n_i\}_{i=0}^{\infty} \subset \{n\}_{n=0}^{\infty}$ such that $\|x_{n_i} - x^*\| \le \|x_{n_i+1} - x^*\|$, $\forall i \in \mathbb{N}$. By Lemma 2.6, there is a nondecreasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $m_k \to \infty$,

$$\|x_{m_k} - z\| \le \|x_{m_k+1} - z\| \text{ and } \|x_k - z\| \le \|x_{m_k+1} - z\|. \tag{3.31}$$

By ***Part 4***, we have

$$\begin{aligned}&\left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_{n_k}\tilde{b}_{n_k}\tilde{\ell}_{n_k}\right)\|y_{m_k} - \tilde{\varphi}_{m_k}\|^2 \le \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 - \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k}\|x_{m_k} - z\|^2\\&\quad + \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k}(\|z\|^2 - \|y_{m_k}\|^2) + 2\tilde{\mu}_{m_k}\tilde{v}_{m_k}\left\langle\tilde{\Gamma}_{m_k}, y_{m_k} - z\right\rangle + \tilde{b}_{m_k}M_2.\end{aligned} \tag{3.32}$$

Combining (3.32), Lemma 3.1, ***Part 3***, $\lim_{k\to\infty}\tilde{b}_{m_k} = 0$, $\lim_{k\to\infty}\tilde{v}_{m_k} = 0$, and $\lim_{k\to\infty}\left(\frac{(1-\mu)^2}{(1+\mu)^2} - \tilde{\mu}_{n_k}\tilde{b}_{n_k}\tilde{\ell}_{n_k}\right) = \frac{(1-\mu)^2}{(1+\mu)^2} > 0$ implies that $\lim_{k\to\infty}\|y_{m_k} - \tilde{\varphi}_{m_k}\| = 0$. According to the proof of **Case I**, we attain $\lim_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| = 0$, $\limsup_{k\to\infty}\langle -z, x_{m_k} - z\rangle \le 0$, $\limsup_{k\to\infty}\langle -z, y_{m_k} - x_{m_k}\rangle \le 0$, $\limsup_{k\to\infty}\frac{2}{\tilde{\ell}_{m_k}}\frac{\tilde{v}_{m_k}}{\tilde{b}_{m_k}}\left\langle\tilde{\Gamma}_{m_k}, y_{m_k} - z\right\rangle \le 0$, and $\limsup_{k\to\infty}\frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_{m_k}}\cdot\frac{\theta_{m_k}}{\tilde{b}_{m_k}}\|x_{m_k} - x_{m_k-1}\| \le 0$. Using (3.32) and the similar proof of ***Part 5***, then for $m_k \ge N_1$, we get

$$\begin{aligned}&\|x_{m_k+1} - z\|^2\\&\quad\le (1 - \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k})\|x_{m_k} - z\|^2 + \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k}\cdot 2(\langle -z, y_{m_k} - x_{m_k}\rangle + \langle -z, x_{m_k} - z\rangle)\\&\quad + \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k}\cdot\frac{2}{\tilde{\ell}_{m_k}}\frac{\tilde{v}_{m_k}}{\tilde{b}_{m_k}}\langle\tilde{\Gamma}_{m_k}, y_{m_k} - z\rangle + \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k}\cdot\frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_{m_k}}\cdot\frac{\theta_{m_k}}{\tilde{b}_{m_k}}\|x_{m_k} - x_{m_k-1}\|.\end{aligned} \tag{3.33}$$

