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Almost automorphic solutions for mean-field SDEs driven by Lévy noise

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Abstract: The paper is dedicated to studying the almost automorphic solutions for mean-field SDEs with Lévy noise. It is proven that if the coefficients satisfy the Lipschitz conditions, the equation admits a unique bounded solution, and the solution can inherit in distribution the almost automorphy of the coefficients. In addition, we investigate the global asymptotic stability of these solutions. We also give an example of the stochastic heat equation to illustrate our work.

Keywords: almost automorphy; mean-field stochastic differential equations; Lévy process; asymptotic stability
Mathematics Subject Classification: 34C27, 34G20, 60G51, 60H10

1. Introduction

Mean-field SDEs are a class of SDEs with coefficients dependent on the distribution, also called McKean-Vlasov equations. In 1956, Kac [14] studied a class of interacting particle systems and found that as the number of particles approaches infinity, the density function of the particles evolving over time satisfies a nonlinear evolution equation. In [21], McKean discussed the propagation of chaos in a class of interacting particle systems and analyzed the McKean-Vlasov equation. Sznitman provided two equivalent conditions for propagation of chaos in his review lecture notes [24] and studied the propagation of chaos and limit equations for weakly interacting random particle systems under different frameworks. Based on these pioneering works, the theory and numerical research of the McKean-Vlasov equation have been a hot topic in several directions over the past few decades, such as the stochastic characterization of nonlinear parabolic PDEs, the propagation of chaos, and the well-posedness problems of associated martingale issues [9, 22, 26]. In a pure stochastic method, Buckdahn et al. [4] obtained mean-field backward SDEs. Lions' series of lectures [18] has pushed the study of mean-field problems to new heights, and mean-field game theory and its applications have also developed rapidly [5, 15, 17].

With the application of statistical software, an increasing number of scholars have begun to focus on the long-term asymptotic behaviors of solutions/trajectories of differential equations and dynamical systems, in particular recurrence. This is because almost all interesting dynamical properties exhibit some form of recurrence, and the complexity of dynamics often concentrates on the set of recurrent points. Periodicity, quasi-periodicity, and almost periodicity are well-known recurrences. The almost automorphic function is the important general notion of the almost periodic function. It was originated by Bochner from the work on differential geometry [2]; for the more basic properties and the follow-up development of almost automorphy on determinate systems, see Bochner [3], Veech [25], Shen and Yi [23], among others.

We know that the stochastic perturbation, or noise, is ubiquitous in nature or social society. When we use deterministic systems to describe these phenomena, they more or less omit some random factors. However, these unrecognized stochastic perturbations may destroy the stability of the systems and generate chaos. For classical differential equations, when considering external forces or stochastic perturbations, we will obtain SDEs. Until now, there are many scholars investigating SDEs and quite a few studying almost automorphy of SDEs. Specifically, Fu and Liu [12] established the uniqueness and existence of almost automorphic solutions for SDEs. Chen and Lin [8] initiated the process of square-mean pseudo almost automorphy and its application to stochastic evolution equations. In [7], Chen and Zhang studied almost automorphic solutions of fractional Brownian motion-driven mean-field SDEs. Liu and Gao [19] investigated the almost automorphic solutions for McKean-Vlasov equations. Meanwhile, parallel research on SDEs with jumps based on Lévy processes have also been in progress, such as Liu and Sun [20] , who introduced the Poisson square-mean almost automorphic functions and proved the existence of almost automorphic solutions for SDEs with Lévy noise. Lévy processes are a class of stochastic processes in which the sample paths are stochastic continuous. They retain the independent increment property of Wiener processes and do not strictly require the sample path to be continuous. They include many important processes, for instance, Poisson processes, Wiener processes, stable and self-decomposing processes, and subordinate processes [1]. Therefore, Lévy noises have been extensively used in physics modelling the movement of particles in a fluid, in finance modelling asset prices, and so on. For the latest work on almost automorphy see, [6, 10, 16], among others.

Motivated by [20], we intend to discuss the existence conditions of almost automorphic solutions for the mean-field type SDEs with infinite-dimensional Lévy noise:

$$\begin{aligned} \mathrm{d}x(\tau) =& Ax(\tau)\mathrm{d}\tau + f_1\left(\tau, x(\tau), \mathbb{P}_{x(\tau)}\right)\mathrm{d}\tau + f_2\left(\tau, x(\tau), \mathbb{P}_{x(\tau)}\right)\mathrm{d}W(\tau) \\ &+ \int_{|p|_\Theta<1} b_1\left(\tau, x(\tau-), \mathbb{P}_{x(\tau-)}, p\right)\widetilde{N}(\mathrm{d}\tau, \mathrm{d}p) + \int_{|p|_\Theta\geq 1} b_2\left(\tau, x(\tau-), \mathbb{P}_{x(\tau-)}, p\right)N(\mathrm{d}\tau, \mathrm{d}p), \end{aligned} \tag{1.1}$$

where the semi-group of the linear operator A satisfies the exponential stable condition, the functions f_1, f_2, b_1, b_2 are almost automorphic in time τ, $W, \widetilde{N}, N$ are the components of Lévy-Itô composition. We establish that if the coefficients f_1, f_2, b_1, b_2 satisfy the given Lipschitz continuous condition, then the SDE (1.1) admits a unique bounded solution, and it is almost automorphic in the distribution sense. It is also proven that the solution is globally asymptotically stable in square-mean.

The structure of the paper is outlined below: Section 2 recalls some definitions and some concepts of Lévy processes and almost automorphic stochastic processes. Section 3 establishes the existence conditions of the almost automorphic solutions of the mean-field SDEs with large jumps driven by

Lévy noise. Section 4 discusses the global asymptotical stability of the almost automorphic solution and proves that any other solution converges to it at an exponential rate. Section 5 gives an example to illustrate the viability of the findings presented in the paper.

2. Preliminaries

We shall denote by $(\mathbb{X}, \|\cdot\|)$ and $(\Theta, |\cdot|_\Theta)$ the real separable Hilbert spaces and by $(L(\Theta,\mathbb{X}), \|\cdot\|_{L(\Theta,\mathbb{X})})$ the set of all linear and bounded operators mapping from Θ to $\mathbb{X}$. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space and by $\mathcal{L}^2(\Omega, \mathbb{X})$ the set of all $\mathbb{X}$-valued random variables ξ satisfying

$$E\|\xi\|^2 = \int_\Omega \|\xi\|^2 \mathrm{d}\mathbb{P} < \infty.$$

For $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$, define

$$\|\xi\|_2 := \left(\int_\Omega \|\xi\|^2 \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}},$$

then $(\mathcal{L}^2(\Omega, \mathbb{X}), \|\cdot\|_2)$ is a Hilbert space. We define the space

$$\mathcal{L}^2(\Omega,\ L(\Theta,\mathbb{X})) := \left\{ \xi : \Omega \to L(\Theta,\mathbb{X}) \middle| E\|\xi\|^2_{L(\Theta,\mathbb{X})} = \int_\Omega \|\xi\|^2_{L(\Theta,\mathbb{X})} \mathrm{d}\mathbb{P} < \infty \right\},$$

equipped with the norm

$$\|\xi\|_{\mathcal{L}^2(\Omega,\ L(\Theta,\mathbb{X}))} := \left(\int_\Omega \|\xi\|^2_{L(\Theta,\mathbb{X})} \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}}.$$

Remark 2.1. *Denote by $L_2(\Theta,\mathbb{X})$ the collection of all Hilbert-Schmidt operators from Θ to $\mathbb{X}$ with inner product $\langle X, Y\rangle_{L_2(\Theta,\mathbb{X})} := \sum_{i\in\mathbb{N}} \langle Xe_i, Ye_i\rangle$, where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of Θ. Note that it is a separable Hilbert space. Let $\Phi \in \mathcal{L}^2(\Omega,\ L(\Theta,\mathbb{X}))$ and let $Q : \Theta \to \Theta$ be a nonnegative and symmetric operator satisfying* $\mathrm{Trace}Q < \infty$*, we have $\Phi Q^{\frac{1}{2}} \in \mathcal{L}^2(\Omega,\ L_2(\Theta,\mathbb{X}))$ and define*

$$\|\Phi Q^{\frac{1}{2}}\|_{\mathcal{L}^2(\Omega,\ L_2(\Theta,\mathbb{X}))} = \left(E\|\Phi Q^{\frac{1}{2}}\|^2_{L_2(\Theta,\mathbb{X})} \right)^{\frac{1}{2}}.$$

Let $P(\mathbb{X})$ be the collection of all Borel probability measures on $\mathbb{X}$ with the metric

$$\beta(\mu_1,\mu_2) := \sup_{\|G\|_{BL}\le 1} \left| \int G \mathrm{d}\mu_1 - \int G \mathrm{d}\mu_2 \right|, \quad \mu_1,\ \mu_2 \in P(\mathbb{X}),$$

where the functions G are real-valued Lipschitz continuous functions on $\mathbb{X}$ with norms

$$\|G\|_{BL} = \|G\|_L + \|G\|_\infty, \quad \|G\|_L = \sup_{z_1\neq z_2} \frac{|G(z_1) - G(z_2)|}{\|z_1 - z_2\|}, \quad \|G\|_\infty = \sup_{z\in\mathbb{X}} |G(z)|.$$

Note that the metric space $(P(\mathbb{X}), \beta)$ is complete. If $\int G\mathrm{d}\mu_k \to \int G\mathrm{d}\mu$ for all $G \in C_b(\mathbb{X})$, the real-valued bounded continuous function space on $\mathbb{X}$, then $\{\mu_k\} \in P(\mathbb{X}) \to \mu$ weakly. Besides, $\{\mu_k\} \to \mu$ weakly equals to $\beta(\mu_k,\mu) \to 0$ as $k \to \infty$. Consider the space

$$P_2(\mathbb{X}) = \left\{ \mu \in P(\mathbb{X}) : \|\mu\|_2^2 := \int_{\mathbb{X}} \|z\|^2 \mu(\mathrm{d}z) < \infty \right\}$$

with Wasserstein distance

$$\mathcal{W}_2(\mu_1,\mu_2) := \inf_{\pi\in C(\mu_1,\mu_2)} \Big(\int\limits_{\mathbb{X}\times\mathbb{X}} \|z-y\|^2 \pi(\mathrm{d}z,\mathrm{d}y) \Big)^{\frac{1}{2}},$$

where $C(\mu_1,\mu_2)$ denotes the set of all couplings of μ_1 and μ_2, i.e., $\pi(\cdot\times\mathbb{X}) = \mu_1$ and $\pi(\mathbb{X}\times\cdot) = \mu_2$. Observe that for $z\in\mathbb{X}$, the Dirac measure $\delta_z\in\mathcal{P}_2(\mathbb{X})$. Let ξ be a random variable and use $\mathbb{P}_\xi$ to denote its distribution. Note that

$$\|E\xi - E\zeta\| \le \mathcal{W}_2(\mathbb{P}_\xi,\mathbb{P}_\zeta) \le \left(E\|\xi-\zeta\|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

2.1. SDEs with Lévy noise

SDEs based on Lévy noise are a class of mathematical models that incorporate random fluctuations with heavy-tailed distributions. These equations extend the traditional Brownian motion-driven SDEs by allowing for jumps and extreme variations, making them suitable for modeling phenomena in finance, physics, and biology where rare events play a significant role. Now we first review some basic definitions and facts of Lévy process; for more details, see the monumental work [1].

