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Research article

Orthogonality preserving transformations on complex projective spaces

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Abstract: It is well known that transformations of $\mathbb{C}^n$ preserving the standard inner product are unitary transformations. In this paper, all bijective transformations of isotropic sets of $\mathbb{C}P^n$ preserving H-orthogonality in both directions, called H-orthogonal transformations, have been determined. This is a generalization of Uhlhorn's version of Wigner's unitary-antiunitary theorem. The group of H-orthogonal transformations on some other sets of $\mathbb{C}P^n$ were also determined.

Keywords: H-orthogonal; unitary group; isotropic vector; orthogonality preserving; Wigner theorem
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1. Introduction

The motivation to study orthogonality preserving maps comes from quantum mechanics. Birkhoff and von Neumann [1] first discovered that the logical structure of quantum mechanics is related to the orthogonal lattices formed by closed subspaces of complex Hilbert spaces. The state is an important type of function defined on every orthogonal lattice, and all states form a convex set whose extreme points are called pure states. By Gleason's theorem [3], the set of pure states of a quantum mechanical system can be identified with the set of rank-one projections, that is, the set of rays in a complex Hilbert space. The classic Wigner theorem [22] describes symmetries of quantum mechanical systems, and it characterizes unitary and anti-unitary operators as symmetries of quantum mechanical systems, that is, every bijective transformation of the set of pure states preserving the transition probability is induced by a unitary or anti-unitary operator. Also, there is a non-bijective version of this result concerning linear and conjugate-linear isometries. From Wigner's theorem one can also derive the Schrödinger equation for conservative physical systems. In [23], Wigner established the foundational role of group theory in quantum mechanics, particularly for analyzing atomic spectra. By leveraging symmetry properties of physical systems, he demonstrated how group representations (especially irreducible representations of the rotation group $SO(3)$ and permutation groups) classify quantum states and predict spectral line splitting. The work bridges abstract algebraic structures with observable phenomena, showing that symmetry operations (e.g., rotations, permutations of electrons) constrain Hamiltonian eigenstates and simplify solving complex atomic systems. Wigner's insights laid the groundwork for understanding angular momentum, selection rules, and degeneracy in quantum systems, profoundly influencing modern theoretical physics and chemistry.

In general, Wigner's theorem includes bijective and non-bijective versions, and each version has a variety of different statements. Various kinds of Wigner-type theorems can be found in [13]. The non-bijective version of Wigner's theorem says that an arbitrary transformation of the Grassmannian formed by rays of a complex Hilbert space, which preserves the angles between any two rays, is induced by a linear or conjugate-linear isometry. The bijective version of Wigner's theorem was first observed by Uhlhorn [19]. Let H be a complex Hilbert space of dimension not less than three. Then every bijective transformation of Grassmannian formed by rays of H preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator. In fact, Uhlhorn's theorem is a simple consequence of the Fundamental Theorem of Projective Geometry. But it reveals the following important relation between the logical structure and the probabilistic structure of quantum mechanical systems: if the logical structure is preserved, then probabilistic structure also is preserved. Since pure states are characterized as extreme points of the convex set of all states, the bijective transformations preserving the convex structure of the set of all quantum states induces a bijective transformation of the set of pure states. These transformations preserve the orthogonality relation in both directions, and this gives rise to a unitary or an anti-unitary operator.

Uhlhorn's theorem has been improved in several directions. Györy [9] and Šemrl [16] independently described bijective transformations of Hilbert Grassmannians preserving the orthogonality relation in both directions. Recently, Pankov [12] studied orthogonality preserving transformations of Hilbert Grassmannians, and Šemrl [17] gave another extension of Wigner's theorem in which the maximal principal angle is replaced by the minimal one. Instead of complex Hilbert spaces one can also treat real and quaternionic inner product spaces. Rodman and Semrl [14,15] studied this kind of problem in indefinite inner product spaces. In this paper, we study the orthogonal invariants in the geometry of a unitary group over $\mathbb{C}$. We use geometric methods in the spirit of Chow's theorem [2].

Let $n \geq 2$ be an integer, and consider $\mathbb{C}^n$ as the n-dimensional row vector space over $\mathbb{C}$. For vectors $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$, let $[\alpha_1, \alpha_2, \ldots, \alpha_s]$ denote their span. For $\alpha = (a_1, a_2, \cdots, a_n), \beta = (b_1, b_2, \cdots, b_n) \in \mathbb{C}^n$, let $(\alpha, \beta) = a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_n\overline{b}_n$ be the *standard inner product* of vectors α and β. Given a nonsingular Hermitian matrix $H \in \mathbb{C}^{n\times n}$, the vector $\alpha \in \mathbb{C}^n$ is said to be *H-orthogonal* to the vector $\beta \in \mathbb{C}^n$ if $(\alpha, \beta H) = 0$. We use the notation $\alpha \perp_H \beta$ to denote the H-orthogonality of α to β. If α is not H-orthogonal to β then it is denoted as $\alpha \not\perp_H \beta$. Clearly we have $\alpha \perp_H \beta$ if and only if $\beta \perp_H \alpha$. Given a nonzero vector $\alpha \in \mathbb{C}^n$, call α *isotropic* (with respect to H) whenever $\alpha \perp_H \alpha$, and call the vector space $[\alpha]$ *isotropic* when α is *isotropic*. Let $\mathbb{C}P^n = \{[\alpha] | \alpha \in \mathbb{C}^{n+1} \backslash (0, \ldots, 0)\}$ be the respective projective space. For $[\alpha], [\beta] \in \mathbb{C}P^n$, we call them *H-orthogonal* whenever $\alpha \perp_H \beta$, and denote it as $[\alpha] \perp_H [\beta]$. For any $[\alpha] \in \mathbb{C}P^n$, let $L_{[\alpha]} = \{[\beta] \in \mathbb{C}P^n | [\beta] \perp_H [\alpha]\}$, and then $L_{[\alpha]}$ is a hyperplane of $\mathbb{C}P^n$.

An $n \times n$ matrix T is called a *unitary matrix* of order n over $\mathbb{C}$ if $TH\overline{T}^t = H$. The set of unitary matrices of order n over $\mathbb{C}$ form a group with respect to the matrix multiplication, which is called the *unitary group* of degree n over $\mathbb{C}$, denoted by $U_n(\mathbb{C}, H)$, or simply $U_n(\mathbb{C})$. Let $S = \{a \in \mathbb{C} | a\overline{a} = 1\}$ be a subgroup of $\mathbb{C}^*$, $Z = \{aI^{(n)} | a \in S\}$, and $U_n(\mathbb{C})/Z$ be denoted by $PU_n(\mathbb{C})$, which is called the *projective unitary group* of degree n over $\mathbb{C}$.

Let ϕ be a transformation of $\mathbb{C}^n$. As is well known, for any $\alpha, \beta \in \mathbb{C}^n$, we have $(\alpha, \beta) = (\phi(\alpha), \phi(\beta))$ if and only if ϕ is a unitary transformation. In this paper, we consider H-orthogonality instead of the standard inner product. Since any $n \times n$ nonsingular Hermitian matrix is necessarily cogredient to

$$H = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \pm I^{(n-2\nu)} \end{pmatrix},$$

for some $\nu \in \mathbb{N}$ and $0 \leq 2\nu \leq n$, we can consider the nonsingular Hermitian matrix of the above form only.

A bijective transformation of $\mathbb{C}P^n$ preserving H-orthogonality in both directions is called an *H-orthogonal transformation* of $\mathbb{C}P^n$. Denote the set of all H-orthogonal transformations by $O(\mathbb{C}P^n)$, which is a group with the multiplication of composition. Let Λ denote the subgroup of $O(\mathbb{C}P^n)$ which consists of the identity transformation and conjugate transformation. By [14], when $n \geq 3$, we have:

Theorem 1.1. *$O(\mathbb{C}P^n) = PU_{n+1}(\mathbb{C}) \cdot \Lambda$.*

For $[\gamma] \in \mathbb{C}P^n$, denote $O(\mathbb{C}P^n)_{[\gamma]}$ the stabilizer subgroup of $O(\mathbb{C}P^n)$ fixing $[\gamma]$. Since $\mathbb{C}P^n \setminus L_{[\gamma]}$ is an open set of $\mathbb{C}P^n$ in the sense of algebraic geometry, similarly, we define and study $O(\mathbb{C}P^n \setminus L_{[\gamma]})$ for every $[\gamma] \in \mathbb{C}P^n$. Clearly $O(\mathbb{C}P^n)_{[\gamma]}$ can act on $\mathbb{C}P^n \setminus L_{[\gamma]}$, and we denote it by $O(\mathbb{C}P^n)_{[\gamma]}|_{\mathbb{C}P^n \setminus L_{[\gamma]}}$. In fact, by [14], when $n \geq 4$, we can obtain $O(\mathbb{C}P^n \setminus L_{[\gamma]}) = O(\mathbb{C}P^n)_{[\gamma]}|_{\mathbb{C}P^n \setminus L_{[\gamma]}}$.

When H is not a positive definite matrix, let $\Phi_0 = \{[\alpha] \in \mathbb{C}P^{n-1} \mid \alpha \perp_H \alpha\}$. For any $[\gamma] \in \mathbb{C}P^{n-1}$, let $\Phi_{[\gamma]} = \Phi_0 \setminus L_{[\gamma]}$. In the same way, we define $O(\Phi_0)$ and $O(\Phi_{[\gamma]})$. Every $T \in U_n(\mathbb{C})$ induces an automorphism of $\mathbb{C}P^{n-1} : [\alpha] \mapsto [\alpha T]$ which will be denoted by σ_T, i.e., $\sigma_T([\alpha]) = [\alpha T]$. Also σ_T induces an H-orthogonal transformation of Φ_0. For any $[\gamma] \in \mathbb{C}P^{n-1}$, there exists $T \in U_n(\mathbb{C})$ such that $[e_1 T] = [\gamma]$ or $[(e_1 + e_{\nu+1})T] = [\gamma]$, according to $[\gamma] \in \Phi_0$ or $[\gamma] \notin \Phi_0$. Then if $[\gamma] \in \Phi_0$, T induces an isomorphism σ_T from $\Phi_{[e_1]}$ to $\Phi_{[\gamma]}$, and if $[\gamma] \notin \Phi_0$, T induces an isomorphism σ_T from $\Phi_{[e_1+e_{\nu+1}]}$ to $\Phi_{[\gamma]}$. Hence $O(\Phi_{[\gamma]})$ is isomorphic to $O(\Phi_{[e_1]})$ or $O(\Phi_{[e_1+e_{\nu+1}]})$. Denote $\Phi_1 = \Phi_{[e_1]}$ and $\Phi_2 = \Phi_{[e_1+e_{\nu+1}]}$. To give a uniform treatment, write $n = 2\nu + \delta$, where $\delta \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Define a matrix

$$H = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \pm I^{(\delta)} \end{pmatrix},$$

and then any $n \times n$ nonsingular Hermitian matrix is necessarily cogredient to H. When $\nu \geq 3$, we will determine $O(\Phi_i), 0 \leq i \leq 2$. When $\delta \geq 1$, we only consider the case

$$H = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & I^{(\delta)} \end{pmatrix},$$

and the other case $H = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & -I^{(\delta)} \end{pmatrix}$ can be considered similarly.

Let $(\mathbb{R}^+)^* = \{x \in \mathbb{R} | x > 0\}$ and $U_\delta(\mathbb{C}) = \{W \in M_{\delta\times\delta}(\mathbb{C}) | W\overline{W}^t = I^{(\delta)}\}$ when $\delta \geq 1$. We define a group Ω_δ of the cartesian product $(\mathbb{R}^+)^* \times \Lambda \times U_\delta(\mathbb{C})$ with the multiplication $*$ defined by

$$(k_1, \pi_1, W_1) * (k_2, \pi_2, W_2) = (k_1k_2, \pi_1\pi_2, \frac{\mu_{k_1}}{\mu_{k_1k_2}}\pi_1(\mu_{k_2}W_2)W_1),$$

where μ_k is one fixed element of the set $\{a \in \mathbb{C} | a\overline{a} = k\}$ for any $k \in (\mathbb{R}^+)^*$, and $\mu_1 := 1$. Let $U_n^{(1)}(\mathbb{C}) = \{T \in U_n(\mathbb{C}) : e_1T = e_1\}$, $U_n^{(2)}(\mathbb{C}) = \{T \in U_n(\mathbb{C}) : (e_1 + e_{\nu+1})T = (e_1 + e_{\nu+1})\}$.

Let E_i be the subset of $O(\Phi_i), i = 1, 2$, which consists of those $\sigma \in O(\Phi_i)$ satisfying

$$\begin{cases} \sigma([e_j + e_{\nu+1}]) = [e_j + e_{\nu+1}] \ \ (j = 2, 3, \ldots, \nu), \\ \sigma([e_{\nu+1}]) = [e_{\nu+1}], \\ \sigma([e_{\nu+1} + e_{\nu+j}]) = [e_{\nu+1} + k_je_{\nu+i}] \ \ (j = 2, 3, \ldots, \nu), \end{cases}$$

where $k_j \in \mathbb{C}^*$, $j = 2, 3, \ldots, \nu$, and $\mathbb{C}^*$ represents the set of nonzero complex numbers. Let E_{Φ_1} be the subset of E_1 such that $k_2 = k_3 = \cdots = k_\nu \in \mathbb{R}^*$. In this paper, we will give the following main results:

Theorem 1.2. *When $\nu \geq 3$ and $\delta \in \mathbb{N}$, we have*

1) $O(\Phi_0) = PU_n(\mathbb{C}) \cdot \Lambda |_{\Phi_0}$.
2) $O(\Phi_1) = U_n^{(1)}(\mathbb{C}) \cdot E_{\Phi_1}$*, E_{Φ_1} is a subgroup of $O(\Phi_1)$, and*

$$E_{\Phi_1} \cong \begin{cases} \mathbb{R}^* \times \Lambda, & \text{when } \delta = 0, \\ \Omega_\delta, & \text{when } \delta \geq 1. \end{cases}$$

3) $O(\Phi_2) = U_n^{(2)}(\mathbb{C}) \cdot \Lambda |_{\Phi_2}$.

2. Preliminaries

In this section, we will introduce some propositions and lemmas that are needed to derive our main results.

