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Research article

Fixed point theorem on $CAT_p(0)$ metric spaces with applications in solving matrix equations and fractional differential equations

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Abstract: This paper aimed to explore fixed point theorems for CMJ generalized mappings in $CAT_p(0)$ metric spaces. To strengthen the established results, we presented a positive example. In applications, we found the existence of the solution to nonlinear matrix equations, and unique solutions of two scale fractal hybrid fractional differential equations in $CAT_p(0)$.

Keywords: fixed point theorems; CMJ mappings; $CAT_p(0)$ metric spaces; nonlinear matrix equation; fractional differential equations
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1. Introduction

Meir and Keeler [1] introduced the concept of a weak contractive condition for mappings, which guarantees the existence of a fixed point in a complete metric space. In 1981, Ćirić [2] used the continuity feature to expand on the findings established by Meir and Keeler [1] in a complete metric space. Later, Matkowski [3] extended the fixed point theorems of Banach and Kannan as well as some results of Boyd and Wong [4], Meir and Keeler [1], Reich [5], and Wong [6] in a complete metric space. Subsequently, Jachymski [7] demonstrated the equivalent requirements to those of Meir and Keeler [1]. The work of Ćirić [2], Matkowski [3], and Jachymski [7] is collectively abbreviated as CMJ.

Gromov [8] introduced the concept of $CAT(0)$ spaces, which was later expanded upon with fixed point results in R-trees and $CAT(0)$ spaces by Kirk [9]. Goebel and Reich [10] contributed significant findings on non-expansive mappings, hyperbolic geometry, and uniform convexity. Reich and Shafrir [11] demonstrated results on nonexpansive iterations in hyperbolic spaces. The $\triangle$-convergence theorems in $CAT(0)$ spaces were given by Dhompongsa and Panyanak [12]. Khamsi and Shukri provided generalized $CAT(0)$ spaces in [13]. Results on monotone non-expansive mappings in $CAT_p(0)$ spaces were provided by Shukri [14]. In $CAT_p(0)$ metric, the fixed points of Suzuki-generalized nonexpansive mappings were established by Darweesh and Shukri [15].

Sun and Agarwal [16] investigated functionals associated with nonlinear differential operators in order to determine whether boundary value issues have solutions. They demonstrated that identifying a fixed point of the corresponding nonlinear operator is comparable to solving such issues. They provided significant theoretical contributions about fixed point existence in the context of partially ordered metric spaces, where their study was conducted. Xie et al. [17] used the Banach contraction principle to show that there are solutions to multi-order nonlinear fractional differential equations over the unbounded interval $[0, \infty)$. Additionally, they developed Ulam-Hyers and Ulam-Hyers-Rassias stability, among other forms of stability, for the comparable initial value problems.

Further, Zhou et al. [18] focused on nonlinear ψ-Hilfer fractional integrodifferential coupled systems defined over a bounded domain. By employing the contraction mapping principle, they proved both the existence and uniqueness of solutions. Additionally, they examined various notions of stability, such as Ulam-Hyers, Ulam-Hyers-Rassias, and semi-Ulam-Hyers-Rassias, within the framework of generalized complete metric spaces. Imran [19] applied the fractal-fractional derivative with a power-law kernel, denoted as ${}_0^{FFP}D_x^{\alpha,\beta}$, to analyze magnetohydrodynamic (MHD) viscous fluid flow between two parallel plates. The study also delved into the chaotic dynamics exhibited by the system.

In modeling complex systems with irregular or hierarchical structures, fractal derivatives offer a robust mathematical tool. Traditionally, hybrid fractional differential equations (HFDEs) utilize standard fractal derivatives characterized by a single scaling exponent, assuming uniform self-similarity. However, many real-world systems display multiscale or heterogeneous behavior, which a single exponent cannot fully capture. To address this, the two-scale fractal derivative extends the standard model by incorporating two distinct fractal dimensions. This approach is especially useful for phenomena like anomalous diffusion, porous media transport, or biological tissue dynamics, where local and global scaling behaviors differ. The two-scale model effectively represents systems with varying fractal dynamics across scales, enabling a more accurate description of localized irregularities in transport or energy processes.