Definition 2.1. *A stochastic process* $D = (D(\tau),\ \tau\ge 0)$ *with values in* Θ *is called Lévy process provided it satisfies*

(1) $D(0) = 0$ *almost surely;*

(2) The increments of D are independent and stationary;

(3) D is continuous in the following sense

$$\lim_{\tau\to\gamma} \mathbb{P}\left(|D(\tau) - D(\gamma)|_\Theta > e\right) = 0 \quad (\forall\, e>0, \forall\, \gamma>0).$$

Given a Lévy process D, the corresponding *jump process* $\Delta D = (\Delta D(\tau),\ \tau\ge 0)$ is defined as $\Delta D(\tau) = D(\tau) - D(\tau-)$, $\forall\,\tau\ge 0$. We define a random counting measure

$$N(\tau,Z)(\omega) := \sharp\{0\le\gamma\le\tau : \Delta D(\gamma)(\omega)\in Z\} = \sum_{0\le\gamma\le\tau} \chi_Z(\Delta D(\gamma)(\omega)),$$

where Z is any Borel set in $\Theta-\{0\}$ and the notation χ_Z denotes the indicator function. We say Z bounded below provided 0 does not belong to the closure of Z. We refer to $\nu(\cdot) = E(N(1,\cdot))$ as the intensity measure (i.m. for short) associated with the Lévy process D. If Z is bounded below, $(N(\tau,Z),\ \tau\ge 0)$ is a Poisson process with $\nu(Z)$. Therefore, N is termed as the Poisson random measure (P.r.m. for short), and the compensated P.r.m. is defined by

$$\widetilde{N}(\tau,Z) = N(\tau,Z) + \tau\nu(Z). \tag{2.2}$$

Remark 2.2. *(Poisson integral [1]) Let Z be bounded below, and let* u_Z *denote the restriction to Z of the measure* ν*. Then for* $f\in L^2(Z,u_Z)$*, we have*

$$E\left(\left|\int_Z f(p)\widetilde{N}(\tau,\mathrm{d}p)\right|^2\right) = \tau\int_Z |f(p)|^2\nu(\mathrm{d}p).$$

Theorem 2.1. *A Lévy process D with values in Θ can be represented as*

$$D(\tau) = g\tau + W(\tau) + \int_{|p|_\Theta<1} p\widetilde{N}(\tau, \mathrm{d}p) + \int_{|p|_\Theta\geq 1} pN(\tau, \mathrm{d}p) \quad (\tau \geq 0), \tag{2.3}$$

where $g \in \Theta$; W is a Q-Brownian motion; the independent P.r.m. N with i.m. ν is defined on $\mathbb{R}^+ \times \Theta'$ and $\widetilde{N}$ is the compensated P.r.m. of N. Here the measure ν satisfies

$$\int_\Theta \left(|p|_\Theta^2 \wedge 1\right) \nu(\mathrm{d}p) < \infty. \tag{2.4}$$

Remark 2.3. *Note that Theorem 2.1 is the well-known Lévy-Itô decomposition theorem. According to (2.4), we have $\int_{|p|_\Theta\geq 1} \nu(\mathrm{d}p) < \infty$, and then we set $c := \int_{|p|_\Theta\geq 1} \nu(\mathrm{d}p)$.*

Consider a two-sided Lévy process as follows:

$$D(\gamma) = \begin{cases} D_1(\gamma), & \gamma \geq 0, \\ -D_2(-\gamma), & \gamma \leq 0, \end{cases}$$

where Lévy processes D_1 and D_2 have the decompositions as in Theorem 2.1, and they are independent and identically distributed. For our convenience, assume that the covariance operator Q of W is trace class, i.e., $TrQ < \infty$ and the two-sided Lévy process D is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_{\tau\in\mathbb{R}}, \mathbb{P})$.

Remark 2.4. *If for a given $\alpha \in \mathbb{R}$, the process $\widetilde{D}$ is defined by $\widetilde{D}(\tau) = D(\tau + \alpha) - D(\alpha)$, then $\widetilde{D}$ is a two-sided Lévy process and it has the same law as D.*

By Theorem 2.1, we consider the mean-field type SDE with Lévy noise

$$\begin{aligned} \mathrm{d}\xi(\tau) =& A\xi(\tau)\mathrm{d}\tau + \varphi\left(\tau, \xi(\tau), \mathbb{P}_{\xi(\tau)}\right)\mathrm{d}\tau + \psi\left(\tau, \xi(\tau), \mathbb{P}_{\xi(\tau)}\right)\mathrm{d}W(\tau) \\ &+ \int_{|p|_\Theta<1} \Phi_1\left(\tau, \xi(\tau-), \mathbb{P}_{\xi(\tau-)}, p\right)\widetilde{N}(\mathrm{d}\tau, \mathrm{d}p) + \int_{|p|_\Theta\geq 1} \Phi_2\left(\tau, \xi(\tau-), \mathbb{P}_{\xi(\tau-)}, p\right)N(\mathrm{d}\tau, \mathrm{d}p). \end{aligned}$$

2.2. Almost automorphic stochastic processes

Definition 2.2. *($\mathcal{L}^2$-continuous and $\mathcal{L}^2$-bounded) Assuming there exists a stochastic process $\zeta : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{X})$,*

(1) The process ζ is called $\mathcal{L}^2$-continuous provided

$$\lim_{\tau\to\gamma} E\|\zeta(\tau) - \zeta(\gamma)\|^2 = 0, \quad \gamma \in \mathbb{R};$$

(2) The process ζ is called $\mathcal{L}^2$-bounded provided

$$\sup_{\tau\in\mathbb{R}} E\|\zeta(\tau)\|^2 < \infty.$$

Definition 2.3. *(almost automorphic)*

(1) *A $\mathcal{L}^2$-continuous stochastic process $\xi : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{X})$ is called square-mean almost automorphic (s.m.a.a. for short). If for any real sequence (seq. for short) $\{\gamma'_k\}$, there exists a subsequence (subseq. for short)$\{\gamma_k\}$ such that for another process $\zeta : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{X})$*

$$\lim_{k\to\infty} E\,\|\xi(\tau+\gamma_k)-\zeta(\tau)\|^2 = 0 \quad \text{and} \quad \lim_{k\to\infty} E\,\|\zeta(\tau-\gamma_k)-\xi(\tau)\|^2 = 0,$$

for any $\tau \in \mathbb{R}$. The set of all s.m.a.a. stochastic processes $\xi : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{X})$ is represented by $AA(\mathbb{R},\ \mathcal{L}^2(\Omega, \mathbb{X}))$;

(2) *A function $\varphi : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega, \mathbb{X})$, $(\tau, \xi, \mathbb{P}_\xi) \mapsto \varphi(\tau, \xi, \mathbb{P}_\xi)$ is called s.m.a.a. in $\tau \in \mathbb{R}$ for $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$ and its corresponding law $\mathbb{P}_\xi \in P_2(\mathbb{X})$ if φ is $\mathcal{L}^2$-continuous, i.e.,*

$$E\|\varphi(\tau, \xi, \mathbb{P}_\xi) - \varphi(\tau', \xi', \mathbb{P}_{\xi'})\|^2 \to 0 \quad as \quad (\tau', \xi', \mathbb{P}_{\xi'}) \to (\tau, \xi, \mathbb{P}_\xi),$$

and for any real seq. $\{\gamma'_k\}$, there exists a subseq. $\{\gamma_k\}$ such that for another function $\widetilde{\varphi} : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega, \mathbb{X})$

$$\lim_{k\to\infty} E\left\|\varphi\left(\tau+\gamma_k, \xi, \mathbb{P}_\xi\right) - \widetilde{\varphi}\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2 = 0,$$

and

$$\lim_{k\to\infty} E\left\|\widetilde{\varphi}\left(\tau-\gamma_k, \xi, \mathbb{P}_\xi\right) - \varphi\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2 = 0,$$

for any $\tau \in \mathbb{R}$, $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$, $\mathbb{P}_\xi \in P_2(\mathbb{X})$.

(3) *A function $\psi : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega,\ L(\Theta, \mathbb{X}))$, $(\tau, \xi, \mathbb{P}_\xi) \mapsto \psi(\tau, \xi, \mathbb{P}_\xi)$ is called s.m.a.a. in $\tau \in \mathbb{R}$ for $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$ and the corresponding law $\mathbb{P}_\xi \in P_2(\mathbb{X})$ if ψ is $\mathcal{L}^2$-continuous, i.e.,*

$$E\|\psi(\tau, \xi, \mathbb{P}_\xi) - \psi(\tau', \xi', \mathbb{P}_{\xi'})\|^2_{L(\Theta,\mathbb{X})} \to 0 \quad as \quad (\tau', \xi', \mathbb{P}_{\xi'}) \to (\tau, \xi, \mathbb{P}_\xi),$$

and for any real seq. $\{\gamma'_k\}$, there exists a subseq. $\{\gamma_k\}$ such that for another function $\widetilde{\psi} : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega,\ L(\Theta, \mathbb{X}))$

$$\lim_{k\to\infty} E\left\|\psi\left(\tau+\gamma_k, \xi, \mathbb{P}_\xi\right) - \widetilde{\psi}\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2_{L(\Theta,\mathbb{X})} = 0,$$

and

$$\lim_{k\to\infty} E\left\|\widetilde{\psi}\left(\tau-\gamma_k, \xi, \mathbb{P}_\xi\right) - \psi\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2_{L(\Theta,\mathbb{X})} = 0,$$

for any $\tau \in \mathbb{R}$, $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$, and $\mathbb{P}_\xi \in P_2(\mathbb{X})$.

(4) *A function $\Phi : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \times \Theta \to \mathcal{L}^2(\Omega, \mathbb{X})$, $(\tau, \xi, \mathbb{P}_\xi, p) \mapsto \Phi(\tau, \xi, \mathbb{P}_\xi, p)$ is called Poisson s.m.a.a. in $\tau \in \mathbb{R}$ for $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$ and corresponding law $\mathbb{P}_\xi \in P_2(\mathbb{X})$ if Φ is $\mathcal{L}^2$-continuous, i.e.,*

$$\int_\Theta E\|\Phi(\tau, \xi, \mathbb{P}_\xi, p) - \Phi(\tau', \xi', \mathbb{P}_{\xi'}, p)\|^2 \nu(\mathrm{d}p) \to 0 \quad as \quad (\tau', \xi', \mathbb{P}_{\xi'}) \to (\tau, \xi, \mathbb{P}_\xi),$$

and for any real seq. $\{\gamma'_k\}$*, there exists a subseq.* $\{\gamma_k\}$ *such that for another* $\widetilde{\Phi}: \mathbb{R}\times\mathcal{L}^2(\Omega,\mathbb{X})\times P_2(\mathbb{X})\times\Theta\to\mathcal{L}^2(\Omega,\mathbb{X})$ *with* $\int_\Theta E\|\widetilde{\Phi}(\tau,\xi,\mathbb{P}_\xi,p)\|^2\nu(\mathrm{d}p)<\infty$

$$\lim_{k\to\infty}\int_\Theta E\left\|\Phi\left(\tau+\gamma_k,\xi,\mathbb{P}_\xi,p\right)-\widetilde{\Phi}\left(\tau,\xi,\mathbb{P}_\xi,p\right)\right\|^2\nu(\mathrm{d}p)=0,$$

and

$$\lim_{k\to\infty}\int_\Theta E\left\|\widetilde{\Phi}\left(\tau-\gamma_k,\xi,\mathbb{P}_\xi,p\right)-\Phi\left(\tau,\xi,\mathbb{P}_\xi,p\right)\right\|^2\nu(\mathrm{d}p)=0,$$

for all $\tau\in\mathbb{R}$, $\xi\in\mathcal{L}^2(\Omega,\mathbb{X})$ *and* $\mathbb{P}_\xi\in P_2(\mathbb{X})$.

