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Research article

Several characterizations of bivariate quantum-Hermite-Appell Polynomials and the structure of their zeros

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Abstract: This paper investigated the fundamental characteristics and uses of a new class of bivariate quantum-Hermite-Appell polynomials. The series representation and generating relation for these polynomials were derived. Also, a determinant representation for these polynomials was derived. Further, important mathematical characteristics were derived, such as q-recurrence relations and q-difference equations. These polynomials' numerical features were methodically examined, providing information on their computational possibilities and the framework of their zeros. A coherent framework was established by extending the study to related families, such as quantum-Hermite Bernoulli, quantum-Hermite Euler, and quantum-Hermite Genocchi polynomials. These discoveries enhance the knowledge of quantum polynomials and their relationships to classical and contemporary special functions.

Keywords: special functions; quantum calculus; explicit form; operational connection; structure of zeros; determinant form
Mathematics Subject Classification: 33E20, 33C45, 33B10, 33E30, 11T23

1. Introduction and preliminaries

In recent years, quantum calculus has emerged as a pivotal study area, garnering substantial interest due to its foundational relevance in applied mathematics, mechanics, and physics. Serving as a bridge between classical mathematics and quantum theories, quantum calculus extends the principles of traditional calculus into quantum realms. This extension has led to numerous breakthroughs, including the discovery of novel notations and significant results in combinatorics and number theory.

At the heart of these advancements lie quantum-special polynomials, with the quantum-binomial coefficient $\binom{\rho}{k}_q$ and the quantum-Pochhammer symbol $(\rho; q)_n$ being particularly significant. These mathematical objects serve as essential tools in studying quantum analogues, extending traditional concepts to the realm of quantum calculus. Their influence spans multiple areas of mathematics and physics, including combinatorics, representation theory, and statistical mechanics. Researchers can uncover deeper relationships and develop new analysis methods through these constructs, facilitating progress in theoretical and applied domains. By providing a unifying framework, quantum-special polynomials enable the translation of complex ideas into more manageable forms, which has profound implications for diverse fields ranging from algebraic combinatorics to quantum physics. Renowned for their intricate algebraic and analytic properties, these polynomials provide powerful tools for investigating quantum-analogue phenomena in diverse domains. While the applications of quantum calculus have predominantly been within advanced physics, its advancement relies heavily on precise and consistent notation.

In the present study, we use the notation $\mathbb{C}$ to represent the set of complex numbers, ρ for the set of natural numbers, and ρ_0 to denote the set of non-negative integers. Additionally, the variable q, which belongs to the complex number set, is subject to the condition that its absolute value satisfies $|q| < 1$. This restriction ensures that q lies within the unit disk of the complex plane.

Notably, the historical foundation of quantum series traces back to Christian Heine [7], who, in the mid-18th century, introduced a series where the normalized factor $n!$ was replaced by the polynomial $(q; q)_n$, a degree-n polynomial in q. This innovation significantly broadened the scope of series expansions, enabling their application to a wider array of mathematical functions and phenomena as follows:

$$(q; q)_\rho := \begin{cases} (1-q)(1-q^2).....(1-q^\rho), & \rho \geq 1, \\ 1, & \rho = 0. \end{cases}$$

The algebraic framework of these series was further advanced by F. H. Jackson in the early 20th century [9], laying the foundation for a systematic exploration of quantum calculus. Jackson's contributions played a pivotal role in formalizing the structure and properties of quantum series, enabling their application across various mathematical and physical contexts. For deformed quantum groups [13, 21], the deformation introduces rich algebraic structures that generalize classical symmetries in quantum integrable systems. These groups are crucial in studying non-commutative geometry and quantum field theories.

The concepts of quantum numbers and quantum factorials, essential tools in quantum calculus, are defined as follows:

$$[\rho]_q \equiv \frac{1-q^\rho}{1-q}\,, \quad q \in \mathbb{C} - \{1\}\,,\ \rho \in \mathbb{C},$$

$$[\rho]_q! \equiv \prod_{k=1}^{\rho} [k]_q, \quad [0]_q! = 1, \ \ \rho \in \rho.$$

The Gauss formula is the cornerstone for defining all significant quantum-binomial coefficients, denoted as $\binom{\rho}{k}_q$. These coefficients play a critical role in quantum calculus and are defined through the following relationship:

$$(\tau_1 + \rho)_q^{\rho} = \sum_{r=0}^{\rho} q^{r(r-1)/2} \binom{\rho}{r}_q \rho^r \, \tau_1^{\rho-k}, \quad \rho \in \rho_0,$$

where τ_1 and ρ are variables, and $(\tau_1 + \rho)_q^{\rho}$ is the quantum-analogue of the binomial expansion.

The quantum-binomial coefficient $\binom{\rho}{k}_q$ is explicitly defined as:

$$\binom{\rho}{k}_q = \frac{\rho_q!}{[k]_q![\rho - k]_q!} \ , \quad k = 0, 1, ..., \rho, \tag{1.1}$$

where $\rho_q!$ represents the quantum-factorial. Further, the formulation

$$\mathfrak{E}_q(\tau_1) = \frac{1}{((1-q)\tau_1; q)_\infty} = \sum_{\rho=0}^{\infty} \frac{\tau_1^{\rho}}{\rho_q!}, \quad |\tau_1| < |1-q|^{-1},$$

generalizes the classical binomial coefficients to the quantum calculus setting, incorporating the deformation parameter q. It finds applications in combinatorics, representation theory, and the study of special functions, providing a powerful framework for analyzing quantum-series and related phenomena.

The quantum-version of the derivative of a function Φ at a given point $\tau_1 \in \mathbb{C}, \ \ \tau_1 \neq 0$, represented by $\mathcal{D}_q\Phi$, is defined as (see [10] for reference):

$$\frac{\Phi(\tau_1) - \Phi(q\tau_1)}{(1-q)\tau_1} = \mathcal{D}_q\Phi(\tau_1).$$

For the special case when $\tau_1 = 0$, the quantum-derivative is defined as $\mathcal{D}_q\Phi(0) = \Phi'(0)$, provided that the conventional derivative $\Phi'(0)$ exists. Importantly, as the parameter q approaches 1, the quantum-derivative converges to the classical derivative, thus bridging quantum calculus with ordinary calculus.

Moreover, for any two arbitrary functions $\Phi(\omega)$ and $\Psi(\omega)$, the q-derivative exhibits the following linearity property:

$$\mathcal{D}_{q,\omega}(\Phi(\omega)) + \varrho\mathcal{D}_{q,\omega}(\Psi(\omega)) = \mathcal{D}_{q,\omega}(\rho\Phi(\omega) + \varrho\Psi(\omega)),$$

where a and b are constants.

Further, we have

$$\Phi(q\omega)\mathcal{D}_{q,\omega}\Psi(\omega) + \Psi(\omega)\mathcal{D}_{q,\omega}\Phi(\omega) = \mathcal{D}_{q,\omega}(\Phi(\omega)\Psi(\omega)) \tag{1.2}$$

and

$$\frac{\Psi(q\omega)\mathcal{D}_{q,\omega}\Phi(\omega) - \Phi(q\omega)\mathcal{D}_{q,\omega}\Psi(\omega)}{\Psi(\omega)\Psi(q\omega)} = \mathcal{D}_{q,\omega}\left(\frac{\Phi(\omega)}{\Psi(\omega)}\right).$$

In the broader context of quantum calculus, the quantum-Jackson integral, which generalizes the concept of integration, is defined over the interval from 0 to $\rho \in \mathbb{R}$. Its formulation (refer to [10]) is given as:

$$\int_0^{\rho} \Phi(\tau_1)\mathcal{D}_q\tau_1 = \rho(1-q)\sum_{k=0}^{\infty}\Phi(\rho q^k)q^k,$$

where Φ is the function to be integrated, and $d_q\tau_1$ denotes the quantum-differential measure.

This integral converges under appropriate conditions on Φ and ρ. As the deformation parameter q approaches 1, the quantum-Jackson integral reduces to the classical Riemann integral, maintaining a consistent connection between quantum calculus and traditional calculus.

The quantum-Jackson integral is a vital tool in quantum calculus, facilitating the study of special functions, quantum-series, and other mathematical constructs within the quantum-analog framework. It has applications in diverse areas, including quantum mechanics, number theory, and combinatorics.

The Hermite polynomials were first introduced in 1836 by Sturm [19]. Later, in 1864, Hermite [8] formally presented these polynomials as $\mathscr{R}_\rho(\tau_1)$, employing tools such as the Rodrigues formula, differential equations, and orthogonality properties. Today, Hermite polynomials are regarded as one of the most fundamental systems of orthogonal polynomials in mathematical analysis.

The quantum-Hermite polynomials, a specific subclass of orthogonal polynomials, extend the classical Hermite polynomials by incorporating the parameter q. These polynomials occupy a foundational position in the hierarchy of classical quantum-orthogonal polynomials [6]. Forming a one-parameter family, the quantum-Hermite polynomials reduce to the classical Hermite polynomials when $q = 1$. They also generalize the heat equation in contexts involving quantum-generalized operators, extending their applicability to broader mathematical and physical frameworks.

The quantum-Hermite polynomials $\mathscr{R}_{\rho,q}(\tau_1)$ are defined through their generating function, expressed as follows:

$$\sum_{\rho=0}^{\infty}\mathscr{R}_{\rho,q}(\tau_1)\frac{\beta^\rho}{\rho_q!} = \mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta) := \mathbb{S}_q(\tau_1,\beta), \tag{1.3}$$

$$\sum_{\rho=0}^{\infty}(-1)^\rho q^{n(\rho-1)/2}\frac{\beta^{2\rho}}{[2\rho]_q!!} := \mathbb{S}_q(\beta), \quad [2\rho]_q[2\rho-2]_q\cdots[2]_q = [2\rho]_q!. \tag{1.4}$$

This generating function encapsulates the key structural properties of quantum-Hermite polynomials, facilitating their use in various applications, including combinatorics, quantum mechanics, and approximation theory.

