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Research article

On paranormed sequence space arising from Riesz Euler Totient matrix

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Abstract: In this paper, we introduce a novel paranormed sequence space $l(R_\Phi, p)$ constructed through the application of the Riesz Euler Totient matrix. We demonstrate that the spaces $l(R_\Phi, p)$ and $l(p)$ are linearly isomorphic. In addition, we identify the dual spaces associated with this sequence space and establish its Schauder basis.

Keywords: α-, β-, γ-duals; paranormed sequence space; Riesz matrix; Euler Totient function
Mathematics Subject Classification: 46B45, 47B06, 47H08

1. Introduction

Let ω represent the set of all real sequences; within this context, a linear subspace of ω is termed a sequence space. In the present study, we denote the null, convergent, and bounded sequence spaces by c_0, c, and l_∞, respectively. Additionally, we utilize the notations bs, cs, l_1, and l_p $(1 < p < \infty)$,to refer to the spaces of all bounded, convergent, absolutely convergent, and $p-$ absolutely convergent series, respectively.

The set of real numbers is denoted by $\mathbb{R}$, and the set of natural numbers is denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$. For ease of writing, we use $\lim_n$, $\sum_n$, $\sup_n$, and $\inf_n$ instead of $\lim_{n\to\infty}$, $\sum_{n=1}^{\infty}$, $\sup_{n\in\mathbb{N}}$, and $\inf_{n\in\mathbb{N}}$, respectively. Throughout the paper, $p_n > 0$, and (p_n) is a bounded sequence in $\mathbb{R}$, where $\sup_{n\in\mathbb{N}} p_n = P$ and $S = \max\{1, P\}$. For any $\zeta \in \mathbb{R}$ and $n \in \mathbb{N}$

$$|\zeta|^{p_n} \leq \max\left\{1, |\zeta|^S\right\} \tag{1.1}$$

is satisfied (see [1]). This is an important inequality that can be used to show that a space is paranormed. Further, the equality $p_n^{-1} + (p_n')^{-1} = 1$ is valid for $1 < \inf_{n\in\mathbb{N}} p_n \leq P$. The set of all finite subsets of $\mathbb{N}$ is denoted by $\aleph$.

Let X be a real linear space, and let σ be a function from X to $\mathbb{R}$. Then, the pair (X, σ) is called a paranormed space over $\mathbb{R}$ if the following axioms are satisfied:

i) σ is sub-additive,

ii) $\sigma(\theta) = 0$ (θ is the zero of X),

iii) $\sigma(x) = \sigma(-x)$ $\forall x \in U$,

iv) $|\zeta_n - \zeta| \to 0$, $\sigma(x_n - x) \to 0$ imply $\sigma(\zeta_n x_n - \zeta x) \to 0$ for every sequence (ζ_n) and (x_n) with $\zeta \in \mathbb{R}$ and $x \in X$.

Complete paranormed sequence space $l(p)$ is introduced by Maddox [1] (see also [2, 3]) as:

$$l(p) = \left\{ x = (x_n) \in \omega : \sum_n |x_n|^{p_n} < \infty \right\}$$

with

$$\sigma_p(x) = \left(\sum_n |x_n|^{p_n} \right)^{1/S}.$$

Let $T = (t_{mn})$ be an infinite matrix of real numbers t_{mn}, and let χ and Υ be any two sequence spaces. $T : \chi \to \Upsilon$ is called a matrix mapping, if $Tx = (T_m(x)) \in \Upsilon$ for all $x = (x_n) \in \chi$. Here

$$T_m(x) = \sum_n t_{mn} x_n \tag{1.2}$$

for every $m \in \mathbb{N}$. The T-transform of x is shown by Tx, which is a sequence. $(\chi : \Upsilon)$ is the notation of the set of all matrices from χ to Υ. T is an element of $(\chi : \Upsilon)$ if and only if $\sum_n t_{mn} x_n$ converges for all $m \in \mathbb{N}$ and $x \in \chi$; additionally, Tx is an element of Υ.

The set $S(\chi, \Upsilon)$ is defined as

$$S(\chi, \Upsilon) = \{ s = (s_n) \in \omega : \ xs = (x_n s_n) \in \Upsilon, \ \forall x = (x_n) \in \chi \}.$$

Then, α-dual, β-dual, and γ-dual of a sequence space χ are denoted by $\chi^\alpha = S(\chi, l_1)$, $\chi^\beta = S(\chi, cs)$, and $\chi^\gamma = S(\chi, bs)$, respectively.