(5) Let $\xi(\tau)$ *be an* $\mathbb{X}$*-valued stochastic process. Then* ξ *is called almost automorphic in distribution provided its law* $\mathbb{P}_{\xi(\tau)}$ *is a* $P(\mathbb{X})$*-valued almost automorphic map, that is, for any real seq.* $\{\gamma'_k\}$*, there exists a subseq.* $\{\gamma_k\}$ *and an* $\mathbb{X}$*-valued stochastic process* $\zeta(\tau)$ *with its* $P(\mathbb{X})$*-valued law* $\mathbb{P}_{\zeta(\tau)}$ *such that for any* $\tau\in\mathbb{R}$

$$\beta(\mathbb{P}_{\xi(\tau+\gamma_k)},\mathbb{P}_{\zeta(\tau)})\to 0\quad\text{and}\quad\beta(\mathbb{P}_{\xi(\tau-\gamma_k)},\mathbb{P}_{\zeta(\tau)})\to 0\quad\text{as } k\to\infty.$$

Remark 2.5. *If the stochastic process* ξ *is s.m.a.a., then it is* $\mathcal{L}^2$*-bounded. By [12], let* $\xi\in AA(\mathbb{R},\ \mathcal{L}^2(\Omega,\mathbb{X}))$,

$$\|\xi\|_\infty:=\sup_{\tau\in\mathbb{R}}\|\xi(\tau)\|_2,$$

then $(AA(\mathbb{R},\ \mathcal{L}^2(\Omega,\mathbb{X})),\|\cdot\|_\infty)$ *is a Banach space.*

Proposition 2.1. *[11] Let* $\psi:\mathbb{R}\times\mathcal{L}^2(\Omega,\mathbb{X})\times P_2(\mathbb{X})\to\mathcal{L}^2(\Omega,\mathbb{X})$, $(\tau,\xi,\mathbb{P}_\xi)\to\psi(\tau,\xi,\mathbb{P}_\xi)$ *be s.m.a.a. in* $\tau\in\mathbb{R}$ *for each* $\xi\in\mathcal{L}^2(\Omega,\mathbb{X})$ *and the corresponding law* $\mathbb{P}_\xi\in P_2(\mathbb{X})$*. Suppose that for all* $\tau\in\mathbb{R}$, $\xi,\zeta\in\mathcal{L}^2(\Omega,\mathbb{X})$ *and* $\mathbb{P}_\xi,\ \mathbb{P}_\zeta\in P_2(\mathbb{X})$,

$$E\|\psi(\tau,\xi,\mathbb{P}_\xi)-\psi(\tau,\zeta,\mathbb{P}_\zeta)\|^2\le L\left(E\|\xi-\zeta\|^2+\mathcal{W}_2^2(\mathbb{P}_\xi,\mathbb{P}_\zeta)\right).$$

Then for all s.m.a.a. process $\xi:\mathbb{R}\to\mathcal{L}^2(\Omega,\mathbb{X})\times P_2(\mathbb{X})$*, the process* $\Psi:\mathbb{R}\to\mathcal{L}^2(\Omega,\mathbb{X})\times P_2(\mathbb{X})$ *given by* $\Psi(\tau):=\psi(\tau,\xi,\mathbb{P}_\xi)$ *is s.m.a.a.*

Lemma 2.1. *If the functions* Φ, Φ_1*, and* $\Phi_2:\ \mathbb{R}\times\mathcal{L}^2(\Omega,\mathbb{X})\times P_2(\mathbb{X})\times\Theta\to\mathcal{L}^2(\Omega,\mathbb{X})$ *are Poisson s.m.a.a. in* $\tau\in\mathbb{R}$ *for* $\xi\in\mathcal{L}^2(\Omega,\mathbb{X})$*, then*

(1) $\Phi_1+\Phi_2$ *is Poisson s.m.a.a.;*

(2) $k\Phi$ *is Poisson s.m.a.a. for every constant* k*;*

(3) For every $\xi\in\mathcal{L}^2(\Omega,\mathbb{X})$ *and the corresponding law* $\mathbb{P}_\xi\in P_2(\mathbb{X})$*, there exists a constant* $C=C(\xi)>0$ *such that*

$$\sup_{\tau\in\mathbb{R}}\int_\Theta E\left\|\Phi\left(\tau,\xi,\mathbb{P}_\xi,p\right)\right\|^2\nu(\mathrm{d}p)\le C.$$

3. Existence of almost automorphic solutions

Consider the mean-field SDE with Lévy process

$$\begin{aligned} \mathrm{d}\xi(\tau) =& A\xi(\tau)\mathrm{d}\tau + \varphi\left(\tau, \xi(\tau), \mathbb{P}_{\xi(\tau)}\right)\mathrm{d}\tau + \psi\left(\tau, \xi(\tau), \mathbb{P}_{\xi(\tau)}\right)\mathrm{d}W(\tau) \\ &+ \int_{|p|_\Theta<1} \Phi_1\left(\tau, \xi(\tau-), \mathbb{P}_{\xi(\tau-)}, p\right)\widetilde{N}(\mathrm{d}\tau, \mathrm{d}p) + \int_{|p|_\Theta\geq 1} \Phi_2\left(\tau, \xi(\tau-), \mathbb{P}_{\xi(\tau-)}, p\right)N(\mathrm{d}\tau, \mathrm{d}p), \end{aligned} \tag{3.1}$$

where an infinitesimal generator A produces a dissipative C_0-semi-group $\{G(\tau)\}_{\tau\geq 0}$ on $\mathbb{X}$ such that

$$\|G(\tau)\| \leq \mathcal{K}e^{-q\tau}, \quad \forall\, \tau \geq 0, \tag{3.2}$$

with $q, \mathcal{K} > 0$; $\varphi : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega, \mathbb{X})$, $\psi : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \to \mathcal{L}^2(\Omega, L(\Theta, \mathbb{X}))$; $\Phi_1,\ \Phi_2 : \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{X}) \times P_2(\mathbb{X}) \times \Theta \to \mathcal{L}^2(\Omega, \mathbb{X})$; N and W are the components of the Lévy-Itô decomposition for a two-sided Lévy process (Theorem 2.1).

Definition 3.1. *An $\mathcal{F}_\tau$-adapted process $\xi(\tau)$ is referred to as a mild solution of SDE (3.1) if*

$$\begin{aligned} \xi(\tau) =& G(\tau - s)\xi(s) + \int_s^\tau G(\tau - \alpha)\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}\alpha \\ &+ \int_s^\tau G(\tau - \alpha)\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}W(\alpha) \\ &+ \int_s^\tau \int_{|p|_\Theta<1} G(\tau - \alpha)\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)\widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p) \\ &+ \int_s^\tau \int_{|p|_\Theta\geq 1} G(\tau - \alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)N(\mathrm{d}\alpha, \mathrm{d}p) \end{aligned} \tag{3.3}$$

holds for all $\tau \geq s$ and every $s \in \mathbb{R}$.

Lemma 3.1. *[13] (a variant of Gronwall's lemma) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$0 \leq g(\tau) \leq h(\tau) + w_1 \int_{-\infty}^\tau e^{-z_1(\tau-\alpha)}g(\alpha)d\alpha + \cdots + w_n \int_{-\infty}^\tau e^{-z_n(\tau-\alpha)}g(\alpha)d\alpha, \quad \tau \in \mathbb{R}, \tag{3.4}$$

for some function $h : \mathbb{R} \to \mathbb{R}$, for some constants $w_1, \ldots, w_n \geq 0$, $z_1, \ldots, z_n > w$ with $w := \sum_{k=1}^n w_k$. Suppose the integrals on the right-hand side of (3.4) *converge. Set $z := \min_{1\leq k\leq n} z_k$. If $\int_{-\infty}^0 e^{\varepsilon\alpha}h(\alpha)d\alpha$, $\varepsilon \in (0, z - w]$ converges, then*

$$g(\tau) \leq h(\tau) + w \int_{-\infty}^\tau e^{-\varepsilon(\tau-\alpha)}h(\alpha)\mathrm{d}\alpha$$

holds for all $\tau \in \mathbb{R}$. If h is a constant, then

$$g(\tau) \leq \frac{hz}{z - w}.$$

Theorem 3.1. *Suppose that*

(1) The semi-group of the linear operator A satisfies the exponential stable condition such that (3.2) *holds;*

(2) φ *and* ψ *are s.m.a.a. in* $\tau \in \mathbb{R}$ *for each* $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$ *and the corresponding law* $\mathbb{P}_\xi \in P_2(\mathbb{X})$*;*

(3) Φ_1 *and* Φ_2 *are Poisson s.m.a.a. in* $\tau \in \mathbb{R}$ *for each* $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$ *and the corresponding law* $\mathbb{P}_\xi \in P_2(\mathbb{X})$*;*

(4) φ, ψ, Φ_1, *and* Φ_2 *satisfy Lipschitz conditions in the following sense:*

$$E\|\varphi(\tau, \xi, \mu) - \varphi(\tau, \zeta, \nu)\|^2 \leq L\left(E\|\xi - \zeta\|^2 + \mathcal{W}_2^2(\mu, \nu)\right), \tag{3.5}$$

$$E\left\|[\psi(\tau, \xi, \mu) - \psi(\tau, \zeta, \nu)]\, Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta, \mathbb{X})} \leq L\left(E\|\xi - \zeta\|^2 + \mathcal{W}_2^2(\mu, \nu)\right), \tag{3.6}$$

$$\int\limits_{|p|_\Theta < 1} E\|\Phi_1(\tau, \xi, \mu, p) - \Phi_1(\tau, \zeta, \nu, p)\|^2 \nu(\mathrm{d}p) \leq L\left(E\|\xi - \zeta\|^2 + \mathcal{W}_2^2(\mu, \nu)\right), \tag{3.7}$$

$$\int\limits_{|p|_\Theta \geq 1} E\|\Phi_2(\tau, \xi, \mu, p) - \Phi_2(\tau, \zeta, \nu, p)\|^2 \nu(\mathrm{d}p) \leq L\left(E\|\xi - \zeta\|^2 + \mathcal{W}_2^2(\mu, \nu)\right), \tag{3.8}$$

for all $\tau \in \mathbb{R}$, $\xi, \zeta \in \mathcal{L}^2(\Omega, \mathbb{X})$ *and* $\mu,\ \nu \in P_2(\mathbb{X})$. *Then we have*

i) If

$$L < \frac{q^2}{8\mathcal{K}^2(1 + 2c + 2q)}, \tag{3.9}$$

the SDE (3.1) has a unique solution in $\mathcal{L}^2(\Omega, \mathbb{X})$*;*

ii) If

$$L < \frac{q^2}{16\mathcal{K}^2(1 + 2c + 4q)}, \tag{3.10}$$

the unique solution is almost automorphic in distribution.