In order to determine $O(\Phi_i), i = 0, 1, 2$, we define a graph Γ_i with Φ_i as the vertex set and the *adjacency* is defined by $[\alpha] \sim [\beta]$ if and only if $[\alpha] \not\perp_H [\beta]$. Then $O(\Phi_i) = \mathrm{Aut}(\Gamma_i)$, where $\mathrm{Aut}(\Gamma_i)$ is the group of automorphisms of Γ_i. In [4–8, 18, 20, 21], the automorphism groups of graphs constructed by symplectic, orthogonal, and unitary groups over finite fields were studied. The methods there can be used to study $\mathrm{Aut}(\Gamma_i)$ now.

Proposition 2.1. *When $\nu \geq 2$, every $T \in U_n(\mathbb{C})$ induces an automorphism σ_T of $\Gamma_0 : [\alpha] \mapsto [\alpha T]$, and for any $T_1, T_2 \in U_n(\mathbb{C})$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = kT_2$, where $k \in S$.*

Proof. It is clear that $\sigma_{T_1} = \sigma_{T_2}$ if $T_1 = kT_2, k \in S$. Conversely, suppose that $\sigma_{T_1} = \sigma_{T_2}$. Then for any $[\alpha] \in \Phi_0$, $\alpha T_1 = k\alpha T_2$ for some $k \in \mathbb{C}^*$.

When $n = 2\nu$, take $\alpha = e_1, e_2, \ldots, e_{2\nu}$, and we get that $T_1 = \mathrm{diag}(k_1, k_2, \ldots, k_{2\nu})T_2$, for some $k_1, k_2, \ldots, k_{2\nu} \in \mathbb{C}^*$. Take $\alpha = e_1 + e_2, e_2 + e_3, \cdots, e_{2\nu-1} + e_{2\nu}$, and we see that $k_1 = k_2 = \cdots = k_{2\nu}$.

When $n = 2\nu + \delta, \delta \geq 1$, take $\alpha = e_1, e_2, \ldots, e_{2\nu}, e_1 + \lambda e_{\nu+1} + e_{2\nu+1}, \ldots, e_1 + \lambda e_{\nu+1} + e_{2\nu+\delta}$ where $\lambda \in \mathbb{C}^*$ such that $\lambda + \overline{\lambda} + 1 = 0$, and we get that $T_1 = MT_2$, where

$$M = \begin{pmatrix} \mathrm{diag}(k_1, k_2, \ldots, k_{2\nu}) & 0 \\ N & \mathrm{diag}(k_{2\nu+1}, \ldots, k_{2\nu+\delta}) \end{pmatrix}$$

for some $k_1, k_2, \ldots, k_{2\nu+\delta} \in \mathbb{C}^*$, and

$$N = \begin{pmatrix} (k_{2\nu+1} - k_1)e_1 + \lambda(k_{2\nu+1} - k_{\nu+1})e_{\nu+1} \\ \vdots \\ (k_{2\nu+\delta} - k_1)e_1 + \lambda(k_{2\nu+\delta} - k_{\nu+1})e_{\nu+1} \end{pmatrix}_{\delta\times 2\nu}.$$

Take $\alpha = e_1 + e_2, e_2 + e_3, \ldots, e_{2\nu-1} + e_{2\nu}, e_1 + e_2 + \lambda e_{\nu+1} + e_{2\nu+1}, \ldots, e_1 + e_2 + \lambda e_{\nu+1} + e_{2\nu+\delta}$, and we see that $k_1 = \cdots = k_{2\nu} = k_{2\nu+1} = \cdots = k_{2\nu+\delta}$.

Thus, $T_1 = k_1 T_2$. Then, $k_1 I = T_1 T_2^{-1} \in U_n(\mathbb{C})$, which implies $(k_1 I)H\overline{(k_1 I)}^t = H$. Therefore, $k_1\overline{k_1}H = H$ and, hence, $k_1\overline{k_1} = 1$, i.e., $k_1 \in S$. □

Proposition 2.2. *Every $T \in U_n^{(i)}(\mathbb{C})$ induces an automorphism σ_T of $\Gamma_i : [\alpha] \mapsto [\alpha T]$ where $i = 1, 2$, and for any $T_1,\ T_2 \in U_n^{(i)}(\mathbb{C})$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = T_2$.*

Proof. We prove only for $i = 1$. Suppose $\sigma_{T_1} = \sigma_{T_2}$. Then, for every vertex $[\alpha]$ of Φ_1, there exists $k \in \mathbb{C}^*$ such that $\alpha T_1 = k\alpha T_2$.

Case (i) $\delta = 0$. Let M be the $2\nu \times 2\nu$ matrix with rows: $e_1, e_2 + e_{\nu+1}, \ldots, e_\nu + e_{\nu+1}, e_{\nu+1}, e_{\nu+1} + e_{\nu+2}, \ldots, e_{\nu+1} + e_{2\nu}$ in order. Since for every vertex $[\alpha]$ of Φ_1, there exists $k \in \mathbb{C}^*$ such that $\alpha T_1 = k\alpha T_2$, there exist $k_2, \ldots, k_{2\nu} \in \mathbb{C}^*$ such that $MT_1 = \mathrm{diag}(1, k_2, \cdots, k_{2\nu})MT_2$. Let $N = M^{-1}\mathrm{diag}(1, k_2, \cdots, k_{2\nu})M$. By computation, we have

$$M^{-1} = \begin{pmatrix} e_1 \\ e_2 - e_{\nu+1} \\ \vdots \\ e_\nu - e_{\nu+1} \\ e_{\nu+1} \\ -e_{\nu+1} + e_{\nu+2} \\ \vdots \\ -e_{\nu+1} + e_{2\nu} \end{pmatrix}_{2\nu\times 2\nu}$$

and

$$N = \begin{pmatrix} e_1 \\ k_2 e_2 + (k_2 - k_{\nu+1})e_{\nu+1} \\ \vdots \\ k_\nu e_\nu + (k_\nu - k_{\nu+1})e_{\nu+1} \\ k_{\nu+1}e_{\nu+1} \\ (k_{\nu+2} - k_{\nu+1})e_{\nu+1} + k_{\nu+2}e_{\nu+2} \\ \vdots \\ (k_{2\nu} - k_{\nu+1})e_{\nu+1} + k_{2\nu}e_{2\nu} \end{pmatrix}_{2\nu\times 2\nu}.$$

But, $N = T_1 T_2^{-1} \in U_{2\nu}(\mathbb{C})$, thus $NH\overline{N}^t = H$, which implies $k_2 = k_3 = \cdots = k_{2\nu} = 1$ and hence $T_1 = T_2$.

Case (ii) $\delta \geq 1$. Let M_1 be the $(2\nu+\delta)\times(2\nu+\delta)$ matrix with rows: $e_1, e_2 + e_{\nu+1}, \ldots, e_\nu + e_{\nu+1}, e_{\nu+1}, e_{\nu+1} + e_{\nu+2}, \ldots, e_{\nu+1} + e_{2\nu}, \lambda e_1 + e_{\nu+1} + e_{2\nu+1}, \ldots, \lambda e_1 + e_{\nu+1} + e_{2\nu+\delta}$ in order, where $\lambda \in \mathbb{C}^*$ satisfies $\lambda + \overline{\lambda} + 1 = 0$.

Since for every vertex $[\alpha]$ of Φ_1, there exists $k \in \mathbb{C}^*$ such that $\alpha T_1 = k\alpha T_2$, there exist $k_2, \ldots, k_{2\nu+\delta} \in \mathbb{C}^*$, such that

$$M_1 T_1 = \mathrm{diag}(1, k_2, \cdots, k_{2\nu+\delta}) M_1 T_2.$$

Let $N_1 = M_1^{-1}\mathrm{diag}(1, k_2, \cdots, k_{2\nu+\delta})M_1$. Similarly, we have

$$N_1 = \begin{pmatrix} e_1 \\ k_2 e_2 + (k_2 - k_{\nu+1})e_{\nu+1} \\ \vdots \\ k_\nu e_\nu + (k_\nu - k_{\nu+1})e_{\nu+1} \\ k_{\nu+1}e_{\nu+1} \\ (k_{\nu+2} - k_{\nu+1})e_{\nu+1} + k_{\nu+2}e_{\nu+2} \\ \vdots \\ (k_{2\nu} - k_{\nu+1})e_{\nu+1} + k_{2\nu}e_{2\nu} \\ \lambda(k_{2\nu+1} - 1)e_1 + (k_{2\nu+1} - k_{\nu+1})e_{\nu+1} + k_{2\nu+1}e_{2\nu+1} \\ \vdots \\ \lambda(k_{2\nu+\delta} - 1)e_1 + (k_{2\nu+\delta} - k_{\nu+1})e_{\nu+1} + k_{2\nu+\delta}e_{2\nu+\delta} \end{pmatrix} \in U_{2\nu+\delta}(\mathbb{C}).$$

Then, $N_1 H \overline{N_1}^t = H$, which implies $k_2 = k_3 = \cdots = k_{2\nu+\delta} = 1$ and hence $T_1 = T_2$. □

Recall that E_i is the subset of $O(\Phi_i), i = 1, 2$, which consists of those $\sigma \in O(\Phi_i)$ satisfying

$$\begin{cases} \sigma([e_j + e_{\nu+1}]) = [e_j + e_{\nu+1}] \ \ (j = 2, 3, \ldots, \nu), \\ \sigma([e_{\nu+1}]) = [e_{\nu+1}], \\ \sigma([e_{\nu+1} + e_{\nu+j}]) = [e_{\nu+1} + k_j e_{\nu+i}] \ \ (j = 2, 3, \ldots, \nu), \end{cases}$$

where $k_j \in \mathbb{C}^*, j = 2, 3, \ldots, \nu$, and $\mathbb{C}^*$ represents the set of nonzero complex numbers. Let $\sigma \in E_i, i = 1, 2, \ \nu \geq 3$, and $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_i$. Suppose $\sigma([\alpha]) = [\alpha']$ and write $[\alpha'] = [a'_1, a'_2, \ldots, a'_{2\nu+\delta}]$. Then, we have:

Lemma 2.1. *$a_j \neq 0$ if and only if $a'_j \neq 0$ for $j = 1, \ldots, \nu, \nu+2, \ldots, 2\nu$.*

Proof. For $j = 1$, $a_1 \neq 0$ if and only if $[\alpha] \not\perp_H [e_{\nu+1}]$, if and only if $[\alpha'] \sim [e_{\nu+1}]$, if and only if $a'_1 \neq 0$. For $j \neq 1$, we prove the Lemma only for the case $j = 2$. Consider first the case $a_1 = 0$, and then $a_2 \neq 0$ if and only if $[\alpha] \not\perp_H [e_{\nu+1} + e_{\nu+2}]$, if and only if $[\alpha'] \not\perp_H [e_{\nu+1} + k_2 e_{\nu+2}]$, if and only if $a'_2 \neq 0$. Similarly, when $a_1 = 0$, we also have $a_j \neq 0$ if and only if $a'_j \neq 0$, $j = 2, \ldots, \nu, \nu+2, \ldots, 2\nu$. Now assume $a_1 \neq 0$. If $a_2 \neq 0$, then $[\alpha] \perp_H [e_{\nu+1} - \overline{a_2}^{-1}\overline{a_1}e_{\nu+2}]$, from which we deduce $a'_2 \neq 0$. On the other hand, if $a_2 = 0$ but $a'_2 \neq 0$, there is an element $a \in \mathbb{C}^*$ such that $\sigma([e_{\nu+1} + ae_{\nu+2}]) = [e_{\nu+1} - \overline{a'_2}^{-1}\overline{a'_1}e_{\nu+2}]$. But $[\alpha] \not\perp_H [e_{\nu+1} + ae_{\nu+2}]$, while $[\alpha'] \perp_H [e_{\nu+1} - \overline{a'_2}^{-1}\overline{a'_1}e_{\nu+2}]$, which is a contradiction. Thus, $a_2 \neq 0$ if and only if $a'_2 \neq 0$. □

Moreover, when $\delta \geq 1$, we have:

Lemma 2.2. *Let $\nu \geq 3$, and suppose any one of the following two conditions is satisfied by $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_i, i = 1, 2$:*
(1) $a_1 = a_2 a_{\nu+2} = \cdots = a_\nu a_{2\nu} = 0$,
(2) $[\alpha] = [xe_1 + ye_j + e_{\nu+1} + ze_{\nu+j}]$, for some $x, y, z \in \mathbb{C}$ and $2 \leq j \leq \nu$.
Then, $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (0, \ldots, 0)$.

Proof. When the condition (1) is satisfied, the conclusion is evident. When the condition (2) is satisfied, first we prove this lemma for $[\alpha] = [ye_j + e_{\nu+1} + ze_{\nu+j}]$ with $y,\ z \in \mathbb{C}$. If not, there are some $y,\ z \in \mathbb{C}$ and $2 \leq j \leq \nu$ such that

$$\sigma([ye_j + e_{\nu+1} + ze_{\nu+j}]) = [y'e_j + e_{\nu+1} + z'e_{\nu+j} + t_1e_{2\nu+1} + \cdots + t_\delta e_{2\nu+\delta}]$$

for some $y', z' \in \mathbb{C}, t_s \in \mathbb{C}^*, 1 \leq s \leq \delta$. Since $t_s \neq 0$, there is $c \in \mathbb{C}^*$ such that $1 + t_s\overline{c} = 0$. Let $[\beta] = [e_1 + be_k + e_{\nu+1} + e_{\nu+k} + ce_{2\nu+s}]$, where $2 \leq j \neq k \leq \nu$ and $b \in \mathbb{C}$ satisfying $2 + b + \overline{b} + c\overline{c} = 0$. Then $[\beta] \in \Phi_i, i = 1, 2$, and $[y'e_j + e_{\nu+1} + z'e_{\nu+j} + t_1e_{2\nu+1} + \cdots + t_\delta e_{2\nu+\delta}] \perp_H [\beta]$. But the preimage $[\gamma]$ of $[\beta]$ is of the form $[\gamma] = [ae_1 + b'e_k + a'e_{\nu+1} + e_{\nu+k} + t'_1e_{2\nu+1} + \cdots + t'_\delta e_{2\nu+\delta}]$ with $a \in \mathbb{C}^*$ by Lemma 2.1, and clearly $[\gamma] \not\perp_H [ye_j + e_{\nu+1} + ze_{\nu+j}]$, which is a contradiction. Our claim is proved.