Recent studies, such as the work by He et al. [20], have highlighted the connection between fractional calculus and fractal geometry and establish the fractal Fick law, the fractal Darcy law, and the fractal Richards equation. According to these theories, the fractional order in differential equations can be interpreted in terms of two-scale fractal dimensions, capturing both the local and global scaling behaviors of heterogeneous materials or systems. This perspective is particularly relevant when modeling media with complex microstructures or hierarchical properties, where classical integer-order models may fall short. By integrating this view, the fractional order becomes more than a fitting parameter; it acquires a geometric and physical interpretation, thereby enriching the modeling framework and enhancing the descriptive power of fractional differential equations where a coupled transport equation can be modeled. Also, it has a wider supplication in MHD (magnetohydrodynamic) fluid flows, where memory and spatial non-locality due to electromagnetic interactions are effectively described through fractional operators. Furthermore, it is applied in

MRI (magnetic resonance imaging).

On the other hand, Bini and Meini [21] focused on solving a class of nonlinear matrix equations that arise in queueing theory, offering analytical insights into the structure and solvability of such equations. Lim [22] presented a solution methodology for the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} M_i X^{\delta_i} M_i^*$ using a contraction principle, further extending the applicability of fixed point theory to nonlinear algebraic systems.

In this study, we employ the two-scale derivative to enhance the descriptive power of HFDEs in modeling multiscale transport phenomena. Its integration provides a more nuanced approach to fractional modeling, especially in media exhibiting dual fractality, such as fractured rock formations, hybrid organic materials, or anomalous diffusion in biological tissues.

This work's novelty lies in establishing fixed point results for CMJ-type mappings in $CAT_p(0)$ metric spaces, with an application to matrix equations and the two-scale fractal fractional hybrid differential equation. It is inspired by the contributions of Darweesh and Shukri [15], Kirk [23], Nanjaras et al. [24], Ćirić [2], Matkowski [3], Jachymski [7], and others.

The structure of the manuscript is as follows. The Introduction provides the motivation and significance of these spaces and mappings in the mathematical analysis in Section 1. Materials and methods covers essential definitions, notations, and auxiliary results necessary for understanding the main findings in Section 2. Section 3 contains the main results section that introduces and proves key fixed point theorems for CMJ mappings, highlighting their generality and improvements over existing results. In Section 4, as an application we investigate the existence of the solution to nonlinear matrix equations in $CAT_p(0)$. In Section 5, we study unique solutions for two-scale fractal hybrid fractional differential equations by utilizing the theorems of Section 3. The paper concludes by summarizing findings and suggesting directions for future research, supported by a comprehensive list of references.

2. Materials and methods

Definitions, lemmas, and some preliminary findings are provided in this section to aid in the development of the primary findings.

According to Bridson and Haefliger [25] and Gromov [8], $CAT(0)$ and $CAT(k)$ spaces have the following properties:

Definition 2.1. Let $(\mathbb{E}, \|.\|)$ be a normed vector space and $(\mathfrak{X}, d)$ be a geodesic metric space. When a comparison triangle $\bar{\triangle}$ exists in $\mathbb{E}$ such that the comparison axioms are satisfied for all $c, s \in \triangle$ and the comparison points $\bar{c}, \bar{s} \in \bar{\triangle}$, then $\mathfrak{X}$ is considered a generalized $CAT(0)$ space.

$$d(c, s) \leq \|\bar{c} - \bar{s}\|.$$

If every triangle in $\mathfrak{X}$ is at least as "thin" as its comparison triangle in the Euclidean plane and the metric space is geodesically connected, it is referred to as a $CAT(0)$ space.

