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Research article

Adaptive state-feedback control for low-order stochastic nonlinear systems with an output constraint and SiISS inverse dynamics

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Abstract: This paper focuses on state-feedback adaptive control for stochastic low-order nonlinear systems with an output constraint and stochastic integral input-to-state stability (SiISS) inverse dynamics. The system with an output constraint was transformed straightforwardly into the equivalent system without a constraint using important coordinate transformations. SiISS was used to characterize unmeasured stochastic inverse dynamics. By introducing Lyapunov functions and using the stochastic systems stability theorem, we constructed a new adaptive state-feedback controller that assures the closed-loop system's trivial solution is stable in probability while fulfilling the requirements of the output constraint and all closed-loop signals are likely to be almost surely bounded. The validity of the control scheme presented in this paper was demonstrated by using simulation outcomes.

Keywords: stochastic low-order nonlinear systems; adaptive state-feedback controller; output constraint; stochastic integral input-to-state stability
Mathematics Subject Classification: 93C40, 93E15

1. Introduction

Extensive research results have been achieved in the study of nonlinear systems, for example, [1] investigated the output feedback resilient control problem of an uncertain system with two quantized signals under hybrid cyber attacks. The problem of adaptive event-triggered security tracking controller design was studied for interval type-2 (IT2) Takagi-Sugeno (T-S) fuzzy-approximation-based nonlinear networked systems in [2]. The study of stochastic nonlinear systems is equally important. It is common knowledge that stochastic nonlinear systems (SNSs) have become indispensable in order to represent many mechanical and physical processes that have stochastic perturbations. One of the key features of the SNSs is described by stochastic stability. Since the establishment and improvement of stochastic stability theory in [3], the study of stabilization issues for SNSs has advanced significantly, for example, [4] studied a finite-time adaptive tracking stability issue of SNSs with state constraints, parametric uncertainties, and input saturation. A unified fuzzy control approach for stochastic high-order nonlinear systems was examined in [5]. The fixed-time synchronization and energy consumption of Kuramoto-oscillator networks with multilayer distributed control were studied in [6]. The finite-time synchronization (FTS) of the prediction of the synchronization time and energy consumption was discussed for multilayer fractional-order networks (MFONs) in [7]. [8] mainly discussed the stabilization issue for a class of stochastic nonlinear delay systems driven by Lévy processes. However, the above references only consider SNSs with order 1 or greater than 1.

Many practical systems, like interactive liquid level systems in [9], leaky bucket systems in [10], and hydraulic control systems in [11], can be described as low-order SNSs because of the existence of stochastic disturbances and signal delays. As a result, it is essential to analyze the stability issue of low-order SNSs. Global stabilization of the low-order SNSs was examined in [12], where states were regulated by multiple time-varying delays. [13] addressed the issue of stability for a family of time-delay low-order SNSs. However, these references do not take the output constraint into account.

It is common knowledge that state/output constraints have been involved in a number of actual systems on account of hardware limitations, performance demands, or safety regulations. During the course of operation, the violation of state/output constraints will result in systems performance degradation and even result in systems becoming unstable. For example, it is necessary to constrain the joint variables of a robotic arm system to maintain its mechanical structure in [14]. Another typical practical example is [15], where the velocity of the non-holonomic vehicle is required to remain within a safe range. That is why the study of output-constrained stability for nonlinear systems is especially significant and imperative. The barrier Lyapunov function (BLF) technique first introduced in [16] was an extremely useful tool for handling state/output constraints. Stability or tracking tasks can be implemented, which assure that the output constraints cannot be violated by keeping the design of BLFs finite during operations. Recently, BLF-based methods for handling output constraints have been progressively generalized to SNSs. A finite-time stability issue about high-order SNSs with an output constraint was studied by [17]. [18] presented a prescribed-time output feedback control algorithm for cyber-physical systems under an output constraint occurring in any finite time interval and malicious attacks. Nevertheless, the approaches characterized in these references are feasible only for remarkably restricted high-order SNSs, since systems' nonlinear terms have to fulfill either a low-order growth or a high-order growth condition. By fully considering these nonlinearity properties, a breakthrough in this regard was achieved in [19], where a state-feedback controller was designed for high-order SNSs. The problem of output feedback stabilization for a class of stochastic switched planar systems (Sto-SPS) subjected to asymmetric output constraints was investigated in [20]. [21] addressed output constraints of the systems by replacing the BLF method with the coordinate transformation method, transforming the SNSs with constraints into an equivalent SNS without constraints and solving the fixed-time stability problem of high-order SNSs with output constraints. The obvious drawback, nevertheless, is that stochastic inverse dynamics is ignored in these references.

Since stochastic inverse dynamics is extensively used in a variety of engineering applications, it is a major cause of the system destabilization and affects the control systems' practical capabilities. Consequently, its examination has played an essential function in the advancement of both control theories and control technologies. For handling inverse dynamics, two of the most typical methods were recognized as input-to-state stability (ISS) proposed by [22] and integral input-to-state stability (iISS) proposed by [23]. For stochastic systems, a new concept about stochastic input-to-state stability (SISS) was described by [24, 25]. Utilising the SISS concept, [26] gave the sufficiency criterion for SISS. [27] was devoted to the global continuous control for stochastic low-order cascade nonlinear systems with time-varying delay and SISS stochastic inverse dynamics. [28] studied the adaptive state feedback stabilization problem of stochastic nonlinear systems with SISS stochastic inverse dynamics. A finite-time stabilization issue for high-order SNSs with finite-time SISS (FT-SISS) inverse dynamics was resolved in [29]. [30] aimed to investigate the global stabilization for a class of stochastic continuous time-delay nonlinear systems involving unknown control coefficients and SISS-like conditions. [31] further examined finite-time stabilization issues of time-varying low-order SNSs with FT-SISS inverse dynamics. Yet, radial unboundedness conditions need to be satisfied for the supplied rate of the SISS, thereby ruling out a number of stochastic systems with convergent properties. For that reason, [32] first expanded iISS into stochastic systems and put forward stochastic integral input to state stability (SiISS) which was rigorously weaker than the SISS. Under this framework, the research on SNSs with stochastic inverse dynamics was expanded significantly. [33] focused on the problem of adaptive state-feedback control for a class of stochastic high-order nonlinearly parameterized systems with SiISS inverse dynamics. [34] provided the research results on adaptive state-feedback control about high-order SNSs with an output constraint and SiISS inverse dynamics. Nonetheless, in low-order SNSs with SiISS inverse dynamics, the above findings do not apply to the case where the systems have output constraints, which gives a significant incentive to our research aim.

On this basis, we solve the adaptive state-feedback control issue of low-order SNSs with an output constraint and SiISS inverse dynamics. Our major contributions are emphasised below:

(i) In comparison with the above results, system models presented in the paper are more universal owing to the fact that low-order SNSs, output constraints, and SiISS inverse dynamics are considered simultaneously. Compared with the low-order SNSs with FT-SISS inverse dynamics in [30, 31], the stochastic inverse dynamics condition is relaxed to SiISS, which is a weaker restriction about the stochastic inverse dynamics. The order of the systems is different compared to the SNSs with output constraints and stochastic inverse dynamics in [21]. In this paper, we investigate the low-order SNSs.

(ii) Without using the commonly available BLFs, a coordinate transformation method is applied to convert output-constrained systems into an equivalent system without an output constraint. For this system without constraint, we construct the adaptive state-feedback controller by employing SiISS to characterize unmeasurable stochastic inverse dynamics. We incorporate the Lyapunov function and utilize stochastic stability theory to ensure that the trivial solution of the closed-loop system is stable probabilistically while satisfying the output constraint and all the closed-loop signals are almost surely bounded.

2. Problem statement and preliminaries

2.1. Problem statement

We will study low-order SNSs in this paper as follows:

$$\mathrm{d}z_0 = f_0(z_0, x_1)\mathrm{d}t + g_0^{\top}(z_0, x_1)\mathrm{d}\omega,$$

$$\begin{aligned} dx_i &= x_{i+1}^{r_i}dt + f_i(\theta, z_0, \bar{x}_i)dt + g_i^\top(\theta, z_0, \bar{x}_i)d\omega, \quad i = 1, \cdots, n-1, \\ dx_n &= u^{r_n}dt + f_n(\theta, z_0, x)dt + g_n^\top(\theta, z_0, x)d\omega, \\ y &= x_1, \end{aligned} \tag{2.1}$$

with the output constraint

$$y \in \Omega_y = \{y \in R : -\epsilon_l < y < \epsilon_l\}, \tag{2.2}$$

where $x = (x_1, \cdots, x_n)^\top \in R^n$ is a measurable state and its initial value is $x(0) = x_0$, $y \in R$ is the system output, and $u \in R$ is the control input. $\bar{x}_i = (x_1, \cdots, x_i)^\top \in R^i, i = 1, \cdots, n, \bar{x}_n = (x_1, \cdots, x_n)^\top = x$, and $z_0 = (z_{01}, \cdots, z_{0d})^\top \in R^d$ are unmeasured stochastic inverse dynamics where the initial value is $z_0(0) = \bar{z}_{0d}$. $\theta \in R^s$ is an unknown constant vector. The system power $r_i \in (0, 1)$ is an odd ratio. ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$. f_i: $R^s \times R^d \times R^i \to R$ and g_i: $R^s \times R^d \times R^i \to R^m$ are Lipschitz locally as well as disappear at the initial point. ϵ_l is a given positive constant.

2.2. Preliminaries

Some notations, definitions, and lemmas are used throughout this paper and are given below.

Notations: R^n stands for the real n-dimensional Euclidean space. For a given vector or matrix A, $A^\top$ denotes its transpose, $\mathrm{Tr}\{A\}$ denotes its trace when A is square, and $|A|$ is the Euclidean norm of a vector A. C^i denotes the set of all functions with continuous ith partial derivatives. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$, which are continuous, strictly increasing, and vanishing at zero. $\mathcal{K}_\infty$ denotes the set of all functions that are of class $\mathcal{K}$ and unbounded.

The following SNS is considered

$$dx = f(x)dt + g^\top(x)d\omega, x(0) = x_0 \in R^n, \forall t \geq 0, \tag{2.3}$$

where $x \in R^n$ is the system state, and ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$. f: $R^n \to R^n$ and g: $R^n \to R^{m\times n}$ are Lipschitz locally.

Definition 1. [24] Given $V(x) \in C^2$, we define the differential operators related to the system (2.3) $\mathcal{L}$ by $\mathcal{L}V(x) = \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\left\{g(x)\frac{\partial^2 V(x)}{\partial x^2} g^\top(x)\right\}$, where $\frac{1}{2}\mathrm{Tr}\{g(x)\frac{\partial^2 V(x)}{\partial x^2} g^\top(x)\}$ is referred to as the Hessian term of $\mathcal{L}$.

Definition 2. [25] The stochastic process $x(t)$ is almost surely bounded if $\sup_{t\geq 0} x(t) < \infty$.

In [32], SiISS was defined using Lyapunov functions. The SNSs described as follows are considered

$$dx = f(x, \upsilon, t)dt + g^\top(x, \upsilon, t)d\omega, \tag{2.4}$$

where $x \in R^n$ is the system state, $\upsilon \in R^r$ is the input, and ω is a m-dimensional standard Wiener process. f: $R^n \times R^r \times R^+ \to R^n$ and g: $R^n \times R^r \times R^+ \to R^{m\times n}$ are Lipschitz locally.

Definition 3. [32] It is said that system (2.4) is SiISS by employing Lyapunov functions, if there are functions $V \in C^2(R^n; R), \alpha, \beta, \gamma \in \mathcal{K}_\infty$, and continuous function $\delta > 0$ such that

$$\alpha(|x|) \leq V(x) \leq \beta(|x|), \mathcal{L}V \leq -\delta(|x|) + \gamma(|\upsilon|). \tag{2.5}$$

The function V fulfilling (2.5) is known as the SiISS-Lyapunov function, and (δ, γ) in (2.5) is referred to as the SiISS supply rate of system (2.4).