Since $\lim_{k\to\infty}\tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k} = 0$, then $\exists\, k_0 \in \mathbb{N}$ ($m_{k_0} \ge N_1$), for $k \ge k_0$, satisfying $1 - \tilde{\mu}_{m_k}\tilde{b}_{m_k}\tilde{\ell}_{m_k} > 0$, for all $k \ge k_0$,

$$\begin{aligned}\|x_{m_k+1} - z\|^2 &\le 2\langle -z, y_{m_k} - x_{m_k}\rangle\\&\quad + 2\langle -z, x_{m_k} - z\rangle + \frac{2}{\tilde{\ell}_{m_k}}\frac{\tilde{v}_{m_k}}{\tilde{b}_{m_k}}\left\langle\tilde{\Gamma}_{m_k}, y_{m_k} - z\right\rangle + \frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_{m_k}}\cdot\frac{\theta_{m_k}}{\tilde{b}_{m_k}}\|x_{m_k} - x_{m_k-1}\|.\end{aligned} \tag{3.34}$$

Note that $\limsup_{k\to\infty}\big(2\langle -z, y_{m_k} - x_{m_k}\rangle$ $+2\langle -z, x_{m_k} - z\rangle + \frac{2}{\tilde{\ell}_{m_k}}\frac{\tilde{v}_{m_k}}{\tilde{b}_{m_k}}\langle\tilde{\Gamma}_{m_k}, y_{m_k} - z\rangle + \frac{2M(1+\mu)^2}{1-\mu}\frac{1}{\tilde{\ell}_{m_k}}\cdot\frac{\theta_{m_k}}{\tilde{b}_{m_k}}\|x_{m_k} - x_{m_k-1}\|\big) \le 0$. This shows that $\lim_{k\to\infty}\|x_{m_k+1} - z\| = 0$. Using (3.31), we deduce that $\lim_{k\to\infty}\|x_{m_k} - z\| = 0$ and $\lim_{k\to\infty}\|x_k - z\| = 0$. Hence, $x_k \to z$.

4. Applications

In this section, we use our proposed algorithm to solve some special cases of the SVIP such as split feasibility and split minimization problems.

4.1. Split minimization problems (SMPs)

$$\text{Find } \tilde{x} \in \chi_1 \text{ such that } \tilde{x} \in \underset{x\in\chi_1}{\operatorname{argmin}} \tilde{f}(x) \text{ and } A\tilde{x} \in \underset{y\in\chi_2}{\operatorname{argmin}} g(y), \tag{4.1}$$

where $\tilde{f} : \chi_1 \to \mathbb{R}$ and $g : \chi_2 \to \mathbb{R}$ are proper lower semi-continuous convex (lsc) functions. We also denote S the solution set of the SMP and assume that $S \neq \emptyset$. In a real Hilbert space χ, we define the proximal operator of $\tilde{f}$ by

$$\operatorname{prox}_{\kappa,\tilde{f}}(x) = \underset{x\in\chi}{\operatorname{argmin}} \left\{ \tilde{f}(x) + \frac{1}{2\tilde{\kappa}} \|x - y\|^2 \right\}, \tilde{\kappa} > 0, \ \forall y \in \chi.$$

Recall that

$$\operatorname{prox}_{\kappa,\tilde{f}}(x) = (I + \tilde{\kappa}\partial\tilde{f})^{-1}(x) = J_{\kappa}^{\partial\tilde{f}}(x),$$

where $\partial\tilde{f}$ is the subdifferential of $\tilde{f}$ defined as

$$\partial\tilde{f}(x) = \left\{ \tilde{x} \in H : \tilde{f}(x) + \langle y - x, \tilde{x} \rangle \leq \tilde{f}(y), \ \forall y \in \chi \right\}.$$

In view of [9], we know that $\partial\tilde{f}$ is a maximal monotone operator and $\operatorname{prox}_{\kappa,\tilde{f}}$ is firmly nonexpansive.