Definition 2.2. Let k be a real number and $(\mathfrak{X}, d)$ be a metric space. Let $\bar{\triangle} \in M_k^2$ be a comparison triangle of $\triangle$, and let $\triangle$ be a geodesic triangle in $\mathfrak{X}$. When all $c, s \in \triangle$ and the comparison points $\bar{c}, \bar{s} \in \bar{\triangle}$ are satisfied, then $\triangle$ is said to meet the $CAT(k)$ inequality.

$$d(c, s) \leq d(\bar{c}, \bar{s}).$$

M_k^n-metric space is a complete, simply linked, Riemannian n-manifold with constant section curvature $k \in \mathbb{R}$. For every integer n, the space M_k^n can be divided into three qualitative classes based on whether k is zero, positive or negative. In order to simplify the notation, we have $\mathbb{E}^n = M_0^n$, $\mathbb{S}^n = M_1^n$, and $\mathbb{H}^n = M_{-1}^n$. The law of cosines and the triangle inequality are shown to be closely related in each instance. The Euclidean space n-space $\mathbb{E}^n$ is a vector space in $\mathbb{R}^n$ that has a scalar product metric.

$$(c/s) = \sum_{i=1}^{n} c_i s_i,$$

where $c = (c_i \dots c_n)$ and $s = (s_i \dots s_n)$. $\mathbb{E}^n$ is a uniquely geodesic space and the geodesic segments in $\mathbb{E}^n$ are the subset of the form

$$[c, s] = \{\lambda_1 s + \lambda_2 c \mid 0 \le \lambda_1 \le 1\},$$

where $\lambda_2 = 1 - \lambda_1$. The n-sphere $\mathbb{S}^n$ belongs to the set $\{c = (c_i \dots c_{n+1}) \in \mathbb{R}^{n+1} | (c|c) = 1\}$, where $(.|.)$ represents the Euclidean scalar product. Assume that $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ is a function that allocates the unique real number $d(C_1, C_2) \in [0, \pi]$ to each pair $(C_1, C_2) \in \mathbb{S}^n \times \mathbb{S}^n$. For example, $\cos d(C_1, C_2) = (C_1|C_2)$. The metric is thus d.

Let $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be a function that allocates the unique nonnegative number $d(C_1, C_2) \ge 0$ to each pair $(C_1, C_2) \in \mathbb{H}^n \times \mathbb{H}^n$ in such a way that $\cosh d(C_1, C_2) = -(C_1|C_2)$. The metric is thus d.

Definition 2.3. [8, 25] For the given real number k, the model spaces M_k^n are defined as:

(i) M_0^n is the Euclidean space $\mathbb{E}^n$ if $k = 0$;

(ii) If $k > 0$ then M_k^n is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$.
Standard n-sphere $\mathbb{S}^n$
The standard n-sphere of radius 1, $\mathbb{S}^n$, is the set of points in $\mathbb{R}^{n+1}$ at the unit distance from the origin.
It comes equipped with the standard round metric, with sectional curvature equal to 1.
When you multiply distances by $\frac{1}{\sqrt{k}}$, you are scaling the Riemannian metric g by: $g_k = \frac{1}{k}g$. This operation does the following: It multiplies all lengths by $\frac{1}{\sqrt{k}}$. It multiplies all areas by $\frac{1}{k}$. It multiplies all sectional curvatures by k. So, if you start with $\mathbb{S}^n$, which has constant sectional curvature 1, and rescale the metric as described, you get a new Riemannian manifold: $M_k^n = \left(\mathbb{S}^n, \frac{1}{k}g\right)$, which is still a sphere (same topology), but now has constant sectional curvature k.