Proof. Let $\xi(\tau)$ be $\mathcal{L}^2$-bounded. Then by (3.2), $G(\tau - s)\xi(s) \to 0$ as $s \to -\infty$. According to Definition 3.1, $\xi(\tau)$ is a mild solution of SDE (3.1) if and only if

$$\begin{aligned}
\xi(\tau) = &\int_{-\infty}^{\tau} G(\tau - \alpha)\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}\alpha + \int_{-\infty}^{\tau} G(\tau - \alpha)\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}W(\alpha) \\
&+ \int_{-\infty}^{\tau} \int\limits_{|p|_\Theta < 1} G(\tau - \alpha)\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)\widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p) \\
&+ \int_{-\infty}^{\tau} \int\limits_{|p|_\Theta \geq 1} G(\tau - \alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)N(\mathrm{d}\alpha, \mathrm{d}p).
\end{aligned} \tag{3.11}$$

Step 1: Existence and uniqueness of the $\mathcal{L}^2$-bounded solution of SDE (3.1). We use the space $C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$ to denote the set of all bounded and continuous maps from $\mathbb{R}$ to $\mathcal{L}^2(\Omega, \mathbb{X})$ with $\|\cdot\|_\infty$. For $\xi \in C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$, a nonlinear operator $\mathcal{S}$ is defined on $C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$ by

$$
\begin{aligned}
(\mathcal{S}\xi)(\tau) = &\int_{-\infty}^{\tau} G(\tau-\alpha)\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) \mathrm{d}\alpha + \int_{-\infty}^{\tau} G(\tau-\alpha)\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) \mathrm{d}W(\alpha) \\
&+ \int_{-\infty}^{\tau} \int_{|p|_\Theta<1} G(\tau-\alpha)\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p) \\
&+ \int_{-\infty}^{\tau} \int_{|p|_\Theta\geq 1} G(\tau-\alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) N(\mathrm{d}\alpha, \mathrm{d}p).
\end{aligned}
$$

If $\xi(\cdot)$ is $\mathcal{L}^2$-bounded, it follows from the conditions of Theorem 3.1, Cauchy-Schwarz inequality, Itô's isometry property, and the property, of Poisson random measures that $(\mathcal{S}\xi)(\cdot)$ is $\mathcal{L}^2$-bounded. Similar to the proof of [20, Theorem 3.2] with minor modifications, we can illustrate that $(\mathcal{S}\xi)(\cdot)$ is $\mathcal{L}^2$-continuous. So we only need to prove the nonlinear operator $\mathcal{S}$ is a contraction on $C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$. For $\xi, \zeta \in C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$ and the corresponding law $\mathbb{P}_\xi, \mathbb{P}_\zeta \in P_2(\mathbb{X})$, we have

$$
\begin{aligned}
&E\|(\mathcal{S}\xi)(\tau) - (\mathcal{S}\zeta)(\tau)\|^2 \\
\leq &4E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \varphi\left(\alpha, \zeta(\alpha), \mathbb{P}_{\zeta(\alpha)}\right)\right] \mathrm{d}\alpha\right\|^2 \\
&+ 4E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \psi\left(\alpha, \zeta(\alpha), \mathbb{P}_{\zeta(\alpha)}\right)\right] \mathrm{d}W(\alpha)\right\|^2 \\
&+ 4E\left\|\int_{-\infty}^{\tau} \int_{|p|_\Theta<1} G(\tau-\alpha)\left[\Phi_1(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p) - \Phi_1(\alpha, \zeta(\alpha-), \mathbb{P}_{\zeta(\alpha-)}, p)\right] \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\
&+ 4E\left\|\int_{-\infty}^{\tau} \int_{|p|_\Theta\geq 1} G(\tau-\alpha)\left[\Phi_2(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p) - \Phi_2(\alpha, \zeta(\alpha-), \mathbb{P}_{\zeta(\alpha-)}, p)\right] N(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\
:= &4(I_1 + I_2 + I_3 + I_4). \qquad (3.12)
\end{aligned}
$$

By the Cauchy-Schwarz inequality, (2.1), (3.2), and (3.5), we have

$$
\begin{aligned}
I_1 \leq &\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} \mathrm{d}\alpha \cdot \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \varphi\left(\alpha, \zeta(\alpha), \mathbb{P}_{\zeta(\alpha)}\right)\right\|^2 \mathrm{d}\alpha \\
\leq &\frac{\mathcal{K}^2}{q^2} \sup_{\alpha\in\mathbb{R}} E\left\|\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \varphi\left(\alpha, \zeta(\alpha), \mathbb{P}_{\zeta(\alpha)}\right)\right\|^2 \\
\leq &\frac{2\mathcal{K}^2 L}{q^2} \sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha) - \zeta(\alpha)\|^2. \qquad (3.13)
\end{aligned}
$$

From Itô's isometry property, (2.1), (3.2), and (3.6), we have

$$
I_2 \leq \int_{-\infty}^{\tau} \mathcal{K}^2 e^{-2q(\tau-\alpha)} E\left\|\left[\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \psi\left(\alpha, \zeta(\alpha), \mathbb{P}_{\zeta(\alpha)}\right)\right] Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})} \mathrm{d}\alpha
$$

$$\leq \frac{\mathcal{K}^2}{2q} \sup_{\alpha\in\mathbb{R}} E\left\|\left[\psi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)-\psi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\right]Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})}$$
$$\leq \frac{\mathcal{K}^2 L}{q} \sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha)-\zeta(\alpha)\|^2. \tag{3.14}$$

According to the Cauchy-Schwarz inequality, (2.1), (3.2), and (3.7), we have

$$I_3 \leq \int_{-\infty}^{\tau}\int_{|p|_\Theta<1} \mathcal{K}^2 e^{-2q(\tau-\alpha)} E\left\|\Phi_1\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_1\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right\|^2 \nu(\mathrm{d}p)\mathrm{d}\alpha$$
$$\leq \frac{\mathcal{K}^2}{2q}\sup_{\alpha\in\mathbb{R}}\int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_1\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right\|^2 \nu(\mathrm{d}p)$$
$$\leq \frac{\mathcal{K}^2 L}{q} \sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha)-\zeta(\alpha)\|^2. \tag{3.15}$$

By the Cauchy-Schwarz inequality, (2.1), (2.2), (3.2), (3.8), and (3.15), we possess

$$I_4 \leq 2E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} G(\tau-\alpha)\left[\Phi_2(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p)-\Phi_2(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p)\right]\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2$$
$$+ 2E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} G(\tau-\alpha)\left[\Phi_2(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p)-\Phi_2(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p)\right]\nu(\mathrm{d}p)\mathrm{d}\alpha\right\|^2$$
$$\leq \frac{2\mathcal{K}^2 L}{q}\sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha)-\zeta(\alpha)\|^2 + 2\mathcal{K}^2\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} e^{-q(\tau-\alpha)}\nu(\mathrm{d}p)\mathrm{d}\alpha$$
$$\cdot\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} e^{-q(\tau-\alpha)} E\|\Phi_2(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p)-\Phi_2(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p)\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha$$
$$\leq \frac{2\mathcal{K}^2 L}{q}\sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha)-\zeta(\alpha)\|^2$$
$$+ \frac{2\mathcal{K}^2 c}{q^2}\sup_{\alpha\in\mathbb{R}}\int_{|p|_\Theta\geq 1} E\left\|\Phi_2\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_2\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)$$
$$\leq \left(\frac{2\mathcal{K}^2 L}{q}+\frac{4\mathcal{K}^2 cL}{q^2}\right)\sup_{\alpha\in\mathbb{R}} E\|\xi(\alpha)-\zeta(\alpha)\|^2, \tag{3.16}$$

recalling that

$$c := \int_{|p|_\Theta\geq 1} \nu(\mathrm{d}p).$$

By (3.12)–(3.16) we can obtain

$$\|(\mathcal{S}\xi)(\tau)-(\mathcal{S}\zeta)(\tau)\|_2^2 \leq \theta\sup_{\alpha\in\mathbb{R}}\|\xi(\alpha)-\zeta(\alpha)\|_2^2$$

with

$$\theta = (1+2c)\frac{8K^2L}{q^2}+\frac{16K^2L}{q}.$$

Since

$$\sup_{\alpha\in\mathbb{R}} \|\xi(\alpha) - \zeta(\alpha)\|_2^2 \le (\sup_{\alpha\in\mathbb{R}} \|\xi(\alpha) - \zeta(\alpha)\|_2)^2,$$

we have

$$\|(\mathcal{S}\xi)(\tau) - (\mathcal{S}\zeta)(\tau)\|_\infty \le \sqrt{\theta}\|\xi(\tau) - \zeta(\tau)\|_\infty.$$

The Lipschitz constant $L < \frac{q^2}{8\mathcal{K}^2(1+2c+2q)}$ in (3.9) implies $\theta < 1$, then $\mathcal{S}$ is a contraction on $C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$. Therefore, there exists a unique fixed point $\bar{\zeta} \in C_b(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{X}))$ satisfying $\mathcal{S}\bar{\zeta} = \bar{\zeta}$, that is to say the SDE (3.1) has a unique $\mathcal{L}^2$-bounded solution.