Corollary 4.1. Let χ_1 and χ_2 be the two infinite-dimensional real Hilbert spaces. Let $\tilde{f}, \ g, \ A$ be the operators defined as above. Assume that assumptions $(a3)$, (a), and (b) of theorem 3.1 hold and the sequence $\{\kappa_n\} \subset [\kappa, \infty)$ for some $\kappa > 0$. The sequence $\{\phi_n\}$ comes from Lemma 2.7. Let $x_0, \ x_1 \in \chi_1$ and $\{x_n\}$ be a sequence produced by

$$\begin{cases} \tilde{\varphi}_n = x_n + \theta_n(x_n - x_{n-1}), \tilde{\Gamma}_n = -A^*(I - \operatorname{prox}_{\kappa_n,g})A\tilde{\varphi}_n + \tilde{\omega}_n\tilde{\Gamma}_{n-1}, \\ L_n = \tilde{\varphi}_n - y_n - \tilde{\ell}_n[A^*(I - \operatorname{prox}_{\kappa_n,g})A\tilde{\varphi}_n - A^*(I - \operatorname{prox}_{\kappa_n,g})Ay_n] \\ y_n = \operatorname{prox}_{\kappa_n,\tilde{f}}\left((1 - \tilde{b}_n\tilde{\ell}_n)\tilde{\varphi}_n - \tilde{\ell}_nA^*(I - \operatorname{prox}_{\kappa_n,g})A\tilde{\varphi}_n + \tilde{\nu}_n\tilde{\Gamma}_n\right), \\ \tilde{\mu}_n = \frac{\langle\tilde{\varphi}_n - y_n, L_n\rangle}{\|L_n\|^2}, x_{n+1} = \tilde{\varphi}_n - \tilde{\mu}_nL_n, \end{cases}$$

where the stepsize $\tilde{\ell}_n$ is defined as

$$\tilde{\ell}_{n+1} = \begin{cases} \min\left\{ \frac{\delta\|\tilde{\varphi}_n - y_n\|^2}{\langle A^*(I - \operatorname{prox}_{\kappa_n,g})A\tilde{\varphi}_n - A^*(I - \operatorname{prox}_{\kappa_n,g})Ay_n, \tilde{\varphi}_n - y_n\rangle}, \phi_n\tilde{\ell}_n \right\}, & \text{if } t_n > 0, \\ \phi_n\tilde{\ell}_n, & \text{otherwise,} \end{cases}$$

where $t_n = \left\langle A^*(I - \operatorname{prox}_{\kappa_n,g})A\tilde{\varphi}_n - A^*(I - \operatorname{prox}_{\kappa_n,g})Ay_n, \tilde{\varphi}_n - y_n \right\rangle$.

Then the iterative sequence $\{x_n\}$ converges strongly to $z \in S$. □

4.2. Split feasibility problem (SFP)

Let C and Q be non-empty closed convex subsets of real Hilbert spaces χ_1 and χ_2, respectively. Let $A : \chi_1 \to \chi_2$ be a bounded linear operator. The *split feasibility problem* (*SFP*) is the following:

$$\text{Find } \tilde{x} \in C \text{ such that } A\tilde{x} \in Q$$

and its solution set is denoted by S. Based on Algorithm 1, we have the following result.

Corollary 4.2. Let χ_1, χ_2, C, Q, A, A^*, and $S \neq \emptyset$ be the same as in the above statement. Assume that assumptions $(a3)$, (a), and (b) of Theorem 3.1 hold and the sequence $\{\phi_n\}$ comes from Lemma 2.7. Let $x_0,\ x_1 \in \chi_1$ and $\{x_n\}$ be a sequence produced by

$$\begin{cases} \tilde{\varphi}_n = x_n + \theta_n(x_n - x_{n-1}), \tilde{\Gamma}_n = -A^*(I - P_Q)A\tilde{\varphi}_n + \tilde{\omega}_n\tilde{\Gamma}_{n-1}, \\ L_n = \tilde{\varphi}_n - y_n - \tilde{\ell}_n[A^*(I - P_Q)A\tilde{\varphi}_n - A^*(I - P_Q)Ay_n] \\ y_n = P_C\left((1 - \tilde{b}_n\tilde{\ell}_n)\tilde{\varphi}_n - \tilde{\ell}_nA^*(I - P_Q)A\tilde{\varphi}_n + \tilde{\nu}_n\tilde{\Gamma}_n\right), \\ \tilde{\mu}_n = \frac{\langle\tilde{\varphi}_n - y_n, L_n\rangle}{\|L_n\|^2}, x_{n+1} = \tilde{\varphi}_n - \tilde{\mu}_nL_n, \end{cases}$$

