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Research article

A study of (m, q)-isometric multimappings in the context of $\mathcal{G}$-metric spaces

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Abstract: This paper introduces and explores new concepts of (m, q)-isometric multimappings in the context of extended metric structures. These newly defined concepts serve as extensions of the existing theory of (m, q)-isometric multimappings in traditional metric spaces, as well as (m, q)-$\mathcal{G}$-isometric single mappings in generalized metric spaces. The study aims to broaden the understanding of isometric multimappings properties and their interactions within these extended spaces.

Keywords: generalized metric space; m-isometric mappings; tuple of mappings
Mathematics Subject Classification: 47B99, 54E40

1. Introduction

The idea of a generalized metric has been previously established in [1] as follows.

Let $\mathcal{E}$ be a set $(\mathcal{E} \neq \emptyset))$ and let $\mathcal{G} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}_+$ be a map for which:

(1) $\mathcal{G}(\omega, \psi, \varphi) = 0$ if $\omega = \psi = \varphi$,
(2) $0 < \mathcal{G}(\omega, \omega, \psi)$ for all $\omega, \psi \in \mathcal{E}$ with $\omega \neq \psi$,
(3) $\mathcal{G}(\omega, \omega, \psi) \leq \mathcal{G}(\omega, \psi, \varphi)$ for all $\omega, \psi, \varphi \in \mathcal{E}$ with $\psi \neq \varphi$,
(4) $\mathcal{G}(\omega, \psi, \varphi) = \mathcal{G}(\omega, \varphi, \psi) = \mathcal{G}(\psi, \varphi, \omega) = ...$
(5) $\mathcal{G}(\omega, \psi, \varphi) \leq \mathcal{G}(\omega, a, a) + \mathcal{G}(a, \psi, \varphi)$ for all $\omega, \psi, \varphi, a \in \mathcal{E}$.

Then the map $\mathcal{G}$ is called a generalized metric, or a $\mathcal{G}$-metric on $\mathcal{E}$, and $(\mathcal{E}, \mathcal{G})$ is called a $\mathcal{G}$-metric space. In the following, we denote by $(\mathcal{E}, d)$ and $(\mathcal{E}, \mathcal{G})$ a metric space and a $\mathcal{G}$-metric space, respectively. In recent years, the class of m-isometric operators and the related class of n-quasi- m-isometric operators in both Hilbert and Banach spaces have attracted significant attention. These operators have been extensively studied by numerous researchers, leading to a wealth of contributions in the literature that explore their structural properties, spectral properties, and various applications in functional analysis [2–4]. Over the years, a significant body of research has emerged, extending classical fixed-point theorems to a wide array of generalized metric spaces, including b-metric spaces, partial metric spaces, G-metric spaces, and others. These generalizations not only enrich the theoretical landscape but also enhance the applicability of fixed-point results to real-world problems where traditional metric assumptions may not hold [5–7].

In this work, our motivation in studying m-isometries within the context of $\mathcal{G}$-metric spaces lies in bridging these two powerful concepts. While m-isometries have been extensively investigated in Hilbert and Banach spaces, their behavior and properties in $\mathcal{G}$-metric settings remain relatively unexplored. By extending the theory of m-isometries to $\mathcal{G}$-metric spaces, we aim to uncover new fixed-point results, deepen the theoretical understanding of operator behavior in generalized metric structures, and open new avenues for applications in nonlinear analysis and related fields.

The concept of m-isometries, originally studied in Banach spaces, has been extended to the setting of metric spaces. This generalization has provided new insights into the geometric and structural properties of metric spaces, leading to further developments in nonlinear analysis and fixed-point theory.

The concept of (m, q)-isometric maps on a metric space was introduced and studied in [8]. A map $\mathcal{W} : (\mathcal{E}, d) \longrightarrow (\mathcal{E}, d)$ is called an (m, q)-isometry if

$$\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} d(\mathcal{W}^{m-k}\omega, \mathcal{W}^{m-k}\psi)^q = 0 \tag{1.1}$$

for all $\omega, \psi \in \mathcal{E}$ and for some $m \in \mathbb{N}$ and $q \in (0, \infty)$). $(1,\ q)$-isometries $\mathcal{W}$ coincide with isometries: $d(\mathcal{W}\omega, \mathcal{W}\psi) = d(\omega, \psi)$ for all $\omega, \psi \in \mathcal{E}$. When $\mathcal{E}$ is a normed space, (1.1) coincides with

$$\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \|\mathcal{W}^{m-k}\omega\|^q = 0. \tag{1.2}$$

This definition has been extended to commutative multivariable mappings on a metric space by the authors Sid Ahmed et al. [9]. Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ be a commutative mapping where $\mathcal{W}_j : \mathcal{E} \to \mathcal{E}$ for $j = 1, \cdots, p$. $\mathcal{W}$ is said to be an (m, q)-isometric commutative tuple of mappings if $\mathcal{W}$ satisfies

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\gamma|=k} \frac{k!}{\gamma!} d(\mathcal{W}^\gamma\omega, \mathcal{W}^\gamma\psi)^q \Big) = 0, \tag{1.3}$$

for all $\omega, \psi \in \mathcal{E}$. When $\mathcal{E}$ is a normed space, (1.3) coincides with

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\gamma|=k} \frac{k!}{\gamma!} \|\mathcal{W}^\gamma\omega\|^q \Big) = 0. \tag{1.4}$$

