Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 11349–11368.
DOI: 10.3934/math.2025516
Received: 22 January 2025
Revised: 27 April 2025
Accepted: 08 May 2025
Published: 19 May 2025

Research article

Statistical inference of the mixed linear model with incorrect stochastic linear restrictions

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Abstract: We considered the general mixed linear model $\mathscr{N}$ subject to two competing stochastic linear restrictions, $\mathscr{M}_0$ and $\mathscr{M}$, where the restrictions $\mathscr{M}$ are the correct information whereas restrictions $\mathscr{M}_0$ may be incorrect. Statistical inference conclusions of using the above two competing restrictions are not necessarily the same, so it is prominent to discuss the relationships between incorrect restrictions $\mathscr{M}_0$ and the corresponding correct restrictions $\mathscr{M}$ in the context of model $\mathscr{N}$. In this article, we first present some properties on the best linear unbiased predictors (BLUPs) under model $\mathscr{N}$ with restrictions $\mathscr{M}$. We then provide necessary and sufficient conditions under which the BLUPs under $\mathscr{N}$ with the incorrect restrictions $\mathscr{M}_0$ continue to be BLUPs associated with correct restrictions.

Keywords: mixed linear model; correct stochastic restrictions; incorrect stochastic restrictions; matrix rank; BLUP
Mathematics Subject Classification: 62H12, 62J05

1. Introduction

The general linear mixed model takes the form

$$\mathscr{N} : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \quad \mathrm{E}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\end{pmatrix} = \mathbf{0}, \tag{1.1}$$

where $\mathbf{y}$ is a response vector, $\mathbf{X} \in \mathbb{R}^{n\times p}$ and $\mathbf{Z} \in \mathbb{R}^{n\times q}$ are both known matrices, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$ is an unknown vector of fixed effects, $\boldsymbol{\gamma} \in \mathbb{R}^{q\times 1}$ is a vector of random effects, $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ is a disturbance vector, and $\mathrm{E}(\cdot)$ represents the expectation.

In practice, in addition to the sample information (1.1), stochastic linear restrictions binding the vector of fixed effects in (1.1) are often encountered, which may come from other studies or some relevant hypothesis testing, among others. In this situation, we must concentrate on handling these restrictions producing higher accuracy for predictors and estimators (see [1,2], among others).

Let stochastic linear restrictions be defined as

$$\mathscr{M}_0 : \mathbf{r} = \mathbf{A}_0\boldsymbol{\beta} + \mathbf{e}_0, \ \ \mathrm{E}(\mathbf{e}_0) = \mathbf{0}, \tag{1.2}$$

where $\mathbf{r} \in \mathbb{R}^{m\times 1}$ and $\mathbf{A}_0 \in \mathbb{R}^{m\times p}$ are both given matrices with any rank, and $\mathbf{e}_0$ is a random error vector with

$$\boldsymbol{\Lambda} = \mathrm{D}\begin{pmatrix} \widetilde{\boldsymbol{\gamma}} \\ \mathbf{e}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 & \boldsymbol{\Lambda}_2 \\ \boldsymbol{\Lambda}_2' & \boldsymbol{\Lambda}_3 \end{pmatrix}, \ \ \widetilde{\boldsymbol{\gamma}} = \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\varepsilon} \end{pmatrix}, \tag{1.3}$$

where $\boldsymbol{\Lambda}$ is postulated as a known matrix of any rank, and $\mathrm{D}(\cdot)$ refers to the dispersion matrix. Assume that $\mathbf{V}_3 \in \mathbb{R}^{m\times m}$ and $\mathbf{A} \in \mathbb{R}^{m\times p}$ are the proper forms of $\boldsymbol{\Lambda}_3$ and $\mathbf{A}_0$, respectively, on account of various reasons. For instance, with the rapid development of the times and changes in the environment, the former result cannot completely reconcile with the current situation. In addition, the restrictions (1.2) are remarkably dependent on the knowledge of $\boldsymbol{\Lambda}_3$, the dispersion matrix of random error vector $\mathbf{e}_0$. Unfortunately, in practice, the matrix $\boldsymbol{\Lambda}_3$ is seldom known, so an incorrect assumption on $\boldsymbol{\Lambda}_3$ is often made. The matrix $\mathbf{A}_0$ may also be misspecified, such as in the data collection and aggregation, in analysis of submodels, or in estimates of experts. In other words, corresponding to the restrictions (1.2), the correct stochastic restrictions in the form

$$\mathscr{M} : \mathbf{r} = \mathbf{A}\boldsymbol{\beta} + \mathbf{e} \ \text{ with } \ \mathrm{E}(\mathbf{e}) = \mathbf{0}$$

and

$$\mathrm{D}\begin{pmatrix} \widetilde{\boldsymbol{\gamma}} \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_2' & \mathbf{V}_3 \end{pmatrix} = \mathbf{V}, \ \ \widetilde{\boldsymbol{\gamma}} = \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\varepsilon} \end{pmatrix}, \tag{1.4}$$

where $\mathbf{A} \in \mathbb{R}^{m\times p}$ and $\mathbf{V}$ are two given matrix of arbitrary rank.

Below, we give some notation utilized in this article. We write $\mathbf{Q} \in \mathbb{R}^{m\times n}$ if $\mathbf{Q}$ is a $m \times n$ real matrix. $(\cdot)^{\dagger}$, $\mathscr{R}(\cdot)$, $r(\cdot)$, and $(\cdot)'$ represent the Moore-Penrose generalized inverse, the column space, the rank, and the transpose of a marix, respectively, and $\mathbf{I}_n$ the identity matrix and $\mathbf{I}_n \in \mathbb{R}^{n\times n}$. In addition, we also use $\mathbf{Q}^{\perp}$ and $\mathbf{F}_{\mathbf{Q}}$ to denote the orthogonal projectors produced by $\mathbf{Q} \in \mathbb{R}^{m\times n}$, $\mathbf{I}_m - \mathbf{Q}\mathbf{Q}^{\dagger}$ and $\mathbf{I}_m - \mathbf{Q}^{\dagger}\mathbf{Q}$, respectively.

Incorporating linear mixed model (1.1) with correct stochastic restrictions (1.4) and its incorrect form (1.2), respectively, yields

$$\mathscr{N}_r : \widehat{\mathbf{y}} = \widehat{\mathbf{X}}\boldsymbol{\beta} + \mathbf{Z}_0\boldsymbol{\gamma} + \widehat{\mathbf{I}}_n\boldsymbol{\varepsilon} + \widehat{\mathbf{I}}_m\mathbf{e} = \widehat{\mathbf{X}}\boldsymbol{\beta} + \widehat{\mathbf{Z}}\widehat{\boldsymbol{\varepsilon}}, \tag{1.5}$$

$$\mathscr{N}_{r_0} : \widehat{\mathbf{y}} = \widehat{\mathbf{X}}_0\boldsymbol{\beta} + \mathbf{Z}_0\boldsymbol{\gamma} + \widehat{\mathbf{I}}_n\boldsymbol{\varepsilon} + \widehat{\mathbf{I}}_m\mathbf{e}_0 = \widehat{\mathbf{X}}_0\boldsymbol{\beta} + \widehat{\mathbf{Z}}\widehat{\boldsymbol{\varepsilon}}_0, \tag{1.6}$$

where

$$\widehat{\mathbf{y}} = \begin{pmatrix} \mathbf{y} \\ \mathbf{r} \end{pmatrix}, \ \widehat{\mathbf{X}} = \begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix}, \ \widehat{\mathbf{X}}_0 = \begin{pmatrix} \mathbf{X} \\ \mathbf{A}_0 \end{pmatrix}, \ \mathbf{Z}_0 = \begin{pmatrix} \mathbf{Z} \\ \mathbf{0} \end{pmatrix}, \ \widehat{\mathbf{I}}_n = \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0} \end{pmatrix},$$

$$\widehat{\mathbf{I}}_m = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_m \end{pmatrix}, \ \widehat{\mathbf{Z}} = (\mathbf{Z}_0, \mathbf{I}_{n+m}), \ \widehat{\boldsymbol{\varepsilon}} = (\boldsymbol{\gamma}', \boldsymbol{\varepsilon}', \mathbf{e}')', \ \widehat{\boldsymbol{\varepsilon}}_0 = (\boldsymbol{\gamma}', \boldsymbol{\varepsilon}', \mathbf{e}_0')'.$$