where the stepsize $\tilde{\ell}_n$ is defined as

$$\tilde{\ell}_{n+1} = \begin{cases} \min\left\{\frac{\delta\|\tilde{\varphi}_n - y_n\|^2}{\left\langle A^*(I-P_Q)A\tilde{\varphi}_n - A^*(I-P_Q)Ay_n, \tilde{\varphi}_n - y_n\right\rangle}, \phi_n\tilde{\ell}_n\right\}, & \text{if } t_n > 0, \\ \phi_n\tilde{\ell}_n, & \text{otherwise,} \end{cases}$$

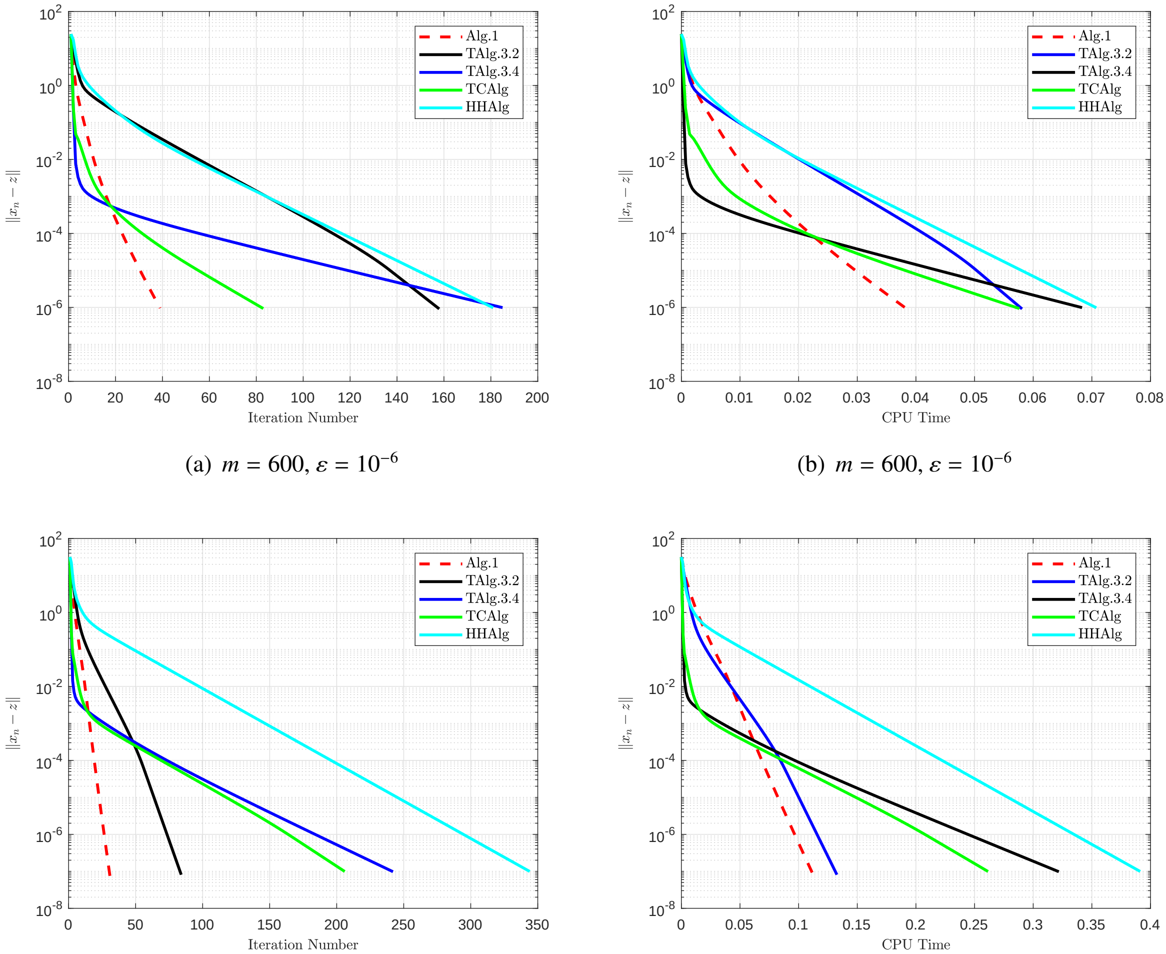
where $t_n = \langle A^*(I - P_Q)A\tilde{\varphi}_n - A^*(I - P_Q)Ay_n, \tilde{\varphi}_n - y_n\rangle$.