Now we prove this lemma for $[\alpha] = [xe_1 + ye_j + e_{\nu+1} + ze_{\nu+j}]$ with $x \in \mathbb{C}^*, y,\ z \in \mathbb{C}$. If not, there are some $x \in \mathbb{C}^*, y,\ z \in \mathbb{C}$, and $2 \leq j \leq \nu$ such that

$$\sigma([xe_1 + ye_j + e_{\nu+1} + ze_{\nu+j}]) = [x'e_1 + y'e_j + ae_{\nu+1} + z'e_{\nu+j} + t_1e_{2\nu+1} + \cdots + t_\delta e_{2\nu+\delta}]$$

for some $y', a, z' \in \mathbb{C}, x', t_s \in \mathbb{C}^*, 1 \leq s \leq \delta$. Since $t_s \neq 0$, there is $c \in \mathbb{C}^*$ such that $x' + t_s\overline{c} = 0$. Let $[\beta] = [be_k + e_{\nu+1} + e_{\nu+k} + ce_{2\nu+s}]$, where $2 \leq j \neq k \leq \nu$ and $b \in \mathbb{C}$ satisfying $b + \overline{b} + c\overline{c} = 0$. Then $[\beta] \in \Phi_i, i = 1, 2$, and $[x'e_1 + y'e_j + ae_{\nu+1} + z'e_{\nu+j} + t_1e_{2\nu+1} + \cdots + t_\delta e_{2\nu+\delta}] \perp_H [\beta]$. But the preimage $[\gamma]$ of $[\beta]$ is of the form $[\gamma] = [b'e_k + a'e_{\nu+1} + e_{\nu+k} + t'_1e_{2\nu+1} + \cdots + t'_\delta e_{2\nu+\delta}]$ with $a' \in \mathbb{C}^*$, and clearly $[\gamma] \not\perp_H [xe_1 + ye_j + e_{\nu+1} + ze_{\nu+j}]$, which is a contradiction. Our claim is proved. □

3. *H*-orthogonal transformations of Φ_0

In this section we will determine $O(\Phi_0)$. By proposition 2.1, $PU_n(\mathbb{C})$ can be regarded as a subgroup of $O(\Phi_0)$. For more works on projective unitary groups refer to [11], in which Pankov explored the interplay between semilinear embeddings (structure-preserving maps between vector spaces over division rings) and their combinatorial applications. He investigated how these embeddings define geometric constraints on incidence structures, such as graphs and codes, particularly in projective and polar spaces. Key results include characterizing embeddings that preserve adjacency or distance properties in graphs (e.g., Grassmann graphs) and their implications for constructing error-correcting codes with optimal parameters.

Let E_{Φ_0} be the subset of $O(\Phi_0)$ which consists of those automorphisms σ satisfying $\sigma([e_i]) = [e_i], 1 \leq i \leq 2\nu$. In order to prove $O(\Phi_0) = PU_n(\mathbb{C}) \cdot \Lambda\ |_{\Phi_0}$ for $\nu \geq 3$ and $\delta \in \mathbb{N}$, we need only to prove:

Theorem 3.1. *Let $\nu \geq 3$, and then $O(\Phi_0) = PU_n(\mathbb{C}) \cdot E_{\Phi_0}$. Let $\sigma \in E_{\Phi_0}$, for any $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_0$, and we have $\sigma([\alpha]) =$*

$$\begin{cases} [k_1\pi(a_1), \ldots, k_\nu\pi(a_\nu), \overline{k_1}^{-1}\pi(a_{\nu+1}), \ldots, \overline{k_\nu}^{-1}\pi(a_{2\nu})], & \text{when } \delta = 0, \\ [k_1\pi(a_1), \ldots, k_\nu\pi(a_\nu), \overline{k_1}^{-1}\pi(a_{\nu+1}), \ldots, \overline{k_\nu}^{-1}\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}], & \text{when } \delta \geq 1, \end{cases}$$

where $k_1, \ldots, k_\nu \in \mathbb{C}^*$, $\pi \in \Lambda$, $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$, and $W \in U_\delta(\mathbb{C})$ such that $W\overline{W}^t = I^{(\delta)}$.

Proof. Suppose $\sigma([a_1, a_2, \ldots, a_{2\nu+\delta}]) = [a'_1, a'_2, \ldots, a'_{2\nu+\delta}]$. Since $\sigma([e_i]) = [e_i], 1 \leq i \leq 2\nu$, we have $a_i = 0$ if and only if $a'_i = 0$ for $1 \leq i \leq 2\nu$. Moreover, if $a_1a_{\nu+1} = a_2a_{\nu+2} = \cdots = a_\nu a_{2\nu} = 0$, we can deduce $(a'_{2\nu+1},\dots,a'_{2\nu+\delta})=(0,\dots,0)$. Then, similar to the proof of Theorems 3.3 and 4.1 in [21], we have $O(\Phi_0)=PU_n(\mathbb{C})\cdot E_{\Phi_0}$ and $\sigma([\alpha])=$

$$\begin{cases}[k_1\pi(a_1),\dots,k_\nu\pi(a_\nu),\overline{k_1}^{-1}\pi(a_{\nu+1}),\dots,\overline{k_\nu}^{-1}\pi(a_{2\nu})], & when\ \delta=0,\\ [k_1\pi(a_1),\dots,k_\nu\pi(a_\nu),\overline{k_1}^{-1}\pi(a_{\nu+1}),\dots,\overline{k_\nu}^{-1}\pi(a_{2\nu}),a'_{2\nu+1},\dots,a'_{2\nu+\delta}], & when\ \delta\geq 1,\end{cases}$$

where $k_1,\dots,k_\nu\in\mathbb{C}^*$, $\pi\in\Lambda$. When $\delta\geq 1$, let $\lambda\in\mathbb{C}^*$ such that $\lambda+\overline{\lambda}+1=0$. Then, $[\gamma_i]=[\lambda e_1+e_{\nu+1}+e_{2\nu+i}]\in\Phi_0$ and $\sigma([\gamma_i])=[k_1\pi(\lambda)e_1+\overline{k_1}^{-1}e_{\nu+1}+\omega_{i1}e_{2\nu+1}+\cdots+\omega_{i\delta}e_{2\nu+\delta}]$, where ω_{ij} satisfies $\pi(\lambda)+\overline{\pi(\lambda)}+\Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}}=0, 1\leq i,j\leq\delta$. By $\lambda+\overline{\lambda}+1=0$, we deduce $\Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}}=1, 1\leq i,j\leq\delta$.

Lemma 3.1. *Let* $[\alpha]=[a_1e_1+\cdots+a_{2\nu}e_{2\nu}+e_{2\nu+i}]\in\Phi_0$, *and* $[\alpha]\perp_H[\gamma_i], 1\leq i\leq\delta$. *Suppose*

$$\sigma([\alpha])=[k_1\pi(a_1),\dots,k_\nu\pi(a_\nu),\overline{k_1}^{-1}\pi(a_{\nu+1}),\dots,\overline{k_\nu}^{-1}\pi(a_{2\nu}),a'_{2\nu+1},\dots,a'_{2\nu+\delta}],$$

and then

$$(a'_{2\nu+1},\dots,a'_{2\nu+\delta})=(\omega_{i1},\dots,\omega_{i\delta}), 1\leq i\leq\delta.$$

Proof. Since $[\alpha]\in\Phi_0$ and $[\alpha]\perp_H[\gamma_i], 1\leq i\leq\delta$, we have $\sum_{j=1}^{\nu}(a_j\overline{a_{\nu+j}}+\overline{a_j}a_{\nu+j})+1=0$ and $a_1+a_{\nu+1}\overline{\lambda}+1=0$. From $\sigma([\alpha])\in\Phi_0$, we have

$$[k_1\pi(a_1),\dots,k_\nu\pi(a_\nu),\overline{k_1}^{-1}\pi(a_{\nu+1}),\dots,\overline{k_\nu}^{-1}\pi(a_{2\nu}),a'_{2\nu+1},\dots,a'_{2\nu+\delta}]\in\Phi_0,$$

i.e.,

$$\sum_{j=1}^{\nu}[\pi(a_j)\overline{\pi(a_{\nu+j})}+\overline{\pi(a_j)}\pi(a_{\nu+j})]+a'_{2\nu+1}\overline{a'_{2\nu+1}}+\cdots+a'_{2\nu+\delta}\overline{a'_{2\nu+\delta}}=0.$$

Since $\pi\in\Lambda$, we deduce

$$\sum_{j=1}^{\nu}\pi(a_j\overline{a_{\nu+j}}+\overline{a_j}a_{\nu+j})+a'_{2\nu+1}\overline{a'_{2\nu+1}}+\cdots+a'_{2\nu+\delta}\overline{a'_{2\nu+\delta}}=0.$$

Since $\sigma([\alpha])\perp_H\sigma([\gamma_i]), 1\leq i\leq\delta$, we have

$$\pi(a_1)+\pi(a_{\nu+1})\overline{\pi(\lambda)}+a'_{2\nu+1}\overline{\omega_{i1}}+\cdots+a'_{2\nu+\delta}\overline{\omega_{i\delta}}=0, 1\leq i\leq\delta.$$

By the above equations, we obtain

$$a'_{2\nu+1}\overline{a'_{2\nu+1}}+\cdots+a'_{2\nu+\delta}\overline{a'_{2\nu+\delta}}=1$$

and $a'_{2\nu+1}\overline{\omega_{i1}}+\cdots+a'_{2\nu+\delta}\overline{\omega_{i\delta}}=1, 1\leq i\leq\delta$. Moreover, since $\Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}}=1, 1\leq i,j\leq\delta$, by the Cauchy-Schwarz inequality, we deduce $(a'_{2\nu+1},\dots,a'_{2\nu+\delta})=(\omega_{i1},\dots,\omega_{i\delta})$. □

Let $W=(\omega_{ij})_{\delta\times\delta}$, and we have the following lemma.

Lemma 3.2. $W\overline{W}^t=I^{(\delta)}$.

Proof. When $\delta = 1$, since $\omega_{11}\overline{\omega_{11}} = 1$, the result is clear. Now we consider the case when $\delta \geq 2$. For any $1 \leq s \neq t \leq \delta$, let $[\alpha] = [\lambda e_1 + e_{\nu+1} + e_{\nu+2} + e_{2\nu+s}], [\beta] = [\lambda e_1 + e_2 + e_{\nu+1} + ie_{\nu+2} + e_{2\nu+t}]$, where $i \in \mathbb{C}$ such that $i^2 = -1$. Then, we have $[\alpha], [\beta] \in \Phi_0, [\alpha] \perp_H [\gamma_s], [\beta] \perp_H [\gamma_t]$, and $[\alpha] \perp_H [\beta]$. Hence, by Lemma 3.1 we obtain

$$\sigma([\alpha]) = [k_1\pi(\lambda)e_1 + \overline{k_1}^{-1}e_{\nu+1} + \overline{k_2}^{-1}e_{\nu+2} + \omega_{s1}e_{2\nu+1} + \cdots + \omega_{s\delta}e_{2\nu+\delta}]$$

and

$$\sigma([\beta]) = [k_1\pi(\lambda)e_1 + k_2e_2 + \overline{k_1}^{-1}e_{\nu+1} + \overline{k_2}^{-1}\pi(i)e_{\nu+2} + \omega_{t1}e_{2\nu+1} + \cdots + \omega_{t\delta}e_{2\nu+\delta}].$$

Since $\sigma([\alpha]) \perp_H \sigma([\beta])$, we deduce that $\Sigma_{j=1}^{\delta}\omega_{sj}\overline{\omega_{tj}} = 0$. In combination with $\Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}} = 1$, we can see that $W\overline{W}^t = I^{(\delta)}$. □

By Lemmas 3.1 and 3.2, similar to the proof of Lemma 3.17 in [8], we have

Lemma 3.3. *Let* $[\alpha] = [a_1e_1 + \cdots + a_{2\nu}e_{2\nu} + e_{2\nu+i}] \in \Phi_0, 1 \leq i \leq \delta$. *Suppose*

$$\sigma([\alpha]) = [k_1\pi(a_1), \ldots, k_\nu\pi(a_\nu), \overline{k_1}^{-1}\pi(a_{\nu+1}), \ldots, \overline{k_\nu}^{-1}\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}],$$

and then

$$(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta}), 1 \leq i \leq \delta.$$

Proof. Consider the case $[\alpha] \perp_H [\gamma_i]$. By Lemma 3.1, we have

$$(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta}), 1 \leq i \leq \delta.$$

Then consider the case when $[\alpha] \not\perp_H [\gamma_i]$. We distinguish the following three cases:
(1) There is some $a_j \neq 0$ where $2 \leq j \leq \nu$. Pick

$$[\beta] = [(-\overline{\lambda} - 1)e_1 + e_{\nu+1} + \overline{((\lambda + 1)a_{\nu+1} - a_1 - 1)a_j^{-1}}e_{\nu+j} + e_{2\nu+i}].$$

Then $[\beta] \in \Phi_0$, $[\alpha] \perp_H [\beta]$, and $[\beta] \perp_H [\gamma_i]$. By Lemma 3.1, we have

$$\sigma([\beta]) = [k_1\pi(-\overline{\lambda} - 1)e_1 + \overline{k_1}^{-1}e_{\nu+1} + \overline{k_j}^{-1}\pi(\overline{((\lambda + 1)a_{\nu+1} - a_1 - 1)a_j^{-1}})e_{\nu+j} + \omega_{i1}e_{2\nu+1} + \cdots + \omega_{i\delta}e_{2\nu+\delta}].$$

From $[\alpha] \perp_H [\beta]$ we deduce $\sigma([\alpha]) \perp_H \sigma([\beta])$. Thus

$$\pi(a_1) + \pi((-\lambda - 1)a_{\nu+1}) + \pi((\lambda + 1)a_{\nu+1} - a_1 - 1) + a'_{2\nu+1}\overline{\omega_{i1}} + \cdots + a'_{2\nu+\delta}\overline{\omega_{i\delta}} = 0,$$

which implies $a'_{2\nu+1}\overline{\omega_{i1}} + \cdots + a'_{2\nu+\delta}\overline{\omega_{i\delta}} = 1$. Since $[\alpha]$ and $\sigma([\alpha]) \in \Phi_0$, we deduce

$$a'_{2\nu+1}\overline{a'_{2\nu+1}} + \cdots + a'_{2\nu+\delta}\overline{a'_{2\nu+\delta}} = 1.$$

Moreover, since $\sum_{k=1}^{\delta} \omega_{ik}\overline{\omega_{ik}} = 1$, by the Cauchy-Schwarz inequality, we deduce that

$$(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta}).$$

Note that, in this case, since $[\alpha] \not\perp_H [\gamma_i]$, we cannot use Lemma 3.1 directly. Hence we introduce the third element $[\beta]$ such that $[\alpha] \perp_H [\beta]$ and $[\beta] \perp_H [\gamma_i]$, and then we can use Lemma 3.1 for $[\beta]$ and deduce the conclusion.