Articles [4–6] are about summability and matrix transformations. Also, for more information related to normed sequence spaces obtained by infinite matrix domains, one can see [7–9]. We refer to [10–12] for detailed information about paranormed Riesz sequence spaces, and [13–15] for more information about different paranormed sequence spaces. These studies have contributed significantly to the ongoing exploration of paranormed sequence spaces within the broader mathematical landscape.

In this study, the Euler Totient function is represented by φ, and the Möbius function is represented by μ. In the following definitions, $\varphi(n)$ and $\mu(n)$ are defined for all $n \in \mathbb{N}$ with $n > 1$. $\varphi(n)$ counts the positive integers up to a given integer n that are relatively prime to n, and $\varphi(1) = 1$. In the sequel some properties of function φ are listed:

i) If the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$, then,

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_l}\right).$$

ii) For all $n \in \mathbb{N}$, the equation

$$n = \sum_{m|n} \varphi(m)$$

is valid.

iii) If two natural numbers n_1 and n_2 are relatively prime, then $\varphi(n_1 n_2) = \varphi(n_1)\varphi(n_2)$ is satisfied [16].

Möbius function μ is defined as:

$$\mu(n) = \begin{cases} (-1)^l, & \text{if } n = p_1 p_2 \dots p_l, \text{ where } p_1, p_2, \dots, p_l \text{ are distinct prime numbers,} \\ 0, & \text{if } p^2 | \text{n for some prime numbers } p, \end{cases}$$

and $\mu(1) = 1$. If the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$, then,

$$\sum_{m|n} m\mu(n) = (1 - p_1)(1 - p_2) \dots (1 - p_l).$$

Additionally, the equation

$$\sum_{m|n} \mu(m) = 0 \tag{1.3}$$

is satisfied excluding $n = 1$. Also, $\mu(n_1 n_2) = \mu(n_1)\mu(n_2)$, where $n_1, n_2 \in \mathbb{N}$ are coprime [16]. [17] is referred for more information about Euler's totient and Möbius functions.

Although in most cases the new sequence space X_A generated by the triangle matrix A from a sequence space X is the expansion or the contraction of the original space X, it may be observed in some cases that those spaces overlap.

The Euler's totient matrix $\Phi = (\phi_{nm})$ is defined as:

$$\phi_{nm} = \begin{cases} \frac{\varphi(m)}{n}, & \text{if } m|n, \\ 0, & m \nmid n, \end{cases}$$

and the inverse of matrix Φ, $\Phi^{-1} = (\phi_{nm}^{-1})$, is obtained [18] as

$$\phi_{nm}^{-1} = \begin{cases} \frac{\mu(\frac{n}{m})}{\varphi(n)} m, & \text{if } m|n, \\ 0, & m \nmid n, \end{cases}$$

for all $n, m \in N$. Lately, in [19], two new Banach sequence spaces are obtained via Euler's totient matrix, namely, $l_p(\Phi)\,(1 \le p < \infty)$ and $l_\infty(\Phi)$ as

$$l_p(\Phi) = \left\{ u = (u_n) \in \omega \colon \sum_m \left| \frac{1}{m} \sum_{n|m} \varphi(n) u_n \right|^p < \infty \right\} \ (1 \le p < \infty),$$

and

$$l_\infty(\Phi) = \left\{ u = (u_n) \in \omega \colon \sup_m \left| \frac{1}{m} \sum_{n|m} \varphi(n) u_n \right| < \infty \right\}.$$

In 2021, İlkhan and Bayraktar [20] introduced the sequence space $l_p(R_\Phi)$ by using the matrix R_Φ, where $1 \le p < \infty$.

The matrix $R_\Phi = (r_{nk})$, which is called the Riesz Euler Totient matrix operator, is defined as:

$$r_{kn} = \begin{cases} \frac{q_n \varphi(n)}{Q_k}, & \text{if } n \mid k, \\ 0, & \text{if } n \nmid k, \end{cases}$$

where $Q_k = q_1 + q_2 + \dots + q_k$.

The inverse matrix of R_Φ, $R_\Phi^{-1} = (r_{kn}^{-1})$, is found as

$$r_{kn}^{-1} = \begin{cases} \frac{\mu(\frac{k}{n})}{\varphi(k)} \frac{Q_n}{q_k}, & \text{if } n \mid k, \\ 0, & \text{if } n \nmid k, \end{cases}$$

for all $k, n \in \mathbb{N}$.