Also,

$$-\beta = \frac{\mathcal{D}_{q,\beta}\mathbb{S}_q(\beta)}{\mathbb{S}_q(q\beta)}$$

and

$$\rho_q\mathscr{R}_{\rho-1,q}(\tau_1) = \mathcal{D}_{q,\tau_1}\mathscr{R}_{\rho,q}(\tau_1).$$

The bivariate Hermite polynomials are crucial in probability theory, quantum mechanics, and solving partial differential equations. Widely used in mathematical physics, signal processing, and approximation theory, they support function representation in multiple variables and model complex phenomena. Motivated by their significance, we extend the quantum-Hermite polynomials in (1.3) to their bivariate form using the following generating relation:

$$\sum_{\rho=0}^{\infty} \mathscr{R}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^{\rho}}{\rho_q!} = \mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2) := \mathbb{S}_q(\tau_1,\tau_2;\beta), \tag{1.5}$$

where $\mathbb{S}_q(\beta)$ is given by expression (1.4).

In 1880, Appell [1] introduced a notable class of polynomial sequences, now known as Appell polynomials. Later, Sharma and Chak [17] developed their quantum analogues using quantum integers. In 1967, Al-Salam [2] and many other researchers [3, 14, 15] proposed a further generalization. The ρ-degree polynomials $\mathbb{L}_{\rho,q}(\tau_1)$ are termed quantum-Appell if they satisfy the following quantum-differential equation:

$$\rho_q\mathbb{L}_{\rho-1,q}(\tau_1) = \mathcal{D}_q(\tau_1)\{\mathbb{L}_{\rho,q}(\tau_1)\}, \qquad \rho = 0,1,2,...... \quad .$$

The quantum-Appell polynomials $\mathbb{L}_{\rho,q}(\tau_1)$ are determined through the subsequent generating function (refer to [2]):

$$\sum_{\rho=0}^{\infty} \mathbb{L}_{\rho,q}(\tau_1)\frac{\beta^{\rho}}{\rho_q!} = \mathbb{L}_q(\beta)\mathfrak{E}_q(\tau_1\beta), \quad 0 < \beta < 1, \tag{1.6}$$

where

$$\sum_{\rho=0}^{\infty} \mathbb{L}_{\rho,q}\frac{\beta^{\rho}}{\rho_q!} := \mathbb{L}_q(\beta), \;\; 1 = \mathbb{L}_{0,q}, \;\; 0 \neq \mathbb{L}_q(\beta) \tag{1.7}$$

is analytic at $\beta = 0$ with $\mathbb{L}_{\rho,q} := \mathbb{L}_{\rho,q}(0)$.

Several characterizations and clarifications regarding the quantum-Appell family of polynomials were extensivel provided. Over the past few decades, the quantum-Appell polynomials have been studied from various perspectives and using different techniques. In 2015, Keleshteri and Mahmudov [11] investigated the determinant representations for the Appell polynomials. A key aspect of quantum-Appell polynomials is deriving quantum-recurrence relations and quantum-difference equations, along with determinant forms for the quantum-Hermite-based Appell family, which aid in computations and uncovering new properties.

By extending the quantum-exponential function $\mathfrak{E}_q(\tau_1\beta)$ in the left-hand side (l.h.s.) of expression (1.6), and substituting the powers of $\tau_1^0, \tau_1^1, \tau_1^2, \cdots, \tau_1^{\rho}$ of τ_1 with the corresponding polynomials $\mathscr{R}_{0,q}(\tau_1,\tau_2), \mathscr{R}_{1,q}(\tau_1,\tau_2), \cdots, \mathscr{R}_{\rho,q}(\tau_1,\tau_2)$ in right-hand side (l.h.s.) of and τ_1 in the polynomials $\mathscr{R}_{1,q}(\tau_1,\tau_2)$ in the r.h.s. of the resulting equation, it follows that

$$\mathbb{L}_q(\beta)\Big[\mathscr{R}_{0,q}(\tau_1,\tau_2) + \mathscr{R}_{1,q}(\tau_1,\tau_2)\frac{\beta}{[1]_q!} + \mathscr{R}_{2,q}(\tau_1,\tau_2)\frac{\beta^2}{[2]_q!} + \cdots + \mathscr{R}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^{\rho}}{\rho_q!} + \cdots\Big] = \sum_{\rho=0}^{\infty} \mathbb{L}_{\rho,q}\Big(\mathscr{R}_{1,q}(\tau_1,\tau_2)\Big)\frac{\beta^{\rho}}{\rho_q!}.$$

Moreover, by using expression (1.5) to sum up the series in the l.h.s. of the aforementioned equation and designating the bivariate quantum-Hermite-Appell polynomials (bivariate quantum-HAP) that results in the r.h.s. by ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, and we discover that

$$\mathbb{L}_{\rho,q}\Big(\mathscr{R}_{1,q}(\tau_1,\tau_2)\Big) = {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2), \tag{1.8}$$

and thus, considering the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, we have the following generating relation:

$$\sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} = \mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2) := \mathbb{Q}_q(\tau_1,\tau_2;\beta), \quad |\beta| < \infty, \tag{1.9}$$

where $\mathscr{R}_q(\beta)$ and $\mathbb{L}_q(\beta)$ are analytic at $\beta = 0$ given by expressions (1.4) and (1.7).

The operational correspondence between the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ and the quantum-Appell polynomials $\mathbb{L}_{\rho,q}(\tau_1)$ is given by expression (1.9).

Further, for $\tau_2 = 0$, the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ simplify to the quantum-Hermite-Appell polynomials ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1)$. These are defined through the following generating function [12]:

$$\sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1)\frac{\beta^\rho}{\rho_q!} = \mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta) := \mathbb{Q}_q(\tau_1;\beta). \tag{1.10}$$

The structure of this article is organized as follows: We begin by introducing the adapted bivariate quantum-HAP, highlighting their relationships with other well-known quantum polynomials such as the quantum-Appell, quantum-Bernoulli, quantum-Euler, and quantum-Genocchi polynomials. Section 2 presents the series definition for these bivariate quantum-HAP and explores their several characteristics. Section 3 is dedicated to deriving determinant representations for theses bivariate quantum-HAP, as well as for selected members of the family. Section 4 investigates the monomiality principle for these polynomials. Also, quantum-recurrence relations and quantum-difference equations govern these polynomials and their specific members. Section 5 establishes zeros and numerical values for specific members of these polynomials. The article concludes with concluding remarks.

2. Explicit forms of the bivariate quantum-HAP and related members

Explicit forms of bivariate special polynomials are essential for theoretical insights and practical computations. They enable direct analysis, property derivation, and application in engineering, physics, and statistics, making them key tools in research and real-world modelling. Therefore, the series expansion for the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ is derived by establishing the following results:

Theorem 2.1. *The following series expansion for bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ holds true:*

$${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho}\binom{\rho}{k}_q \mathbb{L}_{k,q}\, \mathscr{R}_{\rho-k,q}(\tau_1,\tau_2). \tag{2.1}$$

Proof. After inserting expressions (1.5) and (1.7) into the left-hand side of generating function (1.9) and then applying the Cauchy-product rule, it follows that

$$\sum_{\rho=0}^{\infty}\sum_{k=0}^{\rho}\binom{\rho}{k}_q \mathbb{L}_{k,q}\mathscr{R}_{\rho-k,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} = \sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}. \tag{2.2}$$

Equation (2.2) yields assertion (2.1) upon equating the coefficients of $\frac{\beta^\rho}{\rho_q!}$. □

Theorem 2.2. *The following series expansion for bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ holds true:*

$$ {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\,[k]_q!}\tau_2^k\,{}_{\mathscr{R}}\mathbb{L}_{\rho-2k,q}(\tau_1). \tag{2.3} $$

Proof. After inserting expressions (1.1) and (1.10) into the left-hand side of generating function (1.9) and then applying the Cauchy-product rule, it follows that

$$ \sum_{\rho=0}^{\infty}\sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\,[k]_q!}\tau_2^k{}_{\mathscr{R}}\mathbb{L}_{\rho-2k,q}(\tau_1)\frac{\beta^\rho}{\rho_q!} = \sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}. \tag{2.4} $$

Equation (2.4) yields assertion (2.3) upon equating the coefficients of $\frac{\beta^\rho}{\rho_q!}$. □

Theorem 2.3. *The following series expansion for bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ holds true:*

$$ {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2) = \rho_q!\,\sum_{k=0}^{\rho}\sum_{s=0}^{[\frac{k}{2}]} \frac{\mathbb{L}_{\rho-k,q}\,\tau_1^{k-2s}\tau_2^s}{[\rho-k]_q!\,[k-2s]_q!\,[s]_q!}. \tag{2.5} $$

Proof. Inserting expressions (1.1) and (1.10) into the left-hand side of generating function (1.9), it follows that

$$ \sum_{\rho=0}^{\infty}\mathbb{L}_{\rho,q}\frac{\beta^\rho}{\rho_q!}\sum_{k=0}^{\infty}\tau_1^k\frac{\beta^k}{[k]_q!}\sum_{s=0}^{\infty}\tau_2^s\frac{\beta^{2s}}{[s]_q!} = \sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}. $$

Applying the Cauchy-product rule in the second and third terms of the l.h.s. of the preceding expression, we have

$$ \sum_{\rho=0}^{\infty}\mathbb{L}_{\rho,q}\frac{\beta^\rho}{\rho_q!}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{k}{2}]}\tau_1^{k-2s}\tau_2^s\frac{\beta^k}{[k-2s]_q!\,[s]_q!} = \sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}. $$

