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Research article

On projective Ricci curvature of cubic metrics

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Abstract: To study the projective Ricci curvature (**PRic**-curvature) in Finsler geometry is interesting because it reflects the geometric properties that are invariant under the projective transformation. In this paper, we firstly derived an expression of S-curvature for the cubic Finsler metric and proved that this S-curvature vanishes if and only if β is a constant Killing form. Next, we obtain an explicit expression of projective Ricci curvature for the cubic metric. We also proved that for the projective Ricci-flat Finsler space, the 1-form β is closed, and then the Riemannian metric of α is also Ricci-flat. Finally, we show that the cubic Finsler metric is of weak projective Ricci curvature if and only if it is projectively Ricci-flat.

Keywords: Finsler space; cubic metrics; S-curvature; Riemann curvature; Ricci curvature; projective Ricci curvature
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1. Introduction

Finsler geometry extends the classical Riemannian geometry by considering more general metric structures. A very important class of Finsler metrics is known as (α,β)-metrics, which were introduced by M. Matsumoto in 1972. An (α,β)-metric can be expressed as $F = \alpha\phi(s)$, where α is a Riemannian metric and $s = \frac{\beta}{\alpha}$, β is a 1-form. Randers metric, Kropina metric, exponential metric, Matsumoto metric, and cubic metric are important classes of (α,β)-metric [9].

To study the curvature characteristics is a central problem in Finsler geometry. The Ricci curvature and S-curvature are very important non-Riemannian quantities in the Finslerian manifold [2]. The Ricci curvature in Finsler geometry is a natural extension of the Ricci curvature in Riemannian geometry and is defined as the trace of the Riemann curvature [5]. The S-curvature is a mathematical quantity and measures the rate of change of volume form of a Finsler space along the geodesics. Recent studies in differential geometry, such as those on Ricci solitons and conformal structures, have highlighted the importance of Ricci-type curvatures in understanding the geometric flow and structure of manifolds [6–8]. In Finsler geometry, the study of curvature involves understanding the deviation from flatness. The projective Ricci curvature is one aspect of this analysis. The concept of projective Ricci curvature in Finsler geometry is introduced by X. Cheng [1] in 2017. Projective geometry deals with the properties that are invariant under projective transformations. The projective Ricci curvature measures the deviation of the Finsler metric from being projectively flat. Projective Ricci curvature has applications in various areas of mathematics and physics. It plays a crucial role in understanding the geometry of Finsler manifolds and connects to the problems in the calculus of variations, differential equations, and geometric optics.

In 2020, H. Zhu [15] gave an expression of projective Ricci curvature for an (α, β)-metric. Later on, many geometers [4, 12, 13] have studied the geometric properties of projective Ricci curvature. In this article, we obtain the geometric properties and flatness condition of projective Ricci curvature for the cubic Finsler metric, which is defined as $F = \alpha\phi(s)$ with

$$\phi = (1 + s)^3, \tag{1.1}$$

i.e., $F = \frac{(\alpha+\beta)^3}{\alpha^2}$. Cubic metric is Finsler metric for $b^2 < \frac{1}{4}$ [14].

The following notations will be used to state our main result:

$$\begin{aligned} &2s_{jk} = b_{j;k} - b_{k;j}, \quad 2r_{jk} = b_{j;k} + b_{k;j}, \quad s^j_k = a^{jl}s_{kl}, \quad r^j_k = a^{jl}r_{kl}, \quad s_j = b^l s_{lj} = b_k s^k_j, \\ &r_j = b^l r_{lj} = b_k r^k_j, \quad r_{j0} = r_{jk}y^k, \quad r_{00} = r_{jk}y^j y^k, \quad r = r_{jk}b^j b^k = b^j r_j, \quad s_{j0} = s_{jk}y^k, \\ &s_0 = s_j y^j, \quad r_0 = r_j y^j, \quad b^j = a^{jk}b_k, \quad t_{jk} = s_{jm}s^m_k, \quad t_j = b^m t_{mj} = s^i s^i_j, \end{aligned} \tag{1.2}$$

where "; " denotes the covariant derivative with respect to the Levi-Civita connection of the Riemannian metric α.

A 1-form β is said to be a Killing form if $r_{ij} = 0$. The 1-form β is said to be a constant Killing form if it is a Killing form and constant length concerning α, equivalently $r_{ij} = 0$ and $s_i = 0$.