(2) There is some $a_{\nu+j} \neq 0$, where $2 \leq j \leq \nu$. Pick

$$[\beta] = [(-\overline{\lambda} - 1)e_1 + \overline{((\lambda + 1)a_{\nu+1} - a_1 - 1)a_{\nu+j}^{-1}}e_j + e_{\nu+1} + e_{2\nu+i}].$$

Then $[\beta] \in \Phi_0$, $[\alpha] \perp_H [\beta]$, and $[\beta] \perp_H [\gamma_i]$. As subcase (1), we still have

$$(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta}).$$

(3) Suppose $[\alpha] = [a_1e_1 + a_{\nu+1}e_{\nu+1} + e_{2\nu+i}] \in \Phi_0$, where $a_1\overline{a_{\nu+1}} + \overline{a_1}a_{\nu+1} + 1 = 0$. Pick $[\beta] = [a_1e_1+e_2+a_{\nu+1}e_{\nu+1}+e_{2\nu+i}]$, and then $[\beta] \in \Phi_0$ and $[\alpha] \perp_H [\beta]$. By case (1), we have $\sigma([\beta]) = [k_1\pi(a_1)e_1 + k_2e_2 + \overline{k_1}^{-1}\pi(a_{\nu+1})e_{\nu+1} + \omega_{i1}e_{2\nu+1} + \cdots + \omega_{i\delta}e_{2\nu+\delta}]$. From $[\alpha] \perp_H [\beta]$ we deduce $\sigma([\alpha]) \perp_H \sigma([\beta])$. Thus $a'_{2\nu+1}\overline{\omega_{i1}} + \cdots + a'_{2\nu+\delta}\overline{\omega_{i\delta}} = 1$. Since $[\alpha]$ and $\sigma([\alpha]) \in \Phi_0$, we deduce $a'_{2\nu+1}\overline{a'_{2\nu+1}} + \cdots + a'_{2\nu+\delta}\overline{a'_{2\nu+\delta}} = 1$. Moreover, since $\Sigma_{k=1}^{\delta}\omega_{ik}\overline{\omega_{ik}} = 1$, by the Cauchy-Schwarz inequality, we deduce that $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta})$.

Hence in all cases $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\omega_{i1}, \ldots, \omega_{i\delta})$, $1 \leq i \leq \delta$. □

Now we return to the proof of Theorem 3.1. For any $1 \leq s \leq \delta$, take $[\beta_s] = [b_{s1}e_1 + \cdots + b_{s,2\nu}e_{2\nu} + e_{2\nu+s}] \in \Phi_0$ such that $[\alpha] \perp_H [\beta_s]$. By Lemma 3.3, we have

$$\sigma([\beta_s]) = [k_1\pi(b_{s1}), \ldots, k_\nu\pi(b_{s\nu}), \overline{k_1}^{-1}\pi(b_{s,\nu+1}), \ldots, \overline{k_\nu}^{-1}\pi(b_{s,2\nu}), \omega_{s1}, \ldots, \omega_{s\delta}].$$

Since $[\alpha] \perp_H [\beta_s]$ and $\sigma([\alpha]) \perp_H \sigma([\beta_s])$, we have

$$\sum_{j=1}^{\delta}(a_j\overline{b_{s,\nu+j}} + a_{\nu+j}\overline{b_{sj}}) + a_{2\nu+s} = 0$$

and

$$\sum_{j=1}^{\delta}(\pi(a_j)\overline{\pi(b_{s,\nu+j})} + \pi(a_{\nu+j})\overline{\pi(b_{sj})}) + \Sigma_{j=1}^{\delta}a'_{2\nu+j}\overline{\omega_{sj}} = 0.$$

By the two equations above, we deduce

$$\sum_{j=1}^{\delta}a'_{2\nu+j}\overline{\omega_{sj}} = \pi(a_{2\nu+s}), 1 \leq s \leq \delta.$$

Hence $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta})\overline{W}^t = (\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))$. By Lemma 3.2, Theorem 3.1 can be concluded. □

4. H-orthogonal transformations of Φ_1

In this section we will determine $O(\Phi_1)$. By proposition 2.2, $U_n^{(1)}(\mathbb{C})$ can be regarded as a subgroup of $O(\Phi_1)$. First, let us write out some elements of E_{Φ_1}.

Case (1) $\delta = 0$. Let $\pi \in \Lambda$ and $k \in \mathbb{R}^*$. Let $\sigma_{k,\pi}$ be the map which takes any vertex $[a_1, a_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu}]$ of Φ_1 to the vertex

$$[k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu})].$$

Then it is clear that $\sigma_{k,\pi}$ is well defined and $\sigma_{k,\pi} \in E_{\Phi_1}$. Define a map from the direct product $\mathbb{R}^* \times \Lambda$ to E_{Φ_1} by $h : (k, \pi) \longmapsto \sigma_{k,\pi}$. Clearly, h is an injective map of sets. In order to prove E_{Φ_1} is a group and $E_{\Phi_1} \cong \mathbb{R}^* \times \Lambda$, it suffices to show that every elements σ of E_{Φ_1} are of the form $\sigma_{k,\pi}$.

Case (2) $\delta \geq 1$. Denote $\sigma_{k,\pi,W}$ as the map which takes any vertex $[a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+1}]$ of Φ_1 to the vertex $[k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}]$, where $k \in (\mathbb{R}^+)^*$, $\pi \in \Lambda$, $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = \mu_k(\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$, and $W \in U_\delta(\mathbb{C})$ such that $W\overline{W}^t = I^{(\delta)}$. Then it is clear that $\sigma_{k,\pi,W}$ is well defined and $\sigma_{k,\pi,W} \in E_{\Phi_1}$. For any σ_{k_1,π_1,W_1} and σ_{k_2,π_2,W_2}, were $k_1, k_2 \in (\mathbb{R}^+)^*$, $\pi_1, \pi_2 \in \Lambda$, and $W_1, W_2 \in U_\delta(\mathbb{C})$ such that $W\overline{W}^t = I^{(\delta)}$, and we have $\sigma_{k_1,\pi_1,W_1}, \sigma_{k_2,\pi_2,W_2} \in E_{\Phi_1}$, and the composition of them is $\sigma_{k_1,\pi_1,W_1}\sigma_{k_2,\pi_2,W_2} = \sigma_{k_1k_2,\pi_1\pi_2,\frac{\mu_{k_1}}{\mu_{k_1k_2}}\pi_1(\mu_{k_2}W_2)W_1} \in E_{\Phi_1}$. Define a mapping $h : \Omega_\delta \longrightarrow E_{\Phi_1}$ by $(k, \pi, W) \longmapsto \sigma_{k,\pi,W}$. Clearly, h is an injective map of sets. In order to prove E_{Φ_1} is a group and $E_{\Phi_1} \cong \Omega_\delta$, it suffices to show that every elements σ of E_{Φ_1} are of the form $\sigma_{k,\pi,W}$.

Now, in order to prove (2) of Theorem 1.2, we need only to prove:

Theorem 4.1. *Let $\nu \geq 3$, and then $O(\Phi_1) = U_n^{(1)}(\mathbb{C}) \cdot E_{\Phi_1}$. Let $\sigma \in E_{\Phi_1}$, for any $[\alpha] = [a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_1$, and we have*

(1) *if $\delta = 0$, then $\sigma([\alpha]) = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu})]$, where $k \in \mathbb{R}^*, \pi \in \Lambda$;*

(2) *if $\delta \geq 1$, then*

$$\sigma([\alpha]) = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}],$$

where $k \in (\mathbb{R}^+)^*$, $\pi \in \Lambda$, $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = \mu_k(\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$ and $W \in U_\delta(\mathbb{C})$ such that $W\overline{W}^t = I^{(\delta)}$.

Proof. Similar to the proof of Theorem 3.3 in [8], we have $O(\Phi_1) = U_n^{(1)}(\mathbb{C}) \cdot E_{\Phi_1}$. Let $\sigma \in E_{\Phi_1}$, $\nu \geq 3$, and $[\alpha] = [a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_1$. Suppose $\sigma([\alpha]) = [\alpha']$ and write $[\alpha'] = [a'_1, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}]$. By Lemmas 2.1 and 2.2, as in the proof of Theorem 3.3 in [8], we have

1) if $\delta = 0$, then $\sigma([\alpha]) = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu})]$, where $k \in \mathbb{R}^*, \pi \in \Lambda$;

2) if $\delta \geq 1$, then

$$\sigma([\alpha]) = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}],$$

where $k \in (\mathbb{R}^+)^*$, $\pi \in \Lambda$.

When $\delta \geq 1$, let $\lambda \in \mathbb{C}^*$ such that $\lambda + \overline{\lambda} + 1 = 0$. Then $[\gamma_i] = [\lambda e_1 + e_{\nu+1} + e_{2\nu+i}] \in \Phi_1$ and $\sigma([\gamma_i]) = [k\pi(\lambda)e_1 + e_{\nu+1} + \mu_k\omega_{i1}e_{2\nu+1} + \cdots + \mu_k\omega_{i\delta}e_{2\nu+\delta}]$, where ω_{ij} satisfies

$$k\pi(\lambda) + k\overline{\pi(\lambda)} + \sum_{j=1}^{\delta} k\omega_{ij}\overline{\omega_{ij}} = 0, \quad 1 \leq i, j \leq \delta.$$

By $\lambda + \overline{\lambda} + 1 = 0$, we deduce $\Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}} = 1, 1 \leq i, j \leq \delta$. Let $W = (\omega_{ij})_{\delta\times\delta}$, similar to the proof of Theorem 3.1, and we have $W\overline{W}^t = I^{(\delta)}$ and $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = \mu_k(\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$. □

5. H-orthogonal transformations of Φ_2

In this section we will determine $O(\Phi_2)$. By proposition 2.2, $U_n^{(2)}(\mathbb{C})$ can be regarded as a subgroup of $O(\Phi_2)$. Let E_{Φ_2} be the subset of E_2 such that $k_2 = k_3 = \cdots = k_\nu = 1$. In order to prove $O(\Phi_2) = U_n^{(2)}(\mathbb{C}) \cdot \Lambda \mid_{\Phi_2}$ for $\nu \geq 3$ and $\delta \in \mathbb{N}$, we need the following theorem.

Theorem 5.1. *Let $\nu \geq 3$, and then $O(\Phi_2) = U_n^{(2)}(\mathbb{C}) \cdot E_{\Phi_2}$. Let $\sigma \in E_{\Phi_2}$, for any $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_2$, and we have $\sigma([\alpha]) =$*

$$\begin{cases} [\pi(a_1), \pi(a_2), \ldots, \pi(a_{2\nu})], & \text{when } \delta = 0, \\ [\pi(a_1), \pi(a_2), \ldots, \pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}], & \text{when } \delta \geq 1, \end{cases}$$

where $\pi \in \Lambda$, $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$, and $W \in U_\delta(\mathbb{C})$ such that $W\overline{W}^t = I^{(\delta)}$.

Proof. Let $\tau \in O(\Phi_2)$. To extend the domain of τ to $\Phi_2 \cup \{[e_1+e_{\nu+1}]\}$, we define $\tau([e_1+e_{\nu+1}]) = [e_1+e_{\nu+1}]$. Suppose $\tau([e_{\nu+1}]) = [e'_{\nu+1}]$ and $\tau([e_i + e_{\nu+1}]) = [e'_i]$, $i = 2, \ldots, \nu, \nu+2, \ldots, 2\nu$. To be definite, we can assume $(e_1 + e_{\nu+1})H\overline{e'_i}^t = 1$, for $i = 2, \ldots, 2\nu$. Then $e'_i H\overline{e'_j}^t \neq 0$ if $i \equiv j (\text{mod } \nu)$ and $i \neq j$; or it is 0 otherwise. Suppose $e'_i H\overline{e'_{\nu+i}}^t = \overline{k_i} \in \mathbb{C}^*, 2 \leq i \leq \nu$.

Let $A(A'$, respectively) be the $2\nu \times (2\nu + \delta)$ matrix whose rows are $e_1 + e_{\nu+1}, e_2 + e_{\nu+1}, \ldots, e_\nu + e_{\nu+1}, e_{\nu+1}, e_{\nu+1} + e_{\nu+2}, \ldots, e_{\nu+1} + e_{2\nu}(e_1 + e_{\nu+1}, e'_2, \ldots, e'_{2\nu}$, respectively) in order. Let

$$Q_1 = \begin{pmatrix} 1 & & & & -1 & & & \\ & 1 & & & -1 & & & \\ & & \ddots & & \vdots & & & \\ & & & 1 & -1 & & & \\ & & & & 1 & & & \\ & & & & -1 & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & -1 & & & 1 \end{pmatrix}_{2\nu \times 2\nu}$$

and $Q_2 = \text{diag}(I^{(\nu+1)}, k_2^{-1}, \ldots, k_\nu^{-1})Q_1$. One can check that

$$Q_1(AH\overline{A}^t)\overline{Q_1}^t = Q_2(A'H\overline{A'}^t)\overline{Q_2}^t.$$

Then, by Theorem 2 on page 260 of [10], there is some matrix $T \in U_{2\nu+\delta}(\mathbb{C})$, such that $A'T = Q_2^{-1}Q_1A = MA$, where

$$M = Q_2^{-1}Q_1 = Q_1^{-1}(\text{diag}(I^{(\nu+1)}, k_2, \ldots, k_\nu))Q_1 = \begin{pmatrix} I^{(\nu)} & & & & \\ & 1 & & & \\ & 1-k_2 & k_2 & & \\ & \vdots & & \ddots & \\ & 1-k_\nu & & & k_\nu \end{pmatrix}_{2\nu \times 2\nu}.$$

Comparing the first row of both sides of $A'T = MA$, we have $(e_1 + e_{\nu+1})T = e_1 + e_{\nu+1}$, and thus $T \in U_{2\nu+\delta}^{(2)}(\mathbb{C})$. Set $\tau_1 = \sigma_T\tau$, and then $\tau_1([e_i + e_{\nu+1}]) = [e_i + e_{\nu+1}]$, $\tau_1([e_{\nu+1}]) = [e_{\nu+1}]$, $\tau_1([e_{\nu+1} + e_{\nu+i}]) =$

$[e_{\nu+1}+k_ie_{\nu+i}]$, $i=2,\ldots,\nu$. Then $\tau_1\in E_2$. In the following, we will show that $k_2=k_3=\cdots=k_\nu=1$, i.e., $E_2=E_{\Phi_2}$. Thus $\tau=\sigma_T^{-1}\tau_1\in U^{(2)}_{2\nu+\delta}(\mathbb{C})\cdot E_{\Phi_2}$, from which it follows that $O(\Phi_2)=U^{(2)}_{2\nu+\delta}(\mathbb{C})\cdot E_{\Phi_2}$.