Then, they introduce the sequence space $l_p(R_\Phi)$ by

$$l_p(R_\Phi) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k \right|^p < \infty \right\} \quad (1 \leq p < \infty).$$

In some cases, the most general linear operator between two sequence spaces is given by an infinite matrix. So, the theory of matrix transformations has always been of great interest in the study of sequence spaces. The study of the general theory of matrix transformations was motivated by special results in summability theory. The theory of sequence spaces is fundamental to summability. Summability is a wide field of mathematics, mainly in analysis and functional analysis, and has many applications, for instance, in numerical analysis to speed up the rate of convergence, in operator theory, in the theory of orthogonal series, and in approximation theory. The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than the original sequence or series. The reference [13] is a recent study in the field of sequence spaces. They have become the starting point of our study to construct a new paranormed sequence space. By the concept of matrix domain, we have aimed to introduce complete paranormed sequence space $l(R_\Phi, p)$.

The paranormed spaces have more general properties than normed spaces. In this paper, the normed sequence space $l_p(R_\Phi)\,(1 \leq p < \infty)$ is generalized to a new paranormed space $l(R_\Phi, p)$. Also, completeness, α-, β-, γ-duals of this space and the Schauder basis of the space $l(R_\Phi, p)$ are investigated.

2. Materials and methods

2.1. The Paranormed sequence space $l(R_\Phi, p)$

In this section, the paranormed sequence space $l(R_\Phi, p)$ is defined by using the Riesz Euler Totient matrix R_Φ. Then, it is shown that, given paranormed space is complete. Also, the Schauder basis of this space is given.

Throughout the paper, R_Φ-transform of $u = (u_n)$ is denoted by $v = (v_n)$, that is,

$$v_m = \frac{1}{Q_m} \sum_{n|m} q_n \varphi(n) u_n, \quad (\forall m \in \mathbb{N}).$$

$l(R_\Phi, p)$ sequence space is given by

$$l(R_\Phi, p) = \left\{ u = (u_n) \in \omega : \sum_m \left| \frac{1}{Q_m} \sum_{n|m} q_n \varphi(n) u_n \right|^{p_m} < \infty \right\}.$$

$l(R_\Phi, p)$ space can be represented by $l(R_\Phi, p) = (l(p))_{R_\Phi}$ according to the definition of matrix domain.

Theorem 2.1. *With the paranorm given by*

$$\sigma_{R_\Phi}(u) = (\sum_m |\frac{1}{Q_m}\sum_{n|m} q_n\varphi(n)u_n|^{p_m})^{1/S},$$

$l(R_\Phi, p)$ is a complete paranormed space for all $u = (u_n) \in l(R_\Phi, p)$.

Proof. Let $u = (u_n)$, $s = (s_n) \in l(R_\Phi, p)$. According to [21] (p.30), we can write

$$(\sum_m |\frac{1}{Q_m}\sum_{n|m} q_n\varphi(n)(u_n + s_n)|^{p_m})^{1/S} \leq (\sum_m |\frac{1}{Q_m}\sum_{n|m} q_n\varphi(n)u_n|^{p_m})^{1/S} + (\sum_m |\frac{1}{Q_m}\sum_{n|m} q_n\varphi(n)s_n|^{p_m})^{1/S}. \tag{2.1}$$

The linearity of $l(R_\Phi, p)$ relative to scalar multiplication and co-ordinate-wise addition comes from (1.1) and (2.1).

It is trivial that $\sigma_{R_\Phi}(\theta) = 0$ and $\sigma_{R_\Phi}(u) = \sigma_{R_\Phi}(-u)$ for all $u \in l(R_\Phi, p)$. Also, by using (1.1) and (2.1), it is obvious that σ_{R_Φ} is subadditive and $\sigma_{R_\Phi}(\zeta u) \leq \max\{1, |\zeta|\}\, \sigma_{R_\Phi}(u)$ for any $\zeta \in \mathbb{R}$.