In this paper we will use the following lemma:

Lemma 1.1. *If $\alpha^2 = 0(mod\beta)$, that is, $a_{ij}y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case, we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

We first prove the following result:

Theorem 1.1. *For the cubic Finsler metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ on an n-dimensional $(n > 2)$ Finsler manifold M, the S-curvature vanishes if and only if β is a constant Killing form.*

Next, we obtain the flatness condition for the projective Ricci curvature as

Theorem 1.2. *If the n-dimensional $(n > 2)$ Finsler space with cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projective Ricci-flat $(PRic = 0)$, then β is parallel with respect to the Riemannian metric α.*

In view of the above result, we obtain

Corollary 1.1. *If the n-dimensional $(n > 2)$ Finsler space with cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projective Ricci flat then, it vanishes the S-curvature. Therefore, the Riemannian metric of α is Ricci flat $(Ric^\alpha{=}0)$.*

We also prove the following result:

Theorem 1.3. *The n-dimensional $(n > 2)$ Finsler space with cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is weak **PRic**-curvature if and only if it is a **PRic**-flat metric.*

2. Preliminaries

Let F be an n-dimensional Finsler manifold, and let G^j be the geodesic coefficients of F, which are defined as

$$G^j = \frac{1}{4}g^{jl}\Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \qquad y\epsilon T_xM.$$

The geodesic coefficients of an (α,β)-metric are given as [3]

$$G^j = G^j_\alpha + \alpha Q s^j_0 + (r_{00} - 2\alpha Q s_0)(\Psi b^j + \frac{\Theta y^j}{\alpha}), \tag{2.1}$$

where G^i_α denotes the geodesic coefficients of the Riemannian metric α and

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \psi = \frac{\phi''}{2\phi(\phi - s\phi' + (B - s^2)\phi'')}, \quad \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi(\phi - s\phi' + (B - s^2)\phi'')}. \tag{2.2}$$

For any $x\epsilon M$ and $y\epsilon T_xM\backslash\{0\}$, the Riemann curvature R_y is defined as

$$R_y(v) = R^j_k(y)v^k\frac{\partial}{\partial x^j}, v = v^j\frac{\partial}{\partial x^j},$$

where

$$R^j_k = 2\frac{\partial G^j}{\partial x^k} + 2G^i\frac{\partial^2 G^j}{\partial y^i \partial y^k} - \frac{\partial^2 G^j}{\partial x^i \partial y^k}y^i - \frac{\partial G^j}{\partial y^i}\frac{\partial G^i}{\partial y^k}.$$

The trace of Riemann curvature is called Ricci curvature $Ric = R^m_m$, which is a mathematical object that regulates the rate at which a metric ball's volume in a manifold grows. A Finsler metric F is called an Einstein metric if Ricci curvature satisfies the equation $Ric(x,y) = (n-1)\gamma F^2$, where $\gamma = \gamma(x)$ is a scalar function.

In 1997, Z. Shen [11] discussed S-curvature, which measures the average rate of change of $(T_xM; F_x)$ in the direction $y\epsilon T_xM$ and is defined as

$$S(x,y) = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial(log\sigma_F)}{\partial x^m},$$

where σ_F is defined as

$$\sigma_F = \frac{Vol(B^n)}{Vol\{y^i\epsilon\mathbb{R}^n | F(x,y) < 1\}},$$

and Vol denotes the Euclidean volume, and $B^n(1)$ denotes the unit ball in R^n.

The expression of S-curvature for an (α,β)-metric is given as [10]

$$S = (s_0 + r_0)(2\psi - \Pi) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{2.3}$$

where

$$\Pi = \frac{f'(b)}{bf(b)}, \quad \Delta = 1 + sQ + (B - s^2)Q_s, \quad B = b^2,$$
$$\Phi = -(Q - sQ_s)(n\Delta + 1 + sQ) - (B - s^2)(1 + sQ)Q_{ss}. \tag{2.4}$$

The projective Ricci curvature is first defined by X. Cheng [1] as

$$PRic = Ric + \frac{n-1}{n+1}S_{|m}y^m + \frac{n-1}{(n+1)^2}S^2, \tag{2.5}$$

where "|" denotes the horizontal covariant derivative with respect to the Berwald connections of F. A Finsler space F is called weak projective Ricci curvature if

$$PRic = (n-1)\Big[\frac{3\theta}{F} + \gamma\Big]F^2, \tag{2.6}$$

where $\gamma = \gamma(x)$ is a scalar function and $\theta = \theta_i(x)y^i$ is a 1-form. If γ = constant, then F is called constant projective Ricci curvature. If $\theta = 0$, then F is called isotropic projective Ricci curvature **PRic**$=(n-1)\gamma F^2$.