Step 2: Almost automorphy of $\mathcal{L}^2$-bounded solution of SDE (3.1). We denote by $\{r'_n\}$ an arbitrary real seq. Since φ, ψ are s.m.a.a. and Φ_1, Φ_2 are Poisson s.m.a.a., we can extract a subseq. $\{r_n\}$ of $\{r'_n\}$ such that for some functions $\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{\Phi_1}$, $\widetilde{\Phi_2}$

$$\lim_{n\to\infty} E\left\|\varphi\left(\tau + r_n, \xi, \mathbb{P}_\xi\right) - \widetilde{\varphi}\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2 = 0,$$

$$\lim_{n\to\infty} E\left\|\widetilde{\varphi}\left(\tau - r_n, \xi, \mathbb{P}_\xi\right) - \varphi\left(\tau, \xi, \mathbb{P}_\xi\right)\right\|^2 = 0;$$

$$\lim_{n\to\infty} E\left\|\left[\psi\left(\tau + r_n, \xi, \mathbb{P}_\xi\right) - \widetilde{\psi}\left(\tau, \xi, \mathbb{P}_\xi\right)\right] Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})} = 0,$$

$$\lim_{n\to\infty} E\left\|\left[\widetilde{\psi}\left(\tau - r_n, \xi, \mathbb{P}_\xi\right) - \psi\left(\tau, \xi, \mathbb{P}_\xi\right)\right] Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})} = 0;$$

$$\lim_{n\to\infty} \int_{|p|_\Theta<1} E\left\|\Phi_1\left(\tau + r_n, \xi, \mathbb{P}_\xi, p\right) - \widetilde{\Phi_1}\left(\tau, \xi, \mathbb{P}_\xi, p\right)\right\|^2 \nu(\mathrm{d}p) = 0,$$

$$\lim_{n\to\infty} \int_{|p|_\Theta<1} E\left\|\widetilde{\Phi_1}\left(\tau - r_n, \xi, \mathbb{P}_\xi, p\right) - \Phi_1\left(\tau, \xi, \mathbb{P}_\xi, p\right)\right\|^2 \nu(\mathrm{d}p) = 0;$$

and

$$\lim_{n\to\infty} \int_{|p|_\Theta\ge1} E\left\|\Phi_2\left(\tau + r_n, \xi, \mathbb{P}_\xi, p\right) - \widetilde{\Phi_2}\left(\tau, \xi, \mathbb{P}_\xi, p\right)\right\|^2 \nu(\mathrm{d}p) = 0,$$

$$\lim_{n\to\infty} \int_{|p|_\Theta\ge1} E\left\|\widetilde{\Phi_2}\left(\tau - r_n, \xi, \mathbb{P}_\xi, p\right) - \Phi_2\left(\tau, \xi, \mathbb{P}_\xi, p\right)\right\|^2 \nu(\mathrm{d}p) = 0$$

hold for each $\tau \in \mathbb{R}$, each $\xi \in \mathcal{L}^2(\Omega, \mathbb{X})$, and the corresponding law $\mathbb{P}_\xi \in P_2(\mathbb{X})$.

Let $\widetilde{\xi}(\cdot)$ satisfy the equation

$$\begin{aligned}\widetilde{\xi}(\tau) = &\int_{-\infty}^{\tau} G(\tau - r)\widetilde{\varphi}\left(r, \widetilde{\xi}(r), \mathbb{P}_{\widetilde{\xi}(r)}\right)\mathrm{d}r + \int_{-\infty}^{\tau} G(\tau - r)\widetilde{\psi}\left(r, \widetilde{\xi}(r), \mathbb{P}_{\widetilde{\xi}(r)}\right)\mathrm{d}W(r)\\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta<1} G(\tau - r)\widetilde{\Phi_1}\left(r, \widetilde{\xi}(r-), \mathbb{P}_{\widetilde{\xi}(r-)}, p\right)\widetilde{N}(\mathrm{d}r, \mathrm{d}p)\\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta\ge1} G(\tau - r)\widetilde{\Phi_2}\left(r, \widetilde{\xi}(r-), \mathbb{P}_{\widetilde{\xi}(r-)}, p\right)N(\mathrm{d}r, \mathrm{d}p),\end{aligned}$$

and let $\alpha = r - r_n$, then

$$\begin{aligned}\xi(\tau + r_n) = &\int_{-\infty}^{\tau} G(\tau-\alpha)\varphi\left(\alpha + r_n, \xi(\alpha + r_n), \mathbb{P}_{\xi(\alpha+r_n)}\right) \mathrm{d}\alpha \\ &+ \int_{-\infty}^{\tau} G(\tau-\alpha)\psi\left(\alpha + r_n, \xi(\alpha + r_n), \mathbb{P}_{\xi(\alpha+r_n)}\right) \mathrm{d}W_n(\alpha) \\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta<1} G(\tau-\alpha)\Phi_1\left(\alpha + r_n, \xi(\alpha + r_n -), \mathbb{P}_{\xi(\alpha+r_n-)}, p\right) \widetilde{N}_n(\mathrm{d}\alpha, \mathrm{d}p) \\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} G(\tau-\alpha)\Phi_2\left(\alpha + r_n, \xi(\alpha + r_n -), \mathbb{P}_{\xi(\alpha+r_n-)}, p\right) N_n(\mathrm{d}\alpha, \mathrm{d}p),\end{aligned}$$

where W_n, defined as $W_n(\tau) := W(\tau + r_n) - W(r_n)$, $\tau \in \mathbb{R}$, is a $\mathcal{Q}$-Brownian motion, having the same law as W; N_n, defined as $N_n(\tau, p) := N(\tau + r_n, p) - N(r_n, p)$, $\tau \in \mathbb{R}$, have the same law as N with compensated P.r.m. $\widetilde{N}_n$.

Subsequently we consider the process $\xi_n(\cdot)$ satisfying

$$\begin{aligned}\xi_n(\tau) = &\int_{-\infty}^{\tau} G(\tau-\alpha)\varphi\left(\alpha + r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right) \mathrm{d}\alpha \\ &+ \int_{-\infty}^{\tau} G(\tau-\alpha)\psi\left(\alpha + r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right) \mathrm{d}W(\alpha) \\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta<1} G(\tau-\alpha)\Phi_1\left(\alpha + r_n, \xi_n(\alpha -), \mathbb{P}_{\xi_n(\alpha-)}, p\right) \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p) \\ &+ \int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} G(\tau-\alpha)\Phi_2\left(\alpha + r_n, \xi_n(\alpha -), \mathbb{P}_{\xi_n(\alpha-)}, p\right) N(\mathrm{d}\alpha, \mathrm{d}p).\end{aligned}$$

Notice that for every $\tau \in \mathbb{R}$, $\xi(\tau + r_n)$ and $\xi_n(\tau)$ have the same distribution. Like $\widetilde{\xi}(\cdot)$, such $\xi_n(\cdot)$ is also unique and $\mathcal{L}^2$-bounded. We know

$$\begin{aligned}&E\left\|\xi_n(\tau) - \widetilde{\xi}(\tau)\right\|^2 \\ \leq &4E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\varphi\left(\alpha + r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right) - \widetilde{\varphi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}\alpha\right\|^2 \\ &+ 4E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\psi\left(\alpha + r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right) - \widetilde{\psi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}W(\alpha)\right\|^2 \\ &+ 4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta<1} G(\tau-\alpha)\cdot\left[\Phi_1\left(\alpha + r_n, \xi_n(\alpha-), \mathbb{P}_{\xi_n(\alpha-)}, p\right) - \widetilde{\Phi_1}\left(\alpha, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right] \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\ &+ 4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1} G(\tau-\alpha)\cdot\left[\Phi_2\left(\alpha + r_n, \xi_n(\alpha-), \mathbb{P}_{\xi_n(\alpha-)}, p\right) - \widetilde{\Phi_2}\left(\alpha, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right] N(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\ :=&4(J_1 + J_2 + J_3 + J_4). \end{aligned} \tag{3.17}$$

By the Cauchy-Schwarz inequality, (2.1), (3.2), and (3.5), we have

$$\begin{aligned} J_1 \leq & 2E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\varphi\left(\alpha+r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right)-\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}\alpha\right\|^2 \\ & + 2E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\varphi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}\alpha\right\|^2 \\ \leq & 2\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} \mathrm{d}\alpha \cdot \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha+r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right)-\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right\|^2 \mathrm{d}\alpha \\ & + 2\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} \mathrm{d}\alpha \cdot \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\varphi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right\|^2 \mathrm{d}\alpha \\ \leq & \frac{4\mathcal{K}^2 L}{q} \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2 \mathrm{d}\alpha + a_1^n, \end{aligned} \tag{3.18}$$

where

$$a_1^n = \frac{2\mathcal{K}^2}{q} \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\varphi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right\|^2 \mathrm{d}\alpha.$$

Now we show that $a_1^n \to 0$ as $n \to \infty$. Note by (3.5) that we have

$$\begin{aligned} & E\left\|\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right\|^2 \\ \leq & 2E\left\|\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\varphi\left(\alpha+r_n, 0, \delta_0\right)\right\|^2 + 2E\left\|\varphi\left(\alpha+r_n, 0, \delta_0\right)\right\|^2 \\ \leq & 4L \cdot E\|\widetilde{\xi}(\alpha)\|^2 + 2E\left\|\varphi\left(\alpha+r_n, 0, \delta_0\right)\right\|^2. \end{aligned}$$

Since φ is s.m.a.a. in τ and $\widetilde{\xi}(\cdot)$ is $\mathcal{L}^2$-bounded, then by Remark 2.5 we have

$$\sup_{\alpha\in\mathbb{R}} E\|\varphi(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)})\|^2 < \infty.$$

Besides, by Definition 2.3

$$\sup_{\alpha\in\mathbb{R}} E\|\widetilde{\varphi}(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)})\|^2 < \infty.$$

That is,

$$\sup_{\alpha\in\mathbb{R}} E\|\varphi(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)})-\widetilde{\varphi}(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)})\|^2 < \infty.$$

Therefore, by Lebesgue dominated convergence theorem and Definition 2.3, we have

$$\lim_{n\to\infty} \int_{-\infty}^{\tau} e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\varphi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right\|^2 \mathrm{d}\alpha = 0,$$

that is, $a_1^n \to 0$ as $n \to \infty$.

From Itô's isometry formula, (2.1), (3.2), and (3.6), we have

$$\begin{aligned} J_2 \leq & 2E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\psi\left(\alpha+r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right)-\psi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}W(\alpha)\right\|^2 \\ & + 2E\left\|\int_{-\infty}^{\tau} G(\tau-\alpha)\left[\psi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\psi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] \mathrm{d}W(\alpha)\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-2q(\tau-\alpha)} \cdot E\left\|\left[\psi\left(\alpha+r_n, \xi_n(\alpha), \mathbb{P}_{\xi_n(\alpha)}\right)-\psi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] Q^{\frac{1}{2}}\right\|_{L_2(\Theta, \mathbb{X})}^2 \mathrm{d}\alpha \\
&\quad + 2\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-2q(\tau-\alpha)} \cdot E\left\|\left[\psi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\psi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] Q^{\frac{1}{2}}\right\|_{L_2(\Theta, \mathbb{X})}^2 \mathrm{d}\alpha \\
&\leq 4\mathcal{K}^2 L \int_{-\infty}^{\tau} e^{-2q(\tau-\alpha)} E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2 \mathrm{d}\alpha + a_2^n, \qquad (3.19)
\end{aligned}
$$

where

$$
a_2^n = 2\mathcal{K}^2 \int_{-\infty}^{\tau} e^{-2q(\tau-\alpha)} E\left\|\left[\psi\left(\alpha+r_n, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)-\widetilde{\psi}\left(\alpha, \widetilde{\xi}(\alpha), \mathbb{P}_{\widetilde{\xi}(\alpha)}\right)\right] Q^{\frac{1}{2}}\right\|_{L_2(\Theta, \mathbb{X})}^2 \mathrm{d}\alpha.
$$

Similar to a_1^n, using the same argument, we can illustrate $a_2^n \to 0$ as $n \to \infty$.