Again applying the Cauchy-product rule in the l.h.s. of the preceding expression, we have

$$ \sum_{\rho=0}^{\infty}\sum_{k=0}^{\rho}\sum_{s=0}^{[\frac{k}{2}]}\mathbb{L}_{\rho-k,q}\tau_1^{k-2s}\tau_2^s\frac{\rho_q!}{[\rho-k]_q!\,[k-2s]_q!\,[s]_q!}\frac{\beta^\rho}{\rho_q!} = \sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}. \tag{2.6} $$

Equation (2.6) yields assertion (2.5) upon equating the coefficients of $\frac{\beta^\rho}{\rho_q!}$. □

Corollary 2.1. *For $\tau_2 = 0$, the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ reduces to the quantum-Hermite-Appell polynomials ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1)$ and therefore satisfies the series representations:*

$$ {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1) = \sum_{k=0}^{\rho}\binom{\rho}{k}_q\mathbb{L}_{k,q}\,\mathscr{R}_{\rho-k,q}(\tau_1) $$

and

$$ {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2) = \rho_q!\,\sum_{k=0}^{\rho}\frac{\mathbb{L}_{\rho-k,q}\,\tau_1^k}{[\rho-k]_q!\,[k]_q!}. $$

Proof. This result follows directly by substituting $\tau_2 = 0$ into the bivariate generating function

$$\sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} = \mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2),$$

which simplifies to the univariate generating function:

$$\sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1)\frac{\beta^\rho}{\rho_q!} = \mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta),$$

under the setting $\tau_2 = 0$. Expanding both sides in powers of β and comparing coefficients yields the stated representation. □

Corollary 2.2. *For $\mathbb{L}_{\rho,q}(\beta) = 1$, the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ reduces to the bivariate quantum-Hermite polynomials $\mathscr{R}_{n,q}(\tau_1,\tau_2)$ and therefore satisfies the series representations:*

$$\mathscr{R}_{n,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\,[k]_q!}\tau_2^k\,\mathscr{R}_{n-2k,q}(\tau_1)$$

and

$$\mathscr{R}_{n,q}(\tau_1,\tau_2) = \rho_q!\,\sum_{k=0}^{[\frac{\rho}{2}]} \frac{\tau_1^{n-2k}\tau_2^k}{[\rho-2k]_q!\,[k]_q!}.$$

Various members of the quantum-Appell family emerge through the choice of distinct generating functions, denoted as $\mathbb{L}_q(\beta)$. The versatility of this approach allows for the extraction of different polynomials tailored to specific needs or applications. The table below highlights a selection of these members, showcasing the diversity and utility inherent in the quantum-Appell family. From these examples, it becomes evident that by manipulating the generating function, researchers can access a wide range of polynomials with unique properties and characteristics, underscoring the flexibility and richness of the quantum-Appell framework for mathematical analysis and problem-solving.

For $\mathbb{L}_q(\beta) = \frac{\beta}{\mathfrak{E}_q(\beta)-1}$, the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ reduces to the bivariate quantum-Hermite-based Bernoulli polynomials denoted by${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$, satisfying the generating function:

$$\mathbb{Q}_q(\tau_1,\tau_2;\beta) := \frac{\beta}{\mathfrak{E}_q(\beta)-1}\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2) = \sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} \tag{2.7}$$

and operational correspondence between the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ and the quantum-Bernoulli polynomials $\mathcal{B}_{\rho,q}(\tau_1)$ as:

$${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2) = \mathcal{B}_{\rho,q}\Big(\mathscr{R}_{1,q}(\tau_1,\tau_2)\Big). \tag{2.8}$$

Further, the following series expansions for bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ holds true:

$${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho}\binom{\rho}{k}_q \mathcal{B}_{k,q}\,\mathscr{R}_{\rho-k,q}(\tau_1,\tau_2), \tag{2.9}$$

$$ {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\,[k]_q!}\tau_2^k \,{}_{\mathscr{R}}\mathcal{B}_{\rho-2k,q}(\tau_1), $$

and

$$ {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2) = \rho_q! \sum_{k=0}^{\rho}\sum_{s=0}^{[\frac{k}{2}]} \frac{\mathcal{B}_{\rho-k,q}\,\tau_1^{k-2s}\tau_2^{s}}{[\rho-k]_q!\,[k-2s]_q!\,[s]_q!}. $$

Table 1. Distinct polynomials within the quantum-Appell family are derived by selecting various generating functions $\mathbb{L}_q(\beta)$. The subsequent table enumerates specific instances from this family.

S. No.	$\mathbb{L}_q(\beta)$	**Name of the quantum-special polynomial and related number**	**Generating function**	**Series definition**
I.	$\left(\frac{\beta}{\mathfrak{E}_q(\beta)-1}\right)$	quantum-Bernoulli polynomials and numbers [5]	$\left(\frac{\beta}{\mathfrak{E}_q(\beta)-1}\right)\mathfrak{E}_q(\tau_1\beta) = \sum\limits_{\rho=0}^{\infty}\mathcal{B}_{\rho,q}(\tau_1)\frac{\beta^\rho}{\rho_q!}$ $\left(\frac{\beta}{\mathfrak{E}_q(\beta)-1}\right) = \sum\limits_{\rho=0}^{\infty}\mathcal{B}_{\rho,q}\frac{\beta^\rho}{\rho_q!}$ $\mathcal{B}_{\rho,q} := \mathcal{B}_{\rho,q}(0)$	$\mathcal{B}_{\rho,q}(\tau_1) = \sum\limits_{k=0}^{\rho}\binom{\rho}{k}_q\mathcal{B}_{k,q}\,\tau_1^{n-k}$
II.	$\left(\frac{2}{\mathfrak{E}_q(\beta)+1}\right)$	quantum-Euler polynomials and numbers [5]	$\left(\frac{2}{\mathfrak{E}_q(\beta)+1}\right)\mathfrak{E}_q(\tau_1\beta) = \sum\limits_{\rho=0}^{\infty}\mathfrak{E}_{n,q}(\tau_1)\frac{\beta^\rho}{\rho_q!}$ $\left(\frac{2}{\mathfrak{E}_q(\beta)+1}\right) = \sum\limits_{\rho=0}^{\infty}\mathfrak{E}_{n,q}\frac{\beta^\rho}{\rho_q!}$ $\mathfrak{E}_{n,q} := \mathfrak{E}_{n,q}(0)$	$\mathfrak{E}_{n,q}(\tau_1) = \sum\limits_{k=0}^{\rho}\binom{\rho}{k}_q\mathfrak{E}_{k,q}\tau_1^{n-k}$
III.	$\left(\frac{2\beta}{\mathfrak{E}_q(\beta)+1}\right)$	quantum-Genocchi polynomials and numbers	$\left(\frac{2\beta}{\mathfrak{E}_q(\beta)+1}\right)\mathfrak{E}_q(\tau_1\beta) = \sum\limits_{\rho=0}^{\infty}\mathcal{G}_{\rho,q}(\tau_1)\frac{\beta^\rho}{\rho_q!}$ $\left(\frac{2\beta}{\mathfrak{E}_q(\beta)+1}\right) = \sum\limits_{\rho=0}^{\infty}\mathcal{G}_{\rho,q}\frac{\beta^\rho}{\rho_q!}$ $\mathcal{G}_{\rho,q} := \mathcal{G}_{\rho,q}(0)$	$\mathcal{G}_{\rho,q}(\tau_1,\tau_2) = \sum\limits_{k=0}^{\rho}\binom{n}{k}_q\mathcal{G}_{k,q}x^{n-k}$

Also

$$ \mathbb{L}_q(\beta) = \frac{2}{\mathfrak{E}_q(\beta)+1}, $$

and the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ reduces to the bivariate quantum-Hermite-based Euler polynomials denoted by ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$, satisfying the generating function:

$$ \mathbb{Q}_q(\tau_1,\tau_2;\beta) := \frac{2}{\mathfrak{E}_q(\beta)+1}\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2) = \sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} \tag{2.10} $$

and operational correspondence between the bivariate quantum-Hermite-based Euler polynomials

${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2)$ and the quantum-Euler polynomials $\mathcal{E}_{\rho,q}(\tau_1)$ as:

$$ {}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2) = \mathcal{E}_{\rho,q}\Big(\mathscr{R}_{1,q}(\tau_1,\tau_2)\Big). \tag{2.11} $$

Aditionally, the following series expansions for bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2)$ holds true:

$$ {}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathcal{E}_{k,q}\ \mathscr{R}_{\rho-k,q}(\tau_1,\tau_2), \tag{2.12} $$

$$ {}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\ [k]_q!}\tau_2^k\ {}_{\mathscr{R}}\mathcal{B}_{\rho-2k,q}(\tau_1), $$

and

$$ {}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1,\tau_2) = \rho_q!\ \sum_{k=0}^{\rho}\sum_{s=0}^{[\frac{k}{2}]} \frac{\mathcal{E}_{\rho-k,q}\ \tau_1^{k-2s}\tau_2^s}{[\rho-k]_q!\ [k-2s]_q!\ [s]_q!}. $$

Further, for

$$ \mathbb{L}_q(\beta) = \frac{2\beta}{\mathfrak{E}_q(\beta)+1}, $$

the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ reduces to the bivariate quantum-Hermite-based Genocchi polynomials denoted by${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)$, satisfying the generating function:

$$ \mathbb{Q}_q(\tau_1,\tau_2;\beta) := \frac{2\beta}{\mathfrak{E}_q(\beta)+1}\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2) = \sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!} \tag{2.13} $$

and operational correspondence between the bivariate quantum-Hermite-based Genocchi polynomials ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)$ and the quantum-Genocchi polynomials $\mathcal{G}_{\rho,q}(\tau_1)$ as:

$$ {}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2) = \mathcal{G}_{\rho,q}\Big(\mathscr{R}_{1,q}(\tau_1,\tau_2)\Big). $$