In 2020, H. Zhu [15] gave an expression of the projective Ricci curvature for the (α,β)-metrics as

$$\begin{aligned}PRic =& Ric^{\alpha} + \frac{1}{n+1}\Big[\frac{r_{00}^2}{\alpha^2}V_1 - \frac{r_{00}s_0}{\alpha}V_2 - \frac{r_{00}r_0}{\alpha}V_3 + \frac{r_{00|0}}{\alpha}V_4 + s_0^2V_5 + r_{00}rV_6 - 4r_0^2V_7 + 2r_0s_0V_8\Big]\\ &+(r_{00}r_i^i + r_{00|b} - b^ir_{i0|0} - r_{0i}r_0^i)V_9 + r_{0i}s_0^iV_{10} + s_{0|0}V_{11} + s_{0i}s_0^iV_{12} + \alpha rs_0V_{13} + \alpha s_js_0^jV_{14}\\ &+\alpha\Big[\frac{4}{n+1}r_is_0^i - 2s_{0|b} - 2r_i^is_0 + 3r_{0i}s^i + b^is_{|0}^i\Big]V_{15} + \alpha s_{0|i}^iV_{16} + \alpha^2s_is^iV_{17} + \alpha^2s_j^is_i^jV_{18}\\ &+ r_{0|0}V_{19} + \frac{2(n-1)}{n+1}\Big[\frac{\Psi_s}{\alpha}(r_{00} - 2\alpha Qs_0)(B-s^2) + 2\Psi(r_0+s_0)\Big]\rho_0 + (n-1)\Big[-2\Psi(r_{00}\\ &- 2\alpha Qs_0)\rho_b - 2\alpha Q\rho_ks_0^k + \rho_0^2 + \rho_{0|0}\Big],\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}V_1 =& 4s\Psi_s + (4\Psi\Psi_{ss} - \Psi_s^2)(B-s^2)^2 - 2(\Psi_{ss} + 6s\Psi\Psi_s)(B-s^2) - \frac{n^2+1}{n+1}\Psi_s^2(B-s^2)^2,\\ V_2 =& 4[2\Psi(\Psi Q_{ss} + 2Q\Psi_{ss} + Q_s\Psi_s) - Q(\Psi_s)^2](B-s^2)^2 + 4[2(Q-sQ_s)(\Psi_s)^2 - (1+10sQ)\Psi\Psi_s\\ &- 2\Psi Q_{ss} - 2Q\Psi_{ss} - Q_s\Psi_s + \Psi_{Bs}](B-s^2) + 2Q_{ss} + 8\Psi_s - 4Q\Psi + 4s\Psi Q_s + 20sQ\Psi_s\\ &+ (n-1)[4((\Psi)^2Q_{ss} - Q(\Psi_s)^2)(B-s^2)^2 + 4((Q-sQ_s)(\Psi)^2) + \Psi(\Psi_s - Q_{ss})(B-s^2)\\ &+ 2s(Q_s\Psi + Q\Psi_s) - 2Q\Psi + Q_{ss}] + \frac{8}{n+1}\Psi_s[\Psi - Q\Psi_s(B-s^2)](B-s^2),\\ V_3 =& 2\Psi_s - 2(3\Psi\Psi_s - \Psi_{Bs})(B-s^2) - (n-1)(1-2\Psi(B-s^2))\Psi_s + \frac{4}{n+1}\Psi\Psi_s(B-s^2),\\ V_4 =& -2\Psi_s(B-s^2),\\ V_5 =& (n-1)\Big\{4[2Q\Psi^2Q_{ss} - \Psi^2(Q_s)^2 - Q^2(\Psi_s)^2](B-s^2)^2 + 4[2Q\Psi(Q\Psi - 2sQ_s\Psi - Q_{ss} + \Psi_s)\\ &+ \Psi(Q_s)^2 + Q_s\Psi_B](B-s^2) - 4Q\Psi(s^2Q\Psi - 3sQ_s + 2Q) + 4sQ(Q\Psi_s + \Psi_B) + 8Q_s\Psi\\ &+ 2QQ_{ss} - (Q_s)^2 + 4\Psi_B\Big\} + 4[4Q\Psi(\Psi Q_{ss} + Q\Psi_{ss} + Q_s\Psi_s) - 2(Q_s)^2\Psi^2 - Q^2(\Psi_s)^2](B-s^2)^2\\ &+ 8[Q\Psi(-4sQ_s\Psi - 4sQ\Psi_s + 2Q\Psi - 2Q_{ss} - \Psi_s) + (Q_s)^2\Psi - Q^2\Psi_{ss} - QQ_s\Psi_s + Q\Psi_{Bs}\\ &+ Q_s\Psi_B](B-s^2) + 24sQ^2\Psi_s - 8Q\Psi(s^2Q\Psi - 3sQ_s + 2Q) + 8sQ\Psi_B + 4\Psi(\Psi + 4Q_s) + 4QQ_{ss}\\ &+ 16Q\Psi_s - 2(Q_s)^2 - \frac{8}{n+1}[\Psi - Q\Psi_s(B-s^2)]^2,\end{aligned}$$