Let us consider $\{u^m\}$ any sequence in $l(R_\Phi, p)$ satisfying $\sigma_{R_\Phi}(u^m - u) \to 0$, and let (ζ_m) be a sequence in $\mathbb{R}$ with $\zeta_m \to \zeta$. Because σ_{R_Φ} is subadditive, we have

$$\sigma_{R_\Phi}(u^m) \leq \sigma_{R_\Phi}(u) + \sigma_{R_\Phi}(u^m - u).$$

So, $\{\sigma_{R_\Phi}(u^m)\}$ is bounded, and we can obtain:

$$\sigma_{R_\Phi}(\zeta_m u^m - \zeta u) = \left(\sum_m \left|\frac{1}{Q_m}\sum_{n|m} q_n\varphi(n)\,(\zeta_m u_n^m - \zeta u_n)\right|^{p_m}\right)^{1/S}$$

$$\leq |\zeta_m - \zeta|\, \sigma_{R_\Phi}(u^m) + |\zeta|\, \sigma_{R_\Phi}(u^m - u) \to 0 \quad (m \to \infty).$$

Scalar multiplication is hence continuous. So, σ_{R_Φ} is a paranorm on $l(R_\Phi, p)$.

Now, it is time to show that $l(R_\Phi, p)$ space is complete. Let $\left\{u^i\right\}$ be any Cauchy sequence in $l(R_\Phi, p)$, where $u^i = \left\{u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \ldots\right\}$ for each $i \in \mathbb{N}$. For a given $\varepsilon > 0$, there exists an integer $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sigma_{R_\Phi}\left(u^i - u^j\right) < \varepsilon \tag{2.2}$$

for every $i, j \geq n_0(\zeta)$. For every fixed $k \in \mathbb{N}$, we can determine via the definition of σ_{R_Φ} that

$$\left|R_{\Phi k}\left(u^i\right) - R_{\Phi k}(u^j)\right| \leq \left[\sum_k \left|R_{\Phi k}\left(u^i\right) - R_{\Phi k}(u^j)\right|^{p_k}\right]^{1/S} < \varepsilon$$

for all $i, j \geq n_0(\varepsilon)$. This indicates that the sequence $\left\{R_{\Phi k}\left(u^1\right), R_{\Phi k}\left(u^2\right), R_{\Phi k}\left(u^3\right), \ldots\right\}$ is Cauchy in $\mathbb{R}$ for each fixed $k \in \mathbb{N}$. Because $\mathbb{R}$ is complete, it is convergent, say $R_{\Phi k}\left(u^i\right) \to R_{\Phi k}(u)$ as $i \to \infty$

for every fixed $k\in\mathbb{N}$. By using these infinitely many limits $R_{\Phi 1}(u)$, $R_{\Phi 2}(u)$, $R_{\Phi 3}(u)$,..., the sequence $\{R_{\Phi 1}(u), R_{\Phi 2}(u), R_{\Phi 3}(u), \dots\}$ is defined. Then, by using (2.2) for all fixed $m\in\mathbb{N}$ and $i, j \geq n_0(\varepsilon)$

$$\sum_{k=1}^{m} \left| R_{\Phi k}\left(u^i\right) - R_{\Phi k}(u^j) \right|^{p_k} \leq g_{R_\Phi}\left(u^i - u^j\right)^S < \varepsilon^S \tag{2.3}$$

can be written. Let $n_0(\zeta) \leq i$ and $m \to \infty$, $j \to \infty$, respectively. If the limit of (2.3) is taken, then we obtain $\sigma_{R_\Phi}\left(u^i - u\right) \leq \zeta$. Finally, if we assume $\varepsilon = 1$ in (2.3) and let $i \geq n_0$ (1.1), by the use of Minkowski's inequality

$$\left(\sum_{k=1}^{m} |R_{\Phi k}(u)|^{p_k} \right)^{1/S} \leq \sigma_{R_\Phi}\left(u^i - u\right) + \sigma_{R_\Phi}\left(u^i\right) \leq 1 + \sigma_{R_\Phi}\left(u^i\right)$$

is obtained for every fixed $m\in\mathbb{N}$. This means that $u \in l(R_\Phi, p)$. Since $\sigma_{R_\Phi}\left(u^i - u\right) \leq \zeta$ for each $i \geq n_0(\zeta)$, we get that $u^i \to u$ as $i \to \infty$. Due to this, $l(R_\Phi, p)$ is complete.