As to model (1.5), core tasks of statistical inference are to estimate parameter functions of fixed effects $\boldsymbol{\beta}$ and to predict functions of random vectors $\boldsymbol{\gamma}$, $\boldsymbol{\varepsilon}$, and $\mathbf{e}$, separately or simultaneously. However, because (1.2) is a misspecified form of (1.4), it is conceivable that the consequences of statistical inference from these two models $\mathscr{N}_{r_0}$ and $\mathscr{N}_r$ may not be the same. Trivially, the findings from model $\mathscr{N}_{r_0}$ are mostly incorrect, but we would be interested in acquiring valuable information from $\mathscr{N}_{r_0}$. Naturally, this motivates us to compare the two models, $\mathscr{N}_{r_0}$ and $\mathscr{N}_r$, as well as their statistical inference conclusions, particularly to establish the relations of estimators/predictors of unknown parameters under $\mathscr{N}_{r_0}$ and $\mathscr{N}_r$. To acquire more general conclusions, we take into account the function of fixed effects $\boldsymbol{\beta}$ and random vector $\boldsymbol{\gamma}$, $\boldsymbol{\varepsilon}$, and $\mathbf{e}$ as follows

$$\boldsymbol{\xi} = \mathbf{K}\boldsymbol{\beta} + \mathbf{B}_1\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\varepsilon} + \mathbf{B}_3\mathbf{e} = \mathbf{K}\boldsymbol{\beta} + (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\\ \mathbf{e}\end{pmatrix} = \mathbf{K}\boldsymbol{\beta} + \mathbf{B}\widehat{\boldsymbol{\varepsilon}}, \tag{1.7}$$

where $\mathbf{K} \in \mathbb{R}^{k\times p}$, $\mathbf{B}_1 \in \mathbb{R}^{k\times q}$, $\mathbf{B}_2 \in \mathbb{R}^{k\times n}$, and $\mathbf{B}_3 \in \mathbb{R}^{k\times m}$ are four known matrices. Some special situations are given below:

(i) Let $\mathbf{K} = \mathbf{I}_p$ and $\mathbf{B} = \mathbf{0}$. Then, $\boldsymbol{\xi}$ turns into the unknown vector of fixed effects $\boldsymbol{\beta}$.

(ii) Let $\mathbf{K} = \mathbf{0}$ and $\mathbf{B} = \left(\mathbf{I}_q, \mathbf{0}, \mathbf{0}\right)$. Then, $\boldsymbol{\xi}$ turns into the vector of random effects $\boldsymbol{\gamma}$.

(iii) Let $\mathbf{K} = \mathbf{X}$ and $\mathbf{B} = (\mathbf{Z}, \mathbf{I}_n, \mathbf{0})$. Then, $\boldsymbol{\xi}$ turns into the response vector $\mathbf{y}$.

Corresponding to (1.7), we consider the parametric function involving the parameters $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\varepsilon}$, and $\mathbf{e}_0$, which is presented by

$$\boldsymbol{\xi}_0 = \mathbf{K}_0\boldsymbol{\beta} + \mathbf{B}_1\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\varepsilon} + \mathbf{B}_3\mathbf{e}_0 = \mathbf{K}\boldsymbol{\beta} + (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\\ \mathbf{e}_0\end{pmatrix} = \mathbf{K}_0\boldsymbol{\beta} + \mathbf{B}\widehat{\boldsymbol{\varepsilon}}_0, \tag{1.8}$$

where $\mathbf{K}_0 \in \mathbb{R}^{k\times p}$ is a given matrix. In what follows, we first give the definition of estimability and predictability of $\boldsymbol{\xi}$ under $\mathscr{N}_r$.

Definition 1.1. *Assume that there is a matrix* $\mathbf{C}$ *satisfying* $\mathrm{E}(\mathbf{C}\widehat{\mathbf{y}} - \boldsymbol{\xi}) = \mathbf{0}$. *Then, we say that* $\boldsymbol{\xi}$ *in (1.7) is predictable under* $\mathscr{N}_r$. *In this situation, when* $\mathbf{B} = \mathbf{0}$ *in (1.7),* $\boldsymbol{\xi} = \mathbf{K}\boldsymbol{\beta}$ *is also known as estimable under* $\mathscr{N}_r$.

From the above definition, the followings are direct:

(a) $\boldsymbol{\xi}$ in (1.7) is predictable under $\mathscr{N}_r$ if and only if $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\widehat{\mathbf{X}}')$.

(b) $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$ and $\mathbf{e}$ in (1.7) are separately and jointly predictable under $\mathscr{N}_r$.

(c) For any matrix $\mathbf{B}$, $\mathbf{B}\widehat{\boldsymbol{\varepsilon}}$ in (1.7) must be predictable under $\mathscr{N}_r$.

Definition 1.2. *Let* $\boldsymbol{\xi}$ *in (1.7) be predictable under* $\mathscr{N}_r$. *A linear statistic* $\mathbf{C}\widehat{\mathbf{y}}$ *fulfilling the condition* $\mathrm{E}(\mathbf{C}\widehat{\mathbf{y}} - \boldsymbol{\xi}) = \mathbf{0}$ *is called as the best linear unbiased predictor (BLUP) for* $\boldsymbol{\xi}$ *under* $\mathscr{N}_r$, *denoted by* $BLUP(\boldsymbol{\xi}|\mathscr{N}_r)$, *if*

$$\mathrm{D}(\mathbf{C}\widehat{\mathbf{y}} - \boldsymbol{\xi}) \leq \mathrm{D}(\mathbf{L}\widehat{\mathbf{y}} - \boldsymbol{\xi}) \quad \forall\, \mathbf{L}:\ \mathbf{L}\widehat{\mathbf{X}} = \mathbf{K},$$

where $\leq$ *denotes the* Löwner *partial ordering, i.e., the difference*

$$\mathrm{D}(\mathbf{L}\widehat{\mathbf{y}} - \boldsymbol{\xi}) - \mathrm{D}(\mathbf{C}\widehat{\mathbf{y}} - \boldsymbol{\xi})$$

is nonnegative definite. When $\mathbf{B} = \mathbf{0}$ *in (1.7),* $\mathbf{C}\widehat{\mathbf{y}}$ *becomes the best linear unbiased estimator (BLUE) for* $\mathbf{K}\boldsymbol{\beta}$ *expressed by* $BLUE(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_r)$*. Additionally,*

$$\mathbf{C}\widehat{\mathbf{y}} = \mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_r) \Longleftrightarrow \mathbf{C}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right), \tag{1.9}$$

see [3].

As demonstrated by [4], when confronted with models $\mathscr{N}_r$ and $\mathscr{N}_{r_0}$, people often consider the following three questions:

(a) When is a particular expression for the BLUP of predictable $\boldsymbol{\xi}_0$ under $\mathscr{N}_{r_0}$ also a BLUP of predictable $\boldsymbol{\xi}$ under $\mathscr{N}_r$?

(b) When do the BLUPs of predictable $\boldsymbol{\xi}_0$ under $\mathscr{N}_{r_0}$ and BLUPs of predictable $\boldsymbol{\xi}$ under $\mathscr{N}_r$ have a common predictor?