For the third term, on account of Cauchy-Schwarz inequality, (2.1), (3.2), and (3.7), we have

$$
\begin{aligned}
J_3 \leq\, & 2E\left\|\int_{-\infty}^{\tau} \int_{|p|_\Theta<1} G(\tau-\alpha) \cdot\left[\Phi_1\left(\alpha+r_n, \xi_n(\alpha-), \mathbb{P}_{\xi_n(\alpha-)}, p\right)-\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right] \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\
& + 2E\left\|\int_{-\infty}^{\tau} \int_{|p|_\Theta<1} G(\tau-\alpha) \cdot\left[\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)-\widetilde{\Phi_1}\left(\alpha, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right] \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\
\leq\, & 2\mathcal{K}^2 \int_{-\infty}^{\tau} \int_{|p|_\Theta<1} e^{-2q(\tau-\alpha)} \cdot E\left\|\Phi_1\left(\alpha+r_n, \xi_n(\alpha-), \mathbb{P}_{\xi_n(\alpha-)}, p\right)-\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right\|^2 \nu(\mathrm{d}p)\mathrm{d}\alpha \\
& + 2\mathcal{K}^2 \int_{-\infty}^{\tau} \int_{|p|_\Theta<1} e^{-2q(\tau-\alpha)} \cdot E\left\|\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)-\widetilde{\Phi_1}\left(\alpha, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right\|^2 \nu(\mathrm{d}p)\mathrm{d}\alpha \\
\leq\, & 4\mathcal{K}^2 L \int_{-\infty}^{\tau} e^{-2q(\tau-\alpha)} E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2 \mathrm{d}\alpha + a_3^n, \qquad (3.20)
\end{aligned}
$$

where

$$
a_3^n = 2\mathcal{K}^2 \int_{-\infty}^{\tau} \int_{|p|_\Theta<1} e^{-2q(\tau-\alpha)} \cdot E\left\|\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)-\widetilde{\Phi_1}\left(\alpha, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right\|^2 \nu(\mathrm{d}p)\mathrm{d}\alpha.
$$

Now we prove $a_3^n \to 0$ as $n \to \infty$. Note by (3.7) that we have

$$
\begin{aligned}
& \int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)\right\|^2 \nu(\mathrm{d}p) \\
& \leq 2 \int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha+r_n, \widetilde{\xi}(\alpha-), \mathbb{P}_{\widetilde{\xi}(\alpha-)}, p\right)-\Phi_1\left(\alpha+r_n, 0, \delta_0, p\right)\right\|^2 \nu(\mathrm{d}p)+2 \int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha+r_n, 0, \delta_0, p\right)\right\|^2 \nu(\mathrm{d}p) \\
& \leq 4L \cdot E\|\widetilde{\xi}(\alpha)\|^2 + 2 \int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha+r_n, 0, \delta_0, p\right)\right\|^2 \nu(\mathrm{d}p).
\end{aligned}
$$

Since φ is Poisson s.m.a.a. in τ and $\widetilde{\xi}(\cdot)$ is $\mathcal{L}^2$-bounded, then by Lemma 2.1

$$\sup_{\alpha\in\mathbb{R}}\int_{|p|_\Theta<1} E\left\|\Phi_1\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)<\infty.$$

In addition, by Definition 2.3

$$\sup_{\alpha\in\mathbb{R}}\int_{|p|_\Theta<1} E\|\widetilde{\Phi_1}(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p)\|^2\nu(\mathrm{d}p)<\infty,$$

that is

$$\sup_{\alpha\in\mathbb{R}}\int_{|p|_\Theta<1} E\|\Phi_1(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p)-\widetilde{\Phi_1}(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p)\|^2\nu(\mathrm{d}p)<\infty.$$

Therefore, according to Definition 2.3 and the Lebesgue dominated convergence theorem, we can obtain

$$\lim_{n\to\infty}\int_{-\infty}^{\tau}\int_{|p|_\Theta<1} e^{-2q(\tau-\alpha)}\cdot E\left\|\Phi_1\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_1}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha=0,$$

i.e., $a_3^n\to 0$ as $n\to\infty$.

For the fourth term, from (2.1), (2.2), (3.2), (3.8), (3.20), and the Cauchy-Schwarz inequality, we can obtain

$$\begin{aligned}
J_4\leq&4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1} G(\tau-\alpha)\cdot\left[\Phi_2\left(\alpha+r_n,\xi_n(\alpha-),\mathbb{P}_{\xi_n(\alpha-)},p\right)-\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right]\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2\\
&+4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1} G(\tau-\alpha)\cdot\left[\Phi_2\left(\alpha+r_n,\xi_n(\alpha-),\mathbb{P}_{\xi_n(\alpha-)},p\right)-\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right]\nu(\mathrm{d}p)\mathrm{d}\alpha\right\|^2\\
&+4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1} G(\tau-\alpha)\cdot\left[\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right]\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2\\
&+4E\left\|\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1} G(\tau-\alpha)\cdot\left[\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right]\nu(\mathrm{d}p)\mathrm{d}\alpha\right\|^2\\
\leq&8\mathcal{K}^2L\int_{-\infty}^{\tau}e^{-2q(\tau-\alpha)}E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2\mathrm{d}\alpha+4\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1}\mathcal{K}^2e^{-q(\tau-\alpha)}\nu(\mathrm{d}p)\mathrm{d}\alpha\cdot\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1}e^{-q(\tau-\alpha)}\\
&\cdot E\left\|\Phi_2\left(\alpha+r_n,\xi_n(\alpha-),\mathbb{P}_{\xi_n(\alpha-)},p\right)-\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha\\
&+4\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1}\mathcal{K}^2e^{-2q(\tau-\alpha)}\cdot E\left\|\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha\\
&+4\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1}\mathcal{K}^2e^{-q(\tau-\alpha)}\nu(\mathrm{d}p)\mathrm{d}\alpha\cdot\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq1}e^{-q(\tau-\alpha)}
\end{aligned}$$

$$\cdot E\left\|\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha$$
$$\leq 8\mathcal{K}^2L\int_{-\infty}^{\tau}e^{-2q(\tau-\alpha)}E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2\mathrm{d}\alpha+\frac{8\mathcal{K}^2cL}{q}\int_{-\infty}^{\tau}e^{-q(\tau-\alpha)}E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2\mathrm{d}\alpha+a_4^n, \quad (3.21)$$

where

$$a_4^n=4\mathcal{K}^2\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1}e^{-2q(\tau-\alpha)}\cdot E\left\|\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha$$
$$+\frac{4\mathcal{K}^2c}{q}\int_{-\infty}^{\tau}\int_{|p|_\Theta\geq 1}e^{-q(\tau-\alpha)}\cdot E\left\|\Phi_2\left(\alpha+r_n,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)-\widetilde{\Phi_2}\left(\alpha,\widetilde{\xi}(\alpha-),\mathbb{P}_{\widetilde{\xi}(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha.$$

Similar to a_3^n, we can deduce that $a_4^n\to 0$ as $n\to\infty$.

On the basis of (3.17)–(3.21), we have

$$E\left\|\xi_n(\tau)-\widetilde{\xi}(\tau)\right\|^2\leq(1+2c)\frac{16\mathcal{K}^2L}{q}\int_{-\infty}^{\tau}e^{-q(\tau-\alpha)}E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2\mathrm{d}\alpha$$
$$+64\mathcal{K}^2L\int_{-\infty}^{\tau}e^{-2q(\tau-\alpha)}E\left\|\xi_n(\alpha)-\widetilde{\xi}(\alpha)\right\|^2\mathrm{d}\alpha+4\sum_{i=1}^{4}a_i^n. \quad (3.22)$$

From Lemma 3.1 we know

$$E\left\|\xi_n(\tau)-\widetilde{\xi}(\tau)\right\|^2\leq 4\sum_{i=1}^{4}a_i^n+w\int_{-\infty}^{\tau}e^{-\varepsilon(\tau-\alpha)}4\sum_{i=1}^{4}a_i^n\mathrm{d}\alpha \quad (3.23)$$

with $w=(1+2c)\frac{16\mathcal{K}^2L}{q}+64\mathcal{K}^2L$ and $\varepsilon\in(0,q-w)$. Note that in the light of the assumption in (3.10), we have $q-w>0$. Hence it follows from (3.23) and $\sum_{i=1}^4 a_i^n\to 0$ as $n\to\infty$ that

$$\lim_{n\to\infty}E\left\|\xi_n(\tau)-\widetilde{\xi}(\tau)\right\|^2=0,\quad \tau\in\mathbb{R}.$$

It implies that $\xi_n(\tau)\to\widetilde{\xi}(\tau)$ in distribution as $n\to\infty$. Since $\xi(\tau+r_n)$ and $\xi_n(\tau)$ have the identical distribution, we have $\xi(\tau+r_n)\to\widetilde{\xi}(\tau)$ in distribution as $n\to\infty$. Simultaneously using the similar argument, we can also prove $\widetilde{\xi}(\tau-r_n)\to\xi(\tau)$ ($\tau\in\mathbb{R}$) in distribution as $n\to\infty$. We finish the proof. □

4. Stability of the almost automorphic solution

The stability of solutions is a crucial topic in stochastic systems. It has wide applications in various fields, such as finance, biology, and engineering. For example, in financial models, understanding the stability of asset prices under random fluctuations is essential for risk management and investment strategies. Now let us review the definition of stability before we begin our proof.

Definition 4.1. *(square-mean stable)*

(1) A solution $\xi(\tau)$ of SDE (3.1) is called square-mean stable provided for every $\epsilon > 0$, there exists $\delta > 0$ such that when $\|\zeta(0) - \xi(0)\| < \delta$,

$$E\,\|\zeta(\tau) - \xi(\tau)\|^2 < \epsilon, \quad (\tau \geq 0).$$

Here $\zeta(\tau)$ is a solution of SDE (3.1).

(2) A solution $\xi(\tau)$ is called square-mean asymptotically stable provided it is square-mean stable and

$$E\,\|\zeta(\tau) - \xi(\tau)\|^2 \to 0 \quad \text{as } \tau \to \infty. \tag{4.1}$$

(3) A solution $\xi(\tau)$ is called square-mean globally asymptotically stable provided it is square-mean asymptotically stable and the inequality (4.1) holds for arbitrary $\zeta(0) \in \mathcal{L}^2(\Omega, \mathbb{X})$.

First let us investigate the existence interval of the solution of SDE (3.1). We prove that any solution of SDE (3.1) will not blow up for all times $\tau \geq 0$ under the weaker conditions than Theorem 3.1.