Lemma 1. [35] For any $x \in R, y \in R$, if $p \geq 1$, $|x+y|^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}, |x+y|^p \leq 2^{p-1}|x^p + y^p|$ hold; if $p \geq 1$ is an odd ratio, $|x-y|^p \leq 2^{p-1}|x^p - y^p|, |x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}}|x-y|^{\frac{1}{p}}$ hold.
Lemma 2. [36] Assume that there is a radially unbounded non-negative function $V(x) \in C^2$, that is, $\lim_{\|x\|\to\infty} V(x) = \infty$. For any initial value, the system (2.3) has a continuous solution on $[0,\infty)$ if the second-order differential operator $\mathcal{L}$ about (2.3) fulfils $\mathcal{L}V(x) \leq 0, \forall x \in R^n$.
Lemma 3. [33] With respect to system (2.3), there is a series of functions $V(x) \in C^2$, $W(\cdot) \geq 0$, $\alpha, \beta \in \mathcal{K}_\infty$, $c_1 > 0, c_2 \geq 0$ which makes

$$\begin{aligned} &\alpha(|x|) \leq V(x) \leq \beta(|x|), \\ &\mathcal{L}V(x) \leq -c_1 W(x) + c_2. \end{aligned} \tag{2.6}$$

It follows that there is a unique solution almost surely on $[0,\infty)$, when $c_2 = 0$, the equilibrium point $x = 0$ is globally stable in probability, and $P\{\lim_{t\to\infty} W(x) = 0\} = 1$.
Lemma 4. [37] Given $a > 0, b > 0$, for arbitrary real-valued functions $\gamma(x,y)$, $x \in R, y \in R$, it holds that

$$|x|^a |y|^b \leq \gamma(x,y)|x|^{a+b} + (\frac{b}{a+b})(\frac{a+b}{a})^{-\frac{a}{b}} \gamma^{-\frac{a}{b}}(x,y)|y|^{a+b}.$$

Lemma 5. [35] There exists a series of smoothing functions $p_1(x) \geq 0, p_2(y) \geq 0, p_3(x) \geq 1$, and $p_4(y) \geq 1$ for the provided consecutive function $p(x,y)$, which makes $|p(x,y)| \leq p_1(x) + p_2(y), |p(x,y)| \leq p_3(x)p_4(y)$.
Lemma 6. [21] For $i = 1, \cdots, n$, taking into account the known constant $a > 0$ and arbitrary $b_i \in R$, $(\sum_{i=1}^{n} |b_i|)^a \leq d_a(\sum_{i=1}^{n} |b_i|^a)$ holds, where if $a \geq 1$, then $d_a = n^{a-1}$, and if $a < 1$, then $d_a = 1$.
Remark 1. A number of significant features of SiISS have been introduced by [32]: (i) SiISS is rigorously weaker than the SISS applying Lyapunov functions from [26]; and (ii) SiISS has more enhanced minimum phase properties over [38]. Nevertheless, for certain SNSs, there is no dynamic output feedback control law to implement probabilistic global stability only under the assumption of minimum phase. (iii) SiISS implies SISS, but the inverse is not valid. Furthermore, the major distinction between SISS and SiISS lies in allowing δ in (2.5) to denote the continuous positive definite function in SiISS, rather than the $\mathcal{K}_\infty$ function in SISS.

3. Main results

3.1. Assumptions

In the paper, our objective is to construct the adaptive state-feedback controller for the low-order SNSs (2.1) with output constraint (2.2) and SiISS inverse dynamics. To accomplish our goal, some assumptions will be required as below:
Assumption 1. The Order of system (2.1) fulfils $0 < r_n \leq r_{n-1} \leq \cdots \leq r_2 \leq r_1 < 1$.
Assumption 2. There exists two constants $\mu_{ij} \geq 0$, $\bar{\mu}_{ij} \geq 0$, and a series of known smoothing non-negative functions $f_{i1}, f_{i2}, g_{i1}, g_{i2}, i = 1, \cdots, n$, such that

$$|f_i(\theta, z_0, \bar{x}_i)| \leq f_{i1}(|z_0|)|z_0|^{r_i} + \theta f_{i2}(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{r_i + \mu_{ij}},$$

$$|g_i(\theta, z_0, \bar{x}_i)| \le g_{i1}(|z_0|)|z_0|^{\frac{r_i+1}{2}} + \theta g_{i2}(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{r_i+1}{2}+\bar{\mu}_{ij}}. \tag{3.1}$$

Assumption 3. The z_0-subsystem of (2.1) is the SiISS which has x_1 as the input, i.e., there is a function $V_0(z_0) \in C^2$ which makes

$$\alpha_1(|z_0|) \le V_0(z_0) \le \alpha_2(|z_0|), \quad \mathcal{L}V_0(z_0) \le -\alpha_0(|z_0|) + \gamma_0(|x_1|), \tag{3.2}$$

where $\alpha_1, \alpha_2, \gamma_0 \in \mathcal{K}_\infty$, and α_0 is regarded as a continuous positive definite function.
Assumption 4. Some known smoothing non-negative functions ψ_0, ψ_{z_0} exist that make $|g_0(z_0, x_1)| \le \psi_0(|z_0|)$, $\left|\frac{\partial V_0}{\partial z_0}\right| \le \psi_{z_0}(|z_0|)$.
Lemma 7. [26] For the z_0-subsystem satisfying (3.2), if

$$\limsup_{s\to 0^+} \frac{\alpha(s)}{\alpha_0(s)} < \infty, \qquad \limsup_{s\to 0^+} \frac{\psi_{z_0}^2(s)\psi_0^2(s)}{\alpha_0(s)} < \infty, \tag{3.3}$$

$$\int_0^\infty [\varphi(\alpha_1^{-1}(s))]' e^{-\int_0^s [\zeta(\alpha_1^{-1}(\tau))]^{-1} d\tau} ds < \infty, \tag{3.4}$$

where $\alpha(s), \alpha_1(s) \in \mathcal{K}_\infty$, $\zeta(s) > 0$, and $\varphi(s) \ge 0$ are regarded as continuously increasing functions that are determined on $[0, \infty)$, fulfilling

$$\varphi(s)\alpha_0(s) \ge 4\alpha(s), \qquad \zeta(s)\alpha_0(s) \ge 2\psi_{z_0}^2(s)\psi_0^2(s). \tag{3.5}$$

Then one can find a non-decreasing positive function $\varrho(s) \in C^1[0, \infty)$, for any $z_0 \in R^m$, which makes

$$\varrho(V_0(z_0))\alpha_0(|z_0|) \ge 2\varrho'(V_0(z_0))\psi_{z_0}^2(|z_0|)\psi_0^2(|z_0|) + 4\alpha(|z_0|). \tag{3.6}$$

Remark 2. (i) Assumption 1 is the condition of orders, which is similar to that in [30, 31]. With respect to Assumption 2, the power of x_j from f_i can take arbitrary values on (r_i, ∞), and the power of x_j from g_i can take arbitrary values on $(\frac{r_i+1}{2}, \infty)$.

(ii) Assumption 3 indicates that the z_0-subsystem has the SiISS characteristic. In comparison with SISS, since α_0 in (3.2) is just continuous positive definite instead of $\mathcal{K}_\infty$, SiISS is a much weaker restriction on the stochastic inverse dynamics.

(iii) Within Assumption 4, the restriction on $\left|\frac{\partial V_0}{\partial z_0}\right| \le \psi_{z_0}(|z_0|)$ is a common assumption that is easily verified. $|g_0(z_0, x_1)| \le \psi_0(|z_0|)$ is the constraint on the inverse dynamics diffusion vector field, which reflects the fact that the inverse dynamics diffusion vector field is constrained by dynamics themselves, and the influence of controlled subsystem (2.1) is seen to be bounded. This second assumption is required to handle the Itô modification term of the Itô formula, and it is among the most significant distinctions between Itô stochastic systems and deterministic systems.
Remark 3. Stochastic inverse dynamics widely exist in practical systems, which are one of the main sources resulting in instability. Therefore, many stochastic nonlinear systems inevitably have SiISS stochastic inverse dynamics. For example, the z_0-subsystem in the simulation part of [34] is

$$\mathrm{d}z_0 = \left(\frac{-4z_0}{1+z_0^4} + \frac{1}{8}\left(\frac{4z_0}{1+z_0^4} + \frac{3}{2}z_0\right)x_1^4\right)\mathrm{dt} + z_0 \mathrm{d}\omega.$$

For the z_0-subsystem, by choosing $V_0(z_0) = \ln(1+z_0)^4$, then $\mathcal{L}V_0 \le -4\frac{z_0^4}{1+z_0^4} + \frac{14{,}534}{2011}\xi_1^4$. Let $\alpha_0(s) = 4\frac{1+s^4}{s^4}$ and $\gamma_0(s) = \frac{2011}{14{,}534}s^4$ and then Assumption 3 holds. The z_0-subsystem is SiISS with x_1 as the input. Therefore, Assumption 3 is achievable.

3.2. Systems transformation

In this subsection, the equivalent coordinate transformations are described first as follows:

$$x_1 = \lambda_1 \arctan(\xi_1), \quad x_i = \xi_i, \quad i = 2, \cdots, n, \tag{3.7}$$

where $\lambda_1 = \frac{2\epsilon_l}{\pi}$. Particularly, $x_1 = \lambda_1 \arctan(\xi_1)$ satisfies the characteristics below:

$$\begin{aligned} &x_1 \to -\epsilon_l, \quad \text{when } \xi_1 \to -\infty, \\ &x_1 \to \epsilon_l, \quad \text{when } \xi_1 \to \infty. \end{aligned} \tag{3.8}$$

As a result, if $\xi_1(t)$ is bounded almost surely, the constraint (2.2) is almost surely not violated. Applying (3.7), the unconstrained systems can be derived as follows:

$$\begin{aligned} dz_0 &= f_0'(z_0, \xi_1)dt + g_0'^{\top}(z_0, \xi_1)d\omega, \\ d\xi_1 &= D_1(\xi_1)\xi_2^{r_1}dt + f_1'(\theta, z_0, \xi_1)dt + g_1'^{\top}(\theta, z_0, \xi_1)d\omega, \\ d\xi_i &= \xi_{i+1}^{r_i}dt + f_i'(\theta, z_0, \bar{\xi}_i)dt + g_i'^{\top}(\theta, z_0, \bar{\xi}_i)d\omega, \quad i = 2, \cdots, n-1, \\ d\xi_n &= u^{r_n}dt + f_n'(\theta, z_0, \xi)dt + g_n'^{\top}(\theta, z_0, \xi)d\omega, \end{aligned} \tag{3.9}$$

when $D_1 = \frac{1+\xi_1^2}{\lambda_1}$, $f_0' = f_0, g_0' = g_0, f_1' = D_1 f_1 + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} g_1^{\top} g_1, g_1' = D_1 g_1, f_i' = f_i, g_i' = g_i, i = 2, \cdots, n.$

Applying (3.7), $|x_1|^{\varsigma} = |\lambda_1 \arctan(\xi_1)|^{\varsigma} \le \lambda_1^{\varsigma}|\xi_1|^{\varsigma}$, $\varsigma \in \{r_1 + \mu_{11}, \frac{1+r_1}{2} + \bar{\mu}_{11}\}$. By Lemmas 5 and 6, it yields that

$$\begin{aligned} &|f_1'(\xi_1)| \\ &\le |D_1(\xi_1)||f_1(\xi_1)| + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}|g_1(\xi_1)|^2 \\ &\le |D_1|(f_{11}|z_0|^{r_1} + \theta f_{12}|x_1|^{r_1+\mu_{11}}) + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}(g_{11}|z_0|^{\frac{r_1+1}{2}} + \theta g_{12}|x_1|^{\frac{r_1+1}{2}+\bar{\mu}_{11}})^2 \\ &\le |D_1|(f_{11}|z_0|^{r_1} + \theta f_{12}\lambda_1^{r_1}|\xi_1|^{r_1}(1+x_1^2)^{\frac{\mu_{11}}{2}}) + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}(g_{11}^2|z_0|^{r_1}(1+z_0^2)^{\frac{1}{2}} + \theta^2 g_{12}^2\lambda_1^{r_1}|\xi_1|^{r_1}(1+x_1^2)^{\frac{1+2\bar{\mu}_{11}}{2}}) \\ &\le D_{11}(\xi_1)(f_{11}'(|z_0|)|z_0|^{r_1} + \theta f_{12}'(\xi_1)|\xi_1|^{r_1} + \theta^2 f_{13}'(\xi_1)|\xi_1|^{r_1}), \end{aligned}$$

where $D_{11}, f_{11}', f_{12}', f_{13}'$ are regarded as a series of known smoothing non-negative functions. Similarly to $f_1'(\xi_1)$, there exist known smoothing non-negative functions D_{21}, g_{11}', g_{12}', which makes $|g_1'(\xi_1)| \le D_{21}(\xi_1)(g_{11}'(|z_0|)|z_0|^{\frac{r_1+1}{2}} + \theta g_{12}'(\xi_1)|\xi_1|^{\frac{r_1+1}{2}})$.

By (3.1) and Lemmas 5 and 6, we have that

$$\begin{aligned} |f_i'(\bar{\xi}_i)| &\le |f_i(\bar{\xi}_i)| \le f_{i1}'(|z_0|)|z_0|^{r_i} + \theta f_{i2}'(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{r_i}, \\ |g_i'(\bar{\xi}_i)| &\le |g_i(\bar{\xi}_i)| \le g_{i1}'(|z_0|)|z_0|^{\frac{r_i+1}{2}} + \theta g_{i2}'(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{r_i+1}{2}}, \end{aligned} \tag{3.10}$$

where $f_{i1}', f_{i2}', g_{i1}', g_{i2}', i = 2, \cdots, n$, are regarded as a series of known smoothing non-negative functions.