(iii) If $k < 0$, then M_k^n can be obtained from hyperbolic $\mathbb{H}^n$ by multiplying the distance function by $\frac{1}{\sqrt{-k}}$.
The hyperbolic space $\mathbb{H}^n$, which has constant negative sectional curvature.
Standard hyperbolic space $\mathbb{H}^n$
$\mathbb{H}^n$ is the simply connected, complete Riemannian manifold of constant sectional curvature -1.
It can be modeled in several ways (Poincaré disk, upper half-space, hyperboloid model), but they all describe the same geometric structure.
If we multiply all distances in $\mathbb{H}^n$ by $\frac{1}{\sqrt{-k}}$ (where $k < 0$), we are scaling the metric g by:

$$g_k = \frac{1}{-k}g,$$

which has these effects:
Lengths scale by $\frac{1}{\sqrt{-k}}$.
Sectional curvatures scale by $-k$.
So if the original curvature was -1, the new curvature is k.
So, for $k < 0$, we define:

$$M_k^n = \left(\mathbb{H}^n, \frac{1}{-k}g\right),$$

which gives a manifold of constant sectional curvature $k < 0$, the same underlying topology as $\mathbb{H}^n$, and a distance function scaled by $\frac{1}{\sqrt{-k}}$.

Proposition 2.1. *M_k^n is a geodesic metric space.*

(i) If $k \leq 0$, then M_k^n is uniquely geodesic and all balls in M_k^n are convex.
(ii) If $k > 0$, then there is a uniquely geodesic segment joining $c, s \in M_k^n$ if and only if $d(c, s) < \frac{\pi}{\sqrt{k}}$.
(iii) If $k > 0$, closed balls in M_k^n of radius $< \frac{\pi}{\sqrt{k}}$ are convex.

Gromov's study [8] of $CAT(0)$ spaces was an extensive exploration of $CAT_p(0)$ spaces, which was initially addressed by Khamsi and Shukri [13], taking into account that the comparison triangle pertains to a generic Banach space. Specifically, the situation where l_p, $p \geq 2$ is the Banach space. In a geodesic metric space $(\mathfrak{X}, d)$, a geodesic triangle $\triangle(c_1, c_2, c_3)$ in Figure 1, consists of the three vertices c_1, c_2, c_3 in $\mathfrak{X}$ along with the geodesic segments connecting each pair of vertices, which form the edges of $\triangle$. A comparison triangle for the geodesics triangle $c_1, c_2, c_3 = \triangle \in (\mathfrak{X}, d)$ is a triangle $\bar{\triangle}(c_1, c_2, c_3)' = \triangle(\bar{c_1}, \bar{c_2}, \bar{c_3})$ for $p \geq 2$ in the Banach space l_p, such that $\bar{c_i} - \bar{c_j}|| = d(c_i, c_j), \forall\, i, j \in 1, 2, 3$. A point $\bar{c} \in [\bar{c_1}, \bar{c_2}]$ is called a comparison point for $c \in [c_1, c_2]$ if $d(c_1, c) = ||\bar{c_1} - \bar{c}||$.

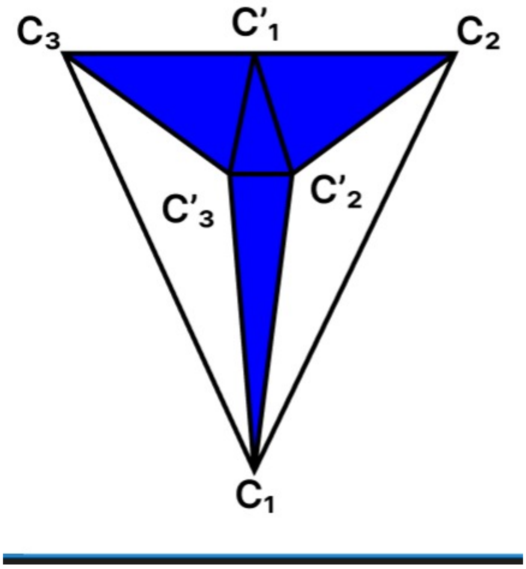

Figure 1. Geodesic triangle.

Definition 2.4. [13] Let $(\mathfrak{X}, d)$ be a geodesic metric space and $(\mathbb{E}, ||.||)$ be a normed vector space. When a comparison triangle $\bar{\triangle}$ in l_p exists for any geodesic triangle $\triangle$ in $\mathfrak{X}$, and the comparison axioms are satisfied (that is, for all $c, s \in \triangle$ and comparison points $\bar{c}, \bar{s} \in \bar{\triangle}$), $d(c, s) \leq \|\bar{c} - \bar{s}\|$ and then $\mathfrak{X}$ is said to be a $CAT_P(0)$ space.