(c) When does every BLUP of predictable $\boldsymbol{\xi}_0$ under $\mathscr{N}_{r_0}$ remain the BLUP of predictable $\boldsymbol{\xi}$ under $\mathscr{N}_r$?

There are many researchers devoted to the investigations of estimators and predictors under correct models and the corresponding incorrect models. For instance, the comparison problems of estimators under two general linear models $\mathscr{M} : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\mathrm{D}(\boldsymbol{\varepsilon}) = \boldsymbol{\Omega}$ and $\mathscr{M}_0 : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_0$ with $\mathrm{E}(\boldsymbol{\varepsilon}_0) = \mathbf{0}$ and $\mathrm{D}(\boldsymbol{\varepsilon}_0) = \boldsymbol{\Omega}_0$ were made by [4–8], etc. The equivalence of predictors/estimators between the model $\mathscr{M}$ and its incorrect model $\mathscr{M}_0 : \mathbf{y} = \mathbf{X}_0\boldsymbol{\beta}_0 + \boldsymbol{\varepsilon}_0$ with $\mathrm{E}(\boldsymbol{\varepsilon}_0) = \mathbf{0}$ and $\mathrm{D}(\boldsymbol{\varepsilon}_0) = \boldsymbol{\Omega}_0$ was dealt with by [9, 10]. Furthermore, the researchers in [11] were concerned with the equivalence of predictors/estimators under true and untrue multivariate general linear models. Alternatively, the researchers in [12, 13] considered the relationships between estimators under the model $\mathscr{M}$ with an exact restriction $\mathbf{r} = \mathbf{A}\boldsymbol{\beta}$ and its mis-specified restriction $\mathbf{r}_0 = \mathbf{A}_0\boldsymbol{\beta}$, which were generalized by [14]. In this paper, we mainly solve the three questions proposed above.

Finally, we provide some lemmas which can be of service to formation of theoretical system in this paper.

Lemma 1.1. *[15] Let* $\mathbf{P}_1 \in \mathbb{R}^{m\times n}$, $\mathbf{P}_2 \in \mathbb{R}^{m\times k}$ *and* $\mathbf{P}_3 \in \mathbb{R}^{l\times n}$*. Then:*

$$r(\mathbf{P}_1, \mathbf{P}_2) = r(\mathbf{P}_1) + r(\mathbf{P}_1^{\perp}\mathbf{P}_2) = r(\mathbf{P}_2) + r(\mathbf{P}_2^{\perp}\mathbf{P}_1), \tag{1.10}$$

$$r\begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_3\end{pmatrix} = r(\mathbf{P}_1) + r(\mathbf{P}_3\mathbf{F}_{\mathbf{P}_1}) = r(\mathbf{P}_3) + r(\mathbf{P}_1\mathbf{F}_{\mathbf{P}_3}). \tag{1.11}$$

If $\mathscr{R}(\mathbf{Q}_1') \subseteq \mathscr{R}(\mathbf{P}_1')$, $\mathscr{R}(\mathbf{O}) \subseteq \mathscr{R}(\mathbf{P}_1)$, $\mathscr{R}(\mathbf{O}') \subseteq \mathscr{R}(\mathbf{P}_2')$ *and* $\mathscr{R}(\mathbf{Q}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$*, then*

$$r\left(\mathbf{Q}_1\mathbf{P}_1^{\dagger}\mathbf{O}\mathbf{P}_2^{\dagger}\mathbf{Q}_2\right) = r\begin{pmatrix}\mathbf{0} & \mathbf{P}_2 & \mathbf{Q}_2\\ \mathbf{P}_1 & \mathbf{O} & \mathbf{0}\\ \mathbf{Q}_1 & \mathbf{0} & \mathbf{0}\end{pmatrix} - r(\mathbf{P}_1) - r(\mathbf{P}_2). \tag{1.12}$$

Lemma 1.2. *[16, 17] Let* $\mathbf{P}_1 \in \mathbb{R}^{n\times m}$ *and* $\mathbf{P}_2 \in \mathbb{R}^{k\times m}$*. Then*

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times k}} r(\mathbf{P}_1 - \mathbf{X}\mathbf{P}_2) = r\begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix} - r(\mathbf{P}_2), \tag{1.13}$$

$$\max_{\mathbf{X}\in\mathbb{R}^{n\times k}} r(\mathbf{P}_1 - \mathbf{X}\mathbf{P}_2) = \min\left\{r\begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}, n\right\}. \tag{1.14}$$

Lemma 1.3. *Assume that* $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *are both collections of matrices of the same dimension. Then*

$$\mathbb{Q}_1 \cap \mathbb{Q}_2 \neq \emptyset \Leftrightarrow \min_{\mathbf{Q}_1 \in \mathbb{Q}_1, \mathbf{Q}_2 \in \mathbb{Q}_2} r(\mathbf{Q}_1 - \mathbf{Q}_2) = 0, \tag{1.15}$$

$$\mathbb{Q}_1 \subseteq \mathbb{Q}_2 \Leftrightarrow \max_{\mathbf{Q}_1 \in \mathbb{Q}_1} \min_{\mathbf{Q}_2 \in \mathbb{Q}_2} r(\mathbf{Q}_1 - \mathbf{Q}_2) = 0. \tag{1.16}$$

Lemma 1.4. *[8] Let* $\mathbf{P}_1 \in \mathbb{R}^{m\times k}$, $\mathbf{Q}_1 \in \mathbb{R}^{p\times k}$, $\mathbf{P}_2 \in \mathbb{R}^{m\times l}$ *and* $\mathbf{Q}_2 \in \mathbb{R}^{p\times l}$ *be known. Then every solution of matrix equation* $\mathbf{XP}_1 = \mathbf{Q}_1$ *continues to be a solution of* $\mathbf{XP}_2 = \mathbf{Q}_2$ *if and only if*

$$r\begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix} = r(\mathbf{P}_1). \tag{1.17}$$

Lemma 1.5. *[18] Let* $\mathbf{0} \leq \mathbf{V} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{X} \in \mathbb{R}^{n\times p}$. *Then*

$$\mathscr{R}\begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix} = \mathscr{R}\begin{pmatrix} \mathbf{V} & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}' \end{pmatrix}. \tag{1.18}$$

In particular,

$$r\begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix} = r(\mathbf{V}, \mathbf{X}) + r(\mathbf{X}). \tag{1.19}$$

2. Some properties on BLUPs

Assume that there exists a matrix $\mathbf{L}$ satisfying $\mathbf{K} = \mathbf{L}\widehat{\mathbf{X}}$, that is to say, $\boldsymbol{\xi}$ in (1.7) is predictable under $\mathscr{N}_r$. Noticing that $\mathscr{R}(\widehat{\mathbf{X}}) \cap \mathscr{R}(\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = \{\mathbf{0}\}$, we have

$$r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} \\ \mathbf{K} & \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} \end{pmatrix} = r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} \\ \mathbf{0} & \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} - \mathbf{L}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} \end{pmatrix} = r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$

implying that the Eq (1.9) is always consistent. Solving Eq (1.9) yields

$$\mathbf{C} = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}, \tag{2.1}$$

and thus

$$\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}}) = \left(\left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.2}$$

where $\mathbf{U}$ is an arbitrary matrix. From the exact algebraic expression (2.2), we immediately have

$$\mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) = \left[(\mathbf{K}, \mathbf{0})\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right]\widehat{\mathbf{y}}, \tag{2.3}$$

$$\mathrm{BLUP}(\mathbf{B}\widehat{\boldsymbol{\varepsilon}}|\mathscr{N}_{\mathrm{r}}) = \left[\left(\mathbf{0}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right]\widehat{\mathbf{y}}, \tag{2.4}$$

$$\mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}) = \left[\left(\mathbf{0}, (\mathbf{B}_1, \mathbf{0}, \mathbf{0})\,\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right]\widehat{\mathbf{y}}, \tag{2.5}$$

$$\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}}) = \left[\left(\mathbf{0}, (\mathbf{0}, \mathbf{B}_2, \mathbf{0})\,\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right]\widehat{\mathbf{y}}, \tag{2.6}$$

$$\mathrm{BLUP}(\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}}) = \left[\left(\mathbf{0}, (\mathbf{0}, \mathbf{0}, \mathbf{B}_3)\,\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger} + \mathbf{U}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\perp}\right]\widehat{\mathbf{y}}, \tag{2.7}$$

where $\mathbf{U}$ is an arbitrary matrix. Moreover,

(a) $\mathscr{R}(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = \mathscr{R}(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}')$ and $r(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = r(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}') = r(\widehat{\mathbf{X}}) + r(\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp})$.