3.3. Controller design

Step 1: Denote $\sigma = \max_{\{1\le i\le n\}}\left\{1, \theta, \theta^2, \theta^{\frac{3+r_1}{r_1-r_i+3}}\right\}$. Selecting $z_1 = \xi_1$ and the first Lyapunov function $V_1(\xi_1, \tilde{\sigma}) = W_1(\xi_1) + \frac{1}{2}\tilde{\sigma}^2 = \frac{1}{4}z_1^4 + \frac{1}{2}\tilde{\sigma}^2$, obviously, V_1 is C^2, positive definite, and radially unbounded. $\hat{\sigma}(t)$ is regarded as the estimate of σ, and $\tilde{\sigma} = \sigma - \hat{\sigma}(t)$ is taken as the estimation error. By Definition 1 and (3.9), one has that

$$\begin{aligned}\mathcal{L}V_1 &= z_1^3(D_1(\xi_1)\xi_2^{r_1} + f_1'(\xi_1)) + \frac{3}{2}z_1^2 g_1'^{\top}(\xi_1)g_1'(\xi_1) - \tilde{\sigma}\dot{\hat{\sigma}} \\ &\le D_1 z_1^3(\xi_2^{r_1} - \xi_2^{*r_1}) + D_1 z_1^3\xi_2^{*r_1} + z_1^3|f_1'| + \frac{3}{2}z_1^2|g_1'|^2 - \tilde{\sigma}\dot{\hat{\sigma}}.\end{aligned} \tag{3.11}$$

According to the above derivation and Lemmas 4 and 5, it follows that

$$\begin{aligned}z_1^3|f_1'| &\le |z_1|^3 D_{11}(\xi_1)(f_{11}'(|z_0|)|z_0|^{r_1} + \theta f_{12}'(\xi_1)|\xi_1|^{r_1} + \theta^2 f_{13}'(\xi_1)|\xi_1|^{r_1}) \\ &\le \sigma\beta_{11}(\xi_1)z_1^{3+r_1} + \kappa_{11}(|z_0|)z_0^{3+r_1},\end{aligned} \tag{3.12}$$

where β_{11}, κ_{11} are regarded as a series of known smoothing non-negative functions. Proceeding similarly to the derivation of (3.12), we obtain known smoothing non-negative functions $\beta_{12}(\xi_1), \kappa_{12}(|z_0|)$ which make

$$\begin{aligned}\frac{3}{2}z_1^2|g_1'|^2 &\le 3|z_1|^2\Big(D_{21}(\xi_1)(g_{11}'(|z_0|)|z_0|^{\frac{r_1+1}{2}} + \theta g_{12}'(\xi_1)|\xi_1|^{\frac{r_1+1}{2}})\Big)^2 \\ &\le \sigma\beta_{12}(\xi_1)z_1^{3+r_1} + \kappa_{12}(|z_0|)z_0^{3+r_1}.\end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13), known function ν_1, and the virtual controller

$$\xi_2^* = -\left(\frac{n + \hat{\sigma}\beta_1(\xi_1) + \varphi(\xi_1)}{D}\right)^{\frac{1}{r_1}} z_1 \triangleq -\alpha_1(\xi_1, \hat{\sigma})z_1 \tag{3.14}$$

into (3.11) yields

$$\mathcal{L}V_1 \le -nz_1^{3+r_1} + D_1 z_1^3(\xi_2^{r_1} - \xi_2^{*r_1}) + (\tilde{\sigma} + \nu_1)(\beta_1 z_1^{3+r_1} - \dot{\hat{\sigma}}) + \kappa_1(|z_0|)z_0^{3+r_1} - \varphi(\xi_1)z_1^{3+r_1}, \tag{3.15}$$

where $D = \frac{1}{\lambda_1}$, $\beta_1 = \beta_{11} + \beta_{12}, \kappa_1 = \kappa_{11} + \kappa_{12}$, $\varphi(\xi_1)$ is the non-negative smoothing function to be identified, and let $\nu_1 = 0$.

Step 2: Set the second Lyapunov function $V_2(\bar{\xi}_2, \tilde{\sigma}) = V_1(\xi_1, \tilde{\sigma}) + W_2(\bar{\xi}_2) = V_1(\xi_1, \tilde{\sigma}) + \frac{1}{r_1-r_2+4}z_2^{r_1-r_2+4}$ and $z_2 = \xi_2 - \xi_2^*$. Clearly, V_2 is C^2, positive definite, and radially unbounded. Applying (3.9) and (3.15), one obtains

$$\begin{aligned}\mathcal{L}V_2 \le& -nz_1^{3+r_1} + (\tilde{\sigma} + \nu_1)(\beta_1 z_1^{3+r_1} - \dot{\hat{\sigma}}) + \kappa_1(|z_0|)z_0^{3+r_1} - \varphi(\xi_1)z_1^{3+r_1} + z_2^{r_1-r_2+3}(\xi_3^{r_2} - \xi_3^{*r_2}) \\ &+ z_2^{r_1-r_2+3}\xi_3^{*r_2} + \frac{\partial W_2}{\partial\hat{\sigma}}\dot{\hat{\sigma}} + \frac{\partial W_2}{\partial\xi_1}(D_1\xi_2^{r_1} + f_1') + z_2^{r_1-r_2+3}f_2' + \frac{1}{2}\frac{\partial^2 W_2}{\partial\xi_1^2}|g_1'|^2 \\ &+ \frac{\partial^2 W_2}{\partial\xi_1\partial\xi_2}|g_1'||g_2'| + \frac{1}{2}\frac{\partial^2 W_2}{\partial\xi_2^2}|g_2'|^2 + D_1 z_1^3(\xi_2^{r_1} - \xi_2^{*r_1}).\end{aligned} \tag{3.16}$$

Applying the definition of $W_2(\bar{\xi}_2)$, we have

$$\frac{\partial W_2}{\partial\xi_1} = -z_2^{r_1-r_2+3}\frac{\partial\xi_2^*}{\partial\xi_1}, \quad \frac{\partial W_2}{\partial\xi_2} = z_2^{r_1-r_2+3}, \quad \frac{\partial W_2}{\partial\hat{\sigma}} = -z_2^{r_1-r_2+3}\frac{\partial\xi_2^*}{\partial\hat{\sigma}},$$

$$\frac{\partial^2 W_2}{\partial \xi_1^2} = (r_1 - r_2 + 3) z_2^{r_1 - r_2 + 2} (\frac{\partial \xi_2^*}{\partial \xi_1})^2 - z_2^{r_1 - r_2 + 3} \frac{\partial^2 \xi_2^*}{\partial \xi_1^2},$$
$$\frac{\partial^2 W_2}{\partial \xi_1 \partial \xi_2} = -(r_1 - r_2 + 3) z_2^{r_1 - r_2 + 2} \frac{\partial \xi_2^*}{\partial \xi_1}, \quad \frac{\partial^2 W_2}{\partial \xi_2^2} = (r_1 - r_2 + 3) z_2^{r_1 - r_2 + 2},$$
$$\left|\frac{\partial \xi_2^*}{\partial \xi_1}\right| = \left|\frac{\partial \alpha_1}{\partial \xi_1}\xi_1 + \alpha_1\right| \le \varpi_{21}(\xi_1),$$
$$\left|\frac{\partial^2 \xi_2^*}{\partial \xi_1^2}\right| = \left|\frac{\partial^2 \alpha_1}{\partial \xi_1^2}\xi_1 + 2\frac{\partial \alpha_1}{\partial \xi_1}\right| \le \varpi_{22}(\xi_1), \tag{3.17}$$

where $\varpi_{21}(\xi_1), \varpi_{22}(\xi_1)$ are regarded as a series of known smoothing non-negative functions.

It is clear by (3.9) and Lemmas 1 and 4, that

$$\begin{aligned}
&\frac{\partial W_2}{\partial \xi_1}(D_1 \xi_2^{r_1} + f_1') \\
&\le |z_2|^{r_1 - r_2 + 3} \varpi_{21}(\xi_1) \Big(D_1 |z_2 - \alpha_1 \xi_1|^{r_1} + D_{11}(\xi_1)(f_{11}'(|z_0|)|z_0|^{r_1} + \theta f_{12}'(\xi_1)|\xi_1|^{r_1} + \theta^2 f_{13}'(\xi_1)|\xi_1|^{r_1}) \Big) \\
&\le (1 + (\varpi_{21} z_2^{r_1 - r_2})2)^{\frac{1}{2}} |z_2|^3 \Big(D_1 |z_2 - \alpha_1 z_1|^{r_1} + D_{11}(\xi_1)(f_{11}'(|z_0|)|z_0|^{r_1} + \theta f_{12}'(\xi_1)|z_1|^{r_1} + \theta^2 f_{13}'(\xi_1)|z_1|^{r_1}) \Big) \\
&\le \frac{1}{7} z_1^{3 + r_1} + \sigma \beta_{21}(\bar{\xi}_2, \hat{\sigma}) z_2^{3 + r_1} + \kappa_{21}(|z_0|) z_0^{3 + r_1},
\end{aligned} \tag{3.18}$$

where β_{21}, κ_{21} are regarded as a series of known smoothing non-negative functions.

In a similar way to the derivation of (3.18), it is apparent that

$$\begin{aligned}
z_2^{r_1 - r_2 + 3} f_2' &\le |z_2|^{r_1 - r_2 + 3} \Big(f_{21}'(|z_0|)|z_0|^{r_2} + \theta f_{22}'(\bar{\xi}_2) \sum_{l=1}^{2} |\xi_l|^{r_2} \Big) \\
&\le |z_2|^{r_1 - r_2 + 3} \Big(f_{21}' |z_0|^{r_2} + \theta f_{22}' \sum_{l=1}^{2} |z_l - \alpha_{l-1} z_{l-1}|^{r_2} \Big) \\
&\le \frac{1}{7} z_1^{3 + r_1} + \sigma \beta_{22}(\bar{\xi}_2, \hat{\sigma}) z_2^{3 + r_1} + \kappa_{22}(|z_0|) z_0^{3 + r_1},
\end{aligned} \tag{3.19}$$

where β_{22}, κ_{22} are regarded as a series of known smoothing non-negative functions.

With (3.10), (3.17) and Lemmas 1, 4, and 5, we can conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2 W_2}{\partial \xi_1^2}|g_1'|^2 \\
&\le \Big((r_1 - r_2 + 3)|z_2|^{r_1 - r_2 + 2} \varpi_{21}^2(\xi_1) + |z_2|^{r_1 - r_2 + 3} \varpi_{22}(\xi_1) \Big) \Big(D_{21}(\xi_1)(g_{11}'(|z_0|)|z_0|^{\frac{r_1 + 1}{2}} + \theta g_{12}'(\xi_1)|\xi_1|^{\frac{r_1 + 1}{2}}) \Big)^2 \\
&\le \rho_{21}(\bar{\xi}_2, z_0) \Big(\varpi_{21}^2 (1 + (z_2^{r_1 - r_2})^2)^{\frac{1}{2}} |z_2|^2 + (1 + (z_2^{r_1 - r_2 + 1} \varpi_{22})^2)^{\frac{1}{2}} |z_2|^2 \Big) \Big(|z_0|^{1 + r_1} + \theta^2 |z_1|^{1 + r_1} \Big) \\
&\le \frac{1}{7} z_1^{3 + r_1} + \sigma \beta_{23}(\bar{\xi}_2, \hat{\sigma}) z_2^{3 + r_1} + \kappa_{23}(|z_0|) z_0^{3 + r_1},
\end{aligned} \tag{3.20}$$

where $\rho_{21}, \beta_{23}, \kappa_{23}$ are regarded as a series of known smoothing non-negative functions.

With the help of (3.10), (3.17) and Lemmas 1, 4, and 5, we are able to deduce that

$$\frac{\partial^2 W_2}{\partial \xi_1 \partial \xi_2}|g_1'||g_2'|$$

$$\le (r_1 - r_2 + 3)|z_2|^{r_1-r_2+2}\varpi_{21}(\xi_1)D_{21}\big(g'_{11}|z_0|^{\frac{r_1+1}{2}} + \theta g'_{12}|\xi_1|^{\frac{r_1+1}{2}}\big)\Big(g'_{21}|z_0|^{\frac{r_2+1}{2}} + \theta g'_{22}\sum_{l=1}^{2}|z_l - \alpha_{l-1}z_{l-1}|^{\frac{r_2+1}{2}}\Big)$$
$$\le \rho_{22}(\bar{\xi}_2, z_0)(1 + (\varpi_{21}z_2^{\frac{r_1+r_2}{2}-r_2})^2)^{\frac{1}{2}}|z_2|^{r_1-\frac{r_1+r_2}{2}+2}\Big(|z_0|^{\frac{r_1+r_2}{2}+1} + \theta|z_1|^{\frac{r_1+r_2}{2}+1} + \theta|z_2|^{\frac{r_1+r_2}{2}+1}\Big)$$
$$\le \frac{1}{7}z_1^{3+r_1} + \sigma\beta_{24}(\bar{\xi}_2, \hat{\sigma})z_2^{3+r_1} + \kappa_{24}(|z_0|)z_0^{3+r_1}, \tag{3.21}$$

where $\rho_{22}, \beta_{24}, \kappa_{24}$ are regarded as a series of known smoothing non-negative functions.