Consider the map $\Lambda_k^{(q)}(\mathcal{W}; ., ., .) : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ by

$$\Lambda_k(\mathcal{W};\ \omega, \psi, \varphi) := \sum_{0\leq j\leq k} (-1)^{k-j} \binom{k}{j} \mathcal{G}(\mathcal{W}^j\omega, \mathcal{W}^j\psi, \mathcal{W}^j\varphi)^q,\ \forall\ \ \omega, \psi, \varphi \in \mathcal{E}. \tag{1.5}$$

In [10] the author considers a generalization of the concept of (m, q)-isometric mappings in metric space to (m, q)-isometric mappings in $\mathcal{G}$-metric space. A map $\mathcal{W} : (\mathcal{E}, \mathcal{G}) \longrightarrow (\mathcal{E}, \mathcal{G})$ is said to be an (m, q)-$\mathcal{G}$-isometric if it satisfies

$$\sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \mathcal{G}(\mathcal{W}^j\omega, \mathcal{W}^j\psi, \mathcal{W}^j\varphi)^q = 0 \quad ,\ \forall\ \ \omega, \psi, \varphi \in \mathcal{E}.$$

The paper introduces a parallel extension of the study of m-isometric commuting mappings from metric spaces to the more general setting of general metric spaces, offering new perspectives on their structural properties. Several results established in Banach and Hilbert spaces to the broader framework of generalized metric spaces. Our motivation stems from the need to explore whether key properties of m-isometries are retained or need to be modified when extended to broader classes of spaces, especially in light of applications in nonlinear analysis, fixed point theory, and geometric group theory, where general metric spaces naturally arise.

2. (m, q)-isometric multimappings in $\mathcal{G}$-metric space

Let $\mathcal{W}_j : (\mathcal{E}, \mathcal{G}) \to (\mathcal{E}, \mathcal{G})$ be a map for $j = 1, \cdots, p$ and consider $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ where $\mathcal{W}_k\mathcal{W}_j = \mathcal{W}_j\mathcal{W}_k$. For $\omega, \psi, \varphi \in \mathcal{E}$, we write

$$\mathcal{Q}_l^{(q)}(\mathcal{W}; \omega, \psi, \varphi) := \sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big). \tag{2.1}$$

We begin by presenting the definition of the multimappings that will be central to the theoretical developments in this paper.

Definition 2.1. The tuple $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ is called an (m, q)-$\mathcal{G}$-isometric multimapping if

$$\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big) = 0, \tag{2.2}$$

for all $\omega, \psi, \varphi \in \mathcal{E}$.

We designate by $[(m, q) - \mathcal{ISO}]_p[(\mathcal{E}, \mathcal{G})]$ the set of (m, q)-$\mathcal{G}$-isometric p-tuples on $(\mathcal{E}.\mathcal{G})$.

Remark 2.1. (i) When $p = 1$, (2.2) was formalized in [10].

Remark 2.2. (i) $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \in [(1, q) - \mathcal{ISO}]_2[(\mathcal{E}, \mathcal{G})]$ if

$$\mathcal{G}(\mathcal{W}_1\omega, \mathcal{W}_1\psi, \mathcal{W}_1\varphi)^q + \mathcal{G}(\mathcal{W}_2\omega, \mathcal{W}_2\psi, \mathcal{W}_2\varphi)^q = \mathcal{G}(\omega, \psi, \varphi)^q \ \text{ for all } \omega, \psi, \varphi \in \mathcal{E}$$

and $\mathcal{W}$ is a $(2, q)$-$\mathcal{G}$-isometric pair if

$$\begin{aligned}&\mathcal{G}(\mathcal{W}_1^2\omega, \mathcal{W}_1^2\psi, \mathcal{W}_1^2\varphi)^q + \mathcal{G}(\mathcal{W}_2^2\omega, \mathcal{W}_2^2\psi, \mathcal{W}_2^2\varphi)^q + 2\mathcal{G}(\mathcal{W}_1\mathcal{W}_2\omega, \mathcal{W}_1\mathcal{W}_2\psi, \mathcal{W}_1\mathcal{W}_2\varphi)^q\\ &-2\mathcal{G}(\mathcal{W}_1\omega, \mathcal{W}_1\psi, \mathcal{W}_1\varphi)^q - 2\mathcal{G}(\mathcal{W}_2\omega, \mathcal{W}_2\psi, \mathcal{W}_2\varphi)^q + \mathcal{G}(\omega, \psi, \varphi)^q = 0\end{aligned}$$

for all $\omega, \psi, \varphi \in \mathcal{E}$.

(ii) $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ is a $(1, q)$-$\mathcal{G}$-isometric pair if

$$\sum_{1\le k\le p} \mathcal{G}(\mathcal{W}_k\omega, \mathcal{W}_k\psi, \mathcal{W}_k\varphi)^q - \mathcal{G}(\omega, \psi, \varphi)^q = 0 \ \text{ for all } \omega, \psi, \varphi \in \mathcal{E}$$

and $\mathcal{W} \in [(2, q) - \mathcal{ISO}]_2[(\mathcal{E}, \mathcal{G})]$ if for all $\omega, \psi, \varphi \in \mathcal{E}$,

$$\sum_{1\le k\le p} G(\mathcal{W}_k^2\omega, \mathcal{W}_k^2\psi, {}_k^2\varphi)^q + 2\sum_{1\le i<k\le p} \mathcal{G}(\mathcal{W}_i\mathcal{W}_k\omega, \mathcal{W}_i\mathcal{W}_k\psi, \mathcal{W}_i\mathcal{W}_k\varphi)^q$$
$$-2\sum_{1\le k\le p} \mathcal{G}(\mathcal{W}_k\omega, \mathcal{W}_k\psi, \mathcal{W}_k\varphi)^q + \mathcal{G}(\omega, \psi\varphi)^q = 0.$$