Then the iterative sequence $\{x_n\}$ converges strongly to $z \in S$. □

5. Numerical experiments

In this section, we make numerical experiments in terms of the SVIP and compare Algorithm 1 (shortly, Alg.1) with Tan et al. [34] (TAlgs.3.2 and 3.4), Hieu et al. [39] (HHAlg), Tong et al. [36] (TCAlg), and Xia et al. [40] (XCD). All the programs are implemented in MATLAB R2017a on a PC Desktop Intel(R) Core(TM) i7-6700 CPU @ 3.40 GHZ computer with RAM 8.00 GB.

Example 5.1. [25] Let matrices $A,\ A_1,\ A_2 : \mathbb{R}^m \to \mathbb{R}^m$ be generated from a normal distribution with mean zero and unit variance. Let $G_1,\ G_2 : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $G_1(x) = A_1^*A_1x$ and $G_2(y) = A_2^*A_2y$. We mainly find a point $z = (z_1, \cdots, z_m)^T \in \mathbb{R}^m$ such that $G_1(z) = (0, \cdots, 0)^T$ and $G_2(Az) = (0, \cdots, 0)^T$. In fact, $z_1 = 0, \cdots, z_m = 0$. Let $\epsilon > 0$ and the stopping criterion is given by $\|x_n - z\| < \epsilon$. For all algorithms, we adopt $x_0 = x_1 = (1, \cdots, 1)^T$, $\tilde{\Gamma}_0 = 0$, $\kappa_n = 1.8$, $\delta = 0.99$, and $\tilde{\ell}_1 = 0.0002$. For Algorithm 1, we choose $\phi_n = \frac{0.001}{(n+1)^{1.1}} + 1$, $\tilde{\omega}_n = \frac{0.0001}{n+1}$, $\tilde{b}_n = \frac{0.01}{n}$, $\tilde{\nu}_n = \frac{0.002}{n}$. As in [36], for TAlg.3.2, we adopt $\tilde{\sigma}_n = \frac{10^{-10}}{n}$, $\phi = 1.2$, $\tilde{\alpha}_n = 0.5(1 - \tilde{\sigma}_n)$, $h(x) = 0.01x$, $\ell = 0.3$, $\sigma = 0.9$, where $\tilde{\ell}, \sigma$ are parameters in the linear search rule and h is a contractive mapping. For TAlg.3.4, we suggest $\tilde{\ell}_n = 1.5$, $\tilde{\sigma}_n = \frac{1}{n^2}$, and $\tilde{\alpha}_n = 0.5(1 - \tilde{\sigma}_n)$, and $\ell_n = 0.05$ in TAlg.3.4. For TCAlg, we choose $\kappa = 0.3$, $\tilde{\ell}_n = \frac{0.9}{\|A\|^2}$, $\tilde{\sigma}_n = \frac{1}{n^2}$, and $h(x) = 0.01x$. For HHAlg, we take $\tilde{\ell}_n = \frac{0.9}{\|A\|^2}$, $\tilde{\alpha}_n = (n + 1)^{-0.5}$, $F(x) = 0.5x - x_0$, $\tilde{\alpha}_0 = 0.1$, and $\phi_n = \frac{5}{(n+1)^{1.1}} + 1$, which is suggested in [39]. The numerical results are given in Figure 1.