By Lemma 2.1, we have that $\sigma([e_1])=[e_1]$ or $\sigma([e_1])=[\lambda_1e_1+e_{\nu+1}]$ and $\sigma([\lambda_2e_1+e_{\nu+1}])=[e_1]$, where $\lambda_1,\lambda_2\in\mathbb{A}\backslash\{0\}$ and $\mathbb{A}\triangleq\{\lambda\in\mathbb{C}|\lambda+\overline{\lambda}=0\}$. In the following we will show that the second case is impossible.

Now we suppose $\sigma([e_1])=[\lambda_1e_1+e_{\nu+1}]$ and $\sigma([\lambda_2e_1+e_{\nu+1}])=[e_1]$, where $\lambda_1,\lambda_2\in\mathbb{A}\backslash\{0\}$. When $a_1=\lambda_2$ and $a_{\nu+1}=1$, since $[\alpha]\perp_H[\lambda_2e_1+e_{\nu+1}]$, we have

$$\sigma([\alpha])=[a_1',a_2',\ldots,a_{2\nu+\delta}']\perp_H\sigma([\lambda_2e_1+e_{\nu+1}])=[e_1],$$

which implies $a'_{\nu+1}=0$.

By Lemmas 2.1 and Lemma 2.2, we have bijectives π_i, $i=2,\ldots,\nu,\nu+2,\ldots,2\nu$, of $\mathbb{C}$ such that for $a\in\mathbb{C}$, $\sigma([ae_i+e_{\nu+1}])=[\pi_i(a)e_i+e_{\nu+1}]$, $\pi_i(0)=0$, and by the definition of E', $\pi_2(1)=\cdots=\pi_\nu(1)=1$, $\pi_{\nu+2}(1)=k_2,\ldots,\pi_{2\nu}(1)=k_\nu$.

By Lemmas 2.1 and 2.2, and our assumption, we have bijective τ from $\mathbb{A}\backslash\{\lambda_2\}$ to $\mathbb{A}\backslash\{\lambda_1\}$ such that for $\lambda\in\mathbb{A}\backslash\{\lambda_2\},\sigma([\lambda e_1+e_{\nu+1}])=[\tau(\lambda)e_1+e_{\nu+1}]$, where $\tau(\lambda)\in\mathbb{A}\backslash\{\lambda_1\},\tau(0)=0$. Thus for any $\lambda\in\mathbb{A}\backslash\{\lambda_2\}$, we have $-\overline{\lambda}=\lambda$. Since $\tau(\lambda)\in\mathbb{A}$, $\tau(-\overline{\lambda})=\tau(\lambda)=-\overline{\tau(\lambda)}$.

We proceed to prove $\pi_2=\cdots=\pi_\nu$ and π_2 is an automorphism of $\mathbb{C}$. As a preparation we prove:

Lemma 5.1. *For any $\lambda\in\mathbb{A}\backslash\{0,\lambda_2\},a\in\mathbb{C}^*$, and $i=2,\ldots,\nu$, we have*

$$\sigma([\lambda e_1+ae_i+e_{\nu+1}])=[\tau(\lambda)e_1-\tau(\lambda)\overline{\pi_{\nu+i}(\lambda\overline{a}^{-1})}^{-1}e_i+e_{\nu+1}]$$

and

$$\sigma([\lambda e_1+e_{\nu+1}+ae_{\nu+i}])=[\tau(\lambda)e_1+e_{\nu+1}-\tau(\lambda)\overline{\pi_i(\lambda\overline{a}^{-1})}^{-1}e_{\nu+i}].$$

For any $a\in\mathbb{C}^*$ and $i=2,\ldots,\nu$, we have $\sigma([\lambda_2e_1+ae_i+e_{\nu+1}])=[e_1-\overline{\pi_{\nu+i}(\lambda_2\overline{a}^{-1})}^{-1}e_i]$ and $\sigma([\lambda_2e_1+e_{\nu+1}+ae_{\nu+i}])=[e_1-\overline{\pi_i(\lambda_2\overline{a}^{-1})}^{-1}e_{\nu+i}]$.

Proof. By Lemmas 2.1 and 2.2, when $\lambda\in\mathbb{A}\backslash\{0,\lambda_2\},a\in\mathbb{C}^*$, we can assume $\sigma([\lambda e_1+ae_i+e_{\nu+1}])=[\lambda'e_1+a'e_i+e_{\nu+1}]$. Otherwise, if $\sigma([\lambda e_1+ae_i+e_{\nu+1}])=[\lambda'e_1+a'e_i]$, then $[\lambda'e_1+a'e_i]\perp_H[e_1]$, hence $[\lambda e_1+ae_i+e_{\nu+1}]\perp_H[\lambda_2e_1+e_{\nu+1}]$, which implies $\lambda=\lambda_2$, which is a contradiction. From $[\lambda e_1+ae_i+e_{\nu+1}]\perp_H[-\overline{\lambda}e_1+e_{\nu+1}]$, we deduce $[\lambda'e_1+a'e_i+e_{\nu+1}]\perp_H[\tau(-\overline{\lambda})e_1+e_{\nu+1}]$, which implies $\lambda'=-\overline{\tau(-\overline{\lambda})}=\tau(\lambda)$. Similarly, from $[\lambda e_1+ae_i+e_{\nu+1}]\perp_H[e_{\nu+1}-\overline{\lambda}\overline{a}^{-1}e_{\nu+i}]$, we deduce $a'=-\tau(\lambda)\overline{\pi_{\nu+i}(\lambda\overline{a}^{-1})}^{-1}$.

By Lemmas 2.1 and 2.2, as above, we can assume $\sigma([\lambda e_1+e_{\nu+1}+ae_{\nu+i}])=[\lambda'e_1+e_{\nu+1}+a'e_{\nu+i}]$. From $[\lambda e_1+e_{\nu+1}+ae_{\nu+i}]\perp_H[-\overline{\lambda}e_1+e_{\nu+1}]$, we deduce $[\lambda'e_1+e_{\nu+1}+a'e_{\nu+i}]\perp_H[\tau(-\overline{\lambda})e_1+e_{\nu+1}]$, which implies $\lambda'=-\overline{\tau(-\overline{\lambda})}=\tau(\lambda)$. Similarly, from $[\lambda e_1+e_{\nu+1}+ae_{\nu+i}]\perp_H[-\overline{\lambda}\overline{a}^{-1}e_i+e_{\nu+1}]$, we deduce $a'=-\tau(\lambda)\overline{\pi_i(\lambda\overline{a}^{-1})}^{-1}$.

Since $[\lambda_2e_1+ae_i+e_{\nu+1}]\perp_H[\lambda_2e_1+e_{\nu+1}]$, we have $\sigma([\lambda_2e_1+ae_i+e_{\nu+1}])\perp_H[e_1]$, hence we can assume $\sigma([\lambda_2e_1+ae_i+e_{\nu+1}])=[e_1+a'e_i]$. From $[\lambda_2e_1+ae_i+e_{\nu+1}]\perp_H[e_{\nu+1}-\overline{\lambda_2}\overline{a}^{-1}e_{\nu+i}]$, we deduce $a'=-\overline{\pi_{\nu+i}(\lambda_2\overline{a}^{-1})}^{-1}$. Similarly we have

$$\sigma([\lambda_2e_1+e_{\nu+1}+ae_{\nu+i}])=[e_1-\overline{\pi_i(\lambda_2\overline{a}^{-1})}^{-1}e_{\nu+i}].$$

□

Lemma 5.2. *Let $2 \le i,\ j \le \nu$, and $i \ne j$. For any $a,\ b \in \mathbb{C}$, we have*

$$\sigma([e_{\nu+1} + ae_{\nu+i} + be_{\nu+j}]) = [e_{\nu+1} + \pi_{\nu+i}(a)e_{\nu+i} + \pi_{\nu+j}(b)e_{\nu+j}].$$

Similarly, $\sigma([ae_i + e_{\nu+1} + be_{\nu+j}]) = [\pi_i(a)e_i + e_{\nu+1} + \pi_{\nu+j}(b)e_{\nu+j}]$ and $\sigma([ae_i + be_j + e_{\nu+1}]) = [\pi_i(a)e_i + \pi_j(b)e_j + e_{\nu+1}]$.

Proof. It suffices to prove the lemma for $a,\ b \in \mathbb{C}^*$. By Lemmas 2.1 and 2.2, we can assume $\sigma([e_{\nu+1} + ae_{\nu+i} + be_{\nu+j}]) = [e_{\nu+1} + a'e_{\nu+i} + b'e_{\nu+j}]$. From $[e_{\nu+1} + ae_{\nu+i} + be_{\nu+j}] \perp_H [\lambda e_1 - \lambda\overline{a}^{-1}e_i + e_{\nu+1}]$, where $\lambda \in \mathbb{A}\backslash\{0, \lambda_2\}$, by Lemma 5.1 we deduce $[e_{\nu+1} + a'e_{\nu+i} + b'e_{\nu+j}] \perp_H [\tau(\lambda)e_1 - \tau(\lambda)\overline{\pi_{\nu+i}(a)}^{-1}e_i + e_{\nu+1}]$, which implies $a' = \pi_{\nu+i}(a)$. Similarly, we have $b' = \pi_{\nu+j}(b)$.

By Lemmas 2.1 and 2.2, we can assume $\sigma([ae_i + e_{\nu+1} + be_{\nu+j}]) = [a'e_i + e_{\nu+1} + b'e_{\nu+j}]$. From $[ae_i + e_{\nu+1} + be_{\nu+j}] \perp_H [\lambda e_1 + e_{\nu+1} - \lambda\overline{a}^{-1}e_{\nu+i}]$, where $\lambda \in \mathbb{A}\backslash\{0, \lambda_2\}$, by Lemma 5.1 we deduce $[a'e_i + e_{\nu+1} + b'e_{\nu+j}] \perp_H [\tau(\lambda)e_1 + e_{\nu+1} - \tau(\lambda)\overline{\pi_i(a)}^{-1}e_{\nu+i}]$, and hence $a' = \pi_i(a)$.

Similarly, we have $b' = \pi_{\nu+j}(b)$, and $\sigma([ae_i + be_j + e_{\nu+1}]) = [\pi_i(a)e_i + \pi_j(b)e_j + e_{\nu+1}]$. □

Lemma 5.3. $k_2 = k_3 = \cdots = k_\nu \in \mathbb{R}^*$. *For any $a \in \mathbb{C}$, $\pi_i(-a) = -\pi_i(a)$, $\pi_{\nu+i}(-a) = -\pi_{\nu+i}(a)$, $2 \le i \le \nu$. Moreover, $\pi_2 = \pi_3 = \cdots = \pi_\nu$ and $\pi_{\nu+2} = \pi_{\nu+3} = \cdots = \pi_{2\nu} = k_2\pi_2$.*

Proof. Let $a,\ b \in \mathbb{C}^*$. For $i = 3, \ldots, \nu$, from $[ae_2 + e_{\nu+1} + be_{\nu+i}] \perp_H [\overline{a}e_i + e_{\nu+1} - \overline{b}e_{\nu+2}]$, by Lemma 5.2 we deduce

$$[\pi_2(a)e_2 + e_{\nu+1} + \pi_{\nu+i}(b)e_{\nu+i}] \perp_H [\pi_i(\overline{a})e_i + e_{\nu+1} + \pi_{\nu+2}(-\overline{b})e_{\nu+2}]$$

which implies

$$\overline{\pi_2(a)}\pi_{\nu+2}(-\overline{b}) = -\pi_i(\overline{a})\overline{\pi_{\nu+i}(b)}.$$

Let $b = -1$ and $a = 1$, and we can get $\pi_{\nu+i}(-1) = -\overline{k_2}$ from the above formula. Hence

$$\pi_{\nu+3}(-1) = \cdots = \pi_{2\nu}(-1) = -\overline{k_2}.$$

Let $b = 1$ and $a = 1$, and we have $\pi_{\nu+2}(-1) = -\overline{k_i}$, hence $k_3 = \cdots = k_\nu$, and

$$\pi_{\nu+2}(-1) = -\overline{k_3}.$$

Let $b = -\overline{a}$ and $a = -1$, and we can get $\overline{\pi_2(-1)}\pi_{\nu+2}(-1) = -\pi_i(-1)\overline{\pi_{\nu+i}(1)}$, where $3 \le i \le \nu$. Hence

$$\pi_3(-1) = \cdots = \pi_\nu(-1).$$

Let $2 \le i \le \nu - 1$ and $a \in \mathbb{C}$. It is easy to verify that

$$[ae_i + e_{\nu+1} + ae_{\nu+i+1}] \perp_H [e_{i+1} + e_{\nu+1} - e_{\nu+i}],$$

$$[ae_i + ae_{i+1} + e_{\nu+1}] \perp_H [e_{\nu+1} - e_{\nu+i} + e_{\nu+i+1}],$$

and

$$[e_{\nu+1} + ae_{\nu+i} + ae_{\nu+i+1}] \perp_H [-e_i + e_{i+1} + e_{\nu+1}].$$

Applying σ to the above non-adjacency relations and using Lemma 5.2, we obtain

$$[\pi_i(a)e_i + e_{\nu+1} + \pi_{\nu+i+1}(a)e_{\nu+i+1}] \perp_H [e_{i+1} + e_{\nu+1} + \pi_{\nu+i}(-1)e_{\nu+i}],$$

$$[\pi_i(a)e_i + \pi_{i+1}(a)e_{i+1} + e_{\nu+1}] \perp_H [e_{\nu+1} + \pi_{\nu+i}(-1)e_{\nu+i} + k_{i+1}e_{\nu+i+1}],$$

and

$$[e_{\nu+1} + \pi_{\nu+i}(a)e_{\nu+i} + \pi_{\nu+i+1}(a)e_{\nu+i+1}] \perp_H [\pi_i(-1)e_i + e_{i+1} + e_{\nu+1}],$$

respectively. From the above non-adjacency relations, we deduce $\pi_i(a)\overline{\pi_{\nu+i}(-1)} = -\pi_{\nu+i+1}(a)$, $\pi_i(a)\overline{\pi_{\nu+i}(-1)} = -\pi_{i+1}(a)\overline{k_{i+1}}$, and $\pi_{\nu+i}(a)\overline{\pi_i(-1)} = -\pi_{\nu+i+1}(a)$. Therefore

$$-\pi_{i+1}(a)\overline{k_{i+1}} = \pi_i(a)\overline{\pi_{\nu+i}(-1)} = -\pi_{\nu+i+1}(a) = \pi_{\nu+i}(a)\overline{\pi_i(-1)}, \tag{5.1}$$

where $2 \le i \le \nu - 1$. Substituting $a = 1,\ i = 2$, into (5.1), we have $-\overline{k_3} = \overline{\pi_{\nu+2}(-1)} = -k_3 = k_2\overline{\pi_2(-1)}$. Hence $k_3 = \overline{k_3}$.