(b) $\mathrm{D}\left(\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})\right) = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\boldsymbol{\Omega}^{\dagger}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\left(\left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\boldsymbol{\Omega}^{\dagger}\right)'$ with $\boldsymbol{\Omega} = \left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)$.

(c) $\mathrm{BLUP}(\mathbf{H}\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}}) = \mathbf{H}\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})$ holds for any $\mathbf{H} \in \mathbb{R}^{p\times k}$.

(d) $\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})$ is unique with probability 1 if and only if $\widehat{\mathbf{y}} \in \mathscr{R}(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}')$ with probability 1. In the case, the model $\mathscr{N}_r$ is said to be consistent (see [19]).

(e) $\mathbf{C}$ in (2.1) is unique if and only if $r(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}') = n + m$. Under the circumstance, one says that $\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})$ is definitely unique.

(f) The BLUP of $\boldsymbol{\xi}$ under $\mathscr{N}_r$ can be decomposed as the following sum

$$\begin{aligned}\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}}) &= \mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\mathbf{B}\widehat{\boldsymbol{\varepsilon}}|\mathscr{N}_{\mathrm{r}})\\ &= \mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\\ &= \mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathbf{B}_1\mathrm{BLUP}(\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}) + \mathbf{B}_2\mathrm{BLUP}(\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}}) + \mathbf{B}_3\mathrm{BLUP}(\mathbf{e}|\mathscr{N}_{\mathrm{r}}).\end{aligned}$$

Lemma 2.1. *Assume that $\boldsymbol{\xi}$ in (1.7) is predictable under $\mathscr{N}_r$. Then*

$$\mathrm{D}\left(\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})\right) = \mathrm{D}\left(\mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}\widehat{\boldsymbol{\varepsilon}}|\mathscr{N}_{\mathrm{r}})\right). \tag{2.8}$$

Proof. From (2.3) and (2.4), we can write

$$\begin{aligned}&\mathrm{Cov}\left\{\mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}), \mathrm{BLUP}(\mathbf{B}\widehat{\boldsymbol{\varepsilon}}|\mathscr{N}_{\mathrm{r}})\right\}\\ &= (\mathbf{K}, \mathbf{0})\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\left[\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'\right]^{\dagger}\left(\mathbf{0}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'.\end{aligned} \tag{2.9}$$

Let us apply (1.12) to (2.9) and use $r(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = r(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}')$. This gives

$$\begin{aligned}&r\left\{(\mathbf{K}, \mathbf{0})\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\left[\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'\right]^{\dagger}\left(\mathbf{0}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'\right\}\\
&= r\begin{pmatrix}\mathbf{0} & \left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)' & \left(\mathbf{0}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'\\ \left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0}\\ (\mathbf{K}, \mathbf{0}) & \mathbf{0} & \mathbf{0}\end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\\
&= r\begin{pmatrix}\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\mathbf{B}'\\ \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0}\\ \mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right)\\
&= r\begin{pmatrix}\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0}\\ \mathbf{0} & -\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\mathbf{B}'\\ \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0}\\ \mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right)\\
&= r\begin{pmatrix}\mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0}\\ \mathbf{0} & -\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\mathbf{B}'\\ \widehat{\mathbf{X}} & \mathbf{0} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0}\\ \mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right)\end{aligned}$$

$$= r\begin{pmatrix} \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' \\ \mathbf{0} & -\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} \\ \widehat{\mathbf{X}} & \mathbf{0} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' \\ \mathbf{K} & \mathbf{0} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right)$$

$$= r\begin{pmatrix} \mathbf{0} & \widehat{\mathbf{X}}' \\ \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' \\ \mathbf{K} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right). \tag{2.10}$$

By virtue of (1.11), we conclude that (2.10) is

$$r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} \\ \mathbf{K} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right). \tag{2.11}$$

Note that

$$\mathscr{R}(\widehat{\mathbf{X}}) \cap \mathscr{R}(\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = \{\mathbf{0}\}.$$

Thereby, (2.11) is equal to zero. Now, the desired identity (2.8) follows. □

Lemma 2.2. *Assume that $\boldsymbol{\xi}$ in (1.7) is predictable under $\mathscr{N}_r$. Then,*

(a) The dispersion matrix equality

$$\mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\right) = \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon} + \mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right)$$

holds if and only if

$$r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0}, \mathbf{B}_2, \mathbf{B}_3\right)' \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}_1, \mathbf{0}, \mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0} \end{pmatrix} = r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right).$$

(b) The dispersion matrix equality

$$\mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon} + \mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right) = \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right)$$

holds if and only if

$$r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0}, \mathbf{0}, \mathbf{B}_3\right)' \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{0}, \mathbf{B}_2, \mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0} \end{pmatrix} = r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right).$$

(c) The dispersion matrix equality

$$\begin{aligned}\mathrm{D}\left(\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})\right) =& \mathrm{D}\left(\mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}})\right) \\ &+ \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right),\end{aligned}$$

holds if and only if

$$r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0}, \mathbf{0}, \mathbf{B}_3\right)' \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{0}, \mathbf{B}_2, \mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0} \end{pmatrix} = r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right),$$

and

$$r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0} \end{pmatrix} = r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right).$$

Proof. Notice that

$$\begin{aligned} &\mathrm{Cov}\left\{\mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}),\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}+\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right\} \\ &=\left(\mathbf{0},\left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)^{\dagger}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\left[\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'\right]^{\dagger}\left(\mathbf{0},\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)'. \end{aligned} \tag{2.12}$$

Applying (1.12) to (2.12) and utilizing $r(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp})=r(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}')$ provide

$$\begin{aligned} &r\left(\mathrm{Cov}\left\{\mathrm{BLUP}(\mathbf{B}_1\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}),\mathrm{BLUP}(\mathbf{B}_2\boldsymbol{\varepsilon}+\mathbf{B}_3\mathbf{e}|\mathscr{N}_{\mathrm{r}})\right\}\right) \\ &= r\begin{pmatrix} \mathbf{0} & \left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)' & \left(\mathbf{0},\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)' \\ \left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} \\ \left(\mathbf{0},\left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \mathbf{0} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) \\ &= r\begin{pmatrix} \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \\ &= r\begin{pmatrix} \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \\ &= r\begin{pmatrix} \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \widehat{\mathbf{X}} & \mathbf{0} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} & \mathbf{0} \end{pmatrix} - 2r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right), \end{aligned}$$

which, by (1.19), (1.10) and (1.11), can be reduced to

$$\begin{aligned} &r\begin{pmatrix} \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{X}}^{\perp}\widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{0} \end{pmatrix} - r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right) \\ &= r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{B}_2,\mathbf{B}_3\right)' \\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}_1,\mathbf{0},\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0} \end{pmatrix} - r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) - r\left(\widehat{\mathbf{X}}\right), \end{aligned}$$

which indicates (a). Similar to proof of (a), we can derive (b). (a) together with (b) and (2.8) results in (c). □

Corresponding to different choices of **K** and **B** in (1.7), we have the following results from the previous lemmas:

Corollary 2.1. *Consider the model $\mathscr{N}_r$. The following three assertions hold.*

(a) The following decomposition holds on BLUPs

$$\begin{aligned}\widehat{\mathbf{y}} &= \mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\widehat{\mathbf{Z}}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\\ &= \mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\mathbf{Z}_0\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\widehat{\mathbf{I}}_{\mathrm{n}}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}}) + \mathrm{BLUP}(\widehat{\mathbf{I}}_{\mathrm{m}}\mathbf{e}|\mathscr{N}_{\mathrm{r}})\\ &= \mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}}) + \mathbf{Z}_0\mathrm{BLUP}(\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}}) + \widehat{\mathbf{I}}_{\mathrm{n}}\mathrm{BLUP}(\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}}) + \widehat{\mathbf{I}}_{\mathrm{m}}\mathrm{BLUP}(\mathbf{e}|\mathscr{N}_{\mathrm{r}}).\end{aligned}$$

(b) $\widehat{\mathbf{y}}$, $\mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}})$ and $\mathrm{BLUP}(\widehat{\mathbf{Z}}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})$ satisfy

$$\mathrm{D}(\widehat{\mathbf{y}}) = \mathrm{D}\left(\mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\widehat{\mathbf{Z}}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\right).$$

(c) The statement

$$\begin{aligned}\mathrm{D}(\widehat{\mathbf{y}}) =&\mathrm{D}\left(\mathrm{BLUE}(\widehat{\mathbf{X}}\boldsymbol{\beta}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\mathbf{Z}_0\boldsymbol{\gamma}|\mathscr{N}_{\mathrm{r}})\right)\\ &+ \mathrm{D}\left(\mathrm{BLUP}(\widehat{\mathbf{I}}_{\mathrm{n}}\boldsymbol{\varepsilon}|\mathscr{N}_{\mathrm{r}})\right) + \mathrm{D}\left(\mathrm{BLUP}(\widehat{\mathbf{I}}_{\mathrm{m}}\mathbf{e}|\mathscr{N}_{\mathrm{r}})|\mathscr{N}_{\mathrm{r}})\right)\end{aligned}$$

holds if and only if

$$r\begin{pmatrix}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\mathbf{0},\widehat{\mathbf{I}}_m\right)'\\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0}\\ \left(\mathbf{0},\widehat{\mathbf{I}}_n,\mathbf{0}\right)\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0}\end{pmatrix} = r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right)$$

and

$$r\begin{pmatrix}\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\left(\mathbf{0},\widehat{\mathbf{I}}_n,\widehat{\mathbf{I}}_m\right)'\\ \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0}\\ (\mathbf{Z}_0,\mathbf{0},\mathbf{0})\,\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{0} & \mathbf{0}\end{pmatrix} = r\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right).$$

Equation (2.2) indicates that the BLUP for $\boldsymbol{\xi}$ under $\mathscr{N}_r$ can be represented by an exact algebraic expression involving some matrices and their Moore-Penrose generalized inverses. One significant superiority of the exact algebraic expression is the accurate analysis of the relationships of relevant statistics, as stated in the preceding part. All the results in the section give a unified theory regarding BLUPs for functions of all unknown parameters, $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\varepsilon}, \mathbf{e}$, and their essential properties under $\mathscr{N}_r$, and can be approached as standard references in the statistical inference of BLUPs. Similar to (2.2), we give an incorrect form of the BLUP of $\boldsymbol{\xi}_0$ in (1.8).

Corollary 2.2. *Let $\boldsymbol{\xi}_0$ in (1.8) be predictable under $\mathscr{N}_{r_0}$, i.e., $\mathscr{R}(\mathbf{K}_0') \subseteq \mathscr{R}(\widehat{\mathbf{X}}_0')$. Then*

$$\mathbf{C}_0\widehat{\mathbf{y}} = \mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0}) \Longleftrightarrow \mathbf{C}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right) = \left(\mathbf{K}_0, \mathbf{B}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right). \tag{2.13}$$

The general solution of Eq (2.13) is

$$\mathbf{C}_0 = \left(\mathbf{K}_0, \mathbf{B}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger} + \mathbf{U}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}, \tag{2.14}$$

where $\mathbf{U}_0$ is an arbitrary matrix. Hence, the BLUP of $\boldsymbol{\xi}_0$ under $\mathscr{N}_{r_0}$ can be written as

$$\mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0}) = \left(\left(\mathbf{K}_0, \mathbf{B}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger} + \mathbf{U}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\right)\widehat{\mathbf{y}}, \tag{2.15}$$

where $\mathbf{U}_0$ is an arbitrary matrix. From expression (2.15), $\mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0})$ *is unique if and only if $\widehat{\mathbf{y}} \in \mathscr{R}(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}')$. Additionally, under the assumption in (1.5),*

$$\mathrm{E}\left[\mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0})\right] = \mathbf{C}_0\widehat{\mathbf{X}}\boldsymbol{\beta} \text{ and } \mathrm{D}\left[\mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0})\right] = \mathbf{C}_0\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\mathbf{C}_0'.$$

3. BLUPs under ξ and ξ_0

In this section, we mainly solve the three questions stated in section one. Because BLUP($\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}}$) in (2.2) and BLUP($\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0}$) in (2.15) are not always unique, we utilize {BLUP($\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}}$)} and {BLUP($\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0}$)} to signify the corresponding sets, respectively. To establish the inclusion relations between the preceding two sets, the following Lemma is essential.

Lemma 3.1. *Assume that $\widehat{\mathbf{y}}$ is given in* (1.5) *and* $\mathbf{C}_j$, $j = 1, 2$, *is a matrix of appropriate size. Then* $\mathbf{C}_1\widehat{\mathbf{y}} = \mathbf{C}_2\widehat{\mathbf{y}}$ *holds with probability 1 if and only if*

$$(\mathbf{C}_1 - \mathbf{C}_2)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \mathbf{0}. \tag{3.1}$$

Furthermore, let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two sets comprised by the matrices of appropriate size. Then,

(a) For a specified $\mathbf{C}_1 \in \mathbb{C}_1$, $\mathbf{C}_1\widehat{\mathbf{y}} \in \{\mathbf{C}_2\widehat{\mathbf{y}}\}$, $\mathbf{C}_2 \in \mathbb{C}_2$, holds with probability 1 if and only if

$$\min_{\mathbf{C}_2\in\mathbb{C}_2} r\left((\mathbf{C}_1 - \mathbf{C}_2)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right) = 0. \tag{3.2}$$

(b) $\{\mathbf{C}_1\widehat{\mathbf{y}}\} \cap \{\mathbf{C}_2\widehat{\mathbf{y}}\} \neq \emptyset$, $\mathbf{C}_1 \in \mathbb{C}_1$, $\mathbf{C}_2 \in \mathbb{C}_2$, holds with probability 1 if and only if

$$\min_{\mathbf{C}_1\in\mathbb{C}_1,\mathbf{C}_2\in\mathbb{C}_2} r\left((\mathbf{C}_1 - \mathbf{C}_2)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right) = 0. \tag{3.3}$$

(c) $\{\mathbf{C}_1\widehat{\mathbf{y}}\} \subseteq \{\mathbf{C}_2\widehat{\mathbf{y}}\}$, $\mathbf{C}_1 \in \mathbb{C}_1$, $\mathbf{C}_2 \in \mathbb{C}_2$, holds with probability 1 if and only if

$$\max_{\mathbf{C}_1\in\mathbb{C}_1} \min_{\mathbf{C}_2\in\mathbb{C}_2} r\left((\mathbf{C}_1 - \mathbf{C}_2)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right) = 0. \tag{3.4}$$

Proof. Observe that obviously $\mathbf{C}_1\widehat{\mathbf{y}} = \mathbf{C}_2\widehat{\mathbf{y}}$ holds with probability 1 if and only if

$$(\mathbf{C}_1 - \mathbf{C}_2)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) = \mathbf{0}. \tag{3.5}$$

Also notice that

$$\mathscr{R}(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}) = \mathscr{R}(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}').$$

Therefore, the equivalence in (3.1) is established. (3.1) together with Lemma 1.3 yields (a)-(c). □

Theorem 3.1. *Consider $\mathscr{N}_{\mathrm{r}}$ and $\mathscr{N}_{\mathrm{r}_0}$ and define*

$$\mathbf{M} = \begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix} \ \textit{and}\ \mathbf{N} = \left(\mathbf{K}, \mathbf{K}_0, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}', \mathbf{B}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\right). \tag{3.6}$$

Then the following conclusions hold.