It should be noted that the absolute property on $l(R_\Phi, p)$ is not satisfied since there can be a sequence u in $l(R_\Phi, p)$ such that $\sigma_{R_\Phi}(u) \neq \sigma_{R_\Phi}(|u|)$, where $|u| = (|u_n|)$. So, $l(R_\Phi, p)$ is a sequence space of non-absolute type. □

Theorem 2.2. *The sequence space $l(R_\Phi, p)$ is linearly isomorphic to the space $l(p)$.*

Proof. For this proof, it should be shown that there is a linear bijection L between $l(R_\Phi, p)$ and $l(p)$. To show this, take the mapping $L : l(R_\Phi, p) \to l(p)$ which is given as $u \to v = Lu = R_\Phi u$. The linearity of L is trivial. Also, L is injective because $u = \theta$, where $Lu = \theta$.

Let us define a sequence $u = (u_n)$ by any $v = (v_n) \in l(p)$:

$$u_n = \frac{1}{q_n\varphi(n)} \sum_{k/n} \mu\left(\frac{n}{k}\right) Q_k v_k \quad (\forall n\in\mathbb{N}).$$

Then, we have that

$$\begin{aligned}
\sigma_{R_\Phi}(u) &= \left(\sum_m \left| \frac{1}{Q_m} \sum_{n/m} q_n\varphi(n) u_n \right|^{p_m} \right)^{1/S} \\
&= \left(\sum_m \left| \frac{1}{Q_m} \sum_{n/m} q_n\varphi(n) \frac{1}{q_n\varphi(n)} \sum_{k/n} \mu\left(\frac{n}{k}\right) Q_k v_k \right|^{p_m} \right)^{1/S} \\
&= \left(\sum_m \left| \frac{1}{Q_m} \sum_{n/m} \sum_{k/n} \mu\left(\frac{n}{k}\right) Q_k v_k \right|^{p_m} \right)^{1/S} \\
&= \left(\sum_m \left| \frac{1}{Q_m} \sum_{n/m} \left(\sum_{k/n} \mu(k) \right) Q_{\frac{m}{n}} v_{\frac{m}{n}} \right|^{p_m} \right)^{1/S} \\
&= \left(\sum_m \left| \frac{1}{Q_m} \mu(1)\, Q_m v_m \right|^{p_m} \right)^{1/S}
\end{aligned}$$

$$= \left(\sum_m |v_m|^{p_m} \right)^{\frac{1}{S}} = h_{R_\Phi}(v) < \infty.$$

This proves that $u \in l(R_\Phi, p)$. So, L preserves the paranorm, and L is surjective. As a result, there is a linear bijection L between the spaces $l(R_\Phi, p)$, and $l(p)$ and these spaces are linearly isomorphic.

It is a well-known fact that for every $n \in \mathbb{N}$ if $1 < p_n \leq s_n$, then $l(p) \subseteq l(s)$. As a consequence, $l(R_\Phi, p) \subseteq l(R_\Phi, s)$. □

A sequence (b_k) of the elements from the paranormed space X, paranormed by p, is known as a Schauder basis of the space X if and only if there exists a sequence (α_k) of scalars such that

$$\lim_{n\to\infty} p\left(u - \sum_{k=0}^{n} \alpha_k b_k\right) = 0 \tag{2.4}$$

holds true for all $u \in X$. Furthermore, it is established that the domain X_T of the matrix T in the space X possesses a Schauder basis if and only if both X possesses a Schauder basis and T is triangular (refer to [22], Theorem 2.3). In light of this observation, we present the Schauder basis for the space $l(R_\Phi, p)$.

Theorem 2.3. *Let* $b^{(m)} = \left\{b_n^{(m)}\right\}_{n\in\mathbb{N}}$ *be a sequence in* $l(R_\Phi, p)$ *given by*

$$b_n^{(m)} = \begin{cases} \dfrac{1}{q_n \varphi(n)} \mu\left(\dfrac{n}{k}\right) Q_k, & \text{if } n|m, \\ 0, n \nmid m, \end{cases}$$

where m is a fixed natural number. Then, $\left\{b^{(m)}\right\}_{m\in\mathbb{N}}$ is a Schauder basis of the space $l(R_\Phi, p)$ and any $u \in l(R_\Phi, p)$ is uniquely represented in the form

$$u = \sum_m \gamma_m b^{(m)},$$

where $\gamma_m = R_{\Phi m}(u)\ \forall m \in \mathbb{N}$.