$$
\begin{aligned}
&V_6 = 4[(n+1)\Psi_B + 2(\Psi)^2], \qquad V_7 = \frac{n^2+n+2}{n+1}\Psi^2 + 2\Psi_B,\\
&V_8 = (n-1)\Big[2(2Q_s\Psi^2 + 2Q\Psi\Psi_s + Q_s\Psi_B)(B-s^2) + 2sQ(2\Psi^2+\Psi_B) - 2Q\Psi_s + 2\Psi_B\Big]\\
&\quad + 4[2Q_s\Psi^2 - 3Q\Psi\Psi_s + Q\Psi_{Bs} + Q_s\Psi_B](B-s^2) + 4sQ(2\Psi^2+\Psi_B) + 4\Psi^2 + 4Q\Psi_s\\
&\quad - 4\Psi_B - \frac{8}{n+1}\Psi[\Psi - Q\Psi_s(B-s^2)],\\
&V_9 = 2\Psi, \qquad V_{10} = 2[-2Q_s\Psi(B-s^2) - 2sQ\Psi - \Psi + Q_s + \frac{4}{n+1}Q\Psi_s(B-s^2)],\\
&V_{11} = 2Q_s\Psi(B-s^2) + 2\Psi(1+2Q) - Q_s + \frac{4}{n+1}[Q\Psi_s(B-s^2) - \Psi],\\
&V_{12} = 2Q_s - 2Q(Q-sQ_s), \qquad V_{13} = -8Q[\Psi_B + \frac{2}{n+1}\Psi^2],\\
&V_{14} = \frac{-2}{n+1}Q[(n-3)\Psi + 4Q\Psi_s(B-s^2)], \qquad V_{15} = 2Q\Psi, \qquad V_{16} = 2Q,\\
&V_{17} = -4Q^2\Psi, \qquad V_{18} = -Q^2, \qquad V_{19} = \frac{2(n-1)}{n+1}\Psi, \qquad \rho = \frac{ln\frac{\sigma_\alpha}{\sigma}}{n+1}, \qquad \rho_0 = \rho_{x^i}y^i.
\end{aligned}
\tag{2.8}
$$

3. S-curvature of cubic metrics

For Eq (1.1), we obtain the following values:

$$
\begin{aligned}
&Q = \frac{3}{1-2s}, \quad Q_s = \frac{6}{(1-2s)^2}, \quad Q_{ss} = \frac{24}{(1-2s)^3}, \quad \psi = \frac{3}{1+6B-s-8s^2},\\
&\psi_s = \frac{3+48s}{(1+6B-s-8s^2)^2}, \quad \psi_{ss} = \frac{18(3+16B+8s+64s^2)}{(1+6B-s-8s^2)^3}, \quad \psi_B = \frac{-18}{(1+6B-s-8s^2)^2},\\
&\psi_{Bs} = \frac{36+576s}{(1+6B-s-8s^2)^2}, \quad \Theta = \frac{3(1-4s))}{2(1+6B-s-8s^2)}, \quad \Theta_s = -\frac{9+72B-48s+96s^2}{2(1+6B-s-8s^2)^2},\\
&\Theta_B = \frac{9(-1+4s)}{2(1+6B-s-8s^2)^2}, \quad \Delta = \frac{1+6B-s-8s^2}{(1-2s)^2},\\
&\Phi = \frac{-(3(1-5s-6s^2+B(8+6n+8s-24ns)+n(1-5s-4s^2+32s^3)}{(1-2s)^4}.
\end{aligned}
\tag{3.1}
$$

By using Eqs (2.1) and (3.1), we obtain the spray coefficient G^j for the cubic metric as

$$
\begin{aligned}
G^j = G^j_\alpha + \frac{1}{2\alpha(1-2s)(1+6B-s-8s^2)}\Big[&(6+36B-6s-48s^2)\alpha^2 s^j_0 + [18\alpha s_0(4s-1)\\
&+ 3r_{00}(1-6s+8s^2)]y^j - 6\alpha b^j[6s_0 + (2s-1)r_{00}]\Big].
\end{aligned}
\tag{3.2}
$$