Furthermore, the subsequent series expansions for the bivariate quantum-Hermite-based Genocchi polynomials, denoted as ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)$, are valid and can be expressed as follows:

$$ {}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathcal{G}_{k,q}\ \mathscr{R}_{\rho-k,q}(\tau_1,\tau_2), \tag{2.14} $$

$$ {}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2) = \sum_{k=0}^{\rho} \frac{\rho_q!}{[\rho-2k]_q!\ [k]_q!}\tau_2^k\ {}_{\mathscr{R}}\mathcal{G}_{\rho-2k,q}(\tau_1), $$

and

$$ {}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2) = \rho_q!\ \sum_{k=0}^{\rho}\sum_{s=0}^{[\frac{k}{2}]} \frac{\mathcal{G}_{\rho-k,q}\ \tau_1^{k-2s}\tau_2^s}{[\rho-k]_q!\ [k-2s]_q!\ [s]_q!}. $$

Furthermore, by setting $\tau_1 = \tau_2 = 0$ in the generating relation (1.9) for the bivariate quantum-HAP ${}_{\mathscr{R}}\mathcal{A}_{n,q}(\tau_1, \tau_2)$, we can deduce that

$$\mathbb{Q}_q(0,0;\beta) := \mathbb{L}_q(\beta)\mathbb{S}_q(\beta) = \sum_{\rho=0}^{\infty} {}_{\mathscr{R}}\mathbb{L}_{\rho,q}\frac{\beta^\rho}{\rho_q!}.$$

Consequently, this simplifies to the bivariate quantum-Hermite-based Appell numbers, which are represented by${}_{\mathscr{R}}\mathcal{A}_{n,q}$ and expressed through the following series formulation:

$${}_{\mathscr{R}}\mathbb{L}_{\rho,q} := {}_{\mathscr{R}}\mathbb{L}_{\rho,q}(0,0) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathbb{L}_{k,q}\, \mathscr{R}_{\rho-k,q}.$$

Therefore, the numbers related to the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2)$, bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2)$, and bivariate quantum-Hermite-based Genocchi polynomials ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1, \tau_2)$ are obtained by taking

$$\tau_1 = \tau_2 = 0$$

in series definitions of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2)$, bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2)$, and bivariate quantum-Hermite-based Genocchi polynomials ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1, \tau_2)$. These numbers are defined as follows:

$$\begin{aligned}
{}_{\mathscr{R}}\mathcal{B}_{\rho,q} : &= {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(0,0) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathcal{B}_{k,q}\, \mathscr{R}_{\rho-k,q} \\
{}_{\mathscr{R}}\mathcal{B}_{\rho,q} : &= {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(0,0) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathcal{E}_{k,q}\, \mathscr{R}_{\rho-k,q} \\
{}_{\mathscr{R}}\mathcal{B}_{\rho,q} : &= {}_{\mathscr{R}}\mathcal{B}_{\rho,q}(0,0) = \sum_{k=0}^{\rho} \binom{\rho}{k}_q \mathcal{E}_{k,q}\, \mathscr{R}_{\rho-k,q}.
\end{aligned}$$

In the next section, we establish the determinant forms for the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1, \tau_2)$, as well as for selected members of this family.

3. Determinant form

The significance of exploring the determinant form of the quantum-Appell polynomials lies in its implications for both computational and practical applications. The determinant form of quantum-Appell polynomials, particularly the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1, \tau_2)$, offers a compact and efficient representation that aids both theoretical analysis and practical applications, as highlighted by Keleshteri and Mahmudov [11]. The determinant definition of the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1, \tau_2)$ is obtained by proving the following result:

Theorem 3.1. *Let ${}_{\mathscr{R}}\mathbb{L}\rho, q(\tau_1, \tau_2)$ denote the bivariate quantum-HAP of degree ρ. Then, the following* *determinant form holds true:*

$$
\begin{aligned}
&{}_{\mathscr{R}}\mathbb{L}_{0,q}(\tau_1,\tau_2) = 1,\\
&{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2) = \frac{(-1)^\rho}{(\delta_{0,q})^{\rho+1}}
\begin{vmatrix}
1 & \mathscr{R}_{1,q}(\tau_1,\tau_2) & \mathscr{R}_{2,q}(\tau_1,\tau_2) & .... & \mathscr{R}_{\rho-1,q}(\tau_1,\tau_2) & \mathscr{R}_{\rho,q}(\tau_1,\tau_2)\\
\delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{\rho-1,q} & \delta_{\rho,q}\\
0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{\rho-1}{1}_q\delta_{\rho-2,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\
0 & 0 & \delta_{0,q} & .... & \binom{\rho-1}{2}_q\delta_{\rho-3,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
0 & 0 & 0 & .... & \delta_{0,q} & \binom{\rho}{\rho-1}_q\delta_{1,q}
\end{vmatrix},
\end{aligned}
\tag{3.1}
$$

where $n = 1, 2, \ldots,$ *and* $\mathscr{R}_{\rho,q}(\tau_1, \tau_2)$ *are the bivariate quantum-HAP of degree* ρ*. Additionally* $\delta_{0,q} \neq 0$ *and are defined as:*

$$
\begin{aligned}
\delta_{0,q} &= \frac{1}{\mathbb{L}_{0,q}},\\
\delta_{\rho,q} &= -\frac{1}{\mathbb{L}_{0,q}}\left(\sum_{k=1}^{\rho}\binom{\rho}{k}_q\mathbb{L}_{k,q}\,\delta_{\rho-k,q}\right), \quad \rho = 1, 2, \cdots, \quad \textit{where } \mathbb{L}_{0,q} \textit{ is given by } (1.7).
\end{aligned}
$$

Proof. When we set $\rho = 0$ in the series definition (2.1) of the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1, \tau_2)$, we obtain

$$
{}_{\mathscr{R}}\mathbb{L}_{0,q}(\tau_1,\tau_2) = 1.
$$

Now, considering the determinant form of the quantum-Appell polynomials $\{\mathbb{L}_{\rho,q}(\tau_1)\}_{\rho=0}^{\infty}$ as given in [11], we have:

$$
\mathbb{L}_{\rho,q}(\tau_1) = \frac{(-1)^\rho}{(\delta_{0,q})^{\rho+1}}
\begin{vmatrix}
1 & \tau_1 & \tau_1^2 & .... & \tau_1^{\rho-1} & \tau_1^{\rho}\\
\delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{\rho-1,q} & \delta_{\rho,q}\\
0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{\rho-1}{1}_q\delta_{\rho-2,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\
0 & 0 & \delta_{0,q} & .... & \binom{\rho-1}{2}_q\delta_{\rho-3,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
0 & 0 & 0 & .... & \delta_{0,q} & \binom{\rho}{\rho-1}_q\delta_{1,q}
\end{vmatrix}
$$

where $\delta_{0,q}, \delta_{1,q}, \delta_{2,q}, \cdots, \delta_{\rho,q} \in \mathbb{R}$, $\delta_{0,q} \neq 0$, and $\rho = 1, 2, 3, \cdots$.

Thus, expanding the above determinant along the first row, we find that

$$\mathbb{L}_{\rho,q}(\tau_1)=\frac{(-1)^{\rho}}{(\delta_{0,q})^{\rho+1}}\begin{vmatrix}\delta_{1,q} & \delta_{2,q} & .... & \delta_{\rho-1,q} & \delta_{\rho,q}\\ \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{\rho-1}{1}_q\delta_{\rho-2,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\ 0 & \delta_{0,q} & .... & \binom{\rho-1}{2}_q\delta_{\rho-3,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\ . & . & .... & . & .\\ . & . & .... & . & .\\ . & . & .... & . & .\\ 0 & 0 & .... & \delta_{0,q} & \binom{\rho}{\rho-1}_q\delta_{1,q}\end{vmatrix}$$

$$+\frac{(-1)^{\rho+1}\tau_1}{(\delta_{0,q})^{\rho+1}}\begin{vmatrix}\delta_{o,q} & \delta_{2,q} & .... & \delta_{\rho-1,q} & \delta_{\rho,q}\\ 0 & \binom{2}{1}_q\delta_{1,q} & .... & \binom{\rho-1}{1}_q\delta_{\rho-2,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\ 0 & \delta_{0,q} & .... & \binom{\rho-1}{2}_q\delta_{\rho-3,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\ . & . & .... & . & .\\ . & . & .... & . & .\\ . & . & .... & . & .\\ 0 & 0 & .... & \delta_{0,q} & \binom{\rho}{\rho-1}_q\delta_{1,q}\end{vmatrix}$$

$$+\frac{(-1)^{\rho+2}\tau_1^2}{(\delta_{0,q})^{\rho+1}}\begin{vmatrix}\delta_{o,q} & \delta_{1,q} & .... & \delta_{\rho-1,q} & \delta_{\rho,q}\\ 0 & \delta_{0,q} & .... & \binom{\rho-1}{1}_q\delta_{\rho-2,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\ 0 & 0 & .... & \binom{\rho-1}{2}_q\delta_{\rho-3,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\ . & . & .... & . & .\\ . & . & .... & . & .\\ . & . & .... & . & .\\ 0 & 0 & .... & \delta_{0,q} & \binom{\rho}{\rho-1}_q\delta_{1,q}\end{vmatrix}$$

$$+......+\frac{(-1)^{2\rho-1}\tau_1^{\rho-1}}{(\delta_{0,q})^{\rho+1}}\begin{vmatrix}\delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{\rho-2,q} & \delta_{\rho,q}\\ 0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-2}{1}_q\delta_{\rho-3,q} & \binom{\rho}{1}_q\delta_{\rho-1,q}\\ 0 & 0 & \delta_{0,q} & .... & \binom{n-2}{2}_q\delta_{\rho-4,q} & \binom{\rho}{2}_q\delta_{\rho-2,q}\\ . & . & . & .... & . & .\\ . & . & . & .... & . & .\\ . & . & . & .... & . & .\\ 0 & 0 & 0 & .... & 0 & \binom{\rho}{\rho-1}_q\delta_{1,q}\end{vmatrix}$$

$$+\frac{(-1)^{2\rho}\tau_1^{\rho}}{(\delta_{0,q})^{\rho+1}}\begin{vmatrix} \delta_{o,q} & \delta_{1,q} & \delta_{2,q} & .... & \delta_{\rho-2,q} & \delta_{\rho-1,q} \\ 0 & \delta_{0,q} & \binom{2}{1}_q\delta_{1,q} & .... & \binom{n-2}{1}_q\delta_{\rho-3,q} & \binom{\rho-1}{1}_q\delta_{\rho-2,q} \\ 0 & 0 & \delta_{0,q} & .... & \binom{n-2}{2}_q\delta_{\rho-4,q} & \binom{\rho-1}{2}_q\delta_{\rho-3,q} \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & .... & 0 & \delta_{0,q} \end{vmatrix}.$$