The following is an important theorem of a $CAT_p(0)$ space [13].

Theorem 2.1. If $\frac{s_1 \oplus s_2}{2}$ is the midpoint of geodesic $[s_1, s_2]$ and c, s_1, s_2 are in $\mathfrak{X}$, then for $p \geq 2$, the comparison axiom entails that

$$d^p(c, \frac{s_1 \oplus s_2}{2}) \leq \frac{1}{2}d^p(c, s_1) + \frac{1}{2}d^p(c, s_2) - \frac{1}{2^p}d^p(s_1, s_2). \tag{2.1}$$

The inequality is known as (CN_p), established by Khamis and Shukra [13]. As for l_p, for $p > 2$, the (CN_p) inequality implies that $(CN_p)(0)$ metric spaces are uniformly convex with $\delta(r, \epsilon) \geq 1 - (1 - \frac{\epsilon^p}{2^p})^{\frac{1}{p}}$, for each $\epsilon > 0$ and each $r > 0$. The traditional (CN) inequality of Bruhat and Tits [26] is reduced to the (CN_p) inequality for $p = 2$. The CAT(0) inequality means that if $z = \frac{s_1 \oplus s_2}{2}$ is the midpoint of the segment $[s_1, s_2]$ and c, s_1, s_2 are points in $CAT(0)$ space, then

$$d(c, z)^2 \leq \frac{1}{2}d(c, s_1)^2 + \frac{1}{2}d(c, s_2)^2 - \frac{1}{4}d(s_1, s_2)^2. \tag{2.2}$$

Recall uniform convexity of a Banach space $\mathfrak{X}$ as follows:

Definition 2.5. [27] For each ϵ, $0 < \epsilon \leq 2$, the inequalities $\|c\| \leq 1, \|s\| \leq 1, \|c - s\| \geq \epsilon$ imply there exists $\delta = \delta(\epsilon) > 0$ such that $\|\frac{c+s}{2}\| \leq 1 - \delta$. The midpoint of c and s lies inside the unit ball $B_{\mathfrak{X}}$ at a distance of at least δ from the unit sphere $\delta_{\mathfrak{X}}$. This indicates that c and s are in the closed ball $B_{\mathfrak{X}} := \{c \in X : \|c\| \leq 1\}$ with $\|c - s\| \geq \epsilon > 0$.

Example 2.1. [27] A uniformly convex space is any Hilbert space H. The parallelogram law provides us with $\|c+s\|^2 = 2(\|c\|^2 + \|s\|^2) - \|c-s\|^2$, for all $c, s \in H$. Suppose $c, s \in B_H$ with $c \neq s$ and $\|c-s\| \geq \epsilon$. Then $\|c - s\|^2 \leq 4 - \epsilon^2$. So it follows that $\|\frac{c+s}{2}\| \leq 1 - \delta(\epsilon)$, where $\delta(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$. Therefore H is uniformly convex.

Let $\mathbb{R}$ be a set of real numbers. A metric space $(\mathfrak{X}, d)$ defined by a mapping $\gamma : \mathbb{R} \to \mathfrak{X}$ in a metric embedding of $\mathbb{R}$ into $\mathfrak{X}$ are the definitions of $CAT_p(0)$ metric spaces with $d(\gamma(c), \gamma(s)) = |c - s|, \forall c, s \in \mathbb{R}$. The image $\mathbb{R}$ under a metric embedding is called a metric line. The image $\gamma([a, b]) \subset \mathfrak{X}$ is called a metric segment.

Definition 2.6. [28, 29] Let $\mathfrak{X}$ be a space with metrics. In $\mathfrak{X}$, a geodesic path is a path $\gamma : [a, b] \to \mathfrak{X}$.