In view of Eqs (2.5) and (3.1) and using *Mathematica* program, we obtain the S-curvature for the cubic Finsler metric as

$$
\begin{aligned}
S = \frac{1}{2(\alpha-2\beta)(\alpha^2+6B\alpha^2-\alpha\beta-8\beta^2)^2}\Big[&-2r_0(\alpha-2\beta)((1+6B)\alpha^2-\alpha\beta-8\beta^2)[-\alpha\beta\Pi - 8\beta^2\Pi\\
&+ \alpha^2(-6+\Pi+6B\Pi)] - 2s_0[3\alpha^2((1+3n+6B(2+3n))\alpha^3 - 3(3+5n+8B(-2+3n))\alpha^2\beta\\
&- 6(1+2n)\alpha\beta^2 + 32(-1+3n)\beta^3) + (\alpha-2\beta)(-(1+6B)\alpha^2+\alpha\beta+8\beta^2)^2\Pi] + 3r_{00}(\alpha-2\beta)[(1+n\\
&+ B(8+6n))\alpha^3 - (5-8B+5n+24Bn)\alpha^2\beta - 2(3+2n)\alpha\beta^2 + 32n\beta^3]\Big].
\end{aligned}
\tag{3.3}
$$

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First we prove the converse part.

Let us assume that β is a constant Killing form i.e., $s_0 = 0$ and $r_{00} = 0$; putting this in Eq (3.3) vanishes the S-curvature.

For the if part, let us take $S = 0$; then Eq (3.3) becomes

$$t_0 + t_1\alpha + t_2\alpha^2 + t_3\alpha^3 + t_4\alpha^4 + t_5\alpha^5 = 0, \tag{3.4}$$

where

$$\begin{aligned}
t_0 &= 64\beta^4(4\beta\Pi r_0 + 4\beta\Pi s_0 - 3nr_{00}), \qquad t_1 = 4\beta^3(-16\beta\Pi r_0 - 16\beta\Pi s_0 + 3(3+10n)r_{00}),\\
t_2 &= (192\beta^3 - 92\beta^3\Pi - 384B\beta^3\Pi)r_0 + (192\beta^3 - 576n\beta^3 - 92384B\beta^3\Pi)s_0\\
&\quad + (12\beta^2 - 48B\beta^2 + 18n\beta^2 + 144Bn\beta^2)r_{00},\\
t_3 &= (-72\beta^2 + 22\beta^2\Pi + 144B\beta^2\Pi)r_0 + (36\beta^2 + 72n\beta^2 + 22\beta^2\Pi + 144B\beta^2\Pi)s_0\\
&\quad + (-21\beta - 24B\beta - 21n\beta - 108Bn\beta)r_{00},\\
t_4 &= (-36\beta - 144B\beta + 8\beta\Pi + 72B\beta\Pi + 144B^2\beta\Pi)r_0 + (54\beta - 288B\beta\\
&\quad + 90n\beta + 432Bn\beta + 8\beta\Pi + 72B\beta\Pi + 144B^2\beta\Pi)s_0 + (3 + 24B + 3n + 18Bn)r_{00},\\
t_5 &= (12 + 72B - 2\Pi - 24B\Pi - 72B^2\Pi)r_0 + (-6 - 72B - 18n - 108Bn - 2\Pi - 24B\Pi - 72B^2\Pi)s_0.
\end{aligned}$$

Taking the rational and irrational parts of Eq (3.4), we obtain

$$t_0 + \alpha^2(t_2 + \alpha^2 t_4) = 0, \tag{3.5}$$

$$t_1 + \alpha^2(t_3 + \alpha^2 t_5) = 0. \tag{3.6}$$

From Eqs (3.5) and (3.6), we can say that α^2 will divide t_0 as well as t_1. In view of Lemma 1.1, α^2 is coprime with β for $n > 2$. Solving Eqs (3.5) and (3.6), we get, respectively,

$$4\beta\Pi(r_0 + s_0) - 3nr_{00} = \gamma_1\alpha^2, \quad \text{for} \quad \gamma_1 = \gamma_1(x),$$

and

$$16\beta\Pi(r_0 + s_0) - 3(10n + 3)r_{00} = \gamma_2\alpha^2, \quad \text{for} \quad \gamma_2 = \gamma_2(x).$$

From the above equations, we obtain

$$r_{00} = c\alpha^2, \quad \text{and then} \quad r_0 = c\beta, \tag{3.7}$$

for some scalar function $c = c(x)$ on M.