Once more, given that each minor is independent of τ_1, we can substitute $\tau_1^1, \tau_1^2, \cdots, \tau_1^{\rho}$ with $\mathscr{R}_{1,q}(\tau_1,\tau_2), \mathscr{R}_{2,q}(\tau_1,\tau_2), \cdots, \mathscr{R}_{\rho,q}(\tau_1,\tau_2)$ respectively. By employing operational relation (1.8) in the left-hand side subsequently consolidating the terms in the right-hand side, we arrive at assertion (3.1). □

Remark 3.1. *Given that the polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ mentioned in expressions* (2.7) *and* (2.9) *are specific members of the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, we can derive the determinant definition for the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ by appropriately selecting the coefficients $\delta_{0,q}$ and $\delta_{i,q}$ in the determinant expression of ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$. Setting $\delta_{0,q} = 1$ and $\delta_{i,q} = \frac{1}{[i+1]_q}$ for $(i = 1,2,...,\rho)$ in expression* (3.1)*, we establish the following determinant definition for ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$:*

$${}_{\mathscr{R}}\mathcal{B}_{0,q}(\tau_1,\tau_2) = 1,$$

$${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2) = (-1)^{\rho}\begin{vmatrix} 1 & \mathscr{R}_{1,q}(\tau_1,\tau_2) & \mathscr{R}_{2,q}(\tau_1,\tau_2) & .... & \mathscr{R}_{\rho-1,q}(\tau_1,\tau_2) & \mathscr{R}_{\rho,q}(\tau_1,\tau_2) \\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & .... & \frac{1}{\rho_q} & \frac{1}{[\rho+1]_q} \\ 0 & 1 & \binom{2}{1}_q\frac{1}{[2]_q} & .... & \binom{\rho-1}{1}_q\frac{1}{[\rho-1]_q} & \binom{\rho}{1}_q\frac{1}{\rho_q} \\ 0 & 0 & 1 & .... & \binom{\rho-1}{2}_q\frac{1}{[\rho-2]_q} & \binom{\rho}{2}_q\frac{1}{[\rho-1]_q} \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & .... & 1 & \binom{\rho}{\rho-1}_q\frac{1}{[2]_q} \end{vmatrix}, \quad \rho = 1,2,\ldots,$$

where $\mathscr{R}_{\rho,q}(\tau_1,\tau_2)$, for $(\rho = 0,1,2,...)$, represents the bivariate quantum-Hermite polynomials of degree ρ.

Remark 3.2. *Given that the polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ mentioned in expressions* (2.10) *and* (2.12) *are specific members of the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, we can derive the determinant definition for the bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ by appropriately selecting the coefficients $\delta_{0,q}$ and $\delta_{i,q}$ in the determinant expression of ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$. Setting $\delta_{0,q} = 1$ and $\delta_{i,q} = \frac{1}{2}$ for $(i = 1,2,...,\rho)$ in expression* (3.1)*, we establish the following determinant definition for ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$:*

$$
{}_{\mathscr{R}}\mathcal{E}_{0,q}(\tau_1,\tau_2)=1,
$$

$$
{}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=(-1)^{\rho}\begin{vmatrix}
1 & \mathscr{R}_{1,q}(\tau_1,\tau_2) & \mathscr{R}_{2,q}(\tau_1,\tau_2) & .... & \mathscr{R}_{\rho-1,q}(\tau_1,\tau_2) & \mathscr{R}_{\rho,q}(\tau_1,\tau_2)\\
1 & \frac{1}{2} & \frac{1}{2} & .... & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \binom{2}{1}_q\frac{1}{2} & .... & \binom{\rho-1}{1}_q\frac{1}{2} & \binom{\rho}{1}_q\frac{1}{2}\\
0 & 0 & 1 & .... & \binom{\rho-1}{2}_q\frac{1}{2} & \binom{\rho}{2}_q\frac{1}{2}\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
0 & 0 & 0 & .... & 1 & \binom{\rho}{\rho-1}_q\frac{1}{2}
\end{vmatrix},\quad \rho=1,2,\cdots,
$$

where $\mathscr{R}_{\rho,q}(\tau_1,\tau_2)$, for $(\rho=0,1,2,...)$, represents the bivariate quantum-Hermite polynomials of degree ρ.

Remark 3.3. *Given that the polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$ mentioned in expression* (2.13) *and* (2.14) *are specific members of the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, we can derive the determinant definition for the bivariate quantum-Hermite-based Genocchi polynomials ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)$ by appropriately selecting the coefficients $\delta_{0,q}$ and $\delta_{i,q}$ in the determinant expression of ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$. Setting*

$$
\delta_{0,q}=1
$$

and

$$
\delta_{i,q}=\frac{1}{2[i+1]_q}
$$

for $(i=1,2,...,\rho)$ in expression (3.1)*, we establish the following determinant definition for ${}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)$:*

$$
{}_{\mathscr{R}}\mathcal{G}_{0,q}(\tau_1,\tau_2)=1,
$$

$$
{}_{\mathscr{R}}\mathcal{G}_{\rho,q}(\tau_1,\tau_2)=(-1)^{\rho}\begin{vmatrix}
1 & \mathscr{R}_{1,q}(\tau_1,\tau_2) & \mathscr{R}_{2,q}(\tau_1,\tau_2) & .... & \mathscr{R}_{\rho-1,q}(\tau_1,\tau_2) & \mathscr{R}_{\rho,q}(\tau_1,\tau_2)\\
1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & .... & \frac{1}{2\rho_q} & \frac{1}{2[\rho+1]_q}\\
0 & 1 & \binom{2}{1}_q\frac{1}{2[2]_q} & .... & \binom{\rho-1}{1}_q\frac{1}{2[\rho-1]_q} & \binom{\rho}{1}_q\frac{1}{2\rho_q}\\
0 & 0 & 1 & .... & \binom{\rho-1}{2}_q\frac{1}{2[\rho-2]_q} & \binom{\rho}{2}_q\frac{1}{2[\rho-1]_q}\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
. & . & . & .... & . & .\\
0 & 0 & 0 & .... & 1 & \binom{\rho}{\rho-1}_q\frac{1}{2[2]_q}
\end{vmatrix},\quad \rho=1,2,\cdots,
$$

where $\mathscr{R}_{\rho,q}(\tau_1,\tau_2)$, for $(\rho=0,1,2,...)$, represents the bivariate quantum-Hermite polynomials of degree ρ.

4. Monomiality principle

The monomiality principle, originating with Steffenson's poweroids in 1941 [18] and refined by Dattoli [4], underpins the structure of special polynomials, ensuring orthogonality and completeness. It facilitates the derivation of recurrence relations and explicit forms, essential in applications like approximation and differential equations. Recently, many authors extended the monomiality principle, for example, [16, 20, 22] extended this principle to many quantum polynomial families, via operators $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$, enhancing the analysis of their quasi-monomial behaviour. These polynomials are governed by the specific forms outlined below:

$$\Phi_{m+1,q}(\tau_1) = \hat{\mathcal{M}}_q\{\Phi_{m,q}(\tau_1)\}$$

and

$$[m]_q\ \Phi_{m-1,q}(\tau_1) = \hat{\mathcal{D}}_q\{\Phi_{m,q}(\tau_1)\}.$$

The collection of operators responsible for manipulating the quasi-monomial sequence $\{\Phi_{m,q}(\tau_1)\}_{m\in\rho}$ must satisfy the following commutative relationship:

$$[\hat{\mathcal{D}}_q, \hat{\mathcal{M}}_q] = \hat{\mathcal{D}}_q\hat{\mathcal{M}}_q - \hat{\mathcal{M}}_q\hat{\mathcal{D}}_q.$$

The properties of the quasi-monomial set $\{\Phi_{m,q}(\tau_1)\}_{m\in\rho}$ stem from the interplay between the operators $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$, which define its structure and transformations through specific axioms:

(i) The function $\Phi_{m,q}(\tau_1)$ satisfies the following differential equation:

$$\hat{\mathcal{M}}_q\hat{\mathcal{D}}_q\{\Phi_{m,q}(\tau_1)\} = [m]_q\ \Phi_{m,q}(\tau_1) \tag{4.1}$$

and

$$\hat{\mathcal{D}}_q\hat{\mathcal{M}}_q\{\Phi_{m,q}(\tau_1)\} = [m+1]_q\ \Phi_{m,q}(\tau_1),$$

provided $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$possesses differential recognitions.

(ii) The explicit form of the function $\Phi_{m,q}(\tau_1)$ is provided by

$$\Phi_{m,q}(\tau_1) = \hat{\mathcal{M}}_q^{\ m}\ \{1\}, \tag{4.2}$$

with $\Phi_{0,q}(\tau_1) = 1$.

(iii) The generating relation for $\Phi_{m,q}(\tau_1)$ in its exponential form can be represented as follows:

$$\mathfrak{E}_q\Big\{\beta\hat{\mathcal{M}}_q\Big\}\{1\} = \sum_{m=0}^{\infty}\Phi_{m,q}(\tau_1)\frac{\beta^m}{[m]_q!}, \quad |\beta| < \infty\ ,$$

by utilizing identity expression (4.2).