(a) $m = 600$, $\varepsilon = 10^{-6}$

(b) $m = 600$, $\varepsilon = 10^{-6}$

(c) $m = 1000$, $\varepsilon = 10^{-7}$

(d) $m = 1000$, $\varepsilon = 10^{-7}$

Figure 1. Numerical results for Example 5.1.

As shown in Figure 1, Algorithm 1 behaves better han TAlgs.3.2 and 3.4, TCAlg, and HHAlg under different m. In detail, TCAlg and HHAlg compute the norm $\|A\|$ when they begin iteration, which make the algorithms cost time and increase the iterative number. TAlg3.2 generally produces an inner iteration with the Armijo-type line search rule, which leads to a lot of time that it takes.

The following numerical example is used to compare our proposed method with self-adaptive algorithms including TAlgs3.2 and 3.4.

Example 5.2. [21] Let $\chi_1 = \chi_2 = \mathbb{R}^N$ and $\tilde{f}(x) = \frac{1}{2}d_C^2(x)$, where $C \subset \mathbb{R}^N$ is a unit ball and $g(x) = \frac{1}{2}\|x\|^2$. Let $\epsilon > 0$ and the stopping criterion is given by $\|x_n - \mathrm{prox}_{\lambda,\tilde{f}}(x_n)\| + \|Ax_n - \mathrm{prox}_{\lambda,g}(Ax_n)\| < \epsilon$. For all algorithms, we adopt $x_0 = (0, 0, 0, \cdots, 0)$, $x_1 = (1, 1, 1, \cdots, 1) \in \mathbb{R}^N$, $A = I$, $\delta = 0.999$, and $\varrho_n = \frac{1}{(n+1)^2}$. the rest of parameters are the following: For Algorithm 1, we suggest $\tilde{\ell}_1 = 0.49$, $\kappa_n = 0.19$, $\phi_n = \frac{0.01}{(n+1)^{1.1}} + 1$, $\tilde{\omega}_n = \frac{0.1}{n+1}$, $\tilde{b}_n = \frac{1}{n}$, *and* $\tilde{\nu}_n = \frac{2}{n+1}$. For TAlg.3.2, we adopt $\tilde{\sigma}_n = \frac{1.99}{n+1}$, $\tilde{\alpha}_n = 0.5(1 - \tilde{\sigma}_n)$, $\tilde{\ell} = 2$, $\sigma = 0.5$, $h(x) = 0.01x$, and $\tilde{\phi} = 1$, where $\tilde{\ell}$, σ are parameters in the linear search rule and h is a contractive mapping.

For TAlg.3.4, we take $\tilde{\ell}_n = 1.6$, $\tilde{\sigma}_n = \frac{1.99}{n+1}$, and $\tilde{\alpha}_n = 0.8(1 - \tilde{\sigma}_n)$.

Table 1 shows the numerical results that we achieve.

Table 1. Comparison of algorithms with $N = 10,000$ for Example 5.2.

	$\epsilon = 10^{-5}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-7}$	
Mehtod	**Iter.**	**CPU**	**Iter.**	**CPU**	**Iter.**	**CPU**
Alg 1	19	4.6260e-04	26	6.0610e-4	28	6.6580e-04
TAlg3.2	66	0.0025	127	0.0022	248	0.0042
TAlg3.4	30	7.1930e-04	36	5.8050e-04	42	6.3130e-04

From Table 1, we observe that Algorithm 1 needs less CPU time and iterative number than TAlg3.2 and TAlg3.4 in different ϵ.

Example 5.3. LASSO problem [36, 37].