(iii) $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ is a $(3, q)$- $\mathcal{G}$-isometric multimappings if

$$\sum_{1\le j\le p} \mathcal{G}(\mathcal{W}_j^3 x, \mathcal{W}_j^3 y, \mathcal{W}_j^3\varphi)^q + 3\sum_{1\le i\neq j\le p} \mathcal{G}(\mathcal{W}_i\mathcal{W}_j^2\omega, \mathcal{W}_i\mathcal{W}_j^2\psi, \mathcal{W}_i\mathcal{W}_j^2\varphi)^q$$
$$+6\sum_{1\le i\neq j\neq r\le p} \mathcal{G}(\mathcal{W}_i\mathcal{W}_j\mathcal{W}_r\omega, \mathcal{W}_i\mathcal{W}_j\mathcal{W}_r\psi, \mathcal{W}_i\mathcal{W}_j\mathcal{W}_r\varphi)^q$$
$$-3\sum_{1\le j\le p} \mathcal{G}(\mathcal{W}_j^2\omega, \mathcal{W}_j^2\psi, \mathcal{W}_j^2\varphi)^q - 6\sum_{1\le i\neq j\le p} \mathcal{G}(\mathcal{W}_j\mathcal{W}_i\omega, \mathcal{W}_j\mathcal{W}_i\psi, \mathcal{W}_j\mathcal{W}_i\varphi)^q$$
$$+3\sum_{1\le j\le p} \mathcal{G}(\mathcal{W}_j\omega, \mathcal{W}_j\psi, \mathcal{W}_j\varphi)^q - \mathcal{G}(\omega, \psi, \varphi)^q = 0 \text{ for all } \omega, \psi, \varphi \in \mathcal{E}.$$

Example 2.1. Let $(\mathbb{R}, \mathcal{G}_0)$ be a $\mathcal{G}$-metric space, where $\mathcal{G}_0(\omega, \psi, \varphi) = |\omega - \psi|^q + |\omega - \varphi|^q + |\psi - \varphi|^q$.

Define the map $\mathcal{W}_0 : (\mathbb{R}, \mathcal{G}_0) \longrightarrow (\mathbb{R}, \mathcal{G}_0)$ to be an (m, q)-$\mathcal{G}$-isometric map. Then the tuple $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$, where $\mathcal{W}_j = \frac{1}{\sqrt[q]{p}}\mathcal{W}_0$ for $q > 0$, $j = 1, \cdots, p$, is an (m, q)-$\mathcal{G}$-isometric multimapping. Indeed, we have

$$\begin{aligned}
&\sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathcal{G}_0(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big)\\
=\ &\sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathcal{G}_0((\frac{1}{\sqrt[q]{p}})^{|\alpha|}\mathcal{W}_0^{|\alpha|}\omega, (\frac{1}{\sqrt[q]{p}})^{|\alpha|}\mathcal{W}_0^{|\alpha|}\psi, (\frac{1}{\sqrt[q]{p}})^{|\alpha|}\mathcal{W}_0^{|\alpha|}\varphi)^q\Big)\\
=\ &\sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}(\frac{1}{\sqrt[q]{p}})^{q|\alpha|}\mathcal{G}_0(\mathcal{W}_0^{|\alpha|}\omega, \mathcal{W}_0^{|\alpha|}\psi, \mathcal{W}_0^{|\alpha|}\varphi)^q\Big)\\
=\ &\sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!}\frac{1}{p^k}\mathcal{G}_0(\mathcal{W}_0^k\omega, \mathcal{W}_0^k\psi, \mathcal{W}_0^k\varphi)^q\Big)\\
=\ &\\
=\ &\sum_{0\le k\le l} (-1)^{l-k}\binom{l}{k}\mathcal{G}_0(\mathcal{W}_0^k\omega, \mathcal{W}_0^k\psi, \mathcal{W}_0^k\varphi)^q\\
=\ &0.
\end{aligned}$$

Remark 2.3. From the above example, it has been observed that the result, which was proven in [8, Proposition 1.4] for a single map, does not hold true in general for a tuple of mappings.

Proposition 2.1. *Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$, where $\mathcal{W}_k : (\mathcal{E}, \mathcal{G}) \to (\mathcal{E}, \mathcal{G})$ is a map. Then for all $m \ge 1$, $q \in (0, \infty)$ and $\omega, \psi, \varphi \in \mathcal{E}$ the following hold.*

$$Q_{m+1}^{(q)}(\mathcal{W}; \omega, \psi, \varphi) = \sum_{1\le k\le n} Q_m^{(q)}(\mathcal{W}; \mathcal{W}_k\omega, \mathcal{W}_k\psi, \mathcal{W}_k\varphi) - Q_m^{(q)}(\mathcal{W}; \omega, \psi, \varphi). \tag{2.3}$$

In particular, if $\mathcal{W} \in [(m, q) - \mathcal{ISO}]_p[(\mathcal{E}, \mathcal{G})]$, then $\mathcal{W} \in [(k, q) - \mathcal{ISO}]_p[(\mathcal{E}, \mathcal{G})]$ for all $k \ge m$.