The quantum-dilatation operator, represented by $\mathcal{T}_\nu$, acts on functions that are linked to the complex variable γ in the following manner:

$$\mathcal{T}_\nu^m(h(\gamma)) = h(q^m\gamma) \tag{4.3}$$

and satisfies the identity:

$$\mathcal{T}_\nu^{-1}\mathcal{T}_\nu^1(h(\gamma)) = h(\gamma),$$

where q represents a fixed complex parameter. This operator acts by scaling the argument of the function by a factor of q, modifying its behavior according to this scaling effect.

The monomiality principle is employed as a fundamental tool in defining both the raising and lowering operators. Within this conceptual structure, we introduce the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$. We demonstrate this by presenting the following key results, which provide insights into the behavior and properties of these polynomials within the established framework.

Theorem 4.1. *For the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, the following multiplicative and derivative operators are valid:*

$$\hat{\mathcal{M}}_{q_{\,\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)} = \tau_1\mathcal{T}_{\tau_2} + \tau_2\,\mathcal{D}_{q,\tau_1} + q\tau_2\,\mathcal{T}_{\tau_2}\mathcal{D}_{q,\tau_1} + \frac{\mathbb{S}'_q(\mathcal{D}_{q,\tau_1})}{\mathbb{S}_q(\mathcal{D}_{q,\tau_1})}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2} + \frac{\mathbb{L}'_q(\mathcal{D}_{q,\tau_1})}{\mathbb{L}_q(\mathcal{D}_{q,\tau_1})}\mathcal{T}_{\beta}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2} \tag{4.4}$$

and

$$\hat{\mathcal{D}}_{q_{\,\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)} = \mathcal{D}_{q,\tau_1}. \tag{4.5}$$

Proof. By applying the quantum-derivative to each component of the expression (1.9) with respect to the variable β, and leveraging the relation given in Eq (1.2), we derive the following result:

$$\begin{aligned}\mathcal{D}_{q,\beta}\Big[\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big] &= \mathbb{L}_q(\beta)\mathcal{D}_{q,\beta}\Big[\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big]\\ &\quad+\mathcal{D}_{q,\beta}\mathbb{L}_q(\beta)\Big[\mathbb{S}_q(q\beta)\mathfrak{E}_q(q\tau_1\beta)\mathfrak{E}_q(q\tau_2\beta^2)\Big],\end{aligned}$$

and by further utilizing expression (1.2), it can be concluded that:

$$\begin{aligned}\mathcal{D}_{q,\beta}\Big[\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big] &= \Big(\tau_1+q\tau_2\beta\Big)\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(q\tau_2\beta^2)\\ &\quad+\tau_2\beta\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\\ &\quad+\frac{\mathbb{S}'_q(\beta)}{\mathbb{S}_q(\beta)}\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(q\tau_1\beta)\mathfrak{E}_q(q\tau_2\beta^2)\\ &\quad+\frac{\mathbb{L}'_q(\beta)}{\mathbb{L}_q(\beta)}\mathbb{L}_q(\beta)\mathbb{S}_q(q\beta)\mathfrak{E}_q(q\tau_1\beta)\mathfrak{E}_q(q\tau_2\beta^2).\end{aligned}$$

By utilizing (4.3), we have

$$\begin{aligned}\mathcal{D}_{q,\beta}\Big[\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big] &= \left(\tau_1\mathcal{T}_{\tau_2}+\tau_2\beta+q\tau_2\beta\mathcal{T}_{\tau_2}+\frac{\mathbb{S}'_q(\beta)}{\mathbb{S}_q(\beta)}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}+\frac{\mathbb{L}'_q(\beta)}{\mathbb{L}_q(\beta)}\mathcal{T}_{\beta}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\right)\\ &\quad\times\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2).\end{aligned} \tag{4.6}$$

By applying the quantum-derivative to each component of the expression (1.9) with respect to the variable τ_1, we derive the following result:

$$\mathcal{D}_{q,\tau_1}\Big[\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big] = \beta\Big[\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2)\Big]. \tag{4.7}$$

By applying the expression provided in (4.6) and equating the coefficients of like powers of β on both sides of the resulting equation, we can establish the validity of assertion (4.4).

Furthermore, utilizing the right-hand side of the expression in (1.9) on both sides of the identity (4.7), and once again comparing the coefficients of the same powers of β in the resulting equation, we can demonstrate the truth of assertion (4.5). □

Theorem 4.2. *For the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$, the following differential equation is valid:*

$$\left(\tau_1\mathcal{T}_{\tau_2}\mathcal{D}_{q,\tau_1}+\tau_2\,\mathcal{D}^2_{q,\tau_1}+q\tau_2\,\mathcal{T}_{\tau_2}\mathcal{D}^2_{q,\tau_1}+\frac{\mathbb{S}'_q(\mathcal{D}_{q,\tau_1})}{\mathbb{S}_q(\mathcal{D}_{q,\tau_1})}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\mathcal{D}_{q,\tau_1}+\frac{\mathbb{L}'_q(\mathcal{D}_{q,\tau_1})}{\mathbb{L}_q(\mathcal{D}_{q,\tau_1})}\mathcal{T}_{\beta}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\mathcal{D}_{q,\tau_1}-\rho_q\right) \tag{4.8}$$
$$\times\,{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)=0.$$

Proof. By substituting expressions (4.4) and (4.5) into the equation (4.1), we derive the result presented in assertion (4.8). □

Subsequently, the quantum-recurrence relation for the bivariate quantum-HAP ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$ is established, with the following result being rigorously proven:

Theorem 4.3. *The following quantum-recurrence relation is valid for the bivariate quantum-HAP polynomials ${}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)$:*

$$\begin{aligned}{}_{\mathscr{R}}\mathbb{L}_{\rho+1,q}(\tau_1,\tau_2)=&\left(\tau_1\mathcal{T}_{\tau_2}+\frac{\mathbb{S}'_q(\beta)}{\mathbb{S}_q(\beta)}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}+\frac{\mathbb{L}'_q(\beta)}{\mathbb{L}_q(\beta)}\mathcal{T}_{\beta}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\right){}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\\&+(2+q\,\mathcal{T}_{\tau_2})\rho\,\tau_2\,{}_{\mathscr{R}}\mathbb{L}_{\rho-1,q}(\tau_1,\tau_2).\end{aligned}$$

Proof. We begin with the generating function identity for the bivariate quantum-HAP polynomials:

$$\sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}=\mathbb{L}_q(\beta)\,\mathbb{S}_q(\beta)\,\mathfrak{E}_q(\tau_1\beta)\,\mathfrak{E}_q(\tau_2\beta^2), \tag{4.9}$$

where $\mathbb{L}_q(\beta)$ and $\mathbb{S}_q(\beta)$ are generating functions defined by analytic series expansions, and $\mathfrak{E}_q(z)$ s the q-exponential function.

Now, apply the q-differential operator $d_{q,\beta}$ to both sides of (4.9). Using the known q-derivative rules for products and compositions (from q-calculus), we obtain:

$$\begin{aligned}D_{q,\beta}\left[\sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^\rho}{\rho_q!}\right]&=D_{q,\beta}\left[\mathbb{L}_q(\beta)\,\mathbb{S}_q(\beta)\,\mathfrak{E}_q(\tau_1\beta)\,\mathfrak{E}_q(\tau_2\beta^2)\right]\\&=\left(\frac{\mathbb{L}'_q(\beta)}{\mathbb{L}_q(\beta)}\mathcal{T}_\beta+\frac{\mathbb{S}'_q(\beta)}{\mathbb{S}_q(\beta)}+\tau_1\mathcal{T}_{\tau_1}+(2+q\mathcal{T}_{\tau_2})\beta\tau_2\right)\\&\quad\times\mathbb{L}_q(\beta)\,\mathbb{S}_q(\beta)\,\mathfrak{E}_q(\tau_1\beta)\,\mathfrak{E}_q(\tau_2\beta^2).\end{aligned}$$

Now let us denote the right-hand side as:

$$\mathbb{Q}_q(\beta;\tau_1,\tau_2):=\mathbb{L}_q(\beta)\mathbb{S}_q(\beta)\mathfrak{E}_q(\tau_1\beta)\mathfrak{E}_q(\tau_2\beta^2),$$

so that the derivative becomes:

$$D_{q,\beta}\left[\mathbb{Q}_q(\beta;\tau_1,\tau_2)\right]=\left(\tau_1\mathcal{T}_{\tau_2}+\frac{\mathbb{S}'_q(\beta)}{\mathbb{S}_q(\beta)}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}+\frac{\mathbb{L}'_q(\beta)}{\mathbb{L}_q(\beta)}\mathcal{T}_\beta\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\right)\mathbb{Q}_q(\beta)+(2+q\mathcal{T}_{\tau_2})\beta\tau_2\mathbb{Q}_q(\beta).$$

Next, recall that:

$$D_{q,\beta}\left[\sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\frac{\beta^{\rho}}{\rho_q!}\right]=\sum_{\rho=0}^{\infty}{}_{\mathscr{R}}\mathbb{L}_{\rho+1,q}(\tau_1,\tau_2)\frac{\beta^{\rho}}{\rho_q!}.$$