(a) For a specified BLUP($\xi_0|\mathscr{N}_{\mathrm{r}_0}$) in (2.15), BLUP($\xi_0|\mathscr{N}_{\mathrm{r}_0}$) $\in$ {BLUP($\xi|\mathscr{N}_{\mathrm{r}}$)} holds with probability 1 if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathbf{U}_0 = \mathbf{G}\mathbf{H}^{\dagger} + \mathbf{F}\mathbf{H}^{\perp}, \tag{3.7}$$

where $\mathbf{F}$ is a fixed matrix corresponding to $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})$,

$$\mathbf{H} = \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$
$$\mathbf{G} = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) - \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right).$$

(b) $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\} \cap \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\} \neq \emptyset$ holds with probability 1 if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}'). \tag{3.8}$$

(c) $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\} \subseteq \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\}$ holds with probability 1 if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathscr{R}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right). \tag{3.9}$$

Proof. With the notation

$$\mathbf{G} = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) - \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$

we note from (2.1) and (2.14) that

$$(\mathbf{C} - \mathbf{C}_0)\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \mathbf{G} - \mathbf{U}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right). \tag{3.10}$$

Set

$$\mathbf{G} - \mathbf{U}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \mathbf{0}, \tag{3.11}$$

which impled that the Eq (3.11) is solvable for $\mathbf{U}_0$ and

$$\mathbf{U}_0 = \mathbf{G}\mathbf{H}^{\dagger} + \mathbf{F}\mathbf{H}^{\perp}, \tag{3.12}$$

with $\mathbf{H} = \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)$ and $\mathbf{F}$ being an any matrix, which means

$$r\begin{pmatrix} \mathbf{G} \\ \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\end{pmatrix} = r\left(\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right). \tag{3.13}$$

Utilizing (1.10) and simplifying, the difference between both sides of the Eq (3.13) is

$$\begin{aligned}
& r\begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\end{pmatrix} - r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right) \\
&= r\begin{pmatrix} \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right) \\ \left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) & \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\end{pmatrix} - r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right) \\
&= r\begin{pmatrix} \mathbf{K} & \mathbf{K}_0 & \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix} - r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix}.
\end{aligned} \tag{3.14}$$

Thus, (3.13) is equivalent to

$$r\begin{pmatrix}\mathbf{M}\\ \mathbf{N}\end{pmatrix}=r(\mathbf{M}), \tag{3.15}$$

i.e.,

$$\mathscr{R}(\mathbf{N}')\subseteq\mathscr{R}(\mathbf{M}'). \tag{3.16}$$

With the help of (a) in Lemma 3.1, we arrive at (a). It follows from (3.10) that

$$\min_{\mathbf{C},\mathbf{C}_0} r\left((\mathbf{C}-\mathbf{C}_0)\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right)=\min_{\mathbf{U}_0} r\left(\mathbf{G}-\mathbf{U}_0\left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right). \tag{3.17}$$

The application of (1.13) to (3.17) gives

$$\begin{aligned}&\min_{\mathbf{U}_0} r\left(\mathbf{G}-\mathbf{U}_0\left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right)\\ =&r\begin{pmatrix}\mathbf{G}\\ \left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\end{pmatrix}-r\left(\left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right).\end{aligned} \tag{3.18}$$

By (3.13) and (3.14), clearly, (3.18) equals to

$$r\begin{pmatrix}\mathbf{M}\\ \mathbf{N}\end{pmatrix}-r(\mathbf{M}), \tag{3.19}$$

implying (b) from (b) in Lemma 3.1. Again using (3.10) and then applying (1.14), we have

$$\begin{aligned}&\max_{\mathbf{C}_0}\min_{\mathbf{C}} r\left((\mathbf{C}-\mathbf{C}_0)\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right)\\ =&\max_{\mathbf{U}_0} r\left(\mathbf{G}-\mathbf{U}_0\left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\right)\\ =&\min\left\{r\begin{pmatrix}\mathbf{G}\\ \left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\end{pmatrix},k\right\}.\end{aligned} \tag{3.20}$$

Analogous to (3.14), we obtain

$$\begin{aligned}&r\begin{pmatrix}\mathbf{G}\\ \left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right)\end{pmatrix}\\ =&r\begin{pmatrix}\mathbf{K}&\mathbf{K}_0&\mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'&\mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\\ \widehat{\mathbf{X}}&\widehat{\mathbf{X}}_0&\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'&\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\\ \mathbf{0}&\mathbf{0}&\widehat{\mathbf{X}}'&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\widehat{\mathbf{X}}_0'\end{pmatrix}-r\left(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right)-r\left(\widehat{\mathbf{X}}_0\right)-r\left(\widehat{\mathbf{X}}\right).\end{aligned} \tag{3.21}$$

In light of

$$\mathscr{R}(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp})=\mathscr{R}(\widehat{\mathbf{X}},\widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}')\ \text{ and }\ \mathscr{R}(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp})=\mathscr{R}(\widehat{\mathbf{X}}_0,\widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'),$$

it is readily seen that

$$\begin{aligned}
& r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix} \\
&= r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' & \mathbf{0} \end{pmatrix} \\
&= r\begin{pmatrix} \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' & \mathbf{0} \end{pmatrix} \\
&= r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}_0\right) + r\left(\widehat{\mathbf{X}}\right).
\end{aligned} \tag{3.22}$$

Combining (3.21) with (3.22) leads to

$$\begin{aligned}
& r\begin{pmatrix} \mathbf{G} \\ \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) \end{pmatrix} \\
&= r\begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} - r(\mathbf{M}) + r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) - r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right).
\end{aligned} \tag{3.23}$$

In view of (c) in Lemma 3.1, substituting (3.23) into (3.20) shows that $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{\mathrm{r}_0})\} \subseteq \{\mathrm{BLUP}(\xi|\mathscr{N}_{\mathrm{r}})\}$ holds with probability 1 if and only if

$$r\begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} + r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) = r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) + r(\mathbf{M}). \tag{3.24}$$

Also observe that

$$r\begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} \geq r(\mathbf{M}) \ \text{ and } \ r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) \geq r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right), \tag{3.25}$$

so that (3.24) is equivalent to

$$r\begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} = r(\mathbf{M}) \ \text{ and } \ r\left(\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) = r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right), \tag{3.26}$$

i.e.,

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathscr{R}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right). \tag{3.27}$$

This completes the proof. □

Equations (3.7)–(3.9) establish a number of vital links between two sets composed by BLUPs under $\mathscr{N}_{r_0}$ and $\mathscr{N}_r$, which are utilized to uncover various new behaviors of BLUPs under different assumptions. Due to no restrictions on the matrices $\mathbf{K}, \mathbf{K}_0, \mathbf{B}, \mathbf{V}, \mathbf{\Lambda}, \mathbf{X}, \mathbf{A}, \mathbf{A}_0, \mathbf{Z}$ in (3.6), the results in Theorem 3.1 can be further simplify for special choices of these matrices. For these two collections, $\{\mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_{\mathrm{r}})\} = \{\mathbf{C}\widehat{\mathbf{y}}\}$ and $\{\mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{\mathrm{r}_0})\} = \{\mathbf{C}_0\widehat{\mathbf{y}}\}$, people also make use of the subsequent criteria describing inclusion relationships of two collections apart from Lemma 3.1.