Similarly, we have

$$[ae_{i+1} + e_{\nu+1} + ae_{\nu+i}] \perp_H [e_i + e_{\nu+1} - e_{\nu+i+1}],$$

$$[e_{\nu+1} + ae_{\nu+i} + ae_{\nu+i+1}] \perp_H [e_i - e_{i+1} + e_{\nu+1}],$$

and

$$[ae_i + ae_{i+1} + e_{\nu+1}] \perp_H [e_{\nu+1} + e_{\nu+i} - e_{\nu+i+1}].$$

Applying σ to the above non-adjacency relations and using Lemma 5.2, we obtain

$$[\pi_{i+1}(a)e_{i+1} + e_{\nu+1} + \pi_{\nu+i}(a)e_{\nu+i}] \perp_H [e_i + e_{\nu+1} + \pi_{\nu+i+1}(-1)e_{\nu+i+1}],$$

$$[e_{\nu+1} + \pi_{\nu+i}(a)e_{\nu+i} + \pi_{\nu+i+1}(a)e_{\nu+i+1}] \perp_H [e_i + \pi_{i+1}(-1)e_{i+1} + e_{\nu+1}],$$

and

$$[\pi_i(a)e_i + \pi_{i+1}(a)e_{i+1} + e_{\nu+1}] \perp_H [e_{\nu+1} + k_ie_{\nu+i} + \pi_{\nu+i+1}(-1)e_{\nu+i+1}],$$

respectively. From the above non-adjacency relations, we deduce $\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)} = -\pi_{\nu+i}(a)$, $-\pi_{\nu+i}(a) = \pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)}$, and $-\pi_i(a)\overline{k_i} = \pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)}$. Therefore

$$\pi_{\nu+i}(a) = -\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)} = \pi_i(a)\overline{k_i} = -\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)}, \tag{5.2}$$

where $2 \le i \le \nu - 1$. Substituting $a = 1,\ i = 2$, into (5.2), we have $k_2 = -\overline{\pi_{\nu+3}(-1)} = \overline{k_2} = -k_3\overline{\pi_3(-1)}$. Hence $k_2 = \overline{k_2}$.

From (5.1) we have $-\pi_{i+1}(a)\overline{k_{i+1}} = \pi_{\nu+i}(a)\overline{\pi_i(-1)}$, and from (5.2) we have $\pi_{\nu+i}(a) = -\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)}$. Substituting the last equation into the previous one we obtain

$$-\pi_{i+1}(a)\overline{k_{i+1}} = -\pi_{i+1}(a)\overline{\pi_{\nu+i+1}(-1)\pi_i(-1)}.$$

Cancelling $-\pi_{i+1}(a)$ and then applying the involutive automorphism, we obtain

$$k_{i+1} = \pi_{\nu+i+1}(-1)\pi_i(-1), \tag{5.3}$$

where $2 \le i \le \nu - 1$.

From (5.1) we have $\pi_i(a)\overline{\pi_{\nu+i}(-1)} = \pi_{\nu+i}(a)\overline{\pi_i(-1)}$, and from (5.2) we have $\pi_{\nu+i}(a) = \pi_i(a)\overline{k_i}$. Substituting the last equation into the previous one we obtain

$$\pi_i(a)\overline{\pi_{\nu+i}(-1)} = \pi_i(a)\overline{k_i\pi_i(-1)}.$$

Cancelling $\pi_i(a)$ and then applying the involutive automorphism, we obtain

$$\pi_{\nu+i}(-1) = k_i\pi_i(-1), \tag{5.4}$$

where $2 \le i \le \nu - 1$.

From (5.1) we have also $-\pi_{\nu+i+1}(a) = \pi_{\nu+i}(a)\overline{\pi_i(-1)}$, and from (5.2) we have $\pi_{\nu+i}(a) = -\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)}$. Substituting the last equation into the previous one we obtain

$$-\pi_{\nu+i+1}(a) = -\pi_{\nu+i+1}(a)\overline{\pi_{i+1}(-1)\pi_i(-1)}.$$

Cancelling $-\pi_{\nu+i+1}(a)$ and then applying the involutive automorphism, we obtain

$$\pi_{i+1}(-1)\pi_i(-1) = 1, \tag{5.5}$$

where $2 \le i \le \nu - 1$.

Substituting $i = 2$ into (5.3) and (5.4), we have $k_3 = \pi_{\nu+3}(-1)\pi_2(-1)$ and $\pi_{\nu+2}(-1) = k_2\pi_2(-1)$, respectively. Since $\pi_{\nu+3}(-1) = -\overline{k_2}$, we obtain $k_3 = -\overline{k_2}\pi_2(-1)$ and $-\overline{k_3} = k_2\pi_2(-1)$, respectively, hence $\pi_2(-1) = \overline{\pi_2(-1)}$.

Since $\overline{\pi_2(-1)}\pi_{\nu+2}(-1) = -\pi_3(-1)\overline{\pi_{\nu+3}(1)}$ and $-\overline{\pi_2(-1)k_3} = -\pi_3(-1)\overline{k_3}$, hence $\overline{\pi_2(-1)} = \pi_3(-1)$. Substituting $i = 2$ into (5.5), we have $\pi_3(-1)\pi_2(-1) = 1$. Moreover, from $\pi_2(-1) = \overline{\pi_2(-1)}$ and $\overline{\pi_2(-1)} = \pi_3(-1)$ we deduce $\pi_2(-1)^2 = 1$ and $\pi_2(-1) = \pi_3(-1)$.

Since π_2 is a bijective of $\mathbb{C}$ and $\pi_2(1) = 1$, we have $\pi_2(-1) = -1$. Since $\pi_2(-1) = \pi_3(-1)$, we have $\pi_2(-1) = \pi_3(-1) = \cdots = \pi_\nu(-1) = -1$. Since $k_3 = -\overline{k_2}\pi_2(-1)$, $k_3 = \overline{k_2}$. From $k_2 = \overline{k_2}$, we obtain $k_2 = k_3 \in \mathbb{R}^*$. From $k_2 = k_3 = \cdots = k_\nu \in \mathbb{R}^*$, $\pi_{\nu+3}(-1) = \cdots = \pi_{2\nu}(-1) = -\overline{k_2}$, and $\pi_{\nu+2}(-1) = -\overline{k_3}$, we obtain $\pi_{\nu+2}(-1) = \cdots = \pi_{2\nu}(-1) = -k_2$.

Substituting $k_{i+1} = k_2 \in \mathbb{R}^*$, $\pi_{\nu+i}(-1) = -k_2$, and $\pi_i(-1) = -1$ into (5.1), we have

$$\pi_{i+1}(a)k_2 = \pi_i(a)k_2 = \pi_{\nu+i+1}(a) = \pi_{\nu+i}(a), \tag{5.6}$$

where $2 \le i \le \nu - 1$, $a \in \mathbb{C}$. By (5.6), we have $\pi_2 = \pi_3 = \cdots = \pi_\nu$, and $\pi_{\nu+2} = \pi_{\nu+3} = \cdots = \pi_{2\nu} = k_2\pi_2$. From $[ae_2 + e_{\nu+1} - ae_{\nu+3}] \perp_H [e_3 + e_{\nu+1} + e_{\nu+2}]$, by Lemma 5.2, we deduce $[\pi_2(a)e_2 + e_{\nu+1} + \pi_{\nu+3}(-a)e_{\nu+3}] \perp_H [e_3 + e_{\nu+1} + k_2e_{\nu+2}]$, which implies $\pi_{\nu+3}(-a) = -k_2\pi_2(a)$. By (5.6), $\pi_2(a)k_2 = \pi_{\nu+3}(a)$. Thus $\pi_{\nu+3}(-a) = -\pi_{\nu+3}(a)$. Writing a for $-a$ in (5.6), we obtain $\pi_2(-a) = -k_2^{-1}\pi_{\nu+3}(a) = -\pi_2(a)$. In virtue of (5.6) we also have $\pi_i(-a) = -\pi_i(a)$, $\pi_{\nu+i}(-a) = -\pi_{\nu+i}(a)$, $2 \le i \le \nu$. □

Lemma 5.4. *For any a, $b \in \mathbb{C}^*$ and $2 \le i \le \nu$, $\pi_i(\overline{a}) = \overline{\pi_i(a)}$, $\pi_i(ab) = \pi_i(a)\pi_i(b)$, and $\pi_i(a^{-1}) = \pi_i(a)^{-1}$.*

Proof. From $[ae_2 + e_{\nu+1} + abe_{\nu+3}] \perp_H [e_3 + e_{\nu+1} - \overline{b}e_{\nu+2}]$, by Lemma 5.2, we deduce $[\pi_2(a)e_2 + e_{\nu+1} + \pi_{\nu+3}(ab)e_{\nu+3}] \perp_H [e_3 + e_{\nu+1} + \pi_{\nu+2}(-\overline{b})e_{\nu+2}]$, which implies

$$\pi_{\nu+3}(ab) = -\pi_2(a)\overline{\pi_{\nu+2}(-\overline{b})} = \pi_2(a)\overline{\pi_{\nu+2}(\overline{b})}.$$

Substituting $a = 1$ into the above formula, we obtain $\pi_{\nu+3}(b) = \overline{\pi_{\nu+2}(\overline{b})}$. By Lemma 5.3 we have $\pi_{\nu+2}(b) = \pi_{\nu+3}(b)$. Thus

$$\pi_{\nu+2}(b) = \overline{\pi_{\nu+2}(\overline{b})}$$

and $\pi_{\nu+2}(\overline{b}) = \overline{\pi_{\nu+2}(b)}$. Hence,

$$\pi_{\nu+3}(ab) = \pi_2(a)\pi_{\nu+2}(b).$$

In Lemma 5.3 we have $\pi_2 = \pi_3 = \cdots = \pi_\nu$, $\pi_{\nu+2} = \pi_{\nu+3} = \cdots = \pi_{2\nu}$, and $\pi_{\nu+2} = k_2\pi_2$. Therefore for $2 \leq i \leq \nu$,

$$\pi_i(\overline{a}) = k_2^{-1}\pi_{\nu+i}(\overline{a}) = k_2^{-1}\pi_{\nu+2}(\overline{a}) = k_2^{-1}\overline{\pi_{\nu+2}(a)} = \overline{k_2^{-1}\pi_{\nu+2}(a)} = \overline{\pi_2(a)} = \overline{\pi_i(a)},$$

and we have

$$\pi_i(ab) = k_2^{-1}\pi_{\nu+i}(ab) = k_2^{-1}\pi_{\nu+3}(ab) = k_2^{-1}\pi_2(a)\pi_{\nu+2}(b) = \pi_2(a)\pi_2(b) = \pi_i(a)\pi_i(b).$$

Replacing b by a^{-1} in $\pi_i(ab) = \pi_i(a)\pi_i(b)$, we have $\pi_i(a^{-1}) = \pi_i(a)^{-1}$. □

Lemma 5.5. *Let* $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_2$ *and* $a_1 \neq 0$. *Suppose* $\sigma([\alpha]) = [a'_1, a'_2, \ldots, a'_{2\nu+\delta}]$. *Then* $a'_j = a'_1 k_2^{-1}\pi_2(a_1)^{-1}\pi_2(a_j)$ *and* $a'_{\nu+j} = a'_1\pi_2(a_1)^{-1}\pi_2(a_{\nu+j})$, *where* $2 \leq j \leq \nu$.

Proof. The case $a_j = 0$ and $a_{\nu+j} = 0$ follows from Lemma 2.1. If $a_j \neq 0$, $2 \leq j \leq \nu$, then $[a_1, a_2, \ldots, a_{2\nu+\delta}] \perp_H [e_{\nu+1} - \overline{a_1 a_j^{-1}} e_{\nu+j}]$, and by Lemmas 5.3 and 5.4, we deduce

$$[a'_1, a'_2, \ldots, a'_{2\nu+\delta}] \perp_H [e_{\nu+1} - k_2\overline{\pi_{\nu+j}(a_1)\pi_{\nu+j}(a_j)^{-1}} e_{\nu+j}],$$

hence, $a'_1 = a'_j k_2 \pi_{\nu+j}(a_1)\pi_{\nu+j}(a_j)^{-1}$. By Lemma 5.3 we have

$$a'_j = a'_1 k_2^{-1}\pi_{\nu+j}(a_1)^{-1}\pi_{\nu+j}(a_j) = a'_1 k_2^{-1}\pi_j(a_1)^{-1}\pi_j(a_j) = a'_1 k_2^{-1}\pi_2(a_1)^{-1}\pi_2(a_j).$$

Similarly, if $a_{\nu+j} \neq 0$, $2 \leq j \leq \nu$, then $[a_1, a_2, \ldots, a_{2\nu+\delta}] \perp_H [-\overline{a_1 a_{\nu+j}^{-1}} e_j + e_{\nu+1}]$, and by Lemmas 5.3 and 5.4, we deduce

$$[a'_1, a'_2, \ldots, a'_{2\nu+\delta}] \perp_H [-\overline{\pi_j(a_1)\pi_j(a_{\nu+j})^{-1}} e_j + e_{\nu+1}],$$

hence $a'_{\nu+j} = a'_1\pi_j(a_1)^{-1}\pi_j(a_{\nu+j}) = a'_1\pi_2(a_1)^{-1}\pi_2(a_{\nu+j})$ by Lemma 5.3. □

Proposition 5.1. π_2 *is an identity mapping or conjugate mapping of* $\mathbb{C}$.