Assume $c, s \in X$. If $\gamma(b) = s$ and $\gamma(a) = c$, then $\gamma([c, s])$ is said to link c and s. This demonstrates the hyperbolic type of $(\mathfrak{X}, d)$.

Lemma 2.1. *[30] Let $\gamma : \mathbb{R} \to \mathfrak{X}$ be a metric embedding $a \leq b \in \mathbb{R}$ and $t \in [0, 1]$. Then*

(i) $d(\gamma(a), \gamma(\lambda_2 a \oplus \lambda_1 b)) = \lambda_1 d(\gamma(a), \gamma(b))$,
(ii) $d(\gamma(b), \gamma(\lambda_2 a \oplus \lambda_1 b)) = \lambda_2 d(\gamma(a), \gamma(b))$.

Khamis et al. [13] introduced a property of a CAT_p metric space. The $CAT(0)$ metric space $(\mathfrak{X}, d)$ is said to be convex whenever $[c, s] \in K$, for any $c, s \in K$. Consider map $\gamma : \mathfrak{X} \to \mathbb{R}$ is a type function if there exists a bounded sequence $\{c_n\}$ in $\mathfrak{X}$ such that $\gamma(c) = \limsup_{i \to \infty} d(c, c_i)$.

Theorem 2.2. [13] Let $p \geq 2$. Consider $(\mathfrak{X}, d)$ as a complete $CAT_p(0)$ metric space, and K is any closed, bounded, convex, non-empty subset of $\mathfrak{X}$. Assume that a type function specified on K is γ. Any γ minimizing sequence is then convergent. Its limit z satisfies and is the unique minimum of γ and

$$\gamma^p(z) + \frac{1}{2^{p-1}} d^p(z, c) \quad \leq \quad \gamma^p(c), \tag{2.3}$$

for any $c \in K$.

Definition 2.7. [29] Let $\mathcal{E}$ be a vector space. An affinely convex subset $\mathfrak{X} \subset \mathcal{E}$ is defined as follows: for any $c, s \in \mathfrak{X}$ and $\lambda_1, \lambda_2 \in (0, 1)$ with $1 - \lambda_1 = \lambda_2$, the affine segment $[c, s] := \left\{\lambda_2 c + \lambda_1 s : \lambda \in [0, 1]\right\}$ is contained in $\mathfrak{X}$.

If $\mathfrak{X}$ has a family of metric segments, then there exists a unique metric line joining c and s for every pair of distinct points c and s in $\mathfrak{X}$. The unique metric segment connecting the two points c and s from $\mathfrak{X}$ is indicated by the notation $[c, s]$ or $[s, c]$. This demonstrates that, for all $c \in \mathfrak{X}$, $[c, c] = \{c\}$.

Proposition 2.2. *[30] Let $(\mathfrak{X}, d)$ be a hyperbolic metric space. Assume $c, s \in \mathfrak{X}$. There exists a unique point $z \in [c, s]$ for every $\lambda \in [0, 1]$, such that $d(c, z) = \lambda_1 d(c, s)$, and $d(s, z) = \lambda_2 d(c, s)$-such points will be denoted by $z = \lambda_2 c \oplus \lambda_1 s$. For $z \in [c, s]$, this imply that $d(c, z) + d(z, s) = d(c, s)$.*

Definition 2.8. [10] $(\mathfrak{X}, d)$ is a hyperbolic metric space if

(i) $d(\lambda_2 c \oplus \lambda_1 s, (\lambda_2 c \oplus \lambda_1 z) \leq \lambda_1 d(s, z)$,
(ii) $d(\lambda_2 c \oplus \lambda_1 s, \lambda_2 z \oplus \lambda w) \leq \lambda_2 d(c, z) + \lambda_1 d(s, w)$,

for any $\lambda_1, \lambda_2 \in [0, 1]$ and all $c, s, z, w \in \mathfrak{X}$. Note that every hyperbolic space is a space of hyperbolic type [30].