Proof. According to (2.1), a straightforward calculation shows that

$$
\begin{aligned}
& \mathcal{Q}^{(q)}_{m+1}(\mathcal{W};\omega,\psi,\varphi) \\
= \;& \sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
= \;& (-1)^{m+1}\mathcal{G}(\omega,\psi,\varphi)^{q} - \sum_{1\le k\le m}(-1)^{m-k}\left[\binom{m}{k}+\binom{m}{k-1}\right]\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
& + \sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}G(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) + \sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
& + \sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) + \sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!(\alpha_1+\cdots+\alpha_n)}{\alpha_1!\cdot\alpha_2!\cdots\alpha_n!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
& + \sum_{|\alpha|=m+1}\frac{m!(\alpha_1+\cdots+\alpha_n)}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\mathcal{W}^{\alpha}\psi,\mathcal{W}^{\alpha}\varphi)^{q} \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) \\
& + \sum_{1\le j\le p}\sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!} \\
& \cdot\mathcal{G}\Big((\underbrace{\mathcal{W}_1^{\alpha_1}\cdots\mathcal{W}_j^{\alpha_j-1}\mathcal{W}_{j+1}^{\alpha_{j+1}}\cdots\mathcal{W}_p^{\alpha_n}}_{\Theta}\mathcal{W}_j\omega),(\mathcal{W}_1^{\alpha_1}\cdots\mathcal{W}_j^{\alpha_j-1}\mathcal{W}_{j+1}^{\alpha_{j+1}}\cdots\mathcal{W}_n^{\alpha_n}\mathcal{W}_j\psi),\Theta\mathcal{W}_j\varphi)\Big)^{q} \\
& + \sum_{1\le j\le p}\sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_p!} \\
& \cdot\mathcal{G}(\underbrace{\mathcal{W}_1^{\alpha_1}\cdots\mathcal{W}_j^{\alpha_j-1}\mathcal{W}_{j+1}^{\alpha_{j+1}}\cdots\mathcal{W}_n^{\alpha_n}}_{\Theta}\mathcal{W}_j\omega,\mathcal{W}_1^{\alpha_1}\cdots\mathcal{W}_j^{\alpha_j-1}\mathcal{W}_{j+1}^{\alpha_{j+1}}\cdots\mathcal{W}_p^{\alpha_p}\mathcal{W}_j\varphi,\Theta\mathcal{W}_j\varphi)^{q} \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) \\
& + \sum_{1\le j\le n}\sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\mathcal{W}_j\omega,\ \mathcal{W}^{\alpha}\mathcal{W}_j\psi,\mathcal{W}^{\alpha}\mathcal{W}_j\varphi)^{q} \\
& + \sum_{1\le j\le n}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\mathcal{W}_j\omega,\mathcal{W}^{\alpha}\mathcal{W}_j\psi,\mathcal{W}^{\alpha}W_j\varphi)^{q} \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) + \sum_{1\le j\le p}\Big(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\mathcal{W}_j\omega,\ \mathcal{W}^{\alpha}\mathcal{W}_j\psi,\mathcal{W}^{\alpha}\mathcal{W}_j\varphi)^{q}\Big) \\
= \;& -\mathcal{Q}^{(q)}_{m}(\mathcal{W};\omega,\psi,\varphi) + \sum_{1\le j\le p}\mathcal{Q}^{(q)}_{m}(\mathcal{W};\mathcal{W}_j\omega,\mathcal{W}_j\psi,\mathcal{W}_j\varphi),
\end{aligned}
$$

and so (3.1) is satisfied. Given that $\mathcal{W}\in[(m,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$, it follows that $\mathcal{W}$ is also in $[(k,q)-$

$\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ for $k \geq m$. This conclusion is directly supported by Eq (3.1). □

The following theorem generalizes [11, Theorem 2.1].

Theorem 2.1. *Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$, where $\mathcal{W}_k : (\mathcal{E},\mathcal{G}) \to (\mathcal{E},\mathcal{G})$ is a map. The following statements are then true:*

(i)

$$\sum_{|\alpha|=r} \frac{r!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q = \sum_{0\leq j\leq r}\binom{r}{j}\mathcal{Q}_j^{(q)}(\mathcal{W};\omega,\psi,\varphi), \tag{2.4}$$

for every $r \geq 1, q > 0$ and $\forall\ \omega,\psi,\varphi \in \mathcal{E}$.

(ii) $\mathcal{W} \in [(m,q) - \mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ *if and only if*

$$\sum_{|\alpha|=n} \frac{n!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q = \sum_{0\leq j\leq m-1}\binom{n}{j}\mathcal{Q}_j^{(q)}(\mathcal{W};\omega,\psi,\varphi) \tag{2.5}$$

for all $n \in \mathbb{N}, q > 0$ and $\forall\ \omega,\psi,\varphi \in \mathcal{E}$.

(iii) *If $\mathcal{W}$ is an (m,q)-$\mathcal{G}$-isometric multimapping , then*

$$\mathcal{Q}_{m-1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) = \lim_{n\to\infty}\frac{1}{\binom{n}{m-1}}\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q. \tag{2.6}$$

In particular, $\mathcal{Q}_{m-1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) \geq 0$ for all $\omega,\psi,\varphi \in \mathcal{E}$.

Proof. (i) Using mathematical induction to prove (2.4). The result is verified for $k = 0$ and $k = 1$. We now assume it holds for $r = k$ and will demonstrate its validity for $k + 1$.