Proof. First, we prove $\pi_2 \in \mathrm{Aut}(\mathbb{C})$, and by Lemma 5.4, it suffices to show that $\pi_2(a+b) = \pi_2(a)+\pi_2(b)$ for $a,\ b \in \mathbb{C}^*$.

Clearly $[e_1 + e_2 + \overline{(a+b)}e_3 + e_{\nu+1} - e_{\nu+2}] \in \Phi_2$. By Lemma 5.5 we can assume

$$\begin{aligned}&\sigma([e_1 + e_2 + \overline{(a+b)}e_3 + e_{\nu+1} - e_{\nu+2}])\\ &= [a'_1 e_1 + a'_1 k_2^{-1} e_2 + a'_1 k_2^{-1}\pi_2(\overline{(a+b)})e_3 + a'_{\nu+1}e_{\nu+1} - a'_1 e_{\nu+2} + a'_{2\nu+1}e_{2\nu+1} + \cdots + a'_{2\nu+\delta}e_{2\nu+\delta}],\end{aligned}$$

where $a'_1 \in \mathbb{C}^*, a'_{\nu+1}, a'_{2\nu+1}, \ldots, a'_{2\nu+\delta} \in \mathbb{C}$. From

$$[e_1 + e_2 + \overline{(a+b)}e_3 + e_{\nu+1} - e_{\nu+2}] \perp_H [e_{\nu+1} + a^{-1}be_{\nu+2} - a^{-1}e_{\nu+3}],$$

by Lemmas 2.2 and 5.2 we deduce

$$[a'_1 e_1 + a'_1 k_2^{-1} e_2 + a'_1 k_2^{-1}\pi_2(\overline{(a+b)})e_3 + a'_{\nu+1}e_{\nu+1} - a'_1 e_{\nu+2} + a'_{2\nu+1}e_{2\nu+1} \cdots + a'_{2\nu+\delta}e_{2\nu+\delta}]$$

$$\perp_H [e_{\nu+1} + \pi_{\nu+2}(a^{-1}b)e_{\nu+2} + \pi_{\nu+3}(-a^{-1})e_{\nu+3}],$$

which implies

$$a_1' + a_1'k_2^{-1}\overline{\pi_{\nu+2}(a^{-1}b)} + a_1'k_2^{-1}\pi_2(\overline{(a+b)})\overline{\pi_{\nu+3}(-a^{-1})} = 0.$$

From Lemmas 5.3 and 5.4, we have $\pi_2(a+b) = \pi_2(a) + \pi_2(b)$. Hence $\pi_2 \in \text{Aut}(\mathbb{C})$.

Since $\pi_2 \in \text{Aut}(\mathbb{C})$ and $\pi_2(\overline{a}) = \overline{\pi_2(a)}$ for any $a \in \mathbb{C}$ by Lemma 5.4, π_2 is an identity mapping or conjugate mapping of $\mathbb{C}$.

□

Denote $\pi_2 \triangleq \pi$, $k_2 \triangleq k$, and then by Lemma 5.3, $\pi_2 = \pi_3 = \cdots = \pi_\nu = \pi$ and $\pi_{\nu+2} = \pi_{\nu+3} = \cdots = \pi_{2\nu} = k\pi$.

We deduce the following result immediately.

Lemma 5.6. *For any $a \in \mathbb{C}$ and $i = 2, \ldots, \nu$, we have $\sigma([\lambda_2 e_1 + ae_i + e_{\nu+1}]) = [e_1 + k^{-1}\pi(\lambda_2)^{-1}\pi(a)e_i]$ and $\sigma([\lambda_2 e_1 + e_{\nu+1} + ae_{\nu+i}]) = [e_1 + \pi(\lambda_2)^{-1}\pi(a)e_{\nu+i}]$.*
$\sigma([e_1 + e_i + e_{\nu+1} - e_{\nu+i}]) = [ke_1 + e_i + \pi(\frac{\lambda_2-1}{\lambda_2})e_{\nu+1} - ke_{\nu+i}]$, where $i = 2, 3, \ldots, \nu$.

Proof. By Lemmas 2.2 and 5.5, we have $\sigma([e_1 + e_i + e_{\nu+1} - e_{\nu+i}]) = [k'e_1 + k'k^{-1}e_i + a'_{\nu+1}e_{\nu+1} - k'e_{\nu+i}]$, for some $a'_{\nu+1} \in \mathbb{C}, k' \in \mathbb{C}^*$ and $k'\overline{a'_{\nu+1}} + \overline{k'}a'_{\nu+1} - 2k'\overline{k'}k^{-1} = 0$, hence $a'_{\nu+1} \neq 0$. By Lemma 5.1, we have

$$\sigma([\lambda_2 e_1 + e_{\nu+1} + (-\lambda_2 - 1)e_{\nu+i}]) = [e_1 + \pi(-1 - \lambda_2^{-1})e_{\nu+i}].$$

From $[e_1 + e_i + e_{\nu+1} - e_{\nu+i}] \perp_H [\lambda_2 e_1 + e_{\nu+1} + (-\lambda_2 - 1)e_{\nu+i}]$, we deduce $a'_{\nu+1} = k'k^{-1}\pi(1 - \lambda_2^{-1})$. Hence

$$\begin{aligned}\sigma([e_1 + e_i + e_{\nu+1} - e_{\nu+i}]) &= [k'e_1 + k'k^{-1}e_i + k'k^{-1}\pi(1 - \lambda_2^{-1})e_{\nu+1} - k'e_{\nu+i}] \\ &= [ke_1 + e_i + \pi(\frac{\lambda_2 - 1}{\lambda_2})e_{\nu+1} - ke_{\nu+i}].\end{aligned}$$

□

Lemma 5.7. *Let $2 \leq i \leq \nu$. For any $\lambda \in \mathbb{A}\backslash\{\lambda_2\}$, $a \in \mathbb{C}$, we have $\tau(\lambda) = \frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2-\lambda)}$ and*

$$\sigma([\lambda e_1 + e_{\nu+1} + ae_{\nu+i}]) = [\frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2 - \lambda)}e_1 + e_{\nu+1} + \frac{k\pi(\lambda_2)\pi(a)}{\pi(\lambda_2 - \lambda)}e_{\nu+i}].$$

Similarly, $\sigma([\lambda e_1 + ae_i + e_{\nu+1}]) = [\frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2-\lambda)}e_1 + \frac{\pi(\lambda_2)\pi(a)}{\pi(\lambda_2-\lambda)}e_i + e_{\nu+1}]$.

Proof. Suppose $\lambda \in \mathbb{A}\backslash\{0, \lambda_2\}$, by Lemmas 5.1 and 5.5, and we have

$$\sigma([\lambda e_1 + e_{\nu+1} + ae_{\nu+i}]) = [\tau(\lambda)e_1 + e_{\nu+1} + \tau(\lambda)\pi(\lambda)^{-1}\pi(a)e_{\nu+i}]$$

and

$$\sigma([\lambda e_1 + ae_i + e_{\nu+1}]) = [\tau(\lambda)e_1 + \tau(\lambda)k^{-1}\pi(\lambda)^{-1}\pi(a)e_i + e_{\nu+1}].$$

In particular, we have

$$\sigma([\lambda e_1 + e_{\nu+1} + (\overline{\lambda} - 1)e_{\nu+i}]) = [\tau(\lambda)e_1 + e_{\nu+1} + \tau(\lambda)\pi(\lambda)^{-1}\pi(\overline{\lambda} - 1)e_{\nu+i}].$$

Since $[e_1 + e_i + e_{\nu+1} - e_{\nu+i}] \perp_H [\lambda e_1 + e_{\nu+1} + (\overline{\lambda} - 1)e_{\nu+i}]$, by Lemma 5.6, we deduce

$$[ke_1 + e_i + \pi(\frac{\lambda_2 - 1}{\lambda_2})e_{\nu+1} - ke_{\nu+i}] \perp_H [\tau(\lambda)e_1 + e_{\nu+1} + \tau(\lambda)\pi(\lambda)^{-1}\pi(\overline{\lambda} - 1)e_{\nu+i}],$$

which implies $k + \overline{\tau(\lambda)}(\overline{\pi(\lambda)})^{-1}\pi(\lambda - 1) + \pi(\frac{\lambda_2-1}{\lambda_2})\overline{\tau(\lambda)} = 0$. Hence $\tau(\lambda) = \frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2-\lambda)}$ and it follows that

$$\sigma([\lambda e_1 + e_{\nu+1} + ae_{\nu+i}]) = [\frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2 - \lambda)}e_1 + e_{\nu+1} + \frac{k\pi(\lambda_2)\pi(a)}{\pi(\lambda_2 - \lambda)}e_{\nu+i}]$$

and $\sigma([\lambda e_1 + ae_i + e_{\nu+1}]) = [\frac{k\pi(\lambda)\pi(\lambda_2)}{\pi(\lambda_2-\lambda)}e_1 + \frac{\pi(\lambda_2)\pi(a)}{\pi(\lambda_2-\lambda)}e_i + e_{\nu+1}]$. □

Lemma 5.8. *Let* $[\alpha] = [a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2, a_1 \neq 0$, *and* $a_1 \neq \lambda_2$. *Suppose* $\sigma([\alpha]) = [a'_1, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}]$. *Then* $a'_1 = \frac{k\pi(a_1)\pi(\lambda_2)}{\pi(\lambda_2-a_1)}$, $a'_j = \frac{\pi(a_j)\pi(\lambda_2)}{\pi(\lambda_2-a_1)}$, $a'_{\nu+j} = \frac{k\pi(a_{\nu+j})\pi(\lambda_2)}{\pi(\lambda_2-a_1)}$ *for* $2 \leq j \leq \nu$.

Proof. We distinguish the following four cases:

(1) Suppose $[\alpha] = [a_1e_1 + e_{\nu+1}]$, where $a_1 \in \mathbb{A}\backslash\{0, \lambda_2\}$. By Lemma 5.7 we have

$$a'_1 = \frac{k\pi(a_1)\pi(\lambda_2)}{\pi(\lambda_2 - a_1)}.$$

(2) There is some $a_j \neq 0$ where $2 \leq j \leq \nu$. Let $b \in \mathbb{A}\backslash\{0, \lambda_2\}$. From

$$[a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [be_1 + e_{\nu+1} - (\overline{a_1} + b)\overline{a_j^{-1}}e_{\nu+j}],$$

by Lemma 5.7 we deduce

$$[a'_1, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(b)\pi(\lambda_2)}{\pi(\lambda_2 - b)}e_1 + e_{\nu+1} - \frac{k\pi(\overline{a_1} + b)\overline{\pi(a_j)}^{-1}\pi(\lambda_2)}{\pi(\lambda_2 - b)}e_{\nu+j}],$$

which implies

$$\overline{\pi(\lambda_2 - b)}a'_1 - a'_jk\pi(a_1 + \overline{b})\pi(a_j)^{-1}\overline{\pi(\lambda_2)} + k\overline{\pi(b)\pi(\lambda_2)} = 0.$$

By Lemma 5.5, we have $a'_j = a'_1k^{-1}\pi(a_1)^{-1}\pi(a_j)$. Hence

$$a'_1 = \frac{k\pi(a_1)\pi(\lambda_2)}{\pi(\lambda_2 - a_1)},\ a'_j = \frac{\pi(a_j)\pi(\lambda_2)}{\pi(\lambda_2 - a_1)}.$$

(3) There is some $a_{\nu+j} \neq 0$ where $2 \leq j \leq \nu$. Let $b \in \mathbb{A}\backslash\{0, \lambda_2\}$. From

$$[a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [be_1 - (\overline{a_1} + b)\overline{a_{\nu+j}^{-1}}e_j + e_{\nu+1}],$$

by Lemma 5.7, we have

$$[a'_1, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(b)\pi(\lambda_2)}{\pi(\lambda_2 - b)}e_1 - \frac{\pi(\lambda_2)\pi(\overline{a_1} + b)\overline{\pi(a_{\nu+j})}^{-1}}{\pi(\lambda_2 - b)}e_j + e_{\nu+1}],$$

which implies

$$\overline{\pi(\lambda_2 - b)}a'_1 - a'_{\nu+j}\pi(a_1 + \overline{b})\pi(a_{\nu+j})^{-1}\overline{\pi(\lambda_2)} + k\overline{\pi(b)\pi(\lambda_2)} = 0.$$

By Lemma 5.5 we have $a'_{\nu+j} = a'_1\pi(a_1)^{-1}\pi(a_{\nu+j})$. Hence

$$a'_1 = \frac{k\pi(a_1)\pi(\lambda_2)}{\pi(\lambda_2 - a_1)},\ a'_{\nu+j} = \frac{k\pi(a_{\nu+j})\pi(\lambda_2)}{\pi(\lambda_2 - a_1)}.$$

(4) When $\delta \geq 1$, suppose $[\alpha] = [ae_1 + e_{\nu+1} + a_{2\nu+1}e_{2\nu+1} + \cdots + a_{2\nu+\delta}e_{2\nu+\delta}] \in \Phi_2$, where $a \in \mathbb{C}^*$ and $a_{2\nu+1}, \ldots, a_{2\nu+\delta} \in \mathbb{C}$ such that

$$a + \overline{a} + \sum_{j=1}^{\delta} a_{2\nu+j}\overline{a_{2\nu+j}} = 0$$

and

$$\sum_{j=1}^{\delta} a_{2\nu+j}\overline{a_{2\nu+j}} \neq 0.$$

Since $[\alpha] \in \Phi_2, a \neq 1$, and $a \neq \lambda_2$. We assume $\sigma([\alpha]) = [a'e_1 + e_{\nu+1} + a'_{2\nu+1}e_{2\nu+1} + \cdots + a'_{2\nu+\delta}e_{2\nu+\delta}]$. By the cases (2) and (3) above, we have

$$\sigma([-\overline{a}e_1 + \overline{a}e_2 + e_{\nu+1} + e_{\nu+2}]) = [-\frac{k\pi(\overline{a})\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_1 + \frac{\pi(\overline{a})\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_2 + e_{\nu+1} + \frac{k\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_{\nu+2}].$$

From

$$[ae_1 + e_{\nu+1} + a_{2\nu+1}e_{2\nu+1} + \cdots + a_{2\nu+\delta}e_{2\nu+\delta}] \perp_H [-\overline{a}e_1 + \overline{a}e_2 + e_{\nu+1} + e_{\nu+2}],$$

we have

$$\sigma([\alpha]) \perp_H [-\frac{k\pi(\overline{a})\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_1 + \frac{\pi(\overline{a})\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_2 + e_{\nu+1} + \frac{k\pi(\lambda_2)}{\pi(\lambda_2 + \overline{a})}e_{\nu+2}],$$

which implies $a' = \frac{k\pi(a)\pi(\lambda_2)}{\pi(\lambda_2 - a)}$. □

Lemma 5.9. *Let* $[\alpha] = [\lambda_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$. *Suppose*

$$\sigma([\alpha]) = [1, \ldots, a'_\nu, 0, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}].$$

Then, $a'_j = k^{-1}\pi(\lambda_2)^{-1}\pi(a_j),\ a'_{\nu+j} = \pi(\lambda_2)^{-1}\pi(a_{\nu+j})$ *for* $2 \leq j \leq \nu$.