Lemma 4.1. *Suppose that the conditions (1)–(4) of Theorem 3.1 hold. Then the solution ξ of SDE (3.1) with $\xi(0) \in \mathcal{L}^2(\Omega, \mathbb{X})$ exists in $\mathcal{L}^2(\Omega, \mathbb{X})$ for $\tau \in [0, +\infty)$.*

Proof. Note that by Remark 2.5 and Proposition 2.1, there exists a positive constant γ such that

$$\begin{aligned}
&\max\Bigg\{\sup_{\tau\in\mathbb{R}} E\|\varphi(\tau, 0, \delta_0)\|^2, \quad \sup_{\tau\in\mathbb{R}} \int_{|p|_\Theta<1} E\|\Phi_1(\tau, 0, \delta_0, p)\|^2 \nu(\mathrm{d}p),\\
&\sup_{\tau\in\mathbb{R}} E\left\|\psi(\tau, 0, \delta_0)Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})}, \quad \sup_{\tau\in\mathbb{R}} \int_{|p|_\Theta\geq 1} E\|\Phi_2(\tau, 0, \delta_0, p)\|^2 \nu(\mathrm{d}p)\Bigg\} \leq \gamma.
\end{aligned}$$

Assume that $\xi(\tau)$ is the solution of SDE (3.1) with $\xi(0) \in \mathcal{L}^2(\Omega, \mathbb{X})$. Then for $\tau \in \mathbb{R}$ we have

$$\begin{aligned}
E\|\xi(\tau)\|^2 =&E\Bigg\|G(\tau)\xi(0) + \int_0^\tau G(\tau-\alpha)\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}\alpha\\
&+ \int_0^\tau G(\tau-\alpha)\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}W(\alpha)\\
&+ \int_0^\tau \int_{|p|_\Theta<1} G(\tau-\alpha)\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)\widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\\
&+ \int_0^\tau \int_{|p|_\Theta\geq 1} G(\tau-\alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)N(\mathrm{d}\alpha, \mathrm{d}p)\Bigg\|^2\\
\leq&5\mathcal{K}^2 e^{-2q\tau} E\|\xi(0)\|^2 + 5E\left\|\int_0^\tau G(\tau-\alpha)\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}\alpha\right\|^2\\
&+ 5E\left\|\int_0^\tau G(\tau-\alpha)\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}W(\alpha)\right\|^2\\
&+ 5E\left\|\int_0^\tau \int_{|p|_\Theta<1} G(\tau-\alpha)\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right)\widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2
\end{aligned}$$

$$+ 5E\left\| \int_0^\tau \int\limits_{|p|_\Theta \geq 1} G(\tau-\alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) N(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2$$

$$:=5\mathcal{K}^2 e^{-2q\tau} E\|\xi(0)\|^2 + 5\sum_{k=1}^{4} i_k. \tag{4.2}$$

It follows from Cauchy-Schwarz inequality, (2.1), (3.2), and (3.5)–(3.8) that

$$\begin{aligned}
i_1 \leq& \frac{\mathcal{K}^2}{q}(1-e^{-q\tau})\int_0^\tau e^{-q(\tau-\alpha)} E\left\|\varphi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \varphi(\alpha, 0, \delta_0) + \varphi(\alpha, 0, \delta_0)\right\|^2 \mathrm{d}\alpha \\
\leq& \frac{\mathcal{K}^2}{q}\int_0^\tau e^{-q(\tau-\alpha)}\left[4LE\|\xi(\alpha)\|^2 + 2E\|\varphi(\alpha, 0, \delta_0)\|^2\right]\mathrm{d}\alpha \\
\leq& \frac{4\mathcal{K}^2 L}{q}\int_0^\tau e^{-q(\tau-\alpha)} E\|\xi(\alpha)\|^2 \mathrm{d}\alpha + \frac{2\mathcal{K}^2\gamma}{q^2};
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
i_2 \leq& \mathcal{K}^2\int_0^\tau e^{-2q(\tau-\alpha)} E\left\|\left[\psi\left(\alpha, \xi(\alpha), \mathbb{P}_{\xi(\alpha)}\right) - \psi(\alpha, 0, \delta_0) + \psi(\alpha, 0, \delta_0)\right] Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta, \mathbb{X})} \mathrm{d}\alpha \\
\leq& \mathcal{K}^2\int_0^\tau e^{-2q(\tau-\alpha)}\left[4LE\|\xi(\alpha)\|^2 + 2E\left\|\psi\left(\alpha, 0, \delta_0\right) Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta, \mathbb{X})}\right]\mathrm{d}\alpha \\
\leq& 4\mathcal{K}^2 L\int_0^\tau e^{-2q(\tau-\alpha)} E\|\xi(\alpha)\|^2 \mathrm{d}\alpha + \frac{\mathcal{K}^2\gamma}{q};
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
i_3 \leq& \mathcal{K}^2\int_0^\tau \int\limits_{|p|_\Theta < 1} e^{-2q(\tau-\alpha)} \cdot E\left\|\Phi_1\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) - \Phi_1(\alpha, 0, \delta_0, p) + \Phi_1(\alpha, 0, \delta_0, p)\right\|^2 \nu(\mathrm{d}p)\mathrm{d}\alpha \\
\leq& \mathcal{K}^2\int_0^\tau e^{-2q(\tau-\alpha)}\Big[4LE\|\xi(\alpha)\|^2 + \int\limits_{|p|_\Theta < 1} 2E\|\Phi_1(\alpha, 0, \delta_0, p)\|^2 \nu(\mathrm{d}p)\Big]\mathrm{d}\alpha \\
\leq& 4\mathcal{K}^2 L\int_0^\tau e^{-2q(\tau-\alpha)} E\|\xi(\alpha)\|^2 \mathrm{d}\alpha + \frac{\mathcal{K}^2\gamma}{q};
\end{aligned} \tag{4.5}$$

and the last one

$$\begin{aligned}
i_4 \leq& 2E\left\| \int_0^\tau \int\limits_{|p|_\Theta \geq 1} G(\tau-\alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) \widetilde{N}(\mathrm{d}\alpha, \mathrm{d}p)\right\|^2 \\
&+ 2E\left\| \int_0^\tau \int\limits_{|p|_\Theta \geq 1} G(\tau-\alpha)\Phi_2\left(\alpha, \xi(\alpha-), \mathbb{P}_{\xi(\alpha-)}, p\right) \nu(\mathrm{d}p)\mathrm{d}\alpha\right\|^2 \\
\leq& 2\mathcal{K}^2\int_0^\tau e^{-2q(\tau-\alpha)}\Big[4LE\|\xi(\alpha)\|^2 + \int\limits_{|p|_\Theta \geq 1} 2E\|\Phi_2(\alpha, 0, \delta_0, p)\|^2 \nu(\mathrm{d}p)\Big]\mathrm{d}\alpha \\
&+ 2\mathcal{K}^2\int_0^\tau \int\limits_{|p|_\Theta \geq 1} e^{-q(\tau-\alpha)}\nu(\mathrm{d}p)\mathrm{d}\alpha \cdot \int_0^\tau e^{-q(\tau-\alpha)}\Big[4LE\|\xi(\alpha)\|^2 + \int\limits_{|p|_\Theta \geq 1} 2E\|\Phi_2(\alpha, 0, \delta_0, p)\|^2 \nu(\mathrm{d}p)\Big]\mathrm{d}\alpha \\
\leq& 8\mathcal{K}^2 L\int_0^\tau e^{-2q(\tau-\alpha)} E\|\xi(\alpha)\|^2 \mathrm{d}\alpha + \frac{8\mathcal{K}^2 cL}{q}\int_0^\tau e^{-q(\tau-\alpha)} E\|\xi(\alpha)\|^2 \mathrm{d}\alpha + \frac{2\mathcal{K}^2\gamma}{q} + \frac{4\mathcal{K}^2 c\gamma}{q^2}.
\end{aligned} \tag{4.6}$$

On the basis of (4.2)–(4.6), we have

$$\begin{aligned}E\|\xi(\tau)\|^2 \leq &(\frac{20\mathcal{K}^2L}{q}(1+2c)+80\mathcal{K}^2L)\int_0^\tau E\|\xi(\alpha)\|^2\mathrm{d}\alpha+5\mathcal{K}^2E\|\xi(0)\|^2+5(\frac{4\mathcal{K}^2\gamma}{q}+\frac{2\mathcal{K}^2\gamma}{q^2}(1+2c))\\ :=&B_1\int_0^\tau E\|\xi(\alpha)\|^2\mathrm{d}\alpha+B_2.\end{aligned}$$

Then by Gronwall's inequality we can obtain

$$E\|\xi(\tau)\|^2 \leq B_2e^{B_1\tau} \quad (\tau\in\mathbb{R}).$$

Hence the solution of SDE (3.1) has an existence duration that can be increased indefinitely, stretching towards positive infinity. We finish the proof. □

Now that we have shown any solution of SDE (3.1) can be extended to $+\infty$, let us study the stability of the solutions.

Theorem 4.1. *Suppose the conditions (1)–(4) of Theorem 3.1 hold. Then,*

i) If

$$L<\frac{q^2}{10\mathcal{K}^2(1+2c+4q)}, \tag{4.7}$$

this unique $\mathcal{L}^2$-bounded solution of SDE (3.1) is square-mean globally asymptotically stable;

ii) If the Lipschitz constant satisfies inequality (3.10), the solution is both square-mean globally asymptotically stable and almost automorphic in distribution.

Proof. Suppose that $\xi(\tau)$ and $\zeta(\tau)$ are two solutions of SDE (3.1) starting from $\xi(0)$ and $\zeta(0)$ at time 0 individually. Then we have

$$\begin{aligned}\xi(\tau)=&G(\tau)\xi(0)+\int_0^\tau G(\tau-\alpha)\varphi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}\alpha+\int_0^\tau G(\tau-\alpha)\psi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)\mathrm{d}W(\alpha)\\ &+\int_0^\tau\int_{|p|_\Theta<1}G(\tau-\alpha)\Phi_1\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\\ &+\int_0^\tau\int_{|p|_\Theta\geq1}G(\tau-\alpha)\Phi_2\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)N(\mathrm{d}\alpha,\mathrm{d}p),\end{aligned}$$

and

$$\begin{aligned}\zeta(\tau)=&G(\tau)\zeta(0)+\int_0^\tau G(\tau-\alpha)\varphi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\mathrm{d}\alpha+\int_0^\tau G(\tau-\alpha)\psi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\mathrm{d}W(\alpha)\\ &+\int_0^\tau\int_{|p|_\Theta<1}G(\tau-\alpha)\Phi_1\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\\ &+\int_0^\tau\int_{|p|_\Theta\geq1}G(\tau-\alpha)\Phi_2\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)N(\mathrm{d}\alpha,\mathrm{d}p).\end{aligned}$$

Afterwards we have

$$
\begin{aligned}
&E\|\xi(\tau)-\zeta(\tau)\|^2 \leq 5E\|G(\tau)[\xi(0)-\zeta(0)]\|^2\\
&+5E\left\|\int_0^\tau G(\tau-\alpha)\left[\varphi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)-\varphi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\right]\mathrm{d}\alpha\right\|^2\\
&+5E\left\|\int_0^\tau G(\tau-\alpha)\left[\psi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)-\psi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\right]\mathrm{d}W(\alpha)\right\|^2\\
&+5E\left\|\int_0^\tau\int_{|p|_\Theta<1} G(\tau-\alpha)\left[\Phi_1\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_1\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right]\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2\\
&+5E\left\|\int_0^\tau\int_{|p|_\Theta\geq1} G(\tau-\alpha)\left[\Phi_2\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_2\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right]N(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2\\
&\leq 5\mathcal{K}^2e^{-2q\tau}E\|\xi(0)-\zeta(0)\|^2+5(K_1+K_2+K_3+K_4). \qquad (4.8)
\end{aligned}
$$