In a similar way to the derivation of (3.21), we can identify a series of known smooth non-negative functions $\rho_{23}, \beta_{25}, \kappa_{25}$ such that

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 W_2}{\partial \xi_2^2}|g'_2|^2 &\le (r_1 - r_2 + 3)|z_2|^{r_1-r_2+2}\Big(g'_{21}|z_0|^{\frac{r_2+1}{2}} + \theta g'_{22}\sum_{l=1}^{2}|z_l - \alpha_{l-1}z_{l-1}|^{\frac{r_2+1}{2}}\Big)^2\\ &\le \rho_{23}(\bar{\xi}_2, z_0)|z_2|^{r_1-r_2+2}(|z_0|^{r_2+1} + \theta^2|z_1|^{r_2+1} + \theta^2|z_2|^{r_2+1})\\ &\le \frac{1}{7}z_1^{3+r_1} + \sigma\beta_{25}(\bar{\xi}_2, \hat{\sigma})z_2^{3+r_1} + \kappa_{25}(|z_0|)z_0^{3+r_1}.\end{aligned} \tag{3.22}$$

By Lemmas 1 and 4, one arrives at

$$D_1 z_1^3(\xi_2^{r_1} - \xi_2^{*r_1}) \le 2^{1-r_1}D_1|z_1|^3|z_2|^{r_1} \le \frac{1}{7}z_1^{3+r_1} + \sigma\beta_{26}(\bar{\xi}_2, \hat{\sigma})z_2^{3+r_1}, \tag{3.23}$$

where β_{26} is a known smoothing non-negative function.

Substituting (3.18)–(3.23) into (3.16) leads to

$$\begin{aligned}\mathcal{L}V_2 \le& -(n - \frac{6}{7})z_1^{3+r} + (\tilde{\sigma} + \nu_1)(\beta_1 z_1^{3+r_1} - \dot{\hat{\sigma}}) + \sum_{j=1}^{2}\kappa_j(|z_0|)z_0^{3+r_1} - \varphi(\xi_1)z_1^{3+r_1} + z_2^{r_1-r_2+3}(\xi_3^{r_2} - \xi_3^{*r_2})\\ &+ z_2^{r_1-r_2+3}\xi_3^{*r_2} + \sigma\beta_2 z_2^{3+r_1} + \frac{\partial W_2}{\partial\hat{\sigma}}\dot{\hat{\sigma}} - \hat{\sigma}\beta_2 z_2^{3+r_1} + \hat{\sigma}\beta_2 z_2^{3+r_1},\end{aligned} \tag{3.24}$$

where $\beta_2 = \sum_{j=1}^{6}\beta_{2j}$, and $\kappa_2 = \sum_{j=1}^{5}\kappa_{2j}$. It is evident by (3.17), and Lemmas 1 and 4–6, that

$$\Big|\frac{\partial W_2}{\partial\hat{\sigma}}\Big(\sum_{j=1}^{2}\beta_j z_j^{3+r_1}\Big)\Big| \le \frac{1}{7}z_1^{3+r_1} + \beta_{27}(\bar{\xi}_2, \hat{\sigma})z_2^{3+r_1}, \tag{3.25}$$

where β_{27} is a known smoothing non-negative function. Thus, we apply the known function ν_2:

$$\begin{aligned}&(\tilde{\sigma} + \nu_1)(\beta_1 z_1^{3+r_1} - \dot{\hat{\sigma}}) + \sigma\beta_2 z_2^{3+r_1} - \hat{\sigma}\beta_2 z_2^{3+r_1} + \frac{\partial W_2}{\partial\hat{\sigma}}\dot{\hat{\sigma}}\\ &= \tilde{\sigma}\beta_1 z_1^{3+r_1} + \tilde{\sigma}\beta_2 z_2^{3+r_1} - \tilde{\sigma}\dot{\hat{\sigma}} + \frac{\partial W_2}{\partial\hat{\sigma}}\dot{\hat{\sigma}} + \frac{\partial W_2}{\partial\hat{\sigma}}\sum_{j=1}^{2}\beta_j z_j^{3+r_1} - \frac{\partial W_2}{\partial\hat{\sigma}}\sum_{j=1}^{2}\beta_j z_j^{3+r_1}\\ &= \tilde{\sigma}\sum_{j=1}^{2}\beta_j z_j^{3+r_1} - \tilde{\sigma}\dot{\hat{\sigma}} - \nu_2\dot{\hat{\sigma}} + \nu_2\sum_{j=1}^{2}\beta_j z_j^{3+r_1} + \frac{\partial W_2}{\partial\hat{\sigma}}\sum_{j=1}^{2}\beta_j z_j^{3+r_1}\\ &= (\tilde{\sigma} + \nu_2)(\sum_{j=1}^{2}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}) + \frac{\partial W_2}{\partial\hat{\sigma}}\sum_{j=1}^{2}\beta_j z_j^{3+r_1}\end{aligned}$$

$$\leq (\tilde{\sigma} + \nu_2)(\sum_{j=1}^{2} \beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}) + \frac{1}{7} z_1^{3+r_1} + \beta_{27}(\bar{\xi}_2, \hat{\sigma}) z_2^{3+r_1}, \tag{3.26}$$

where $\nu_2 = -\frac{\partial W_2}{\partial \hat{\sigma}}$. Substituting (3.26) into (3.24) and using the virtual controller

$$\xi_3^* = -\Big(n - 1 + \hat{\sigma}\beta_2 + \beta_{27}\Big)^{\frac{1}{r_2}} z_2 \triangleq -\alpha_2(\bar{\xi}_2, \hat{\sigma}) z_2, \tag{3.27}$$

we have

$$\begin{aligned} \mathcal{L}V_2 \ \leq & -(n-1) \sum_{j=1}^{2} z_j^{3+r_1} + (\tilde{\sigma} + \nu_2)\Big(\sum_{j=1}^{2} \beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}\Big) + \sum_{j=1}^{2} \kappa_j(|z_0|) z_0^{3+r_1} \\ & - \varphi(\xi_1) z_1^{3+r_1} + z_2^{r_1-r_2+3}(\xi_3^{r_2} - \xi_3^{*r_2}). \end{aligned} \tag{3.28}$$

Inductive step ($3 \leq k \leq n$): Assume that in step $k-1$, $V_{k-1}(\bar{\xi}_{k-1}, \hat{\sigma}) \in C^2$ exists, where $V_{k-1}(\bar{\xi}_{k-1}, \hat{\sigma})$ is clearly positive definite and radially unbounded. The virtual controllers $\xi_1^*, \cdots, \xi_k^*$ are determined by

$$\begin{aligned} & \xi_1^* = 0, \quad z_1 = \xi_1 - \xi_1^* = \xi_1, \quad \xi_j^* = -\alpha_{j-1}(\bar{\xi}_{j-1}, \hat{\sigma}) z_{j-1}, \\ & z_j = \xi_j - \xi_j^* = \xi_j + \alpha_{j-1}(\bar{\xi}_{j-1}, \hat{\sigma}) z_{j-1}, \quad j = 2, \cdots, k, \end{aligned} \tag{3.29}$$

and then, the inequality exists as below:

$$\begin{aligned} \mathcal{L}V_{k-1} \ \leq & -(n-k+2) \sum_{j=1}^{k-1} z_j^{3+r_1} + (\tilde{\sigma} + \nu_{k-1})\Big(\sum_{j=1}^{k-1} \beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}\Big) + \sum_{j=1}^{k-1} \kappa_j(|z_0|) z_0^{3+r_1} \\ & - \varphi(\xi_1) z_1^{3+r_1} + z_{k-1}^{r_1-r_{k-1}+3}(\xi_k^{r_{k-1}} - \xi_k^{*r_{k-1}}), \end{aligned} \tag{3.30}$$

where α_j, $j = 1, \cdots, k-1$, are regarded as known smoothing non-negative functions and known function $\nu_{k-1} = -\sum_{j=2}^{k-1} \frac{\partial W_j}{\partial \hat{\sigma}}$.

Next we demonstrate that (3.30) remains applicable to step k.

Choose $V_k(\bar{\xi}_k, \hat{\sigma}) = V_{k-1}(\bar{\xi}_{k-1}, \hat{\sigma}) + W_k(\bar{\xi}_k) = V_{k-1}(\bar{\xi}_{k-1}, \hat{\sigma}) + \frac{1}{r_1 - r_k + 4} z_k^{r_1-r_k+4}$, where, obviously, V_k is C^2, positive definite, and radially unbounded. Using (3.9) and (3.30), we have

$$\begin{aligned} & \mathcal{L}V_k \\ & \leq -(n-k+2) \sum_{j=1}^{k-1} z_j^{3+r_1} + (\tilde{\sigma} + \nu_{k-1})(\sum_{j=1}^{k-1} \beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}) + \sum_{j=1}^{k-1} \kappa_j(|z_0|) z_0^{3+r_1} - \varphi(\xi_1) z_1^{3+r_1} \\ & + z_k^{r_1-r_k+3}(\xi_{k+1}^{r_k} - \xi_{k+1}^{*r_k}) + z_k^{r_1-r_k+3} \xi_{k+1}^{*r_k} + \frac{\partial W_k}{\partial \hat{\sigma}} \dot{\hat{\sigma}} + z_k^{r_1-r_k+3} f_k' + \Big(\frac{\partial W_k}{\partial \xi_1}(D_1 \xi_2^{r_1} + f_1') + \sum_{j=2}^{k-1} \frac{\partial W_k}{\partial \xi_j}(\xi_{j+1}^{r_j} + f_j') \Big) \\ & + \frac{1}{2} \sum_{i,j=1}^{k-1} \frac{\partial^2 W_k}{\partial \xi_i \partial \xi_j} |g_j'||g_i'| + \frac{1}{2} \sum_{j=1}^{k-1} \frac{\partial^2 W_k}{\partial \xi_j^2} |g_j'|^2 + \sum_{j=1}^{k-1} \frac{\partial^2 W_k}{\partial \xi_j \partial \xi_k} |g_j'||g_k'| + \frac{1}{2} \frac{\partial^2 W_k}{\partial \xi_k^2} |g_k'|^2 + z_{k-1}^{r_1-r_{k-1}+3}(\xi_k^{r_{k-1}} - \xi_k^{*r_{k-1}}). \end{aligned} \tag{3.31}$$

In addition, $W_k(\bar{\xi}_k) = \frac{1}{r_1 - r_k + 4} z_k^{r_1-r_k+4}$ is C^2, and a simple calculation yields

$$\frac{\partial W_k}{\partial \xi_j} = -z_k^{r_1-r_k+3} \frac{\partial \xi_k^*}{\partial \xi_j}, \quad \frac{\partial W_k}{\partial \xi_k} = z_k^{r_1-r_k+3}, \quad \frac{\partial W_k}{\partial \hat{\sigma}} = -z_k^{r_1-r_k+3} \frac{\partial \xi_k^*}{\partial \hat{\sigma}}$$

$$\frac{\partial^2 W_k}{\partial\xi_i\partial\xi_j} = (r_1 - r_k + 3)z_k^{r_1-r_k+2}\frac{\partial\xi_k^*}{\partial\xi_i}\frac{\partial\xi_k^*}{\partial\xi_j} - z_k^{r_1-r_k+3}\frac{\partial^2\xi_k^*}{\partial\xi_i\partial\xi_j}, \quad \frac{\partial^2 W_k}{\partial\xi_j\partial\xi_k} = -(r_1 - r_k + 3)z_k^{r_1-r_k+2}\frac{\partial\xi_k^*}{\partial\xi_j},$$
$$\frac{\partial^2 W_k}{\partial\xi_j^2} = (r_1 - r_k + 3)z_k^{r_1-r_k+2}(\frac{\partial\xi_k^*}{\partial\xi_j})^2 - z_k^{r_1-r_k+3}\frac{\partial^2\xi_k^*}{\partial\xi_j^2}, \quad \frac{\partial^2 W_k}{\partial\xi_k^2} = (r_1 - r_k + 3)z_k^{r_1-r_k+2}. \tag{3.32}$$

From (3.29) and Lemma 5,

$$\Big|\frac{\partial\xi_k^*}{\partial\xi_j}\Big| = \Big|\sum_{s=1}^{k-1}\frac{\partial(\prod_{l=s}^{k-1})\alpha_l}{\partial\xi_j}\xi_s + \alpha_{k-1}\cdots\alpha_j\Big| \le \varpi_{k1}(\bar{\xi}_{k-1}),$$
$$\Big|\frac{\partial^2\xi_k^*}{\partial\xi_j^2}\Big| = \Big|\sum_{s=1}^{k-1}\frac{\partial^2(\prod_{l=s}^{k-1})\alpha_l}{\partial\xi_j^2}\xi_s + 2\frac{\partial(\alpha_{k-1}\cdots\alpha_j)}{\partial\xi_j}\Big| \le \varpi_{k2}(\bar{\xi}_{k-1}),$$
$$\Big|\frac{\partial^2\xi_k^*}{\partial\xi_i\partial\xi_j}\Big| = \Big|\sum_{s=1}^{k-1}\frac{\partial^2(\prod_{l=s}^{k-1})\alpha_l}{\partial\xi_i\partial\xi_j}\xi_s + \frac{\partial(\alpha_{k-1}\cdots\alpha_j)}{\partial\xi_j} + \frac{\partial(\alpha_{k-1}\cdots\alpha_i)}{\partial\xi_j}\Big| \le \varpi_{k3}(\bar{\xi}_{k-1}), \tag{3.33}$$

where $\varpi_{k1}(\bar{\xi}_{k-1}), \varpi_{k2}(\bar{\xi}_{k-1}), \varpi_{k3}(\bar{\xi}_{k-1})$ are regarded as known smoothing non-negative functions.