Proof. If we have some $a_j \neq 0$ where $2 \leq j \leq \nu$, from

$$[\lambda_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [e_1 + e_{\nu+1} - (\overline{\lambda_2} + 1)\overline{a_j^{-1}}e_{\nu+j}],$$

by Lemma 5.7 we deduce

$$[1, \ldots, a'_\nu, 0, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_1 + e_{\nu+1} - \frac{k\pi(\overline{\lambda_2} + 1)\overline{\pi(a_j)}^{-1}\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_{\nu+j}],$$

which implies $a'_j = k^{-1}\pi(\lambda_2)^{-1}\pi(a_j)$.

If we have some $a_{\nu+j} \neq 0$ where $2 \leq j \leq \nu$, from

$$[\lambda_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [e_1 - (\overline{\lambda_2} + 1)\overline{a_{\nu+j}^{-1}}e_j + e_{\nu+1}],$$

by Lemma 5.7, we have

$$[1, \ldots, a'_\nu, 0, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_1 - \frac{\pi(\lambda_2)\pi(\overline{\lambda_2} + 1)\overline{\pi(a_{\nu+j})}^{-1}}{\pi(\lambda_2 - 1)}e_j + e_{\nu+1}],$$

which implies $a'_{\nu+j} = \pi(\lambda_2)^{-1}\pi(a_{\nu+j})$. □

Lemma 5.10. *Let* $[\alpha] = [0, a_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$. *Suppose*

$$\sigma([\alpha]) = [0, a'_2, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}].$$

Then, $a'_j = \pi(a_j),\ a'_{\nu+j} = k\pi(a_{\nu+j})$ *for* $2 \leq j \leq \nu$.

Proof. Let $b \in \mathbb{A}\backslash\{0, \lambda_2\}$. If there is an $a_j \neq 0,\ 2 \leq j \leq \nu$, from

$$[0, a_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [be_1 + e_{\nu+1} - b\overline{a_j^{-1}}e_{\nu+j}],$$

by Lemma 5.7, we deduce

$$[0, a'_2, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(b)\pi(\lambda_2)}{\pi(\lambda_2 - b)}e_1 + e_{\nu+1} - \frac{k\pi(\lambda_2)\pi(b)\overline{\pi(a_j)}^{-1}}{\pi(\lambda_2 - b)}e_{\nu+j}],$$

which implies $a'_j = \pi(a_j)$.

If there is an $a_{\nu+j} \neq 0,\ 2 \leq j \leq \nu$, from

$$[0, a_2, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [be_1 - b\overline{a_{\nu+j}^{-1}}e_j + e_{\nu+1}],$$

by Lemma 5.7, we deduce

$$[0, a'_2, \ldots, a'_\nu, 1, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(b)\pi(\lambda_2)}{\pi(\lambda_2 - b)}e_1 - \frac{\pi(\lambda_2)\pi(b)\overline{\pi(a_{\nu+j})}^{-1}}{\pi(\lambda_2 - b)}e_j + e_{\nu+1}],$$

which implies $a'_{\nu+j} = k\pi(a_{\nu+j})$. □

By Lemmas 5.8–5.10 we have:

Lemma 5.11. *Let* $[\alpha] = [a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$. *Suppose* $\sigma([\alpha]) = [a'_1, a'_2, \ldots, \ldots, a'_{2\nu+\delta}]$. *Then* $a'_1 = k\pi(a_1), a'_{\nu+1} = \pi(1 - a_1\lambda_2^{-1}),\ a'_j = \pi(a_j),\ a'_{\nu+j} = k\pi(a_{\nu+j})$ *for* $2 \leq j \leq \nu$.

Lemma 5.12. *Let* $[\alpha] = [1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$. *Suppose* $\sigma([\alpha]) = [1, a'_2, \ldots, a'_{2\nu+\delta}]$. *Then* $a'_{\nu+1} = \lambda_1^{-1},\ a'_j = (k^{-1}\pi(\lambda_2^{-1} + 1) + \lambda_1^{-1})\pi(a_j),\ a'_{\nu+j} = (\pi(\lambda_2^{-1} + 1) + k\lambda_1^{-1})\pi(a_{\nu+j})$ *for* $2 \leq j \leq \nu$.

Proof. From $[\alpha] \perp_H [e_1]$, we deduce $[1, a'_2, \ldots, a'_{2\nu+\delta}] \perp_H [\lambda_1 e_1 + e_{\nu+1}]$, which implies $a'_{\nu+1} = \lambda_1^{-1}$.

If we have some $a_j \neq 0$ where $2 \leq j \leq \nu$, from

$$[1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [e_1 + e_{\nu+1} - \overline{a_j^{-1}}e_{\nu+j}],$$

by Lemma 5.7 we deduce

$$[1, a'_2, \ldots, a'_\nu, \lambda_1^{-1}, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_1 + e_{\nu+1} - \frac{k\overline{\pi(a_j)}^{-1}\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_{\nu+j}],$$

which implies $a'_j = (k^{-1}\pi(\lambda_2^{-1} + 1) + \lambda_1^{-1})\pi(a_j)$.

If we have some $a_{\nu+j} \neq 0$ where $2 \leq j \leq \nu$, from

$$[1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \perp_H [e_1 - \overline{a_{\nu+j}^{-1}}e_j + e_{\nu+1}],$$

by Lemma 5.7, we have

$$[1, a'_2, \ldots, a'_\nu, \lambda_1^{-1}, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}] \perp_H [\frac{k\pi(\lambda_2)}{\pi(\lambda_2 - 1)}e_1 - \frac{\pi(\lambda_2)\overline{\pi(a_{\nu+j})}^{-1}}{\pi(\lambda_2 - 1)}e_j + e_{\nu+1}],$$

which implies $a'_{\nu+j} = (\pi(\lambda_2^{-1} + 1) + k\lambda_1^{-1})\pi(a_{\nu+j})$. □

Lemma 5.13. $\pi(\lambda_2^{-1}) + k\lambda_1^{-1} = 0$.

Proof. For any $a \in \mathbb{C}\backslash\{-1\}$, let $[\alpha] = [ae_1+e_2-ae_3+e_{\nu+1}+e_{\nu+3}]$, and then $[\alpha] \in \Phi_2$ and $[\alpha] \perp_H [e_1-e_{\nu+2}]$. By Lemmas 5.11 and 5.12, we have $\sigma([\alpha]) = [k\pi(a)e_1 + e_2 - \pi(a)e_3 + \pi(1 - a\lambda_2^{-1})e_{\nu+1} + ke_{\nu+3}]$ and $\sigma([e_1 - e_{\nu+2}]) = [e_1 + \lambda_1^{-1}e_{\nu+1} - (\pi(\lambda_2^{-1} + 1) + k\lambda_1^{-1})e_{\nu+2}]$. From $\sigma([\alpha]) \perp_H \sigma([e_1 - e_{\nu+2}])$, we have $\overline{\pi(a)}(\pi(\lambda_2^{-1}) + k\lambda_1^{-1}) = \pi(\lambda_2^{-1}) + k\lambda_1^{-1}$, which implies $\pi(\lambda_2^{-1}) + k\lambda_1^{-1} = 0$. □

By Lemmas 5.12 and 5.13, we have:

Lemma 5.14. *Let* $[\alpha] = [1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$. *Suppose* $\sigma([\alpha]) = [1, a'_2, \ldots, a'_{2\nu+\delta}]$. *Then* $a'_{\nu+1} = \lambda_1^{-1}$, $a'_j = k^{-1}\pi(a_j)$, $a'_{\nu+j} = \pi(a_{\nu+j})$ *for* $2 \le j \le \nu$.

We can now complete the proof of Theorem 5.1. Let $[\alpha] \in \Phi_2$ and write $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}]$, where $a_{\nu+1} = 1$ or $a_{\nu+1} = 0$ and $a_1 = 1$, and by Lemmas 5.11, 5.13, and 5.14, we have

$$\sigma([\alpha]) = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), \delta_{1,a_{\nu+1}} + k\lambda_1^{-1}\pi(a_1), k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}],$$

where $\delta_{1,a_{\nu+1}} = \begin{cases} 1, & a_{\nu+1} \neq 0 \\ 0, & a_{\nu+1} = 0 \end{cases}$.

Since $k \in \mathbb{R}^*$, $\lambda_1 \in \mathbb{A}\backslash\{0\}$, we have $\frac{\lambda_1 k^{-1}}{\lambda_1+1} \neq 1$. Let

$$[\alpha] = [-\pi^{-1}(\frac{\lambda_1 k^{-1}}{\lambda_1 + 1})e_1 + \pi^{-1}(\frac{\lambda_1 k^{-1}}{\lambda_1 + 1})e_2 + e_{\nu+1} + e_{\nu+2}],$$

and then $[\alpha] \in \Phi_2$ and $\sigma([\alpha]) \perp_H [e_1 + e_{\nu+1}]$, hence $\sigma([\alpha]) \notin \Phi_2$, which is a contradiction. So $\sigma([e_1]) = [\lambda_1 e_1 + e_{\nu+1}]$ is impossible.

Hence we have $\sigma([e_1]) = [e_1]$. By the proof of Theorem 3.1 in [8], similarly for any $[\alpha] = [a_1, \ldots, a_\nu, 1, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$, we have

$$\sigma[\alpha] = [k\pi(a_1), \pi(a_2), \ldots, \pi(a_\nu), 1, k\pi(a_{\nu+2}), \ldots, k\pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}].$$

Now we show $k = 1$. If $k \neq 1$, then $\pi^{-1}(k^{-1}) \neq 1$. Let

$$[\alpha] = [-\pi^{-1}(k^{-1})e_1 + \pi^{-1}(k^{-1})e_2 + e_{\nu+1} + e_{\nu+2}],$$

and then $[\alpha] \in \Phi_2$ and $\sigma([\alpha]) \perp_H [e_1 + e_{\nu+1}]$, hence $\sigma([\alpha]) \notin \Phi_2$, which is a contradiction. Hence $k = 1$.

For any $[\alpha] = [1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$, since $\sigma([e_1]) = [e_1]$ we can assume $\sigma([\alpha]) = [1, a'_2, \ldots, a'_\nu, 0, a'_{\nu+2}, \ldots, a'_{2\nu+\delta}]$. For any $a_i \neq 0, 2 \le i \le \nu$, from $[\alpha] \perp_H [e_{\nu+1} - \overline{a_i}^{-1}e_{\nu+i}]$, we deduce $a'_i = \pi(a_i)$. For any $a_{\nu+i} \neq 0, 2 \le i \le \nu$, from $[\alpha] \perp_H [-\overline{a_{\nu+i}}^{-1}e_i + e_{\nu+1}]$, we deduce $a'_{\nu+i} = \pi(a_{\nu+i})$. Hence for any $[\alpha] = [1, a_2, \ldots, a_\nu, 0, a_{\nu+2}, \ldots, a_{2\nu+\delta}] \in \Phi_2$, we have

$$\sigma[\alpha] = [1, \pi(a_2), \ldots, \pi(a_\nu), 0, \pi(a_{\nu+2}), \ldots, \pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}].$$

Hence for any $[\alpha] = [a_1, a_2, \ldots, a_{2\nu+\delta}] \in \Phi_2$, we have

$$\sigma[\alpha] = [\pi(a_1), \ldots, \pi(a_\nu), \pi(a_{\nu+1}), \pi(a_{\nu+2}), \ldots, \pi(a_{2\nu}), a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}].$$

Moreover we have $E_2 = E_{\Phi_2}$, and Theorem 5.1 is proved for the case where $\delta = 0$. Now we consider the case where $\delta \ge 1$. Let $\lambda \in \mathbb{C}^*\backslash\{-1\}$ such that $\lambda + \overline{\lambda} + 1 = 0$. Then $[\gamma_i] = [\lambda e_1 + e_{\nu+1} + e_{2\nu+i}] \in \Phi_2$

and $\sigma([\gamma_i]) = [\pi(\lambda)e_1 + e_{\nu+1} + \omega_{i1}e_{2\nu+1} + \cdots + \omega_{i\delta}e_{2\nu+\delta}]$, where ω_{ij} satisfies $\pi(\lambda) + \overline{\pi(\lambda)} + \Sigma_{j=1}^{\delta}\omega_{ij}\overline{\omega_{ij}} = 0, 1 \leq i, j \leq \delta$. By $\lambda + \overline{\lambda} + 1 = 0$ we deduce

$$\sum_{j=1}^{\delta} \omega_{ij}\overline{\omega_{ij}} = 1, 1 \leq i, j \leq \delta.$$

Let $W = (\omega_{ij})_{\delta\times\delta}$, and similar to the proof of Theorem 3.1, we have $W\overline{W}^t = I^{(\delta)}$ and $(a'_{2\nu+1}, \ldots, a'_{2\nu+\delta}) = (\pi(a_{2\nu+1}), \ldots, \pi(a_{2\nu+\delta}))W$. □

Author contributions

Kai Zhou: Conceptualization, formal analysis, writing & editing; Zhenhua Gu: Formal analysis, Review; Hongfeng Wu: Methodology, Validation, writing-review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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