Proof. The previous proof showed the surjectivity of the isomorphism $L : l(R_\Phi, p) \to l(p)$. Therefore, the inverse image of the Schauder basis of the space $l(p)$ is a Schauder basis of the space $l(R_\Phi, p)$. This ends the proof. □

2.2. Dual spaces of $l(\boldsymbol{R_\Phi}, \boldsymbol{p})$

In this section, dual spaces of the space $l(R_\Phi, p)$ are given. In the sequel, there are some lemmas that are essential for proving dual spaces in the following proofs.

Lemma 2.4. *[23]*

(i) Let $1 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then, $T = (t_{mn}) \in (l(p) : l_1)$ if and only if there exists an integer $K > 1$ such that

$$\sup_{N\in\aleph} \sum_n \left| \sum_{m\in N} t_{mn} K^{-1} \right|^{p'_n} < \infty. \tag{2.5}$$

(ii) Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then, $T = (t_{mn}) \in (l(p) : l_1)$ if and only if

$$\sup_{N\in\aleph} \sup_n \left| \sum_{m\in N} t_{mn} \right|^{p_n} < \infty. \tag{2.6}$$

Lemma 2.5. *[23]*

(i) Let $1 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then, $T = (t_{mn}) \in (l(p) : l_\infty)$ if and only if there exists an integer $K > 1$ such that

$$\sup_m \sum_n \left| t_{mn} K^{-1} \right|^{p'_n} < \infty. \tag{2.7}$$

(ii) Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then, $T = (t_{mn}) \in (l(p) : l_\infty)$ if and only if

$$\sup_{m,n \in \mathbb{N}} |t_{mn}|^{p_n} < \infty. \tag{2.8}$$

Lemma 2.6. *[23]*

Let $0 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then, $T = (t_{mn}) \in (l(p) : c)$ if and only if (2.7) and (2.8) hold, and

$$\lim_m t_{mn} = c_n, \quad (n \in \mathbb{N}) \tag{2.9}$$

also holds.

Theorem 2.7. *Let $N \in \aleph$ and $K > 1$. The sets D_1^α and D_2^α are defined as:*

$$D_1^\alpha = \left\{ s = (s_n) \in \omega : \sup_{N \in \aleph} \sup_n \left| \frac{1}{q_m \varphi(m)} \sum_{n/m} \mu\left(\frac{m}{n}\right) Q_n s_m \right|^{p_n} < \infty \right\},$$

and

$$D_2^\alpha = \bigcup_{K>1} \left\{ s = (s_n) \in \omega : \sup_{N \in \aleph} \sum_n \left| \frac{1}{q_m \varphi(m)} \sum_{n/m} \mu\left(\frac{m}{n}\right) Q_n s_m K^{-1} \right|^{p'_n} < \infty \right\}.$$

$$\{l(R_\Phi, p)\}^\alpha = \begin{cases} D_1^\alpha, & \text{if } 0 < p_n \leq 1 \text{ for all } n \in \mathbb{N}; \\ \\ D_2^\alpha, & \text{if } 1 < p_n \leq P < \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Proof. Only the second case has been proved since the proof of the first case is similar. Take any $s = (s_n) \in \omega$. It is trivial from the relation between $u = (u_n)$ and $v = (v_n)$,

$$\begin{aligned} s_m u_m = & \quad s_m \frac{1}{q_m \varphi(m)} \textstyle\sum_{n/m} \mu\left(\frac{m}{n}\right) Q_n v_n \\ = & \left(\frac{1}{q_m \varphi(m)} \textstyle\sum_{n/m} \mu\left(\frac{m}{n}\right) Q_n s_m \right) v_n \\ = & \qquad B_m(v), \quad (m \in \mathbb{N}), \end{aligned} \tag{2.10}$$

where B is a matrix defined as:

$$B = b_{mn} = \begin{cases} \dfrac{1}{q_m \varphi(m)} \mu\left(\dfrac{m}{n}\right) Q_n s_m, & if \quad n|m; \\ 0, \quad otherwise, & (\forall n, m \in \mathbb{N}). \end{cases}$$

We obtain that $su = (s_n u_n) \in l_1$, where $u = (u_n) \in l(R_\Phi, p)$ if and only if $Bv \in l_1$, where $v = (v_n) \in l(p)$. This gives us $s = (s_n) \in \{l(R_\Phi, p)\}^\alpha$ if and only if $B \in (l(p) : l_1)$. Hence, from Eq (7) of Lemma 2.4 (i), we obtain $\{l(R_\Phi, p)\}^\alpha = D_2^\alpha$. □