Definition 3.1. *Suppose that* $\widehat{\mathbf{y}}$ *is given in* (1.5)*, and* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *are two sets composed by the matrices of appropriate size. Then*

(a) For a specified $\mathbf{C}_1 \in \mathbb{C}_1$, the statement $\mathbf{C}_1\widehat{\mathbf{y}} \in \{\mathbf{C}_2\widehat{\mathbf{y}}\}$, $\mathbf{C}_2 \in \mathbb{C}_2$ is defined to hold definitely if $\mathbf{C}_1 \in \mathbb{C}_2$ holds.

(b) $\{\mathbf{C}_1\widehat{\mathbf{y}}\} \cap \{\mathbf{C}_2\widehat{\mathbf{y}}\} \neq \emptyset$, $\mathbf{C}_1 \in \mathbb{C}_1$, $\mathbf{C}_2 \in \mathbb{C}_2$, is defined to hold definitely if $\mathbb{C}_1 \cap \mathbb{C}_2 \neq \emptyset$.

(c) $\{\mathbf{C}_1\widehat{\mathbf{y}}\} \subseteq \{\mathbf{C}_2\widehat{\mathbf{y}}\}$, $\mathbf{C}_1 \in \mathbb{C}_1$, $\mathbf{C}_2 \in \mathbb{C}_2$, is defined to hold definitely if $\mathbb{C}_1 \subseteq \mathbb{C}_2$.

According to Definition 3.1, we intend to solve the three problems in Section 1.

Theorem 3.2. *Consider* $\mathscr{N}_r$ *and* $\mathscr{N}_{r_0}$ *and define*

$$\mathbf{M} = \begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix} \text{ and } \mathbf{N} = \left(\mathbf{K}, \mathbf{K}_0, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}', \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right). \tag{3.28}$$

Then, the following three statements hold.

(a) For a specified $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})$ in (2.15), $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0}) \in \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\}$ holds definitely if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathbf{U}_0 = \mathbf{G}\mathbf{H}^{\dagger} + \mathbf{F}\mathbf{H}^{\perp},$$

where $\mathbf{F}$ is a fixed matrix corresponding to $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})$,

$$\mathbf{H} = \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$

$$\mathbf{G} = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) - \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right).$$

(b) $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\} \cap \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\} \neq \emptyset$ holds definitely if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}').$$

(c) $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\} \subseteq \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\}$ holds definitely if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathscr{R}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right).$$

Proof. According to (a) in Definition 3.1, from (1.9) and (2.14) we find that

$$\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0}) \in \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\},$$

holds definitely if and only if

$$\mathbf{C}_0\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right), \tag{3.29}$$

i.e.,

$$\mathbf{U}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \mathbf{G}, \tag{3.30}$$

with

$$\mathbf{G} = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) - \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right).$$

In terms of (3.11)–(3.16), (3.30) holds if and only if (3.12) and (3.16) hold, implying (a). Trivially,

$$\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\} \cap \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\} \neq \emptyset,$$

holds definitely if and only if there exists a $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})$, such that

$$\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0}) \in \{\mathrm{BLUP}(\xi|\mathscr{N}_r)\},$$

holds definitely, which is in turn equivalent to it, so that (3.30) is solvable for $\mathbf{U}_0$ by proof of (a). Again, making use of (3.11)–(3.16), we derive that (3.30) is solvable for $\mathbf{U}_0$ if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}').$$

As for (c), notice these two expressions

$$\mathbf{C}\widehat{\mathbf{y}} = \mathrm{BLUP}(\boldsymbol{\xi}|\mathscr{N}_r) \text{ and } \mathbf{C}_0\widehat{\mathbf{y}} = \mathrm{BLUP}(\boldsymbol{\xi}_0|\mathscr{N}_{r_0}), \tag{3.31}$$

where $\mathbf{C}$ and $\mathbf{C}_0$ respectively satisfy

$$\mathbf{C}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) = \left(\mathbf{K}, \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) \text{ and } \mathbf{C}_0\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right) = \left(\mathbf{K}_0, \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right). \tag{3.32}$$

Utilizing Lemma 1.4, any solution of the second equation in (3.32) is a solution of the first equation in (3.32) if and only if

$$r\begin{pmatrix} \widehat{\mathbf{X}} & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp} \\ \mathbf{K} & \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp} & \mathbf{K}_0 & \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp} \end{pmatrix} = r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right), \tag{3.33}$$

which by (1.11) becomes

$$r\begin{pmatrix} \mathbf{K} & \mathbf{K}_0 & \mathbf{B}\mathbf{V}\widehat{\mathbf{Z}}' & \mathbf{B}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \widehat{\mathbf{X}} & \widehat{\mathbf{X}}_0 & \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}' & \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}' \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_0' \end{pmatrix} = r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right) + r\left(\widehat{\mathbf{X}}\right) + r\left(\widehat{\mathbf{X}}_0\right). \tag{3.34}$$

From (3.21)–(3.27), the identity (3.34) holds if and only if

$$\mathscr{R}(\mathbf{N}') \subseteq \mathscr{R}(\mathbf{M}') \text{ and } \mathscr{R}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right).$$

This completes the proof. □

It is amazing that the results of statistical inference are the same in Theorems 3.1 and 3.2 even though the criteria introduced in Lemma 3.1 and Definiton 3.1 are different. In some usual assumptions, the former conclusions can be further simplifed, for instance, if $\mathbf{K} = \mathbf{K}_0 = \mathbf{0}$, $\mathbf{B}_1 = \mathbf{I}_q$, $\mathbf{B}_2 = \mathbf{0}$, and $\mathbf{B}_3 = \mathbf{0}$, then we have the following corollaries.

Corollary 3.1. *Use the above notation and define*

$$\mathbf{N}_1 = \left(\mathbf{0}, \mathbf{0}, \widehat{\mathbf{I}}_q\mathbf{V}\widehat{\mathbf{Z}}', \widehat{\mathbf{I}}_q\mathbf{\Lambda}\widehat{\mathbf{Z}}'\right),$$

where $\widehat{\mathbf{I}}_q = \left(\mathbf{I}_q, \mathbf{0}, \mathbf{0}\right)$. *Then*

(a) The following results are equivalent:

(i) For a specified BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$), BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$) $\in$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} holds with probability 1.

(ii) For a specified BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$), BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$) $\in$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} holds definitely.

(iii) $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}')$ and $\mathbf{U}_0 = \mathbf{G}\mathbf{H}^{\dagger} + \mathbf{F}\mathbf{H}^{\perp}$, where $\mathbf{F}$ is a fixed matrix corresponding to BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$),

$$\mathbf{H} = \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$

$$\mathbf{G} = \left(\mathbf{K}, \widehat{\mathbf{I}}_q\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right) - \left(\mathbf{K}_0, \widehat{\mathbf{I}}_q\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right).$$

(b) The following results are equivalent:

(i) {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$)} $\cap$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} $\neq \emptyset$ holds with probability 1.

(ii) {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$)} $\cap$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} $\neq \emptyset$ holds definitely.

(iii) $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}')$.

(c) The following results are equivalent:

(i) {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$)} $\subseteq$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} holds with probability 1.

(ii) {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r_0}$)} $\subseteq$ {BLUP($\boldsymbol{\gamma}|\mathscr{N}_{r}$)} holds definitely.

(iii) $\mathscr{R}(\mathbf{N}_1') \subseteq \mathscr{R}(\mathbf{M}')$ and $\mathscr{R}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\right) \subseteq \mathscr{R}\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\right)$.