Putting the above values in Eq (3.4) and simplifying, we get

$$256\Pi\beta^5(c\beta + s_0) = \alpha^2\ (....),$$

where (...) denotes the polynomial term in α and β. Here also α^2 does not divide β^5 and $(c\beta + s_0)$. Therefore, $c\beta + s_0 = 0$. Differentiating it with respect to y^i, we obtain $cb_i + s_i = 0$, which, on contracting by b^i, gives $c = 0$, implying $s_0 = 0$ and $r_{00} = 0$. Which means β is a constant Killing form.

This completes, the proof of Theorem 1.1. □

4. Ricci curvature of cubic metric

In this section we obtain the projective Ricci curvature for the aforesaid metric.

Proof of Theorem 1.2. For this, we first obtain all the values of Eq (2.8) by using Eq (3.1) and the *Mathematica* program as

$$V_1 = \frac{-1}{(1+n)(-1-6B+s+8s^2)^4}[3(6B(6(1+n)+8(1+n)s+(36-(-37+n)n)s^2 - 4(45+n(37+8n))s^3 - 256(4+n(3+n))s^4) + s(-4(1+n) - 92(1+n)s + 92(1+n)s^2 + (-238+n(-241+3n))s^3 + 32(23+n(20+3n))s^4 + 256(14+n(11+3n))s^5) + 3B^2(66+24s(1+32s)+(n+16ns)^2+n(65+8s(-1+64s))))],$$

$$V_2 = \frac{-1}{(1+n)(-1+2s)(-1-6B+s+8s^2)^4}[6(-5+192B+1224B^2-6n+186Bn+1206B^2n - n^2 - 18Bn^2 - 54B^2n^2 - 6(3(5+6n+n^2)+12B^2(-10+n+9n^2)+B(16+41n+51n^2))s + 3(-99-104n-n^2-36B(-12-5n+n^2)+384B^2(8+n+3n^2))s^2 + 2(508+675n+245n^2 + 12B(-244-125n+105n^2))s^3 - 6(330+183n-45n^2+64B(70+23n+15n^2))s^4 - 96(-36-11n+23n^2)s^5 + 2048(8+3n+n^2)s^6)],$$

$$V_3 = \frac{3(1+16s)(6B(3+5n)+n(-2+2s-26s^2)+3(-1+s-4s^2)-n^2(-1+s+2s^2))}{(1+n)(-1-6B+s+8s^2)^3},$$

$$V_4 = \frac{6(1+16s)(-B+s^2)}{(-1-6B+s+8s^2)^2},$$

$$V_5 = \frac{1}{(1+n)(-1+2s)^3(-1-6B+s+8s^2)^4}[36(-6-7n+n^2-(63+n(82+35n))s+15(8 + 3n(3+n))s^2 + (849+n(1354+785n))s^3 - (2562+n(2351+1123n))s^4 - 6(-398+495n + 771n^2)s^5 + 64(45+n(-7+100n))s^6 + 2048(-3+n(7+2n))s^7 - 216B^3(1+n)^2(-1+8s) + 9B^2(74+9n(9+n)-144s-6n(47+37n)s+48(1+n(-11+8n))s^2 + 160(2+3n(5+n))s^3) + 6B(13+13n+2n^2-(70+n(109+89n))s+(151+55n(1+2n))s^2 + 2(-178+n(317+505n))s^3 - 8(47+n(-136+199n))s^4 - 1024(-1+n(5+n))s^5))],$$

$$V_6 = \frac{-72n}{(1+6B-s-8s^2)^2}, \qquad V_7 = \frac{9(-2-3n+n^2)}{(1+n)(1+6B-s-8s^2)^2},$$

$$V_8 = \frac{1}{(1+n)(-1+2s)(1+6B-s-8s^2)^3}[18(-7+n(-8+3n)-33s+3n(-4+3n)s + 6(10+(7-5n)n)s^2 - 256(1+2n)s^3 - 6B(1+n-2n^2+4(-14+(-23+n)n)s))],$$

$$V_9 = \frac{6}{1+6B-s-8s^2},$$

$$V_{10} = \frac{-6(1+n+6B(3+n))+18(1+4B(-15+n)+n)s+36(3+n)s^2-96(-11+n)s^3}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2},$$

$$V_{11} = \frac{12(1+3B-3s-60Bs-3s^2+64s^3)}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2}, \qquad V_{12} = \frac{6(1-8s)}{(-1+2s)^3},$$
$$V_{13} = \frac{-432n}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2},$$
$$V_{14} = \frac{18(3+6B-n-6Bn+3(-3+4B(-19+n)+n)s+6(-1+n)s^2-16(-15+n)s^3)}{(1+n)(-1+2s)^2(-1-6B+s+8s^2)^2}, \tag{4.1}$$
$$V_{15} = \frac{18}{(-1+2s)(-1-6B+s+8s^2)}, \qquad V_{16} = \frac{6}{1-2s},$$
$$V_{17} = \frac{108}{(-1+2s)^2(-1-6B+s+8s^2)}, \quad V_{18} = \frac{-9}{(-1+2s)^2}, \quad V_{19} = \frac{6(-1+n)}{(1+n)(1+6B-s-8s^2)}.$$