Next, we recall that Hadamard manifolds [10], the Hilbert open unit ball [23], and $CAT_p(0)$ metric spaces [13] are some examples of normed spaces which are hyperbolic metric spaces. Opial [31] introduced an inequality for a well-convergent sequence characterizing its limits as follows:

Definition 2.9. Suppose there are Banach spaces $\mathfrak{X}$. If every sequence $\{c_n\}$ weakly converges to c for every c in $\mathfrak{X}$, then $\mathfrak{X}$ meets Opial's condition:

$$\liminf_{n \to \infty} \|c_n - s\| \quad > \quad \liminf_{n \to \infty} \|c_n - c\|, \tag{2.4}$$

which holds for $s \neq c$.

Browder [32] obtained an equivalent definition by replacing (2.4) by

$$\limsup_{n \to \infty} \|c_n - s\| \quad > \quad \limsup_{n \to \infty} \|c_n - c\|. \tag{2.5}$$

The following is the extension of p-uniform convexity to the set of geodesic space by Naor and Silberman [33]:

Definition 2.10. Let $1 < p < \infty$ be fixed. If $(\mathfrak{X}, d)$ is geodesic, it is called p-uniformly convex if there exist a constant $\mu > 0$ such that $c, s, z \in X$ with every $\lambda_i \in [0, 1]$, and for all $i = 1, 2$, we have:

$$d^p(\lambda_2 c \oplus \lambda_1 s, z) \quad \leq \quad \lambda_2 d^p(c, z) + \lambda_1 d^p(s, z) - \frac{\mu}{2} \lambda_1 \lambda_2 d^p(c, s). \tag{2.6}$$

One should bear in mind that the space $\mathfrak{X}$ is confirmed to be uniquely geodesic based on the inequality stated above. Darweesh et al. [15] proved the following theorem on $CAT_p(0)$ space.

Theorem 2.3. [15] Given a complete $CAT_p(0)$ metric space $\mathfrak{X}$ with $p \geq 2$, let K be a non-empty bounded, closed, convex subset of that space. Given a Suzuki generalized non-expansive mapping $\mathfrak{T} : K \to K$, $\mathfrak{T}$ has a fixed point.

Lemma 2.2. *[34] Let K be a non-empty subset of a metric space $\mathfrak{X}$. Suppose $\mathfrak{T} : K \to K$ is a Suzuki generalized non-expansive mapping. Then $d(c, \mathfrak{T}s) \leq 3d(\mathfrak{T}c, c) + d(c, s)$, for all $c, s \in K$.*

Lemma 2.3. *[34] Assume that the generalized metric space is $(\mathfrak{X}, d)$. If we have $\mathfrak{T}c_n \to \mathfrak{T}z$ such that $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ is continuous at $z \in \mathfrak{X}$, then $\lim_{n\to\infty} d(\mathfrak{T}c_n, \mathfrak{T}z) = 0$, for any sequence $\{c_n\}$ in $\mathfrak{X}$, converges to $z \in \mathfrak{X}$. That is $c_n \to z$.*

The following is the relationship between the aforementioned lemma and the Ćirić [2], Jachymski [7], and Matkowski [3] (CMJ) fixed point theorem.

Theorem 2.4. [35] Let $\mathfrak{X}$ be a CMJ contraction on $\mathfrak{X}$, and let (X, d) be a complete v-generalized metric space. This means that the following is true:

(i) For every $\epsilon > 0$, there exists $\delta > 0$ such that $d(c, s) < \epsilon + \delta$ implies $d(\mathfrak{T}c, \mathfrak{T}s) \leq \epsilon$ for any $c, s \in \mathfrak{X}$.
(ii) $c \neq s$ implies that $d(\mathfrak{T}c, \mathfrak{T}s) < d(c, s)$ for any $c, s \in \mathfrak{X}$.

Then $\mathfrak{T}$ has a unique fixed point z of $\mathfrak{T}$. Moreover $\lim_{n\to\infty} d(\mathfrak{T}^n c, z) = 0$ for any $c, s \in \mathfrak{X}$.