For these four terms, by (2.1), (3.2), and the Lipschitz conditions (3.5)–(3.8), we have for each $\tau \geq 0$

$$
\begin{aligned}
K_1 &\leq \int_0^\tau \mathcal{K}^2e^{-q(\tau-\alpha)}d\alpha\int_0^\tau e^{-q(\tau-\alpha)}E\left\|\varphi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)-\varphi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\right\|^2\mathrm{d}\alpha\\
&\leq \frac{2\mathcal{K}^2L}{q}(1-e^{-q\tau})\int_0^\tau e^{-q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha\\
&\leq \frac{2\mathcal{K}^2L}{q}\int_0^\tau e^{-q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha, \qquad (4.9)\\
K_2 &\leq \int_0^\tau \mathcal{K}^2e^{-2q(\tau-\alpha)}E\left\|\left[\psi\left(\alpha,\xi(\alpha),\mathbb{P}_{\xi(\alpha)}\right)-\psi\left(\alpha,\zeta(\alpha),\mathbb{P}_{\zeta(\alpha)}\right)\right]Q^{\frac{1}{2}}\right\|^2_{L_2(\Theta,\mathbb{X})}\mathrm{d}\alpha\\
&\leq 2\mathcal{K}^2L\int_0^\tau e^{-2q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha, \qquad (4.10)\\
K_3 &\leq \int_0^\tau\int_{|p|_\Theta<1}\mathcal{K}^2e^{-2q(\tau-\alpha)}E\left\|\Phi_1\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_1\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right\|^2\nu(\mathrm{d}p)\mathrm{d}\alpha\\
&\leq 2\mathcal{K}^2L\int_0^\tau e^{-2q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha, \qquad (4.11)
\end{aligned}
$$

and the last one

$$
\begin{aligned}
K_4 \leq\,& 2E\left\|\int_0^\tau\int_{|p|_\Theta\geq1} G(\tau-\alpha)\left[\Phi_2(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p)-\Phi_2(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p)\right]\widetilde{N}(\mathrm{d}\alpha,\mathrm{d}p)\right\|^2\\
&+2E\left\|\int_0^\tau\int_{|p|_\Theta\geq1} G(\tau-\alpha)\left[\Phi_2(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p)-\Phi_2(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p)\right]\nu(\mathrm{d}p)\mathrm{d}\alpha\right\|^2\\
&\leq 4\mathcal{K}^2L\int_0^\tau e^{-2q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha+2\int_0^\tau\int_{|p|_\Theta\geq1}\mathcal{K}^2e^{-q(\tau-\alpha)}\nu(\mathrm{d}p)\mathrm{d}\alpha
\end{aligned}
$$

$$\cdot\int_0^\tau\int_{|p|_\Theta\geq 1} e^{-q(\tau-\alpha)}E\left\|\Phi_2\left(\alpha,\xi(\alpha-),\mathbb{P}_{\xi(\alpha-)},p\right)-\Phi_2\left(\alpha,\zeta(\alpha-),\mathbb{P}_{\zeta(\alpha-)},p\right)\right\|^2 v(\mathrm{d}p)\mathrm{d}\alpha$$

$$\leq 4\mathcal{K}^2L\int_0^\tau e^{-2q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha+\frac{4\mathcal{K}^2cL}{q}\int_0^\tau e^{-q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha. \tag{4.12}$$

From (4.8)–(4.12) and $e^{-2q\tau}\leq e^{-q\tau}$ for $\tau\geq 0$, we can obtain

$$E\|\xi(\tau)-\zeta(\tau)\|^2\leq 5\mathcal{K}^2e^{-q\tau}E\|\xi(0)-\zeta(0)\|^2+((1+2c)\frac{10\mathcal{K}^2L}{q}+40\mathcal{K}^2L)\int_0^\tau e^{-q(\tau-\alpha)}E\|\xi(\alpha)-\zeta(\alpha)\|^2\mathrm{d}\alpha.$$

Let $O(\tau):=E\|\xi(\tau)-\zeta(\tau)\|^2$ and $\lambda:=(1+2c)\frac{10\mathcal{K}^2L}{q}+40\mathcal{K}^2L$, then

$$O(\tau)\leq 5\mathcal{K}^2e^{-q\tau}O(0)+\lambda\int_0^\tau e^{-q(\tau-\alpha)}O(\alpha)\mathrm{d}\alpha. \tag{4.13}$$

Let

$$\widetilde{O}(\tau)=5\mathcal{K}^2e^{-q\tau}O(0)+\lambda\int_0^\tau e^{-q(\tau-\alpha)}\widetilde{O}(\alpha)\mathrm{d}\alpha,$$

and $\widetilde{O}(0)=5\mathcal{K}^2O(0)$, we have

$$\widetilde{O}(\tau)=e^{-q\tau}\widetilde{O}(0)+\lambda\int_0^\tau e^{-q(\tau-\alpha)}\widetilde{O}(\alpha)\mathrm{d}\alpha. \tag{4.14}$$

Hence $O(\tau)\leq\widetilde{O}(\tau)$ ($\tau\in\mathbb{R}$). By taking the derivative of both sides of (4.14) with respect to τ, we have

$$\frac{\mathrm{d}\widetilde{O}(\tau)}{\mathrm{d}\tau}=(\lambda-q)\widetilde{O}(\tau). \tag{4.15}$$

Solving the Eq (4.15) with $\widetilde{O}(0)=5\mathcal{K}^2O(0)$, we can obtain

$$\widetilde{O}(\tau)=5\mathcal{K}^2O(0)e^{(\lambda-q)\tau}.$$

By the assumption of the Lipschitz constant in (4.7) and the definition of λ, we have $\lambda-q<0$. That is, if (4.7) holds, $O(\tau)\to 0$ exponentially fast as $\tau\to\infty$. If the Lipschitz constant (3.10) in Theorem 3.1 holds, then Theorem 4.1ii) holds. We finish the proof. □

5. Application

In this part, we provide an example to demonstrate the findings presented in our work.

Example 5.1. Consider a stochastic heat equation within the range from 0 to 1, subject to the Dirichlet boundary condition:

$$\begin{aligned}\frac{\partial u}{\partial\tau}(\tau,x)=&\frac{\partial^2u}{\partial x^2}(\tau,x)+\frac{(\cos 2\tau+\sin\sqrt{5}\tau)Eu}{8(1+u^2(\tau,x))}+\frac{\sin\sqrt{5}\tau\cdot Eu}{3(2+\cos 2\tau)}\frac{\partial W}{\partial\tau}(\tau,x)\\&+\frac{\cos\sqrt{3}\tau\sin Eu}{5(1+V^2(\tau,x))}\frac{\partial V}{\partial\tau}(\tau,x),\qquad\tau>0,\ x\in(0,1)\end{aligned} \tag{5.1}$$

$$=: \frac{\partial^2 u}{\partial x^2} + f(\tau, u, \mathbb{P}_u) + g(\tau, u, \mathbb{P}_u)\frac{\partial W}{\partial \tau} + h(\tau, u, \mathbb{P}_u, V)\frac{\partial V}{\partial \tau},$$
$$u(\tau, 0) = u(\tau, 1) = 0, \qquad \tau > 0.$$

Here W with $TrQ < \infty$ is a Q-Brownian motion on $L^2(0, 1)$, and V independent of W is a Lévy pure jump process on $L^2(0, 1)$. Let A be a Laplace operator, then $A : D(A) = H_0^1(0, 1) \cap H^2(0, 1) \to L^2(0, 1)$. Let $\Theta = \mathbb{X} := L^2(0, 1)$. Then the stochastic heat equation can be transformed into an abstract evolution equation

$$d\xi = (A\xi + \varphi\left(\tau, \xi, \mathbb{P}_\xi\right))d\tau + \psi\left(\tau, \xi, \mathbb{P}_\xi\right)dW + \int_{|p|_\Theta < 1} \Phi\left(\tau, \xi, \mathbb{P}_\xi, p\right)\widetilde{N}(d\tau, dp) + \int_{|p|_\Theta \geq 1} \Phi\left(\tau, \xi, \mathbb{P}_\xi, p\right)N(d\tau, dp) \quad (5.2)$$

on the Hilbert space $\mathbb{X}$, where

$$\xi := u, \quad \varphi(\tau, \xi, \mathbb{P}_\xi) := f(\tau, u, \mathbb{P}_u), \quad \psi(\tau, \xi, \mathbb{P}_\xi) := g(\tau, u, \mathbb{P}_u),$$

$$\int_{|p|_\Theta < 1} \Phi\left(\tau, \xi, \mathbb{P}_\xi, p\right)\widetilde{N}(d\tau, dp) + \int_{|p|_\Theta \geq 1} \Phi\left(\tau, \xi, \mathbb{P}_\xi, p\right)N(d\tau, dp) := h(\tau, u, \mathbb{P}_u, V)dV$$

with

$$V(\tau, x) = \int_{|p|_\Theta < 1} p\widetilde{N}(\tau, dp) + \int_{|p|_\Theta \geq 1} pN(\tau, dp), \quad \Phi\left(\tau, \xi, \mathbb{P}_\xi, p\right) = h(\tau, u, \mathbb{P}_u, V)p.$$

Here for simplicity we assume that by Lévy-Itô decomposition, Lévy pure jump process on $L^2(0, 1)$ is decomposed as above.

The eigenvalues of operator A are $\{-k^2\pi^2\}$ with $k = 1, 2, \cdots$, and A produces a C_0-semi-group $G(\tau)$ on $\mathbb{X}$ such that $\|G(\tau)\| \leq e^{-\pi^2\tau}$ holds for $\tau \geq 0$, i.e., $q = \pi^2$ and $\mathcal{K} = 1$. We respectively chose $\frac{1}{4}, \frac{1}{3}, \frac{1}{5}$ as the Lipschitz constants of f, g, h, then the conditions (3.5)–(3.8) in Theorem 3.1 are given by

$$L = \max\left\{\frac{1}{32}, \frac{\|Q\|_{L(\Theta,\Theta)}}{18}, \frac{\nu(B_1(0))}{50}, \frac{c}{50}\right\}$$

with $B_1(0)$ denotes a ball in Θ with a radius of 1 that is centered at the origin. If $L < \frac{\pi^4}{8(1+2c+2\pi^2)}$ (i.e., condition (3.9) holds), then by Theorem 3.1i), Eq (5.2) (and hence Eq (5.1)) has a unique bounded solution. If $L < \frac{\pi^4}{10(1+2c+4\pi^2)}$ (i.e., condition (4.7) holds), then by Theorem 4.1, the unique bounded solution is square-mean asymptotically stable. Note that φ and ψ are s.m.a.a. in $\tau \in \mathbb{R}$, Φ is Poisson s.m.a.a. in $\tau \in \mathbb{R}$. If $L < \frac{\pi^4}{16(1+2c+4\pi^2)}$ (i.e., condition (3.10) holds), then by Theorem 3.1ii), the solution is both almost automorphic in distribution and globally asymptotically stable in square-mean.

Author contributions

Xin Liu: conceptualization, funding acquisition, methodology, supervision, validation, writing-review and editing; Yongqi Hou: formal analysis, investigation, methodology, validation, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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