Plugging all the values of the above Eq (4.1) into Eq (2.7) and simplifying by the using *Mathematica* program, we obtain the projective Ricci curvature for the aforesaid metric as

$$PRic = \frac{1}{(1+n)^2(\alpha-2\beta)^3(-(1+6B)\alpha^2+\alpha\beta+8\beta^2)^4}\sum_{i=0}^{i=13}\alpha^i t_i',$$

where

$$\begin{aligned} t_0' &= -2048\beta^9(-3r_{00}^2(14+n(11+3n))+8(1+n)\beta(3r_{00|0}+2(1+n)Ric^{\alpha}\beta)\\ &\quad +8\beta(2(-1+n)(1+n)^2\beta(\rho_0^2+\rho_{0|0}))-3(-1+n^2)\rho_0 r_{00}),\\ t_1' &= 256\beta^8(-3r_{00}^2(145+n(112+33n))+4(1+n)\beta(57r_{00|0}+32(1+n)Ric^{\alpha}\beta)\\ &\quad +4\beta(32(-1+n)(1+n)^2\beta(\rho_0^2+\rho_{0|0}))-57(-1+n^2)\rho_0 r_{00}),\\ &\vdots\\ t_{13}' &= -9(1+6B)^3(12s_k s^k+s^i_k s^k_i+6Bs^i_k s^k_i)(1+n)^2. \end{aligned} \tag{4.2}$$

Next, we obtain the flatness condition under which the projective Ricci curvature vanishes.

Let the projective Ricci curvature $PRic = 0$, which implies $U(\alpha,\beta) = 0$, where

$$U(\alpha,\beta) = t_0' + \alpha t_1' + \alpha^2 t_2' + ... + \alpha^{13} t_{13}'. \tag{4.3}$$

Using *Mathematica*, we can see that

$$U(\alpha,\beta) = \frac{1}{4}(-6-7n+n^2)(\alpha-2\beta)^3(\alpha+\beta)^2(\alpha+16\beta)^2(r_{00}(\alpha-2\beta)-6s_0\alpha^2)^2 \mod[(1+6B)\alpha^2-\alpha\beta-8\beta^2].$$

Therefore

$$(...)[(1+6B)\alpha^2-\alpha\beta-8\beta^2])-\frac{1}{4}(-6-7n+n^2)(\alpha-2\beta)^3(\alpha+\beta)^2(\alpha+16\beta)^2(r_{00}(\alpha-2\beta)-6s_0\alpha^2)^2 = 0,$$

where (....) are polynomial in α and β. As $B < \frac{1}{4}$, therefore $((1+6B)\alpha^2-\alpha\beta-8\beta^2)$ does not divide $(\alpha-2\beta)^3$ or $(\alpha+\beta)^2$ or $(\alpha+16\beta)^2$. Therefore $((1+6B)\alpha^2-\alpha\beta-8\beta^2)$ will divide $(r_{00}(\alpha-2\beta)-6s_0\alpha^2)^2$; then $((1+6B)\alpha^2-\alpha\beta-8\beta^2)$ will also divide $(r_{00}(\alpha-2\beta)-6s_0\alpha^2)$, i.e.,

$$(r_{00}(\alpha-2\beta)-6s_0\alpha^2) = (c_1+\alpha c_0)((1+6B)\alpha^2-\alpha\beta-8\beta^2),$$

where c_1 is a 1-form and c_0 is a scalar. Taking the rational and irrational parts of the above equation, we obtain

$$-2\beta r_{00} - 6\alpha^2 s_0 = c_1\alpha^2(1+6B) - 8\beta^2 c_1 - c_0\alpha^2\beta, \tag{4.4}$$

and

$$r_{00} = c_0\alpha^2(1+6B) - \beta c_1 - 8c_0\beta^2. \tag{4.5}$$

Solving the above equations, we get $c_1 = -\frac{8}{5}\beta c_0$, and then (4.5) gives