In this section, we employ the SFP to model a real problem, which is the recovery of a sparse signal. We take advantage of the well-known LASSO problem whose form is the following:

$$\min\left\{\frac{1}{2}\|Ax-b\|^2 : x \in \mathbb{R}^N, \|x\|_1 \le \kappa\right\}, \tag{5.1}$$

where $A \in \mathbb{R}^{M\times N}$, $M < N$, $b \in \mathbb{R}^M$, and $\kappa > 0$. This problem is devoted to finding a sparse solution of the SFP. The system A is generated from a standard normal distribution with mean zero and unit variance. We generate the true sparse signal z^* from uniform distribution in the interval $[-2, 2]$ with random k at a nonzero position while the rest is kept at zero. The sample data $b = Az^*$.

Under certain conditions on matrix A, the solution of the minimization problem (5.1) is equivalent to the ℓ_0- norm solution of the underdetermined linear system. For the SFP, we define $C = \{z|\|z\|_1 \le \kappa\}$, $\kappa = k$, and $Q = \{b\}$. Since the projection onto the closed convex set C does not have a closed-form solution, we employ the subgradient projection. Thus, we define a convex function $c(z) = \|z\|_1 - \kappa$ and denote C_n by

$$C_n = \{z : c(w_n) + \langle \varepsilon_n, z - w_n\rangle \le 0\},$$

where $\varepsilon_n \in \partial c(w_n)$. Also, the orthogonal projection of a point $z \in \mathbb{R}^N$ onto C_n can be computed via:

$$P_{C_n}(z) = \begin{cases} z, & \text{if } c(w_n) + \langle \varepsilon_n, z - w_n\rangle \le 0, \\ z - \frac{c(w_n)+\langle \varepsilon_n, z-w_n\rangle}{\|\varepsilon_n\|^2}\varepsilon_n, & \text{otherwise.} \end{cases}$$

The subdifferential ∂c at w_n is

$$\partial c(w_n) = \begin{cases} 1, & \text{if } w_n > 0, \\ [-1, 1], & \text{if } w_n = 0, \\ -1, & \text{if } w_n < 0. \end{cases}$$

To implement our method in this example, we initialize the algorithms at the original and define

$$E_n = \frac{\|x_n - z^*\|}{\max\{1, \|x_n\|\}}.$$

We test the numerical behavior of all algorithms with the same iteration error E_n in different M, N, and k, limit the number of iterations to 4000, and report E_n in Tables 2 and 3.

Table 2. Results of all algorithms for Example 5.3.

(M, N, k)	Alg.1		TAlg.3.2		TAlg.3.4	
	E_n	time	E_n	time	E_n	time
(240,1024,30)	0.0188	0.8279	0.0159	10.3333	2.1090e-10	10.6364
(480,2048,60)	0.0188	3.6698	0.0156	36.2253	2.1103e-10	37.5464

Table 3. Results of all algorithms for Example 5.3.

(M, N, k)	HHAlg		XCD		TCAlg	
	E_n	time	E_n	time	E_n	time
(240,1024,30)	7.6579e-05	27.7955	1.8297e-10	28.6027	3.3554e-10	19.2763
(480,2048,60)	7.7373e-05	100.8832	1.7278e-10	105.6597	3.2833e-10	69.2340