By doing so, applying (2.1) and (2.4), we obtain

$$\begin{aligned}
&\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq j\leq k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq j\leq k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{0\leq r\leq j}\binom{j}{r}\mathcal{Q}_r^{(q)}(\mathcal{W}\omega,\psi,\varphi)\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq r\leq k}\mathcal{Q}_r^{(q)}(\mathcal{W};\omega,\psi,\varphi)\sum_{r\leq j\leq k}(-1)^{k+1-j}\binom{k+1}{j}\binom{j}{r}\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq r\leq k}\mathcal{Q}_r^{(q)}(\mathcal{W};\omega,\psi,\varphi)\Big(\sum_{r\leq j\leq k}(-1)^{k+1-j}\binom{k+1}{r}\binom{k+1-r}{j-r}\Big)\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq r\leq k}\binom{k+1}{r}\mathcal{Q}_r^{(q)}(\mathcal{W};\omega,\psi,\varphi)\Big(\sum_{r\leq j\leq k}(-1)^{k+1-j}\binom{k+1-r}{j-r}\Big)\\
=\;& \mathcal{Q}_{k+1}^{(q)}(\mathcal{W};\omega,\psi,\varphi) - \sum_{0\leq r\leq k}\binom{k+1}{r}\mathcal{Q}_r^{(q)}(\mathcal{W};\omega,\psi,\varphi)\Big(-1 + \underbrace{\sum_{0\leq r\leq k+1-j}(-1)^{k+1-j-r}\binom{k+1-r}{r}}_{=0}\Big)
\end{aligned}$$

$$= \sum_{0\le r\le k+1} \binom{k+1}{r} Q_r^{(q)}(\mathcal{W}; \omega, \psi, \varphi).$$

(ii) The 'only if' part of statement (ii) can be deduced from (2.4), as

$$[(m,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})] \subset [(k,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})] \quad \text{for } k \ge m.$$

(iii) From statement (ii), it can be easily observed that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} d(\mathcal{W}^\alpha x, \mathcal{W}^\alpha \psi, \mathcal{W}^\alpha \varphi)^q = \sum_{0\le j\le m-2} \binom{k}{j} Q_j^{(q)}(\mathcal{W}; \omega, \psi, \varphi) + \binom{k}{m-1} Q_{m-1}^{(q)}(\mathcal{W}; \omega, \psi, \varphi).$$

Given $\frac{\binom{k}{j}}{\binom{k}{m-1}} \to 0$ as $k \to \infty$), for $0 \le j \le m-2$. The goal is to show that dividing both sides of some equation by $\binom{k}{m-1}$ leads to the desired result. □

We extend [11, Corollary 1 and Corollary 2] to $\mathcal{G}$-metric space. The proof is similar to the existing proofs and directs readers to the original source for more information.

Proposition 2.2. *Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ be a multimapping where each $\mathcal{W}_k$ is a self-map on a $\mathcal{G}$-metric space $\mathcal{E}$. The following properties are true.*

(i) $\mathcal{W} \in [(m,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ if and only if

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathcal{G}(\mathcal{W}^\alpha \omega, \mathcal{W}^\alpha \psi, \mathcal{W}^\alpha \varphi)^q = \sum_{0\le j\le m-1} \Big(\sum_{j\le p\le m-1} (-1)^{p-j} \binom{k}{p}\binom{p}{j} \Big) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \mathcal{G}(\mathcal{W}^\alpha \omega, \mathcal{W}^\alpha \psi, \mathcal{W}^\alpha \varphi)^q. \tag{2.7}$$

(ii) If $\mathcal{W} \in [(m,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ and $k \in \mathbb{N}$, then

$$\sum_{0\le j\le k} (-1)^j \binom{k}{j} j^p \Big(\sum_{|\alpha|=k-j} \frac{(k-j)!}{\alpha!} G(\mathcal{W}^\alpha \omega, \mathcal{W}^\alpha \psi, \mathcal{W}^\alpha \varphi)^q \Big) = 0 \tag{2.8}$$

for $k \ge m$, $p = 0, 1, \cdots, k-m$.

Corollary 2.1. *Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ be multimappings where each $\mathcal{W}_k$ is a self-map on $\mathcal{G}$-metric space $\mathcal{E}$. The following properties are true.*

(1) $\mathcal{W} \in [(2,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ *if and only if* $\mathcal{W}$ *satisfies*

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathcal{G}(\mathcal{W}^\alpha \omega, \mathcal{W}^\alpha \psi, \mathcal{W}^\alpha \varphi)^q = (1-k)\mathcal{G}(\omega, \psi, \varphi)^q + k \sum_{1\le j\le n} \mathcal{G}(\mathcal{W}_j \omega, \mathcal{W}_j \psi, \mathcal{W}_j \varphi)^q \tag{2.9}$$

$\forall\ k \in \mathbb{N}$ *and* $\omega, \psi, \varphi \in \mathcal{E}$.

(2) *If* $\mathcal{W} \in [(2,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$. *The following property is true.*

$$\sum_{1\le j\le n} \mathcal{G}(\mathcal{W}_j \omega, \mathcal{W}_j \psi, \mathcal{W}_j \varphi)^q \ge \frac{k-1}{k} \mathcal{G}(\omega, \psi, \varphi)^q \quad (\forall\ k \in \mathbb{N} \text{ and } \forall\ \omega, \psi, \varphi \in \mathcal{E}), \tag{2.10}$$

$$\sum_{1\le j\le p} \mathcal{G}(\mathcal{W}_j\omega, \mathcal{W}_j\psi, \mathcal{W}_j\varphi)^q \ge \mathcal{G}(\omega,\psi,\varphi)^q \quad (\forall\ \omega,\psi,\varphi \in \mathcal{E}), \tag{2.11}$$

$$\lim_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big)^{\frac{1}{k}} = 1 \qquad (\ \forall\ \omega,\psi,\varphi \in \mathcal{E},\ \ \omega \ne \psi, \omega \ne \varphi, \psi \ne \varphi). \tag{2.12}$$

Proof. (1) Connect the results obtained from Proposition 2.1(ii) and Remark 2.3(ii) and then link them to establish the equivalence.