$$r_{00} = c_0[\alpha^2(1+6B) - \frac{32}{5}\beta^2]. \tag{4.6}$$

Substituting the above values into Eq (4.4), we obtain

$$(4B-1)c_0\beta + 10s_0 = 0. \tag{4.7}$$

Differentiating the above equation with respect to y^i gives $(4B-1)c_0 b_i + s_i = 0$, which, on contracting by b^i, we obtain $c_0 = 0$. Then from Eqs (4.6) and (4.7), we obtain

$$r_{00} = 0, \qquad s_0 = 0. \tag{4.8}$$

In view of (4.8), Eq (4.3) becomes

$$\begin{aligned}&3\alpha^2(-2s_{0k}s^k_0(\alpha-8\beta) + (-3s^i_k s^k_i\alpha^2 + 2s^k_{0;k}(\alpha-2\beta))(\alpha-2\beta)) + Ric^\alpha(\alpha-2\beta)^3\\ &-(n-1)(\alpha-2\beta)^2(-(\alpha-2\beta)\rho_0^2 + 6\alpha^2 s^k_0\rho_k - (\alpha-2\beta)\rho_{0|0}) = 0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&(\alpha-2\beta)\Big\{6\alpha^2 s_{0k}s^k_0 + 9\alpha^4 s^i_k s^k_i - 6\alpha^2 s^k_{0;k}(\alpha-2\beta) - Ric^\alpha(\alpha-2\beta)^2\\ &+(n-1)(\alpha-2\beta)[6\alpha^2 s^k_0\rho_k - (\alpha-2\beta)\rho_0^2 - (\alpha-2\beta)\rho_{0|0}]\Big\} = 36s_{0k}s^k_0\alpha^2\beta.\end{aligned}$$

Since $(\alpha-2\beta)$ does not divide α^2 or β, therefore $(\alpha-2\beta)$ will divide $s_{0k}s^k_0$. Thus

$$s_{0k}s^k_0 = (d_1 + \alpha d_0)(\alpha - 2\beta),$$

where d_1 is a 1-form and d_0 is a scalar. Taking the rational and irrational parts of the above equation and solving, we obtain

$$s_{0k}s^k_0 = d_0(\alpha^2 - 4\beta^2). \tag{4.9}$$

If $d_0 \neq 0$ then one can conclude by the above equation that α is not positive definite, which is not possible. Therefore, $d_0 = 0$. This implies that

$$s_{ik} = 0, \tag{4.10}$$

i.e., β is closed. In view of Eqs (4.8) and (4.10), we obtain $b_{i;k} = 0$, then 1-form β is parallel with respect to α.

This completes the proof of Theorem 1.2. □

Now, we obtain the condition for the weak projective Ricci curvature of a cubic Finsler metric.

Proof of Theorem 1.3. Let F be a cubic Finsler metric with weak projective Ricci curvature. Then from Eq (2.6) we obtain

$$(n-1)[3\theta(\alpha+\beta)^3\alpha^2 + \gamma(\alpha+\beta)^6] = \frac{\alpha^4}{(1+n)^2(\alpha-2\beta)^3((1+6B)\alpha^2-\alpha\beta-8\beta^2)^4}\sum_{i=0}^{i=13}\alpha^i t_i'. \tag{4.11}$$

For the cubic metric, we have $B < \frac{1}{4}$, which implies that α^4 does not divide $(\alpha-2\beta)^3$ or $((1+6B)\alpha^2-\alpha\beta-8\beta^2)^4$ or $3\theta(\alpha+\beta)^3\alpha^2$. Consequently, it follows that α^2 must divide $\gamma(\alpha+\beta)^6$. However, such division is only possible if $\gamma = 0$. Combining this result with Eq (4.11), then we deduce that $3\theta(\alpha+\beta)^3$ is divided by α^2. This is impossible unless $\theta = 0$. Then F reduces to a projective Ricci-flat metric.

The converse is obvious. This completes the proof. □

Example 4.1. *The Finsler metric* $\frac{1}{|y|^2}(|y| + < a, y >)^3$ *for a =constant is projectively Ricci flat.*

5. Conclusions

Projective Ricci curvature is a concept in differential geometry that generalizes the notion of Ricci curvature. It has various applications in the fields of general relativity, optimal transformation theory, complex geometry, Weyl geometry, Einstein metrics, and many more. In this article, we have proved that if the cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projective Ricci flat ($PRic = 0$), then β is parallel with respect to Riemannian metric α, and then from Eq (2.3), the S-curvature vanishes. Therefore, from Eq (2.5), we obtain that the Riemannian metric α is also Ricci-flat, which is Corollary 1.1.

Author contributions

M. K. Gupta and S. Sharma wrote the framework and the original draft of this manuscript. Y. Li and Y. Xie reviewed and validated the manuscript. All authors have read and agreed to the final version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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