Besides, it is interesting to consider the situation where the identity $\mathbf{V}_3 = \boldsymbol{\Lambda}_3 = \mathbf{0}$ is assumed, i.e., $\mathscr{M}$ and $\mathscr{M}_0$ become the exact restrictions $\mathbf{r} = \mathbf{A}\boldsymbol{\beta}$ and $\mathbf{r} = \mathbf{A}_0\boldsymbol{\beta}$, respectively. In this situation, the comparison problems of estimators were discussed by [13] under the assumption $\mathbf{Z} = \mathbf{0}$ in (1.1) associated with the dispersion matrix criterion, and extended by [14].

Corollary 3.2. *Consider the set-up presented above and suppose that* $\mathbf{V}_3 = \boldsymbol{\Lambda}_3 = \mathbf{0}$, $\mathscr{R}(\mathbf{X}') \cap \mathscr{R}(\mathbf{A}') = \{\mathbf{0}\}$ *and* $\mathscr{R}(\mathbf{X}') \cap \mathscr{R}(\mathbf{A}_0') = \{\mathbf{0}\}$. *Then,*

(a) The following results are equivalent:

(i) For a specified BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$), BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$) $\in$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} holds with probability 1.

(ii) For a specified BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$), BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$) $\in$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} holds definitely.

(iii) $\mathbf{U}_0 = \mathbf{G}\mathbf{H}^{\dagger} + \mathbf{F}\mathbf{H}^{\perp}$, where $\mathbf{F}$ is a fixed matrix corresponding to BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$),

$$\mathbf{H} = \left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\perp}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right),$$

$$\mathbf{G} = (\mathbf{X}, \mathbf{0}) - (\mathbf{X}, \mathbf{0})\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}_0^{\perp}\right)^{\dagger}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Z}}\mathbf{V}\widehat{\mathbf{Z}}'\widehat{\mathbf{X}}^{\perp}\right).$$

(b) {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$)} $\cap$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} $\neq \emptyset$ holds with probability 1.

(c) {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$)} $\cap$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} $\neq \emptyset$ holds definitely.

(d) The following results are equivalent:

(i) {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$)} $\subseteq$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} holds with probability 1.

(ii) {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r_0}$)} $\subseteq$ {BLUE($\mathbf{X}\boldsymbol{\beta}|\mathscr{N}_{r}$)} holds definitely.

(iii) $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{A}_0)$.

From the derivation of the primary conclusions, it can be seen that matrix inertia and rank methodology plays a crucial role in simplifying the complex matrix expressions, especially in removing the Moore-Penrose generalized inverses. As is known, when making statistical inferences in the framework of a linear model, we would encounter complex calculations of matrices and their Moore-Penrose generalized inverses. It has been challenging work, but maybe now one can manipulate them with the development of the matrix theory in recent decades (see [1, 3, 8–11, 14, 20–22]).

4. Concluding remarks

We provide deep insights into the connections between BLUPs under $\mathscr{N}_r$ and $\mathscr{N}_{r_0}$, which is a subject of linear regression model. These kinds of connections help evaluate the performance of BLUPs under $\mathscr{N}_{r_0}$, or, more precisely, the necessary and sufficient conditions appearing in Section 3 give the judgement of effectiveness of BLUPs under $\mathscr{N}_{r_0}$; for example, if the conditions of Corollary 3.2 and $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{A}_0)$ hold, then all BLUEs for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{N}_{r_0}$ remain BLUEs for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{N}_r$, as stated in Corollary 3.2.

It should be emphasized that the core findings in this article can be extensively applied to specific statistical theory and practice and present a comprehensive picture of BLUPs under $\mathscr{N}_{r_0}$ by reason of the generality of conclusions. Alternatively, when $\boldsymbol{\Lambda}$ is positive definite, or rather

$$r\left(\widehat{\mathbf{X}}_0, \widehat{\mathbf{Z}}\boldsymbol{\Lambda}\widehat{\mathbf{Z}}'\right) = n + m,$$

the BLUP of $\boldsymbol{\xi}_0$ under $\mathscr{N}_{r_0}$ has a unique expression. At this point, the three questions posed in section one unite into one.

To explain the previous consequences, we present a real data example of model (1.1) utilized by [23], and then by [24, 25]. The example comes from a study about the first lactation yields of dairy cows with sire additive genetic merits and herd effects. The sire additive genetic merits are regarded as random effects denoted by $\gamma_i, i = 1, 2, 3, 4$, which correspond to sires A_1, A_2, A_3, and A_4, and herd effects are regarded as fixed effects denoted by β_j, $j = 1, 2, 3$, where β_j is the environmental influence of the jth herd on the yields. Moreover, y_{ji} is taken to be the yield of the dairy cow with the ith sire and jth herd. Assume that the corresponding data is recorded in Table 1.

Table 1. The data of first lactation yields.

Herd	1	1	2	2	2	3	3	3	3
Sire	A_1	A_4	A_2	A_4	A_4	A_3	A_3	A_4	A_4
Yield	110	100	110	100	100	110	110	100	100

Now, we can give the mixed linear model

$$\mathscr{N} : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \tag{4.1}$$

where

$$\mathbf{X}=\begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\\0&0&1\\0&0&1\end{pmatrix},\mathbf{Z}=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\\0&0&1&0\\0&0&0&1\\0&0&0&1\end{pmatrix},\mathbf{y}=\begin{pmatrix}y_{11}\\y_{14}\\y_{22}\\y_{24}\\y_{24}\\y_{33}\\y_{33}\\y_{34}\\y_{34}\end{pmatrix}=\begin{pmatrix}110\\100\\110\\100\\100\\110\\110\\100\\100\end{pmatrix},\boldsymbol{\beta}=\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix},\boldsymbol{\gamma}=\begin{pmatrix}\gamma_1\\\gamma_2\\\gamma_3\\\gamma_4\end{pmatrix}. \tag{4.2}$$

Set

$$\mathbf{V}_1=\begin{pmatrix}0.1\mathbf{I}_q&\mathbf{0}\\\mathbf{0}&\mathbf{I}_n\end{pmatrix}$$

as in [23]. Two competing stochastic linear restrictions are given by

$$\mathscr{M}_0:\mathbf{r}=\mathbf{A}_0\boldsymbol{\beta}+\mathbf{e}_0\ \text{ and }\ \mathscr{M}:\mathbf{r}=\mathbf{A}\boldsymbol{\beta}+\mathbf{e},$$

where $\mathbf{A}=(1,0,0),\mathbf{A}_0=(0,1,0),\mathbf{V}_2=\boldsymbol{\Lambda}_2=\mathbf{0}$, V_3 and Λ_3 are any two real numbers. The assumption $\mathbf{V}_2=\boldsymbol{\Lambda}_2=\mathbf{0}$ emphasizes the extrinsic character of the stochastic linear restrictions. Also, suppose that $\mathbf{K}=\mathbf{K}_0=(1,0,0)$, $\mathbf{B}_1=(1,0,0,0)$, $\mathbf{B}_2=(1,0,0,\ldots,0)$, and $\mathbf{B}_3=\mathbf{0}$. Moreover, we wish to establish the relationships of $\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})$ and $\mathrm{BLUP}(\xi|\mathscr{N}_r)$. Now, we can easily see that (3.9) holds. According to (c) in Theorem 3.1, the set conclusion $\{\mathrm{BLUP}(\xi_0|\mathscr{N}_{r_0})\}\subseteq\{\mathrm{BLUP}(\xi|\mathscr{N}_r)\}$ holds with probability 1, i.e., although an incorrect stochastic restriction is used, the BLUP remains the correct BLUP.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by Natural Science Foundation of Anhui Provincial Education Department (2024AH050299), the key discipline construction project of Anhui Science and Technology University (XK-XJGY002).

Conflict of interest

The author declares no conflict of interest.

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