Via (3.10) and Lemma 4, a number of known smoothing non-negative functions β_{k1}, κ_{k1} exist that make

$$\begin{aligned} z_k^{r_1-r_k+3}f_k' &\le |z_k|^{r_1-r_k+3}\Big(f_{k1}'(|z_0|)|z_0|^{r_k} + \theta f_{k2}'(\bar{\xi}_k)\sum_{j=1}^{k}|\xi_j|^{r_k}\Big) \\ &\le \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \sigma\beta_{k1}((\bar{\xi}_k, \hat{\sigma}))z_k^{3+r_1} + \kappa_{k1}(|z_0|)z_0^{3+r_1}. \end{aligned} \tag{3.34}$$

It is deduced from (3.10), (3.32), (3.33), and Lemmas 4–6 that

$$\begin{aligned} &\Big(\frac{\partial W_k}{\partial\xi_1}(D_1\xi_2^{r_1} + f_1') + \sum_{j=2}^{k-1}\frac{\partial W_k}{\partial\xi_j}(\xi_{j+1}^{r_j} + f_j')\Big) \\ &\le |z_k|^{r_1-r_k+3}\varpi_{k1}(\bar{\xi}_{k-1})\Big(D_1|z_2 - \alpha_1 z_1|^{r_1} + D_{11}(f_{11}'|z_0|^{r_1} + \theta f_{12}'|z_1|^{r_1} + \theta^2 f_{13}'|z_1|^{r_1})\Big) \\ &\quad + \sum_{j=2}^{k-1}|z_k|^{r_1-r_k+3}\varpi_{k1}(\bar{\xi}_{k-1})\Big(|z_{j+1} - \alpha_j z_j|^{r_j} + f_{j1}'|z_0|^{r_j} + \theta f_{j2}'\sum_{l=1}^{j}|z_l - \alpha_{l-1}z_{l-i}|^{r_j}\Big) \\ &\le \rho_{k1}(\xi_k, z_0)\Big((1 + (\varpi_{k1}z_k^{r_1-r_k})^2)^{\frac{1}{2}}|z_k|^3(|z_0|^{r_1} + \theta|z_1|^{r_1} + \theta^2|z_1|^{r_1}) \\ &\quad + \sum_{j=2}^{k-1}(1 + (\varpi_{k1}z_k^{r_j-r_k})^2)^{\frac{1}{2}}|z_k|^{r_1-r_j+3}(|z_0|^{r_j} + \theta\sum_{l=1}^{j}|z_l|^{r_j} + |z_{j+1}|^{r_j})\Big) \\ &\le \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \sigma\beta_{k2}(\bar{\xi}_k, \hat{\sigma})z_k^{3+r_1} + \kappa_{k2}(|z_0|)z_0^{3+r_1}, \end{aligned} \tag{3.35}$$

where $\rho_{k1}, \beta_{k2}, \kappa_{k2}$ are regarded as known smoothing non-negative functions.

With the help of (3.10), (3.32), (3.33), and Lemmas 4–6, it is possible to identify known smoothing non-negative functions $\rho_{k2}, \beta_{k3}, \kappa_{k3}$ such that

$$\frac{1}{2}\sum_{i,j=1}^{k-1}\frac{\partial^2 W_k}{\partial\xi_i\partial\xi_j}|g_j'||g_i'|$$

$$
\begin{aligned}
&\le \sum_{i,j=1}^{k-1}\Big((r_1-r_k+3)|z_k|^{r_1-r_k+2}\varpi_{k1}^2+|z_k|^{r_1-r_k+3}\varpi_{k3}\Big)\\
&\quad\times\left(g'_{j1}|z_0|^{\frac{r_j+1}{2}}+\theta g'_{j2}\sum_{l=1}^{j}|\xi_l|^{\frac{r_j+1}{2}}\right)\left(g'_{i1}|z_0|^{\frac{r_i+1}{2}}+\theta g'_{i2}\sum_{l=1}^{i}|\xi_l|^{\frac{r_i+1}{2}}\right)\\
&\le \rho_{k2}(\bar{\xi}_k,z_0)\sum_{i,j=1}^{k-1}\left(\varpi_{k1}^2(1+(z_k^{\frac{r_i+r_j}{2}-r_k})^2)^{\frac12}|z_k|^{r_1-\frac{r_i+r_j}{2}+2}+(1+(\varpi_{k3}z_k^{\frac{r_i+r_j}{2}-r_k+1})^2)^{\frac12}|z_k|^{r_1-\frac{r_i+r_j}{2}+2}\right)\\
&\quad\times\left(|z_0|^{\frac{r_i+r_j}{2}+1}+\theta|z_1|^{\frac{r_i+r_j}{2}+1}+\cdots+\theta|z_j|^{\frac{r_i+r_j}{2}+1}+\cdots+\theta|z_i|^{\frac{r_i+r_j}{2}+1}\right)\\
&\le \frac18\sum_{j=1}^{k-1}z_j^{3+r_1}+\sigma\beta_{k3}(\bar{\xi}_k,\hat{\sigma})z_k^{3+r_1}+\kappa_{k3}(|z_0|)z_0^{3+r_1}. \qquad (3.36)
\end{aligned}
$$

In a similar way to (3.36), there exist known smoothing non-negative functions $\rho_{k3}, \beta_{k4}, \kappa_{k4}$ such that

$$
\begin{aligned}
&\frac12\sum_{j=1}^{k-1}\frac{\partial^2 W_k}{\partial\xi_j^2}|g'_j|^2\\
&\le \sum_{j=1}^{k-1}\Big((r_1-r_k+3)|z_k|^{r_1-r_k+2}\varpi_{k1}^2+|z_k|^{r_1-r_k+3}\varpi_{k2}\Big)\left(g'_{j1}|z_0|^{\frac{r_j+1}{2}}+\theta g'_{j2}\sum_{l=1}^{j}|\xi_l|^{\frac{r_j+1}{2}}\right)^2\\
&\le \rho_{k3}(\bar{\xi}_k,z_0)\sum_{j=1}^{k-1}\left(\varpi_{k1}^2(1+(z_k^{r_j-r_k})^2)^{\frac12}|z_k|^{r_1-r_j+2}+(1+(\varpi_{k2}z_k^{r_j-r_k+1})^2)^{\frac12}|z_k|^{r_1-r_j+2}\right)\\
&\quad\times\left(|z_0|^{r_j+1}+\theta\sum_{l=1}^{j}|z_l|^{r_j+1}\right)\\
&\le \frac18\sum_{j=1}^{k-1}z_j^{3+r_1}+\sigma\beta_{k4}(\bar{\xi}_k,\hat{\sigma})z_k^{3+r_1}+\kappa_{k4}(|z_0|)z_0^{3+r_1}. \qquad (3.37)
\end{aligned}
$$

It is clear by (3.10), (3.32), (3.33), and Lemmas 4–6 that

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\frac{\partial^2 W_k}{\partial\xi_j\partial\xi_k}|g'_j||g'_k|\\
&\le \sum_{j=1}^{k-1}(r_1-r_k+3)|z_k|^{r_1-r_k+2}\varpi_{k1}\left(g'_{j1}|z_0|^{\frac{r_j+1}{2}}+\theta g'_{j2}\sum_{l=1}^{j}|\xi_l|^{\frac{r_j+1}{2}}\right)\left(g'_{k1}|z_0|^{\frac{r_k+1}{2}}+\theta g'_{k2}\sum_{l=1}^{k}|\xi_l|^{\frac{r_k+1}{2}}\right)\\
&\le \rho_{k4}(\bar{\xi}_k,z_0)\sum_{j=1}^{k-1}(1+(\varpi_{k1}z_k^{\frac{r_j+r_k}{2}-r_k})^2)^{\frac12}|z_k|^{r_1-\frac{r_j+r_k}{2}+2}\\
&\quad\times\left(|z_0|^{\frac{r_j+r_k}{2}+1}+\theta|z_1|^{\frac{r_j+r_k}{2}+1}+\cdots+\theta|z_j|^{\frac{r_j+r_k}{2}+1}+\cdots+\theta|z_k|^{\frac{r_j+r_k}{2}+1}\right)\\
&\le \frac18\sum_{j=1}^{k-1}z_j^{3+r_1}+\sigma\beta_{k5}(\bar{\xi}_k,\hat{\sigma})z_k^{3+r_1}+\kappa_{k5}(|z_0|)z_0^{3+r_1}, \qquad (3.38)
\end{aligned}
$$

where $\rho_{k4}, \beta_{k5}, \kappa_{k5}$ are regarded as known smoothing non-negative functions.

With the help of (3.10), (3.32), and Lemmas 4 and 5, there exist a series of known smoothing non-negative functions β_{k3}, κ_{k6} that make

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2 W_k}{\partial \xi_k^2}|g_k'|^2 \\
&\le (r_1 - r_k + 3)|z_k|^{r_1 - r_k + 2}\Big(g_{k1}'|z_0|^{\frac{r_k+1}{2}} + \theta g_{k2}' \sum_{l=1}^{k} |\xi_l|^{\frac{r_k+1}{2}}\Big)^2 \\
&\le \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \sigma\beta_{k6}(\bar{\xi}_k, \hat{\sigma}) z_k^{3+r_1} + \kappa_{k6}(|z_0|) z_0^{3+r_1}.
\end{aligned} \tag{3.39}$$

We observe from (3.29) and Lemmas 1 and 4 that

$$\begin{aligned}
z_{k-1}^{r_1 - r_{k-1} + 3}(\xi_k^{r_{k-1}} - \xi_k^{*r_{k-1}}) &\le 2^{1 - r_{k-1}}|z_{k-1}|^{r_1 - r_{k-1} + 3}|\xi_k - \xi_k^*|^{r_{k-1}} \le 2^{1 - r_{k-1}}|z_{k-1}|^{r_1 - r_{k-1} + 3}|z_k|^{r_{k-1}} \\
&\le \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \sigma\beta_{k7}(\bar{\xi}_k, \hat{\sigma}) z_k^{3+r_1},
\end{aligned} \tag{3.40}$$

where β_{k7} is a known smoothing non-negative function.

Substituting (3.34)–(3.40) into (3.31) yields

$$\begin{aligned}
\mathcal{L}V_k &\le -(n - k + \frac{9}{8})\sum_{j=1}^{k-1} z_j^{3+r_1} + (\tilde{\sigma} + \nu_{k-1})(\sum_{j=1}^{k-1}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}) + \sum_{j=1}^{k}\kappa_j(|z_0|) z_0^{3+r_1} - \varphi(\xi_1) z_1^{3+r_1} \\
&\quad + z_k^{r_1 - r_k + 3}(\xi_{k+1}^{r_k} - \xi_{k+1}^{*r_k}) + z_k^{r_1 - r_k + 3}\xi_{k+1}^{*r_k} + \frac{\partial W_k}{\partial \hat{\sigma}}\dot{\hat{\sigma}} + \sigma\beta_k z_k^{3+r_1} + \hat{\sigma}\beta_k z_k^{3+r_1} - \hat{\sigma}\beta_k z_k^{3+r_1} \\
&\le -(n - k + \frac{9}{8})\sum_{j=1}^{k-1} z_j^{3+r_1} + (\tilde{\sigma} + \nu_{k-1})(\sum_{j=1}^{k-1}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}) + \sum_{j=1}^{k}\kappa_j(|z_0|) z_0^{3+r_1} - \varphi(\xi_1) z_1^{3+r_1} \\
&\quad + z_k^{r_1 - r_k + 3}(\xi_{k+1}^{r_k} - \xi_{k+1}^{*r_k}) + z_k^{r_1 - r_k + 3}\xi_{k+1}^{*r_k} + \frac{\partial W_k}{\partial \hat{\sigma}}\dot{\hat{\sigma}} + \tilde{\sigma}\beta_k z_k^{3+r_1} + \hat{\sigma}\beta_k z_k^{3+r_1},
\end{aligned} \tag{3.41}$$

where $\beta_k = \sum_{j=1}^{7}\beta_{kj}, \kappa_k = \sum_{j=1}^{6}\kappa_{kj}$. Applying (3.32), and Lemmas 1 and 4–6, a known non-negative smoothing function β_{k8} exists such that