Therefore, equating the coefficients of $\frac{\beta^{\rho}}{\rho_q!}$ on both sides of the expression yields the recurrence:

$$\begin{aligned}{}_{\mathscr{R}}\mathbb{L}_{\rho+1,q}(\tau_1,\tau_2)&=\left(\tau_1\mathcal{T}_{\tau_2}+\frac{\mathbb{S}_q'(\beta)}{\mathbb{S}_q(\beta)}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}+\frac{\mathbb{L}_q'(\beta)}{\mathbb{L}_q(\beta)}\mathcal{T}_{\beta}\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\right){}_{\mathscr{R}}\mathbb{L}_{\rho,q}(\tau_1,\tau_2)\\&\quad+(2+q\mathcal{T}_{\tau_2})\rho\tau_2{}_{\mathscr{R}}\mathbb{L}_{\rho-1,q}(\tau_1,\tau_2),\end{aligned}$$

which completes the proof. □

5. Distribution of zeros and graphical representation

For the bivariate quantum-Hermite-based Bernoulli polynomials denoted by ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)$, given by expressions (2.7) and (2.8), the few numerical values are given by

$$\begin{aligned}{}_{\mathscr{R}}\mathcal{B}_{1,q}(\tau_1,\tau_2)&=\tau_1-\frac{1}{[2]_q!},\\{}_{\mathscr{R}}\mathcal{B}_{2,q}(\tau_1,\tau_2)&=-\tau_1+\tau_1^2+\frac{1}{[2]_q!}+\tau_2[2]_q!-\frac{[2]_q!}{[3]_q!},\\{}_{\mathscr{R}}\mathcal{B}_{3,q}(\tau_1,\tau_2)&=-\tau_1+\tau_1^3+\frac{2}{[2]_q!}+\tau_1\tau_2[3]_q!-\frac{[3]_q!}{[2]_q!^3}+\frac{\tau_1[3]_q!}{[2]_q!^2}\\&\quad-\frac{\tau_1^2[3]_q!}{[2]_q!^2}-\frac{\tau_2[3]_q!}{[2]_q!}-\frac{[3]_q!}{[4]_q!},\end{aligned}$$

$$\begin{aligned}{}_{\mathscr{R}}\mathcal{B}_{4,q}(\tau_1,\tau_2)&=-\tau_1+\tau_1^4+\frac{2}{[2]_q!}+\frac{[4]_q!}{[2]_q!^4}-\frac{\tau_1[4]_q!}{[2]_q!^3}+\frac{\tau_1^2[4]_q!}{[2]_q!^3}+\frac{\tau_2[4]_q!}{[2]_q!^2}-\frac{\tau_1\tau_2[4]_q!}{[2]_q!}\\&\quad-\frac{\tau_1\tau_2[4]_q!}{[2]_q!}+\frac{\tau_1^2\tau_2[4]_q!}{[2]_q!}+\frac{\tau_2^2[4]_q!}{[2]_q!}+\frac{[4]_q!}{[3]_q!^2}-\frac{\tau_2[4]_q!}{[3]_q!}-\frac{3[4]_q!}{[2]_q!^2[3]_q}+\frac{2\tau_1[4]_q!}{[2]_q![3]_q!}\\&\quad-\frac{\tau_1^2[4]_q!}{[2]_q![3]_q!}-\frac{\tau_1^3[4]_q!}{[2]_q![3]_q!}-\frac{[4]_q!}{[5]_q!},\\{}_{\mathscr{R}}\mathcal{B}_{5,q}(\tau_1,\tau_2)&=-\tau_1+\tau_1^5+\frac{2}{[2]_q!}-\frac{[5]_q!}{[2]_q!^5}+\frac{\tau_1[5]_q}{[2]_q!^4}-\frac{\tau_1^2[5]_q!}{[2]_q!^4}-\frac{\tau_2[5]_q!}{[2]_q!^3}\\&\quad+\frac{\tau_1\tau_2[5]_q!}{[2]!^2}-\frac{\tau_1^2\tau_2[5]_q!}{[2]_q!^2}-\frac{\tau_2^2[5]_q!}{[2]_q!^2}+\frac{\tau_1\tau_2^2[5]_q!}{[2]_q!}+\frac{\tau_1[5]_q!}{[3]_q!^2}-\frac{\tau_1^3[5]_q!}{[3]_q!^2}-\frac{3[5]_q!}{[2]_q![3]_q!^2}\\&\quad-\frac{\tau_1\tau_2[5]_q!}{[3]_q!}+\frac{\tau_1^3\tau_2[5]_q!}{[3]_q!}+\frac{4[5]_q!}{[2]_q!^3[3]_q!}-\frac{3\tau_1[5]_q!}{[2]_q!^2[3]_q!}+\frac{2\tau_1^2[5]_q!}{[2]_q!^2[3]_q!}+\frac{\tau_1^3[5]_q!}{[2]_q!^2[3]_q!}\\&\quad+\frac{2\tau_2[5]_q!}{[2]_q![3]_q!}-\frac{\tau_2[5]_q!}{[4]_q!}-\frac{3[5]_q!}{[2]_q!^2[4]_q!}+\frac{2\tau_1[5]_q!}{[2]_q![4]_q!}-\frac{\tau_1^2[5]_q!}{[2]_q![4]_q!}-\frac{\tau_1^4[5]_q!}{[2]_q![4]_q!}\\&\quad+\frac{2[5]_q!}{[3]_q![4]_q!}-\frac{[5]_q!}{[6]_q!}.\end{aligned}$$

We numerically explore and plot the zeros of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$ for $\rho=30$ (Figure 1).

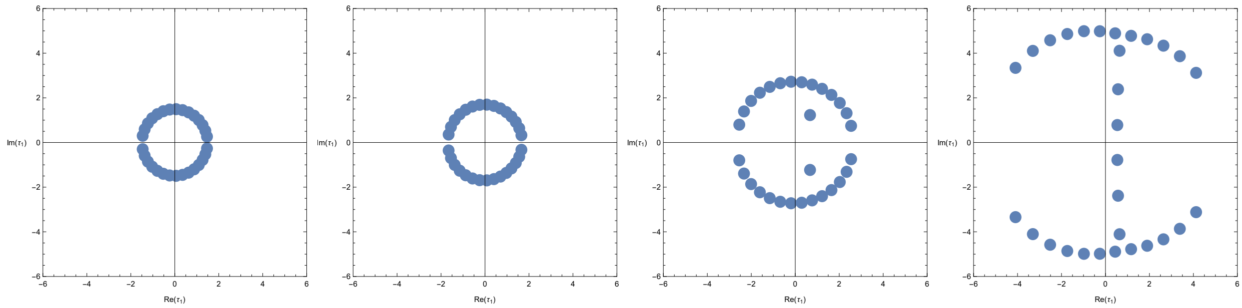

Figure 1. Zeros of ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$.

In Figure 1 (from left to right): 1st, $\tau_2=2$, $q=\frac{1}{10}$; in 2nd, $\tau_2=2$, $q=\frac{3}{10}$; in 3rd, $\tau_2=2$, $q=\frac{7}{10}$; in 4th, $\tau_2=2$, $q=\frac{9}{10}$.

Stacks of zeros of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$ for $1\leq\rho\leq 30$ form a 3D structure, are shown in Figure 2.

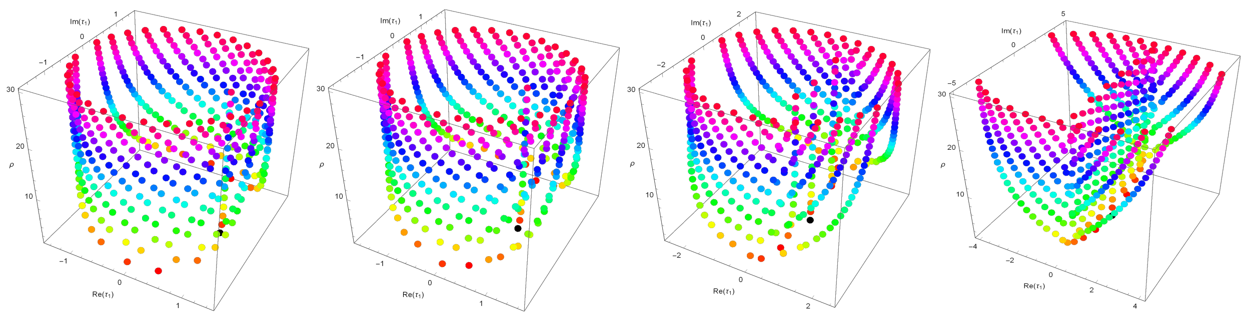

Figure 2. Zeros of ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$.

In Figure 2 (from left to right): 1st, $\tau_2=2$ and $q=\frac{1}{10}$; 2nd, $\tau_2=2$ and $q=\frac{3}{10}$; 3rd, $\tau_2=2$ and $q=\frac{7}{10}$; 4th, $\tau_2=2$ and $q=\frac{9}{10}$.

We plot the zeros of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$, for $\tau_2=2$ and $q=\frac{9}{10}$ (Figure 3).

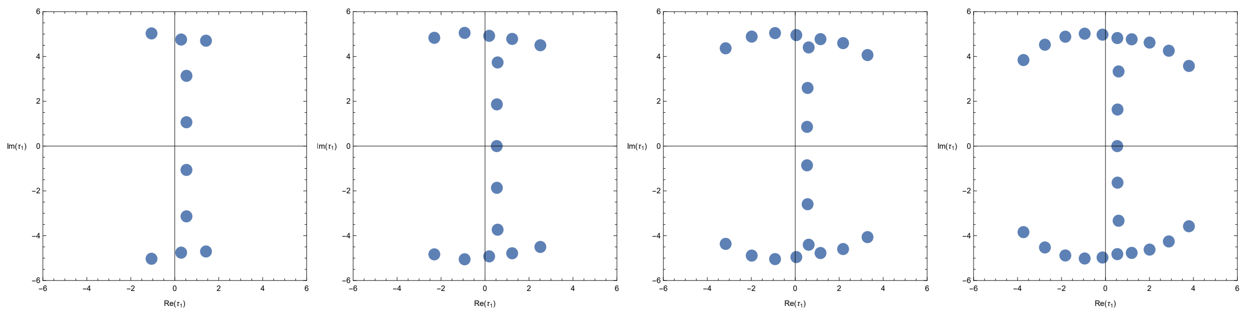

Figure 3. Zeros of ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1,\tau_2)=0$.