(2) We have (2.9) $\Longrightarrow$ (2.10) and (2.10) $\Longrightarrow$ (2.11) ($k \to \infty$). By following these steps, we can rigorously verify the validity of the inequalities.

We need to verify (2.12). Let $\omega,\psi,\varphi \in \mathcal{E}$ for which $\omega \ne \psi, \omega \ne \varphi, \psi \ne \varphi$. It follows from (2.9) that

$$\limsup_{r\to\infty}\Big(\sum_{|\alpha|=r}\frac{r!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big)^{\frac{1}{r}} \le 1.$$

However, according to (2.11), the sequence $\Big(\sum_{|\alpha|=r}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big)_{r\in\mathbb{N}}$ is monotonically increasing, so

$$\liminf_{r\to\infty}\Big(\sum_{|\alpha|=r}\frac{r!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\Big)^{\frac{1}{r}} \ge \lim_{r\to\infty}\Big(\mathcal{G}(\omega,\psi,\varphi)^q\Big)^{\frac{1}{r}} = 1.$$

□

Definition 2.2. Let $\mathcal{W} = (\mathcal{W}_1,\cdots,\mathcal{W}_p)$ be a tuple of mappings on $(\mathcal{E},\mathcal{G})$. $\mathcal{W}$ is a $\mathcal{G}$ - power bounded tuple if

$$\sup\Big\{\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q\ ,\ \forall\ k \in \mathbb{N}\Big\} < \infty$$

for all $\omega,\psi,\varphi \in \mathcal{E}$.

Theorem 2.2. *Let $\mathcal{W} = (\mathcal{W}_1,\cdots,\mathcal{W}_p) \in [(m,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$ and $\mathcal{G}$-power bounded tuple. Then*

$$\Big(\sum_{1\le i\le n}\mathcal{G}(\mathcal{W}_i\omega, \mathcal{W}_i\psi, \mathcal{W}_i\varphi)^q\Big)^{\frac{1}{q}} = \mathcal{G}(\omega,\psi,\varphi),$$

that is $\mathcal{W} \in [(1,q)-\mathcal{ISO}]_p[(\mathcal{E},\mathcal{G})]$.

Proof. According to that $\mathcal{W}$ is an (m,q)-$\mathcal{G}$isometric and (2.5) we obtain

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^\alpha\omega, \mathcal{W}^\alpha\psi, \mathcal{W}^\alpha\varphi)^q = \sum_{0\le j\le m-1}\binom{k}{j}\mathcal{Q}_j^{(q)}(\mathcal{W};\omega,\psi,\varphi),\quad \forall\ k \in \mathbb{N}.$$

Consequently, real numbers $\delta_0(\omega,\psi,\varphi), \delta_1(\omega,\psi,\varphi),\cdots,\delta_{m-1}(\omega,\psi,\varphi)$ exist such that

$$\sum_{1\le i\le n}\mathcal{G}(\mathcal{W}_i^k\omega,\ \mathcal{W}_i^k\psi, \mathcal{W}_i^k\varphi)^q = \sum_{0\le j\le m-1}\delta_j(\omega,\psi,\varphi)\,k^j. \tag{2.13}$$

Since $\mathcal{W}$ is power bounded, we put

$$M = \sup\Big\{ \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathcal{G}(\mathcal{W}^{\alpha}\omega,\ \mathcal{W}^{\alpha}\psi, \mathcal{W}^{\alpha}\varphi)^q,\ \ k = 0, 1, 2, ...\Big\} < \infty$$

for $\omega, \psi, \varphi \in \mathcal{E}$. We now obtain

$$0 \leq \sup\Big\{ \sum_{0\leq j\leq m-1} \delta_j(\omega, \psi, \varphi)k^j \ :\ k = 0, 1, 2, ... \Big\} \leq M^q.$$

Since k is arbitrary, we have $\delta_1(\omega, \psi, \varphi) = \delta_2(\omega, \psi, \varphi) = \cdots = \delta_{m-1}(\omega, \psi, \varphi) = 0$. Therefore

$$\sum_{0\leq i\leq p} \mathcal{G}(\mathcal{W}_i^k\omega,\ \mathcal{W}_i^k\psi, \mathcal{W}_i^k\varphi)^q = \mathcal{G}(\omega, \psi, \varphi)^q.$$

Since k is arbitrary, we choose $k = 1$ to get that $\mathcal{W}$ is a $(1, q)$-$\mathcal{G}$-isometric multimapping. □

3. Generalized metric in the context of (m, q)-$\mathcal{G}$-isometries

By using these generalized metrics and their associated isometries, one can explore some $\mathcal{G}$-metrics-preserving transformations and their applications. Our results extend those given in [12–14].

Let $\mathcal{W}$ be an (m, q)-$\mathcal{G}$-isometric multimapping. We set

$$\Delta_{\mathcal{W}}(\omega, \psi, \varphi) = \Big(Q_{m-1}^{(q)}(\mathcal{W}; \omega, \psi, \varphi)\Big)^{\frac{1}{q}},\ \forall\ \omega, \psi, \varphi \in \mathcal{E}, \quad m \geq 1,\ q \geq 1.$$

Theorem 3.1. *Let $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ be an (m, q)-$\mathcal{G}$-isometric multimapping. Then*

$$\Delta_{\mathcal{W}}(\omega, \psi, \varphi) = \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty} \frac{1}{\sqrt[q]{n^{(m-1)}}} \sum_{|\gamma|=n} \frac{n!}{\gamma!}\mathcal{G}(\mathcal{W}^{\gamma}\omega,\ \mathcal{W}^{\gamma}\psi, \mathcal{W}^{\gamma}\varphi)^q. \tag{3.1}$$

Moreover $\Delta_{\mathcal{W}}$ is a semi-$\mathcal{G}$-metric on $\mathcal{E}$.