$$\Big| -\nu_{k-1}\beta_k z_k^{3+r_1} + \frac{\partial W_k}{\hat{\sigma}}\sum_{j=1}^{k}\beta_j z_j^{3+r_1}\Big| \le \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \beta_{k8}(\bar{\xi}_k, \hat{\sigma}) z_k^{3+r_1}, \tag{3.42}$$

and

$$\begin{aligned}
&(\tilde{\sigma} + \nu_{k-1})\Big(\sum_{j=1}^{k-1}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}\Big) + \frac{\partial W_k}{\partial \hat{\sigma}}\dot{\hat{\sigma}} + \tilde{\sigma}\beta_k z_k^{3+r_1} \\
&= \tilde{\sigma}\sum_{j=1}^{k-1}\beta_j z_j^{3+r_1} + \nu_{k-1}\sum_{j=1}^{k-1}\beta_j z_j^{3+r_1} - \tilde{\sigma}\dot{\hat{\sigma}} - \nu_{k-1}\dot{\hat{\sigma}} + \frac{\partial W_k}{\partial \hat{\sigma}}\dot{\hat{\sigma}} + \tilde{\sigma}\beta_k z_k^{3+r_1} \\
&\quad + \nu_{k-1}\beta_k z_k^{3+r_1} - \nu_{k-1}\beta_k z_k^{3+r_1} + \frac{\partial W_k}{\partial \hat{\sigma}}\sum_{j=1}^{k}\beta_j z_j^{3+r_1} - \frac{\partial W_k}{\partial \hat{\sigma}}\sum_{j=1}^{k}\beta_j z_j^{3+r_1}
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\sigma}\sum_{j=1}^{k}\beta_j z_j^{3+r_1} + \nu_k\sum_{j=1}^{k}\beta_j z_j^{3+r_1} - \tilde{\sigma}\dot{\hat{\sigma}} - \nu_k\dot{\hat{\sigma}} - \nu_{k-1}\beta_k z_k^{3+r_1} + \frac{\partial W_k}{\hat{\sigma}}\sum_{j=1}^{k}\beta_j z_j^{3+r_1} \\
&\le (\tilde{\sigma}+\nu_k)\Big(\sum_{j=1}^{k}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}\Big) + \frac{1}{8}\sum_{j=1}^{k-1} z_j^{3+r_1} + \beta_{k8}(\bar{\xi}_k,\hat{\sigma})z_k^{3+r_1},
\end{aligned} \tag{3.43}
$$

where known function $\nu_k = -\sum_{j=2}^{k}\frac{\partial W_j}{\partial\hat{\sigma}}$. Substituting (3.43) and the virtual controller

$$
\xi_{k+1}^{*} = -\Big(n-k+1+\hat{\sigma}\beta_k+\beta_{k8}\Big)^{\frac{1}{r_k}} z_k \triangleq -\alpha_k(\bar{\xi}_k,\hat{\sigma})z_k \tag{3.44}
$$

into (3.41) leads to

$$
\begin{aligned}
\mathcal{L}V_k \le& -(n-k+1)\sum_{j=1}^{k} z_j^{3+r_1} + (\tilde{\sigma}+\nu_k)\Big(\sum_{j=1}^{k}\beta_j z_j^{3+r_1} - \dot{\hat{\sigma}}\Big) + \sum_{j=1}^{k}\kappa_j(|z_0|)z_0^{3+r_1} \\
&- \varphi(\xi_1)z_1^{3+r_1} + z_k^{r_1-r_k+3}(\xi_{k+1}^{r_k} - \xi_{k+1}^{*r_k}).
\end{aligned} \tag{3.45}
$$

As a result, (3.30) remains applicable to step k.

For step n, choose $V_n(\xi,\tilde{\sigma}) = V_{n-1}(\bar{\xi}_{n-1},\tilde{\sigma}) + W_n(\xi)$, which is C^2, positive definite, and radially unbounded. With the adoption of the adaptive controller

$$
\dot{\hat{\sigma}} = \sum_{j=1}^{n}\beta_j z_j^{3+r_1}, \tag{3.46}
$$

$$
u = \xi_{n+1}^{*}(\xi,\hat{\sigma}) = -\alpha_n(\xi,\hat{\sigma})z_n, \tag{3.47}
$$

we have

$$
\mathcal{L}V_n \le -\sum_{j=1}^{n} z_j^{3+r_1} + \kappa(|z_0|)z_0^{3+r_1} - \varphi(\xi_1)z_1^{3+r_1}, \tag{3.48}
$$

where $\kappa(|z_0|) = \sum_{j=1}^{k}\kappa_j(|z_0|)$.

Remark 4. In the theoretical derivation process of this article, we set the system parameters to a series of known non-negative smooth functions, such as $\beta(\cdot),\kappa(\cdot),\nu(\cdot),\rho(\cdot),\varpi(\cdot)$, and so on. To simplify the derivation process, functions such as $\beta(\cdot),\kappa(\cdot),\nu(\cdot),\rho(\cdot),\varpi(\cdot)$ are set as abstract expressions without providing specific expressions. In the actual derivation process, these functions can provide specific expressions that satisfy the conditions, such as examples in simulation. If the specific expressions of these functions are too complex, it will reduce the performance of the systems, thereby slowing down the convergence speed of the systems. Therefore, in practice, we usually set simple function expressions for such functions to improve the convergence speed of the systems.

3.4. Stability analysis

Next we use a theorem to declare the major consequence of the paper.

Theorem 1. For system (2.1) with the constraint (2.2), if Assumptions 1–4, (3.4), (3.5), $\liminf_{s\to\infty}\alpha_0(s)=\infty$, $\limsup_{s\to 0^+}\frac{\kappa(s)s^{3+r_1}}{\alpha_0(s)}<\infty$, $\limsup_{s\to 0^+}\frac{\bar{\gamma}_0(s)}{s^{3+r_1}}<\infty$, and $\limsup_{s\to 0^+}\frac{\psi_{z_0}^2(s)\psi_0^2(s)}{\alpha_0(s)}<\infty$ hold, then the adaptive controller (3.46)–(3.47) exists, which makes, for any initial value $(z_0^\top(0),\sum_{j=1}^{n}x_j^\top(0))^\top\in R^d\times\Omega_x$, where $\bar{\gamma}_0(s)=\gamma_0(\lambda_1\cdot s)$, $\Omega_x=\{x:x\in R^n \text{ with } -\epsilon_l<x_1=y<\epsilon_l\}$:

(1) The systems (2.1), (3.7), (3.46), and (3.47) have the continuously unique solution almost surely on $[0,\infty)$;

(2) All signals are almost surely bounded, and the constraints $y(t)$ are almost surely not violated;

(3) The closed-loop system's equilibrium point is stable in probability, $P\{\lim_{t\to\infty}(|z_0(t)|+\sum_{j=1}^{n}|x_j(t)|)=0\}=1$, and $P\{\lim_{t\to\infty}\hat{\sigma}$ exists and is finite$\}=1$.

Proof. (1) Assume that $\varrho(s)\in C^1[0,\infty)$ is a non-decreasing positive function as defined in Lemma 7 and select

$$V_{z_0}(z_0)=\int_0^{V_0(z_0)}\varrho_(s)ds. \tag{3.49}$$

It is clear by the tangent function's definition that

$$\left|\arctan(\xi_1)\right|\le\left|\xi_1\right|,\forall\xi_1\in R. \tag{3.50}$$

Using (3.7), (3.50), and Assumption 3, one has

$$\mathcal{L}V_0\le-\alpha_0(|z_0|)+\gamma_0(|x_1|)\le-\alpha_0(|z_0|)+\bar{\gamma}_0(|\xi_1|), \tag{3.51}$$

where $\bar{\gamma}_0(s)=\gamma_0(\lambda_1\cdot s)$. According to Itô's formula, (3.49), (3.51), and Assumption 4, it is evident that

$$\begin{aligned}\mathcal{L}V_{z_0}&=\varrho(V_0(z_0))\frac{\partial V_0}{\partial z_0}f_0+\frac{1}{2}\varrho'(V_0(z_0))\Big|\frac{\partial V_0}{\partial z_0}g_0\Big|^2+\frac{1}{2}\varrho(V_0(z_0))\frac{\partial^2 V_0}{\partial z_0^2}g_0^\top g_0\\&=\varrho(V_0(z_0))\mathcal{L}V_0+\frac{1}{2}\varrho'(V_0(z_0))\Big|\frac{\partial V_0}{\partial z_0}g_0\Big|^2\\&\le\varrho(V_0(z_0))(-\alpha_0(|z_0|)+\bar{\gamma}_0(|\xi_1|))+\frac{1}{2}\varrho'(V_0(z_0))\psi_{z_0}(|z_0|)^2\psi_0^2(|z_0|).\end{aligned} \tag{3.52}$$

Since $\alpha_0(s)$ satisfies $\liminf_{s\to\infty}\alpha_0(s)=\infty$, there exists a function $\bar{\alpha}_0(s)\in\mathcal{K}_\infty$, such that $\bar{\alpha}_0(s)\le\alpha_0(s),\forall s\ge 0$. Now, we justify the inequality below in two cases:

$$\varrho(V_0(z_0))(-\alpha_0(|z_0|)+\bar{\gamma}_0(|\xi_1|))\le\varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|)-\frac{1}{2}\varrho(V_0(z_0))\alpha_0(|z_0|), \tag{3.53}$$

where $\eta(|\xi_1|)=\alpha_2(\bar{\alpha}_0^{-1}(2\bar{\gamma}_0(|\xi_1|)))$.

Case (i): When $\bar{\gamma}_0(|\xi_1|)\le\frac{1}{2}\alpha_0(|z_0|)$, one has

$$\begin{aligned}&\varrho(V_0(z_0))(-\alpha_0(|z_0|)+\bar{\gamma}_0(|\xi_1|))\\&\le\varrho(V_0(z_0))(-\frac{1}{2}\alpha_0(|z_0|)-\frac{1}{2}\alpha_0(|z_0|)+\bar{\gamma}_0(|\xi_1|))\end{aligned}$$

$$
\begin{aligned}
&\le \varrho(V_0(z_0))(-\frac{1}{2}\alpha_0(|z_0|) - \bar{\gamma}_0(|\xi_1|) + \bar{\gamma}_0(|\xi_1|)) \\
&\le -\varrho(V_0(z_0))\frac{1}{2}\alpha_0(|z_0|) \\
&\le \varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|) - \frac{1}{2}\varrho(V_0(z_0))\alpha_0(|z_0|).
\end{aligned}
$$

Case (ii): When $\bar{\gamma}_0(|\xi_1|) \ge \frac{1}{2}\alpha_0(|z_0|)$, it is easy to get $|z_0| \le \alpha_0^{-1}(2\bar{\gamma}_0(|\xi_1|))$ and $V_0(z_0) \le \alpha_2(|z_0|) \le \alpha_2(\alpha_0^{-1}(2\bar{\gamma}_0(|\xi_1|))) \le \alpha_2(\bar{\alpha}_0^{-1}(2\bar{\gamma}_0(|\xi_1|))) = \eta(|\xi_1|)$. By the monotonicity of ϱ, we have

$$
\begin{aligned}
&\varrho(V_0(z_0))(-\alpha_0(|z_0|) + \bar{\gamma}_0(|\xi_1|)) \\
&\le \varrho(\alpha_2(\bar{\alpha}_0^{-1}(2\bar{\gamma}_0(|\xi_1|))))\bar{\gamma}_0(|\xi_1|)) - \varrho(V_0(z_0))\alpha_0(|z_0|) \\
&\le \varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|) - \frac{1}{2}\varrho(V_0(z_0))\alpha_0(|z_0|).
\end{aligned}
$$

By combing these two cases, (3.53) holds. It can be deduced from (3.52) and (3.53) that

$$
\mathcal{L}V_{z_0} \le \varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|) - \frac{1}{2}\varrho(V_0(z_0))\alpha_0(|z_0|) + \frac{1}{2}\varrho'(V_0(z_0))\psi_{z_0}(|z_0|)^2\psi_0^2(|z_0|). \tag{3.54}
$$

Set $V(z_0, \xi, \tilde{\sigma}) = V_n(\xi, \tilde{\sigma}) + V_{z_0}(z_0)$, which is C^2, positive definite, and radially unbounded. Using (3.48) and (3.54), it is clear that

$$
\begin{aligned}
\mathcal{L}V \le &-\sum_{j=1}^{n} z_j^{3+r_1} + \kappa(|z_0|)z_0^{3+r_1} - \varphi(\xi_1)|\xi_1|^{3+r_1} + \varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|) \\
&- \frac{1}{4}\varrho(V_0(z_0))\alpha_0(|z_0|) + \frac{1}{2}\varrho'(V_0(z_0))\psi_{z_0}(|z_0|)^2\psi_0^2(|z_0|) - \frac{1}{4}\varrho(V_0(z_0))\alpha_0(|z_0|).
\end{aligned} \tag{3.55}
$$

Since $\limsup_{s\to 0^+} \frac{\bar{\gamma}_0(s)}{s^{3+r_1}} < \infty$, there is a smoothing non-negative function $\varphi(s)$ such that

$$
\varrho(\eta(|\xi_1|))\bar{\gamma}_0(|\xi_1|) \le \varphi(\xi_1)|\xi_1|^{3+r_1}. \tag{3.56}
$$

Due to $\limsup_{s\to 0^+} \frac{\kappa(s)s^{3+r_1}}{\alpha_0(s)} < \infty$, $\limsup_{s\to 0^+} \frac{\psi_{z_0}^2(s)\psi_0^2(s)}{\alpha_0(s)} < \infty$, (3.4), and (3.5), by Lemma 7, one has

$$
\frac{1}{4}\varrho(V_0(z_0))\alpha_0(|z_0|) \ge \frac{1}{2}\varrho'(V_0(z_0))\psi_{z_0}^2(|z_0|)\psi_0^2(|z_0|) + \kappa(|z_0|)z_0^{3+r_1}. \tag{3.57}
$$

Substituting (3.56) and (3.57) into (3.55) results in

$$
\mathcal{L}V \le -\sum_{j=1}^{n} z_j^{3+r_1} - \frac{1}{4}\varrho(V_0(z_0))\alpha_0(|z_0|) \le 0. \tag{3.58}
$$

Denote $\chi(t) = [z_0(t)^\top, \xi(t)^\top, \tilde{\sigma}]$, and $V(\chi)$ is C^2, positive definite, and radially unbounded. The existence of two functions $\alpha, \beta \in \mathcal{K}_\infty$ makes

$$
\alpha(|\chi|) \le V(\chi) \le \beta(|\chi|). \tag{3.59}
$$

We can derive using (3.58), (3.59), and Lemmas 2 and 3 that the closed-loop systems (2.1), (3.7), (3.46), and (3.47) almost surely have a continuous unique solution on $[0, \infty)$.