In Figure 3 (from left to right), in 1st, we choose $\rho = 10$, in 2nd, $\rho = 15$, in 3rd, $\rho = 20$, and in 4th, $\rho = 25$.

Figure 2 illustrates the intricate 3D configuration formed by the zero sets of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2) = 0$ for $1 \leq \rho \leq 30$. This layered distribution of zeros not only reveals the rich structural complexity of these polynomials but also provides visual insight into their oscillatory and geometric behavior across degrees.

Figure 4 presents plots of real zeros of the bivariate quantum-Hermite-based Bernoulli polynomials ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2) = 0$ for $1 \leq \rho \leq 30$.

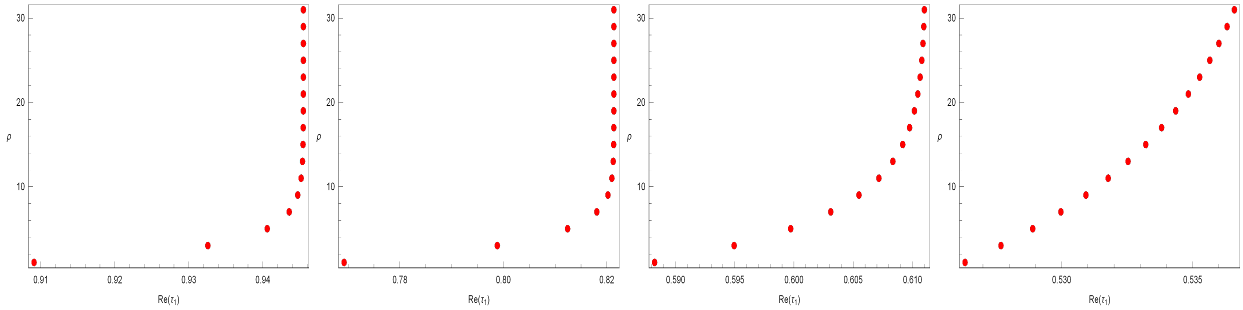

Figure 4. Real zros of ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2) = 0$.

In Figure 4 (from left to right), we choose in 1st, $\tau_2 = 2$, $q = \frac{1}{10}$, in 2nd, $\tau_2 = 2$ and $q = \frac{3}{10}$, in 3rd, $\tau_2 = 2$ and $q = \frac{7}{10}$, and in 4th, $\tau_2 = 2$ and $q = \frac{9}{10}$.

We approximated a solution to ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2) = 0$ for $\tau_2 = 2$, $q = \frac{9}{10}$; see Table 2.

Table 2. Approximate solutions of ${}_{\mathscr{R}}\mathcal{B}_{\rho,q}(\tau_1, \tau_2) = 0$.

degree ρ	τ_1
1	0.52632
2	0.5000 - 1.9254 i, 0.5000 + 1.9254 i
3	0.4493 - 3.1699i, 0.4493 + 3.1699 i, 0.52769
4	0.3879 - 3.9955 i, 0.3879 + 3.9955i, 0.5171 - 1.4921 i, 0.5171 + 1.4921 i
5	0.3150 - 4.5101 i, 0.3150 + 4.5101 i, 0.4982 - 2.6750 i , 0.4982 + 2.6750 i, 0.52890
6	0.2078 - 4.7489 i, 0.2078 + 4.7489 i, 0.5013 - 3.6470 i ,
	0.5013 + 3.6470 i, 0.52395 - 1.28063 i, 0.52395 + 1.28063 i
7	-0.1142 - 4.7873 i, -0.1142 + 4.7873 i, 0.5160 - 2.3823 i ,
	0.5160 + 2.3823 i, 0.52997, 0.7061 - 4.4124 i, 0.7061 + 4.4124 i
8	-0.4619 - 4.9359 i, -0.4619 + 4.9359 i, 0.5174 - 3.3519 i, 0.5174 + 3.3519 i,
	0.52791 - 1.15177 i, 0.52791 + 1.15177 i , 0.9153 - 4.6974 i, 0.9153 + 4.6974 i
9	-0.7555 - 5.0033 i, -0.7555 + 5.0033 i, 0.4847 - 4.2532 i , 0.4847 + 4.2532 i,
	0.5259 - 2.1878 i, 0.5259 + 2.1878 i , 0.53093, 1.0914 - 4.6876 i, 1.0914 + 4.6876 i
10	1.0581 - 5.0293 i, -1.0581 + 5.0293 i, 0.2897 - 4.7560 i ,
	0.2897 + 4.7560 i, 0.53059 - 1.06409 i, 0.53059 + 1.06409 i,
	0.5324 - 3.1360 i, 0.5324 + 3.1360 i, 1.4194 - 4.7041 i , 1.4194 + 4.7041 i

For the bivariate quantum-Hermite-based Euler polynomials denoted by ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1, \tau_2)$, given by expressions (2.10) and (2.11). We numerically explore and plot the zeros of the bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1, \tau_2) = 0$ for $\rho = 34$ (Figure 5).

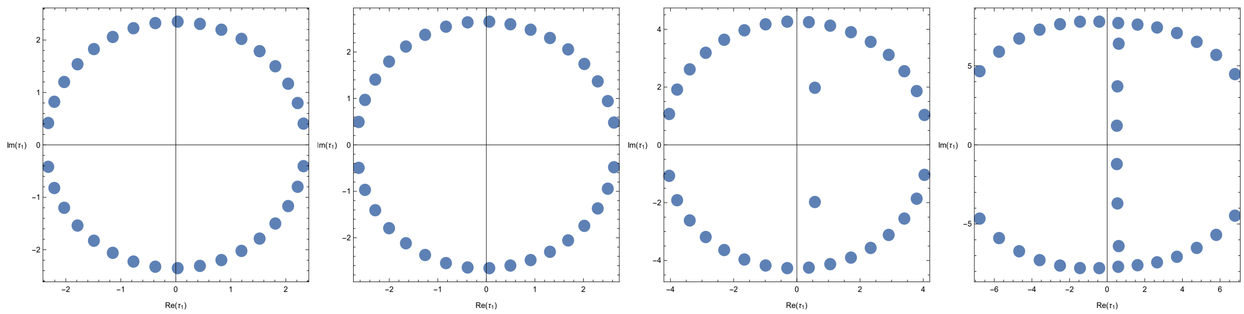

Figure 5. Zeros of ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1, \tau_2) = 0$.

In Figure 5 (from left to right), In 1st, we choose $\tau_2 = 5$ and $q = \frac{1}{10}$, in 2nd, we choose $\tau_2 = 5$ and $q = \frac{3}{10}$, in 3rd, $\tau_2 = 5$ and $q = \frac{7}{10}$, and in 4th, $\tau_2 = 5$ and $q = \frac{9}{10}$.

The distribution of zeros of the bivariate quantum-Hermite-based Euler polynomials ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1, \tau_2) = 0$, for degrees $1 \leq \rho \leq 35$, unveils an intricate three-dimensional configuration, revealing underlying structural symmetries and patterns, as illustrated in Figure 6.

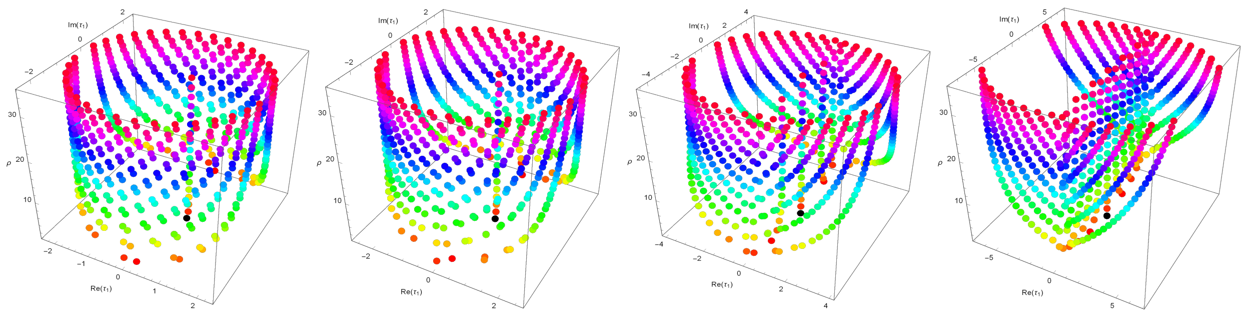

Figure 6. Zeros of ${}_{\mathscr{R}}\mathcal{E}_{\rho,q}(\tau_1, \tau_2) = 0$.

In Figure 6 (left to right), in 1st, we choose $\tau_2 = 5$ and $q = \frac{1}{10}$, in 2nd, we choose $\tau_2 = 5$ and $q = \frac{3}{10}$, in 3rd, we choose $\tau_2 = 5$ and $q = \frac{7}{10}$, and in 4th we choose $\tau_2 = 5$ and $q = \frac{9}{10}$.

6. Concluding remarks

This paper investigates the bivariate quantum-HAP, examining their relationships with other notable quantum-special polynomials, including the quantum-Appell, quantum-Bernoulli, quantum-Euler, and quantum-Genocchi polynomials. Through the presentation of comprehensive series definitions, determinant representations, quantum-recurrence relations, and quantum-difference equations, this work provides a robust framework for understanding these polynomials' intricate connections and characteristics. This foundational approach enhances theoretical perspectives and opens doors to practical applications in various mathematical and scientific domains.

Future research may focus on practical applications of modified polynomials in quantum mechanics and integrable systems, multivariate extensions for higher-dimensional problems, and deeper exploration of quasi-monomiality and operator methods to uncover new polynomial families and structures.

Author contributions

All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Professor William Ramírez is a guest editor of the special issue "Orthogonal polynomials and related applications" of AIMS Mathematics. Professor William Ramírez was not involved in the editorial review and the decision to publish this article.

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