Theorem 2.8. *Let us define the sets D_1^β, D_2^β, and D_3^β as follows:*

$$D_1^\beta = \bigcup_{K>1}\left\{ s=(s_n)\in\omega:\ \sup_m \sum_{n=1}^{m}\left|\frac{1}{q_k\varphi(k)}\sum_{k=n,n/k}^{m}\mu\left(\frac{k}{n}\right)Q_n s_k K^{-1}\right|^{p_n'}\right\},$$

$$D_2^\beta = \left\{ s=(s_n)\in\omega:\ \sup_{m,n\in\mathbb{N}}\left|\frac{1}{q_k\varphi(k)}\sum_{k=n,n/k}^{m}\mu\left(\frac{k}{n}\right)Q_n s_k\right|^{p_n}\right\},$$

and

$$D_3^\beta = \left\{ s=(s_n)\in\omega:\lim_m \frac{1}{q_k\varphi(k)}\sum_{k=n,n/k}^{m}\mu\left(\frac{k}{n}\right)Q_n s_k \text{ exists for } n\in\mathbb{N}\right\}.$$

Then, $\{l(R_\Phi,p)\}^\beta = D_1^\beta\cup D_2^\beta\cup D_3^\beta$.

Proof. Choose any $s=(s_n)\in\omega$. Since $v=(v_n)$ is the R_Φ-transform of the sequence $u=(u_n)$, we write

$$\sum_{n=1}^{m} s_n u_n = \sum_{n=1}^{m} s_n\left(\frac{1}{q_k\varphi(k)}\sum_{n/k}\mu\left(\frac{k}{n}\right)Q_n v_k\right) \tag{2.11}$$

$$= \sum_{n=1}^{m}\left(\frac{1}{q_k\varphi(k)}\sum_{k=n,n/k}^{m}\mu\left(\frac{k}{n}\right)Q_n s_k\right)v_k = D_m(v), \tag{2.12}$$

where the matrix D is a matrix defined as:

$$D = d_{mn} = \begin{cases} \dfrac{1}{q_k\varphi(k)}\sum_{k=n,n/k}^{m}\mu\left(\dfrac{k}{n}\right)Q_n s_k, & 1\le n\le m,\\ 0, & n>m,\end{cases}$$

for all $n,m\in\mathbb{N}$. From Lemma 2.6 with (2.11), it follows that $su=(s_nu_n)\in cs$, where $u=(u_n)\in l(R_\Phi,p)$ if and only if $Dv\in c$, where $v=(v_n)\in l(p)$. This gives us $s=(s_n)\in\{l(R_\Phi,p)\}^\beta$ if and only if $D\in(l(p):c)$. Hence, from (2.7)–(2.9), we conclude that

$$\{l(R_\Phi,p)\}^\beta = D_1^\beta\cup D_2^\beta\cup D_3^\beta.$$

□

Theorem 2.9.

$$\{l(R_\Phi,p)\}^\gamma = \begin{cases} D_2^\beta, & 0<p_n\le 1,\quad \forall n\in\mathbb{N};\\ D_1^\beta, & 1<p_n\le P<\infty,\quad \forall n\in\mathbb{N}.\end{cases}$$

Proof. If we use Lemma 2.5 instead of Lemma 2.6 in the proof of Theorem 2.8, we obtain the result. So, we omit the details. □

3. Conclusions

In continuation of the work presented by İlkhan and Bayrakdar [20], our focus centered on an extensive exploration of the domains governed by the Riesz-Euler totient matrix R_Φ. In this article, we introduce a novel paranormed sequence space $l(R_\Phi, p)$ constructed through the application of the Riesz-Euler totient matrix. We show that the spaces $l(R_\Phi, p)$ and $l(p)$ are linearly isomorphic. Further, we identify the dual spaces associated with this sequence space and establish its Schauder basis.

Let $p \in [1, +\infty)$. $l(R_\Phi, p)$ is a generalization of the spaces $l_p(R_\Phi)$ and $l_p(\Phi)$. In special cases, for all m and n in $\mathbb{N}$:

i) if $p_m = p$, $l(R_\Phi, p)$ is reduced to $l_p(R_\Phi)$;

ii) if $p_m = p$ and $q_n = 1$, $l(R_\Phi, p)$ is reduced to $l_p(\Phi)$.

In the future, studies can be done on the geometric properties of these spaces. Also, a more general space can be obtained by using the Jordan totient matrix instead of Riesz-Euler totient matrix.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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