(2) Set the stopping time $\tau_k = \inf\{t \geq 0; |\chi(t)| \geq k\}$, $k \in \{2, 3, 4, \cdots\}$. Utilizing (3.58) and Itô's formula, we obtain an expression as follows:

$$EV(\chi(\tau_k \wedge t)) = V(\chi(0)) + E\int_0^{\tau_k \wedge t} \mathcal{L}V(\chi(s))ds \leq V(\chi(0)). \tag{3.60}$$

With the help of the definition of τ_k, one gets

$$\begin{aligned} EV(\chi(\tau_k \wedge t)) &\geq \int_{\{\sup_{0\leq s\leq t}|\chi(s)|>k\}} V(\chi(\tau_k \wedge t))\mathrm{d}P \\ &= \int_{\{\sup_{0\leq s\leq t}|\chi(s)|>k\}} V(\chi(\tau_k))\mathrm{d}P \\ &\geq P\{\sup_{0\leq s\leq t} \left|\chi(s)\right| > k\} \inf_{|\chi|\geq k} V(\chi) \\ &\geq P\{\sup_{0\leq s\leq t} \left|\chi(s)\right| > k\} \inf_{|\chi|\geq k} \alpha(|\chi|), \quad \forall t > 0. \end{aligned} \tag{3.61}$$

Substituting (3.61) into (3.60) yields

$$P\{\sup_{0\leq s\leq t} \left|\chi(s)\right| > k\} \leq \frac{V(\chi(0))}{\inf_{|\chi|\geq k} \alpha(|\chi|)}, \quad \forall t > 0. \tag{3.62}$$

Let $t \to \infty$ and $k \to \infty$, and utilize the radial unboundedness of $\alpha(|\chi|)$ to obtain $P\{\sup_{t\geq 0} |\chi(t)| < \infty\} = 1$. Consequently, $\chi(t)$, $\sum_{j=1}^{n} \xi_j(t), \tilde{\sigma}(t), z_0(t)$ are almost surely bounded, in the same way that $\sum_{j=1}^{n} x_j(t)$ is bounded. Remembering this, and employing the definitions of $\sum_{j=2}^{n} \xi_j^*(t)$, $u(t)$, we can demonstrate the boundedness of $\sum_{j=2}^{n} \xi_j^*(t), u(t)$. The constraint (3.8) is almost surely fulfilled on the basis of (2.2) and the almost surely boundedness of $\xi_1(t)$.

(3) By (3.58), (3.59), and Lemma 3, $P\{\lim_{t\to\infty}(|z_0(t)| + \sum_{j=1}^{n} |x_j(t)| = 0\} = 1$ holds. Since the fact that (3.7) is an equivalent coordinate transformation, the closed-loop system's equilibrium point is stable in probability and $P\{\lim_{t\to\infty}(|z_0(t)| + \sum_{j=1}^{n} |x_j(t)| = 0\} = 1$. Through (3.46) and Theorem 1's proof in [39], it is available that $P\{\lim_{t\to\infty} \hat{\sigma}$ exists and is finite $\} = 1$.

Remark 5. In comparison with the high-order ($r_i \geq 1$) SNSs with stochastic inverse dynamics in [29], one of the major distinctions lies in constraint conditions on f_j and g_j, as well as the selection of the Lyapunov functions. Throughout the paper, to ensure that V_n is C^2, ξ_j^* of formula (3.29) ought to be C^2. We cannot assure that V_n is C^2 if we use the assumptions of nonlinear functions and Lyapunov functions of high-order SNSs in [29]. Therefore, the stability issues of high-order SNSs and low-order SNSs are two completely distinct issues. In the paper, the stability issue of low-order SNSs with an output constraint and stochastic inverse dynamics can be solved by employing new nonlinear function assumptions, choosing new Lyapunov functions, and using the stability theorem of stochastic systems.

Remark 6. We should show that radial unboundedness about the Lyapunov function V is essential, that is, the existence of function $V \in C^2$ and two functions $\alpha, \beta \in \mathcal{K}_\infty$ with $\alpha \leq V \leq \beta$. Using the barrier Lyapunov function $V_1(z_1) = \log(\frac{k_{b_1}^4}{k_{b_1}^4 - z_1^4})$ from [4] as an example, it can be easily seen that V_1

is not a radial unbounded function. Although the BLF V_1 can efficiently resolve the issue of output constraint control, it makes the entire Lyapunov function V not a radially unbounded function. The stability analysis cannot be performed using Theorem 1 in [4] since the BLF is used in the controlling scheme.

In order to address the fatal issue, this paper uses (3.7) to convert the original system (2.1) with output constraints into the system (3.9) without constraints. The output-constrained $y(t) \in (-\epsilon_l, \epsilon_l)$ is not violated by showing the almost certain boundedness for ξ_1. Even more significantly, the radial unboundedness of the entire Lyapunov function V ensures the stability in probability of the original solution of the closed-loop system (2.1) by employing Lemmas 2 and 3.

Remark 7. Referring to [34], the block diagram of this control scheme is shown in Figure 1. Specifically, by introducing a coordinate transformation and using SiISS to characterize unmeasurable stochastic inverse dynamics, the systems with an output constraint are transformed into equivalent unconstrained systems, guiding us to construct a state feedback stabilizer for stochastic low-order nonlinear systems with SiISS inverse dynamics, while preventing the violation of a prespecified output constraint during operation.

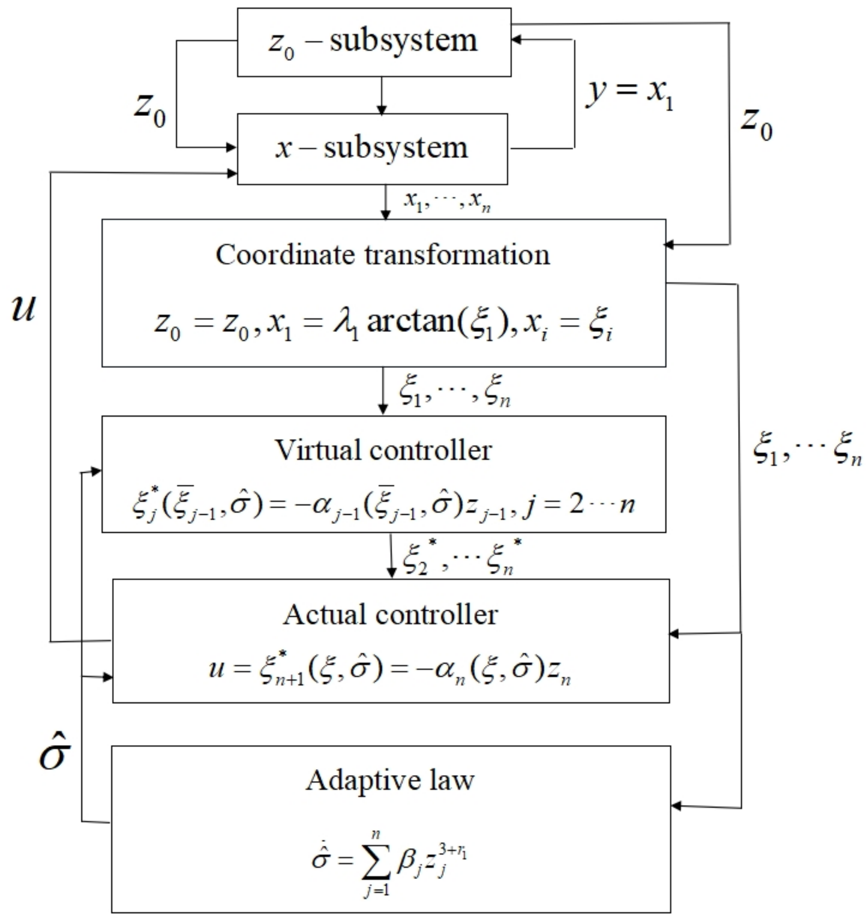

Figure 1. The block diagram of this scheme.

4. A simulation example

The output-constrained SNS is considered as follows:

$$
\begin{aligned}
\mathrm{d}z_0 &= (-z_0^{\frac{3}{5}} + x_1^{\frac{2}{3}})\mathrm{d}t + 0.05\sin z_0^{\frac{3}{5}}\mathrm{d}\omega,\\
\mathrm{d}x_1 &= x_2^{\frac{7}{9}}\mathrm{d}t + 0.2\sin(z_0x_1)\mathrm{d}t + 0.5\sin x_1\cos z_0\mathrm{d}\omega,\\
\mathrm{d}x_2 &= u^{\frac{3}{5}}\mathrm{d}t + z_0^{\frac{2}{5}}x_2^{\frac{3}{5}}\mathrm{d}t + x_1^{\frac{4}{5}}\mathrm{d}\omega,\\
y &= x_1,
\end{aligned}
\tag{4.1}
$$

with an output constraint:

$$
y \in \Omega_y = \{y \in R : -2 < y < 2\},
\tag{4.2}
$$

where $r_1 = \frac{7}{9}, r_2 = \frac{3}{5}, f_0 = -z_0^{\frac{3}{5}} + x_1^{\frac{2}{3}}, g_0 = 0.05\sin z_0^{\frac{3}{5}}, f_1 = 0.2\sin(z_0x_1) \le 0.2|x_1|^{\frac{7}{9}}, g_1 = 0.5\sin x_1\cos z_0 \le 0.5|x_1|^{\frac{8}{9}}, f_2 = z_0^{\frac{2}{5}}x_2^{\frac{3}{5}} \le |z_0|^{\frac{3}{5}} + |x_2|^{\frac{3}{5}}, g_2 = x_1^{\frac{4}{5}} \le |x_1|^{\frac{4}{5}}$. Obviously, Assumption 2 holds. By introducing

$$
\xi_1 = \tan\Big(\frac{x_1}{\lambda_1}\Big), \xi_2 = x_2,
\tag{4.3}
$$

where $\lambda_1 = \frac{4}{\pi}$, then (4.1) may be reconstructed as below:

$$
\begin{aligned}
\mathrm{d}z_0 &= f_0'(z_0,\xi_1)\mathrm{d}t + g_0'^{\top}(z_0,\xi_1)\mathrm{d}\omega,\\
\mathrm{d}\xi_1 &= D_1(\xi_1)\xi_2^{r_1}\mathrm{d}t + f_1'(\theta,z_0,\xi_1)\mathrm{d}t + g_1'^{\top}(\theta,z_0,\xi_1)\mathrm{d}\omega,\\
\mathrm{d}\xi_2 &= u^{r_2}\mathrm{d}t + f_2'(\theta,z_0,\bar{\xi}_2)\mathrm{d}t + g_2'^{\top}(\theta,z_0,\bar{\xi}_2)\mathrm{d}\omega,
\end{aligned}
\tag{4.4}
$$

where $D_1 = \frac{\pi(1+\xi_1^2)}{4}, f_0' = f_0, g_0' = g_0, f_1' = D_1f_1 + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}g_1^{\top}g_1, g_1' = D_1g_1, f_2' = f_2, g_2' = g_2$.