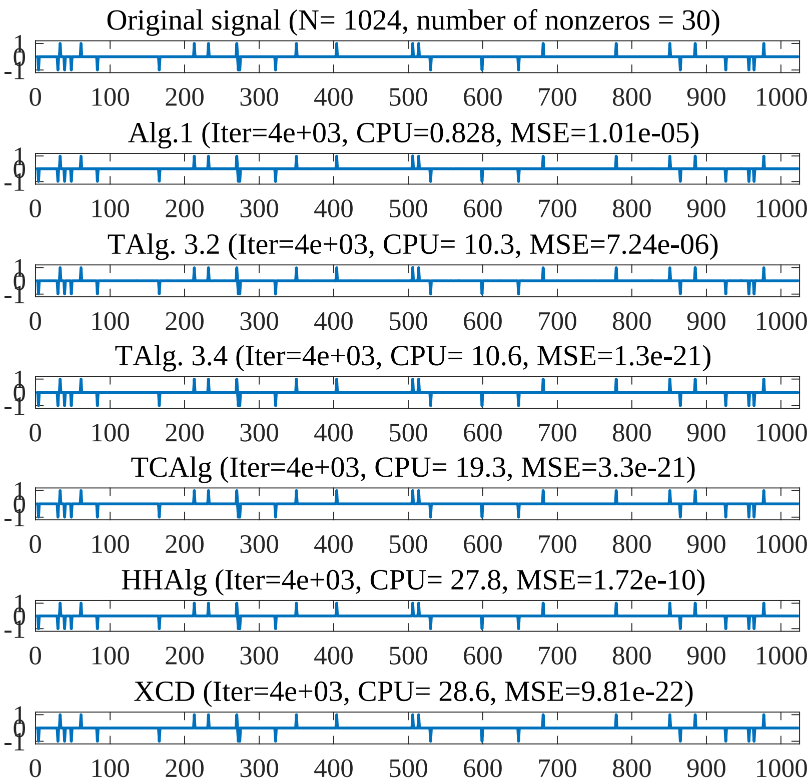


Figure 2. Comparison of signal processing.

The second problem is the recovery of the signal z^* when $M = 240$, $N = 1024$, $k = 30$, and $M \times N$ matrix A is randomly obtained with independent samples of standard Gaussian distribution.

The original signal z^* contains 30 randomly placed ±1 spikes. The iterative process is started with $x_0 = 0$ and the following method of mean squared error is used for measuring the recovery accuracy:

$$\text{MSE} = \frac{1}{N}\|x_n - z^*\|^2.$$

For Algorithm 1, we choose $\phi_n = \frac{0.1}{(n+1)^{1.1}} + 1,\ \tilde{\omega}_n = \frac{1}{n},\ \tilde{b}_n = \frac{1}{n},\ \tilde{\nu}_n = \frac{1}{n(1+\frac{1}{n})}, \delta = 0.3, \tilde{\Gamma}_0 = 0, \tilde{\ell}_1 = 0.1$. For XCD, we choose $\alpha = 0.9, \tilde{\ell}_1 = 0.1, \delta = 0.49, \phi_n = \frac{0.1}{(n+1)^{1.1}} + 1, \tilde{\sigma}_n = \frac{1e-7}{n+1}$, and $h(x) = 0.01x$, which is suggested in [40]. The choices of the parameters of the other methods are the same as in Example 5.1.

The recovery results of all algorithms are summarized in Figure 2, which represents the original signal, the mean squared error (MSE) of the restored signal, and the computing time taken for the iterative process.

Remark 5.1. From Tables 2 and 3 and Figure 2, we see that Algorithm 1 takes less execution time than TAlgs.3.2 and 3.4, TCAlg, HHAlg and XCD, but its E_n and MSE are both larger than that in other compared algorithms under the same iterative numbers.

6. Conclusions

In our work, the inertial term is introduced to accelerate the convergent rate of the conjugate method, and a non-Lipschitz stepsize is proposed to avoid computing first the operator norm. A strong convergence theorem is obtained under weaker conditions. Finally, some examples show the numerical efficiency of our methods.

Author contributions

Yu Zhang: Conceptualization, Methodology, Writing–original draft, Formal analysis, Writing–review and editing; Xiaojun Ma: Writing–original draft, Software, Validation, Formal analysis, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author was supported by the Natural Science Foundation of Hubei Province (No. 2025AFC080), the Scientific Research Project of Hubei Provincial Department of Education (No. 23Q166), and the Scientific Research Foundation of Hubei University of Education for Talent Introduction (No. ESRC20220008). The second author was supported by the National Natural Science Foundation of China (No. 12172266) and the Fundamental Research Program of Shanxi Province, China (No. 202303021222208, No.2023-B-06 and No.606-02010609). This work was supported by the Startup Foundation for Newly Recruited Employees of Suqian University and sponsored by the Qing Lan Project of Jiangsu Province in China (No. 106-CK00042/126).

Conflict of interest

There are no conflicts of interest in this work.

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