Proof. According to statement (iii) of Theorem 2.1, we may write

$$Q_{m-1}^{(q)}(\mathcal{W}; \omega, \psi, \varphi) = \lim_{n\longrightarrow\infty} \frac{1}{\binom{n}{m-1}}\Big(\sum_{|\gamma|=n} \frac{n!}{\gamma!}\mathcal{G}(\mathcal{W}^{\gamma}\omega,\ \mathcal{W}^{\gamma}\psi, \mathcal{W}^{\gamma}\varphi)^q\Big). \tag{3.2}$$

This means that

$$\begin{aligned} \rho_{\mathcal{W}}(\omega, \psi, \varphi) &= Q_{m-1}^{(q)}(\mathcal{W}; \omega, \psi, \varphi) \\ &= \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty} \frac{1}{\sqrt[q]{n^{(m-1)}}}\Big(\sum_{|\gamma|=n} \frac{n!}{\gamma!}\mathcal{G}(\mathcal{W}^{\gamma}\omega,\ \mathcal{W}^{\gamma}\psi, \mathcal{W}^{\gamma}\varphi)^q\Big)^{\frac{1}{q}}. \end{aligned}$$

We show that $\Delta_{\mathcal{W}}$ satisfied the following conditions:

- $\Delta_{\mathcal{W}}(\omega, \psi, \varphi \geq 0$, by the statement (iii) of Theorem 2.1.
- $\Delta_{\mathcal{W}}(\omega, \omega, \omega) = 0$ for all $\omega \in \mathbb{E}$.

- $\Delta_{\mathcal{W}}(\omega,\omega,\psi)\leq\Delta_{\mathcal{W}}(\omega,\psi,\varphi)$ for all $\omega,\psi,\varphi\in\mathcal{E}$ with $\psi\neq\varphi$.
- $\Delta_{\mathcal{W}}(\omega,\psi,\varphi)=\Delta_{\mathcal{W}}(\omega,\varphi,\psi)=\Delta_{\mathcal{W}}(\psi,\varphi,\omega)=...$ (symmetry in all three variables).
- $\Delta_{\mathcal{W}}(\omega,\psi,\varphi)\leq\Delta_{\mathcal{W}}(\omega,a,a)+\Delta_{\mathcal{W}}(a,\psi,\varphi)$ for all $\omega,\psi,\varphi,a\in\mathcal{E}$ (rectangle inequality).

□

Remark 3.1. It should be noted that if $\mathcal{W}$ is an (m,q)-$\mathcal{G}$-isometry, then in view of Proposition 2.1,

$$\sum_{1\leq k\leq n}\mathcal{Q}_m^{(q)}(\mathcal{W};\mathcal{W}_k\omega,\mathcal{W}_k\psi,\mathcal{W}_k\varphi)=\mathcal{Q}_m^{(q)}(\mathcal{W};\omega,\psi,\varphi).$$

This means that

$$\Delta_{\mathcal{W}}(\omega,\psi,\varphi)=\sum_{1\leq k\leq m}\Delta_{\mathcal{W}}(\mathcal{W}_k\omega,\mathcal{W}_k\psi,\mathcal{W}_k\varphi)$$

and therefore

$$\mathcal{W}:(\mathcal{E},\ \Delta_{\mathcal{W}})\longrightarrow(\mathcal{E},\Delta_{\mathcal{W}}),$$

is an $(1,q)$-$\mathcal{G}$-isometry.

By observing that

$$\begin{aligned}
&\mathcal{Q}_m^{(q)}(\mathcal{W},\omega,\psi,\varphi)\\
=\ &\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathcal{G}(\mathcal{W}^\gamma\omega,\ \mathcal{W}^\gamma\psi,\mathcal{W}^\gamma\varphi)^q\\
=\ &\sum_{\substack{0\leq k\leq m\\ k\text{ (even)}}}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathcal{G}(\mathcal{W}^\gamma\omega,\ \mathcal{W}^\gamma\psi,\mathcal{W}^\gamma\varphi)^q\\
&-\sum_{\substack{0\leq k\leq m\\ k\text{ (odd)}}}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathcal{G}(\mathcal{W}^\gamma\omega,\ \mathcal{W}^\gamma\psi,\mathcal{W}^\gamma\varphi)^q\\
=\ &\sum_{\substack{0\leq k\leq m\\ k\text{ (even)}}}\binom{m}{k}\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathcal{G}(\mathcal{W}^\gamma\omega,\ \mathcal{W}^\gamma\psi,\mathcal{W}^\gamma\varphi)^q\\
&-\sum_{1\leq k\leq p}\sum_{\substack{0\leq k\leq m\\ k\text{ (odd)}}}\binom{m}{k}\sum_{|\gamma'|=k}\frac{k!}{\gamma'!}\mathcal{G}(\mathcal{W}^{\gamma'}W_k\omega,\ \mathcal{W}^{\gamma'}\mathcal{W}_k\psi,\mathcal{W}^{\gamma'}\mathcal{W}_k\varphi)^q\\
=\ &\widetilde{\Delta_{\mathcal{W}}}(\omega,\psi,\varphi)-\sum_{1\leq k\leq p}\widetilde{\Delta_{\mathcal{W}}}'(\mathcal{W}_k\omega,\mathcal{W}_k\psi,\mathcal{W}_k\varphi).
\end{aligned}$$

Lemma 3.1. *$(\mathcal{E},\widetilde{\Delta_{\mathcal{W}}})$ and $(\mathcal{E},\widetilde{\Delta_{\mathcal{W}}}')$ are both $\mathcal{G}$- metric spaces.*