By setting $\sigma = \max_{\{1\le i\le 2\}}\Big\{1,\theta,\theta^2,\theta^{\frac{3+r_1}{r_1-r_i+3}}\Big\}, V_1 = \frac{1}{4}z_1^4 + \frac{1}{2}\tilde{\sigma}^2$ with $\xi_1^* = 0$ and $z_1 = \xi_1$, the virtual controller $\xi_2^* = -(\frac{4(2+\hat{\sigma}\beta_1+\varphi(\xi_1))}{\pi})^{\frac{9}{7}}z_1 \triangleq -\alpha_1z_1$ guarantees that $\mathcal{L}V_1 \le -2z_1^{\frac{34}{9}} + D_1z_1^3(\xi_2^{\frac{7}{9}} - \xi_2^{*\frac{7}{9}}) + \tilde{\sigma}(\beta_1z_1^{\frac{34}{9}} - \dot{\hat{\sigma}}) - \varphi(\xi_1)z_1\frac{34}{9}$, where $\beta_1 = \frac{2}{5}\frac{\pi}{4}^{\frac{2}{9}}\xi_1^{\frac{16}{9}}(1+\xi_1^2)(1+x_1^2)^{\frac{1}{2}} + 0.75\frac{\pi}{4}^{\frac{2}{9}}\xi_1^{\frac{34}{9}}(1+\xi_1^2), \varphi(\xi_1) = 0.67\xi_1^{\frac{81}{3}} + \frac{2}{3}\xi_1^{\frac{1}{3}}$. Set $z_2 = \xi_2 - \xi_2^*$ and $V_2 = V_1 + \frac{45}{188}z_2^{\frac{188}{45}}$. The adaptive controller

$$
\begin{aligned}
\dot{\hat{\sigma}} &= \beta_1z_1^{\frac{34}{9}} + \beta_2z_2^{\frac{34}{9}},\\
u &= -(1 + \hat{\sigma}\beta_2 + \beta_{27})^{\frac{5}{3}}z_2
\end{aligned}
\tag{4.5}
$$

leads to $\mathcal{L}V_2 \le -\sum_{j=1}^{2} z_j^{\frac{34}{9}} - \varphi(\xi_1)z_1^{\frac{34}{9}} + \frac{27}{170}z_0^{\frac{34}{9}}$, where $\beta_2 = \beta_{21} + \beta_{22} + \cdots + \beta_{26}, \beta_{21} = \frac{90}{34}D_1\alpha_1(1+\alpha_1)^{\frac{7}{9}}\Big(\frac{34}{27}(\frac{(10D_1\alpha_1(1+\alpha_1)^{\frac{7}{9}})^{-1}}{7} - 1)\Big)^{-\frac{27}{7}}, \beta_{22} = \frac{143}{170} + (1+\alpha_1)^{\frac{3}{5}}(\frac{170}{189}(1+\alpha_1)^{-\frac{3}{5}})^{-\frac{27}{143}}, \beta_{23} = 7^{\frac{8}{9}}(\frac{1568}{176}\frac{\pi}{4}^{\frac{2}{9}}\alpha_1^2(1+z_2^{\frac{106}{45}})^{\frac{1}{2}}(1+\xi_1^2)^2)^{\frac{16}{9}}, \beta_{24} = \frac{38{,}038}{3825}^{\frac{38}{47}}\frac{3861}{3825}(\frac{4}{\pi}^{\frac{31}{45}}\alpha_1(1+\xi_1^2)(1+z_2^{\frac{8}{45}})^{\frac{1}{2}})^{\frac{85}{47}}, \beta_{25} = \frac{1275}{12{,}012}^{-\frac{36}{49}}\frac{7007}{3825}\frac{4}{\pi}^{\frac{136}{49}}, \beta_{26} = \frac{34}{189}^{-\frac{27}{7}}\frac{7}{34}(2^{\frac{2}{9}}D_1)^{\frac{34}{7}}, \beta_{27} = \frac{\beta_1}{7\beta_2}$.

For the z_0-subsystem, by choosing $V_0(z_0) = z_0^2, \mathcal{L}V_0 \le -\frac{1}{2}z_0^{\frac{8}{5}} + \frac{1}{3}x_1^4, |\frac{\partial V_0}{\partial z_0}| \le 2z_0, |g_0| \le 0.05$. Assumptions 3 and 4 are satisfied with $\alpha_0(s) = \frac{1}{2}s^{\frac{8}{5}}, \gamma_0(s) = \frac{1}{3}s^4, \psi_{z_0}(s) = 2s, \psi_0(s) = 0.05$, and then

$\limsup_{s\to 0^+} \frac{\frac{27}{170}s^{\frac{34}{9}}}{\alpha_0(s)} < \infty, \limsup_{s\to 0^+} \frac{\psi_{z_0}^2(s)\psi_0^2(s)}{\alpha_0(s)} < \infty, \limsup_{s\to 0^+} \frac{\gamma_0(s)}{s^{\frac{34}{9}}} < \infty$. We have to look for the function $\varrho(s)$ that satisfies (3.57), that is, $0.125\varrho(z_0^2)z_0^{\frac{8}{5}} \geq 0.005\varrho'(z_0^2)z_0^2 + \frac{27}{170}z_0^{\frac{34}{9}}$. In this simulation, we select $\varrho(s) = 2s^4 + 1$. Via (3.56), we select $\varphi(s) = 0.67s^{\frac{81}{3}} + \frac{2}{3}s^{\frac{1}{3}}$. First let $V = V_2 + \int_0^{z_0^2}(2s^4 + 1)ds$, then we have $\mathcal{L}V \leq -\sum_{j=1}^{2} z_j^{\frac{34}{9}} - \frac{1}{8}(2z_0^8 + 1)z_0^{\frac{8}{5}}$.

The initial values $(z_0(0), x_1(0), x_2(0)) = (0.8, 0.5, -0.6)$ and $\hat{\sigma}(0) = 1$ are selected, the and results of the simulation are shown in Figures 2–6. In particular, Figures 2–4 show the trajectories of z_0, x_1, and x_2, and we can clearly see that the trajectories of z_0, x_1, and x_2 tend to zero after two seconds, indicating that z_0, x_1, and x_2 are stable. Figure 3 shows that the trajectory of $y = x_1$ is restricted within the pre-specified output constraint range (4.2), and after two seconds, the trajectory of $y = x_1$ tends to zero, indicating that $y = x_1$ is stable. As shown in Figures 5–6, the range of $\sigma(t), u(t)$ are almost certainly bounded, and the trajectory of $u(t)$ tends to zero after two seconds, indicating that $u(t)$ is stable. Therefore, through the trajectory curves in Figures 2–6, it can be ensured that the system in the simulation is stable and does not violate the output constraints. All signals are almost certainly bounded.

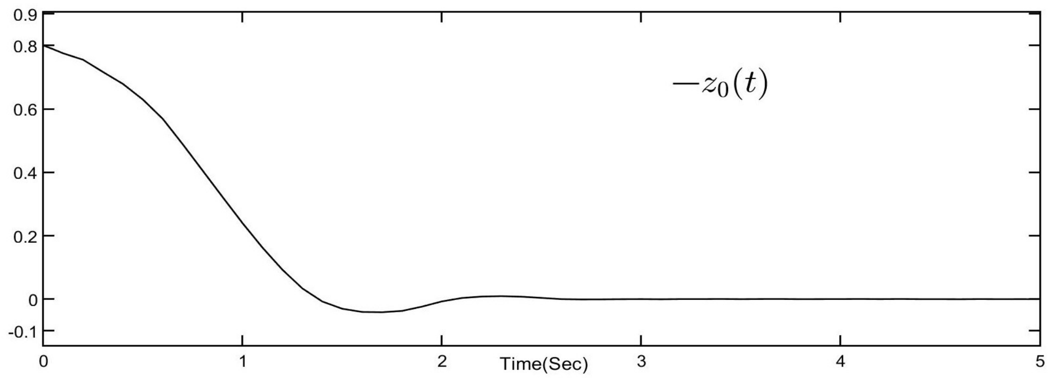

Figure 2. The curve of state $z_0(t)$.

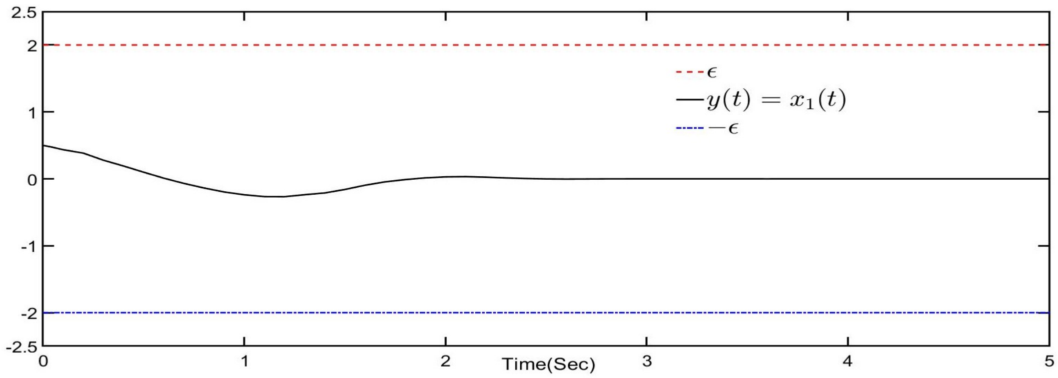

Figure 3. The curves of state $x_1(t)$.

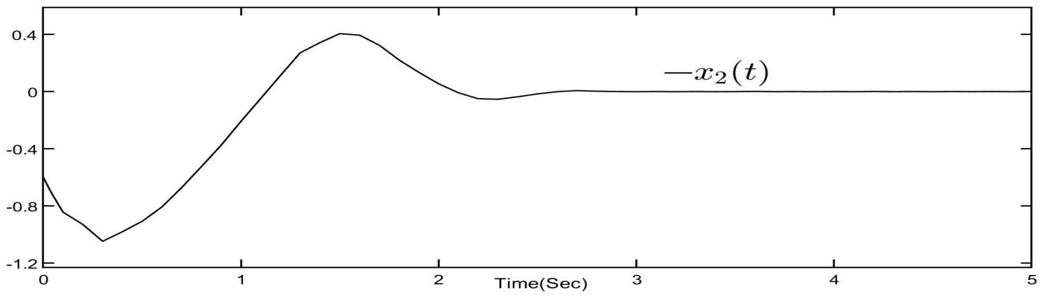

Figure 4. The curve of state $x_2(t)$.

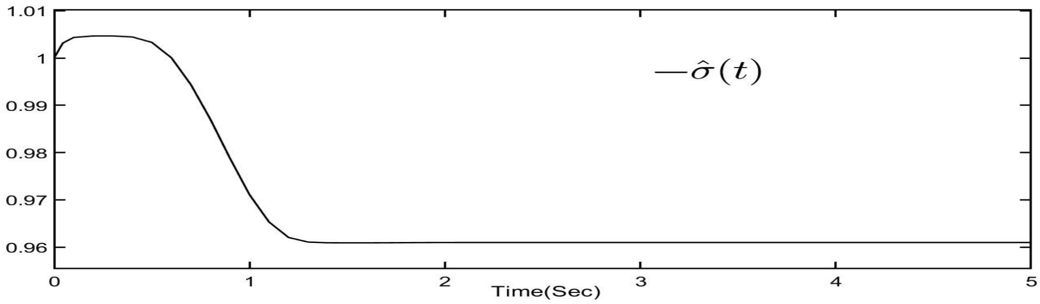

Figure 5. The curve of $\hat{\sigma}(t)$.

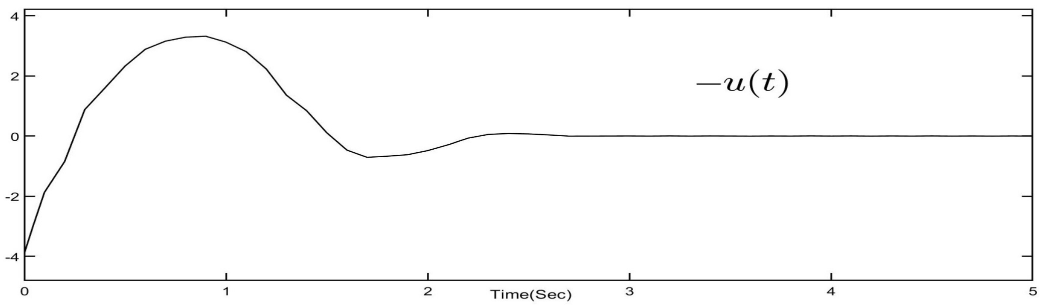

Figure 6. The curve of controller $u(t)$.

5. Conclusions

The adaptive state-feedback control issue of low-order SNSs with output constraints and SiISS inverse dynamics has been researched in this paper.

Some problems are still remaining as follows: (i) What is the best way to devise adaptive output feedback controllers of low-order SNSs with an output constraint to achieve the systems' finite-time stabilization? (ii) For low-order SNSs with asymmetric output constraints, what is the best way to devise controllers to maintain the systems' finite-time stabilization? (iii) The paper deals with constant output constraints. Can the proposed method be extended to time-varying output constraints?

Author contributions

Mengmeng Jiang: Conceptualization, Validation, Data curation, Methodology, Writing-review, editing; Qiqi Ni: Writing-original draft, Software, Conceptualization; Methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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