Theorem 3.2. *Let $\mathcal{W}$ be a self multimaping on a general metric space $(\mathcal{E},\ \mathcal{G})$ and $q \geq 1$. Then following statements are equivalent.*

(1) $\mathcal{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_d)$ *where* $\mathcal{W}_k : (\mathcal{E},\ \mathcal{G}) \longrightarrow (\mathcal{E},\ \mathcal{G})$ *is an* (m, q)*-* $\mathcal{G}$*-isometry.*

(2) $\mathcal{W} : (\mathcal{E},\ \widetilde{\Delta_{\mathcal{W}}}) \longrightarrow (\mathcal{E},\ \widetilde{\Delta_{\mathcal{W}}}')$ *is an* $(1, \mathcal{G})$*-isometry.*

Proof. In view of Proposition 2.1, we obtain

$$\begin{aligned}
& \mathcal{W} \text{ is an } (m,q) - \mathcal{G} - \text{isometric tuple} \\
\Leftrightarrow \quad & \sum_{\substack{0 \leq k \leq m \\ k \text{ (even)}}} \binom{m}{k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} \mathcal{G}(\mathcal{W}^{\gamma}\omega,\ \mathcal{W}^{\gamma}\psi, \mathcal{W}^{\gamma}\varphi)^q \\
= \quad & \sum_{\substack{0 \leq k \leq m \\ k \text{ (odd)}}} \binom{m}{k} \sum_{|\gamma|=k} \frac{k!}{\gamma!} \mathcal{G}(\mathcal{W}^{\gamma}\omega,\ \mathcal{W}^{\gamma}\psi, \mathcal{W}^{\gamma}\varphi)^q, \ \forall\ \omega, \psi, \varphi \in \mathcal{E} \\
\Leftrightarrow \quad & \widetilde{\Delta_{\mathcal{W}}}(\omega, \psi, \varphi) = \sum_{1 \leq k \leq p} \widetilde{\Delta_{\mathcal{W}}}'(\mathcal{W}_k\omega, \mathcal{W}_k\psi, \mathcal{W}_k\varphi), \ \forall\ \omega, \psi, \varphi \in \mathcal{E} \\
\Leftrightarrow \quad & \mathcal{W} \text{ is an } (1,q) - \mathcal{G} - \text{isometry tuple.}
\end{aligned}$$

□

In general, if either of the statements (1) or (2) in Theorem 3.4 is not satisfied, then the other is also not satisfied, as illustrated in the following example.

Example 3.4.

Let (E, d) be a $\mathcal{G}$-metric space where $\mathcal{E} = \{0, 1, 2\}$ and

$$\mathcal{G}(\omega, \psi, \varphi) = \max\{|\omega - \psi|, |\omega - \varphi|, |\varphi - \psi|\} \forall\ \omega, \psi, \varphi \in E.$$

Define mappings $\mathcal{W}_1$ and $\mathcal{W}_2 : \mathcal{E} \to \mathcal{E}$ by

$$\begin{aligned}
\mathcal{W}_1(0) = 2, \quad \mathcal{W}_1(1) = \mathcal{W}_1(2) = 0, \\
\mathcal{W}_2(0) = 0, \quad \mathcal{W}_2(1) = \mathcal{W}_2(2) = 2.
\end{aligned}$$

Then, a straightforward calculation shows that $\mathcal{W}_1\mathcal{W}_2 = \mathcal{W}_2\mathcal{W}_1$, and

$$\begin{aligned}
& \mathcal{G}(\mathcal{W}_1^2\omega, \mathcal{W}_1^2\psi, \mathcal{W}_1^2\varphi)^q + \mathcal{G}(\mathcal{W}_2^2\omega, \mathcal{W}_2^2\psi, \mathcal{W}_2^2\varphi)^q \\
& +2\mathcal{G}(\mathcal{W}_1\mathcal{W}_2\omega, \mathcal{W}_1\mathcal{W}_2\psi, \mathcal{W}_1\mathcal{W}_2\varphi)^q + \mathcal{G}(\omega, \psi, \varphi)^q \\
\neq \ & 2\mathcal{G}(\mathcal{W}_1\omega, \mathcal{W}_1\psi, \mathcal{W}_1\varphi)^q + 2\mathcal{G}(\mathcal{W}_2\omega, \mathcal{W}_2\psi, \mathcal{W}_2\varphi)^q
\end{aligned}$$

for all $\omega, \psi, \varphi \in \mathcal{E}$. Therefore, we conclude that the pair $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ is not a $(2, q)$-isometric pair of mappings. Moreover,

$$\widetilde{\Delta_{\mathcal{W}}}(\omega, \psi, \varphi) \neq \sum_{1 \leq k \leq 2} \widetilde{\Delta_{\mathcal{W}}}'(\mathcal{W}_k\omega, \mathcal{W}_k\psi, \mathcal{W}_k\varphi), \ \forall\ \omega, \psi, \varphi \in \mathcal{E}.$$

4. Conclusions

(m, q)-isometric multimappings in $\mathcal{G}$-metric spaces provide a novel framework for analyzing distance-preserving transformations. This research extends classical isometric concepts, offering a broader perspective on metric structures and their properties.

Author contributions

Hadi Obaid Alshammari developed the main theoretical framework and proofs. Abdulrahman Obaid Alshammari contributed to the refinement of the results and the preparation of illustrative examples. All authors have contributed equally to the research. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the writing of this manuscript. All content was written entirely by the authors.

Conflict of interest

The authors declare that they have no conflicts of interest.

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