A weak contractive condition that grants the existence of a fixed point in all metric space was developed by Meir et al. [1]. In their investigation, they considered $\epsilon > 0$, where there exists $\delta > 0$ such that $\epsilon \leq d(c, s) < \epsilon + \delta$ implies $d(\mathfrak{T}c, \mathfrak{T}s) < \epsilon$.

Theorem 2.5. [1] Assume that $\mathfrak{T}$ is a mapping from $\mathfrak{X}$ into itself and that $(\mathfrak{X}, d)$ is a full metric space. Then, there is a single fixed point ζ for $\mathfrak{T}$. Additionally, $\lim_{n\to\infty} \mathfrak{T}^n c = \zeta$ for any $c \in \mathfrak{X}$.

The following theorem is proved in metric space by Ćirić [2] using the preceding principles for contractive mappings with the continuity property.

Theorem 2.6. [2] Given $\epsilon > 0$, there exists $\delta > 0$ such that, given $(\mathfrak{X}, d)$, a full metric space, and $\mathfrak{T}$ a self mapping of $\mathfrak{X}$ into itself, satisfying the condition $\epsilon < d(c, s) < \epsilon + \delta$ implies $d(\mathfrak{T}c, \mathfrak{T}s) \leq \epsilon$. Then $\mathfrak{T}$ has a unique fixed point $\zeta \in \mathfrak{X}$ and $\lim_{n\to\infty} \mathfrak{T}^n c = \zeta$ for each $\mathring{x} \in \mathfrak{X}$.

Additionally, Matkowski [3] established the subsequent theorem.

Theorem 2.7. [3] Let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a full metric space, and let $(\mathfrak{X}, d)$ be its metric space. Assume that for each $c, s \in \mathfrak{X}$ and $\epsilon > 0$,

$$0 \leq \max\left\{d(\mathfrak{T}c, s), d(s, \mathfrak{T}s), d(c, s), \frac{d(\mathfrak{T}c, s) + d(c, \mathfrak{T}s)}{2}\right\} \leq \epsilon \Rightarrow\ d(\mathfrak{T}c, \mathfrak{T}s) < \epsilon.$$

If for every $\epsilon > 0$ there is a $\delta > 0$ such that for $c, s \in \mathfrak{X}$:

$$\left.\begin{array}{l}\epsilon \le d(c,s) < \epsilon+\delta \\ 0 < \max\left\{d(\mathfrak{T}c,s), \frac{d(\mathfrak{T}c,s)+d(c,\mathfrak{T}s)}{2}\right\} \le \epsilon \\ d(s,\mathfrak{T}s) < \epsilon+\delta\end{array}\right\} \Rightarrow d(\mathfrak{T}c,\mathfrak{T}s) < \epsilon,$$ then for every $c \in \mathfrak{X}$, the sequence $\{\mathfrak{T}^n c\}$ converges. Moreover, if $\mathfrak{T}$ is continuous, or, given $\epsilon > 0$, there is a, μ, $0 < \mu < \epsilon$ such that for every $c, s \in \mathfrak{X}$,

$$\left.\begin{array}{l}0 < \max\left\{d(\mathfrak{T}c,c), \frac{d(\mathfrak{T}c,s)+d(c,\mathfrak{T}s)}{2}\right\} \le \epsilon \\ 0 < d(c,s), d(s,\mathfrak{T}s) < \mu\end{array}\right\} \Rightarrow d(\mathfrak{T}c,\mathfrak{T}s) < \epsilon,$$ and then $\mathfrak{T}$ has a unique fixed point $\zeta \in \mathfrak{X}$ and $\lim_{n\to\infty} \mathfrak{T}^n c = \zeta$ for each $c \in \mathfrak{X}$.

Furthermore, the following Mier Keeler-type theorem was demonstrated by Jachymski [7].

Theorem 2.8. [7] Let $(\mathfrak{X}, d)$ be a complete metric space and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a self map. If $\mathfrak{T}$ satisfies

(i) for every $\epsilon > 0$ and $\delta > 0$, for any $c, s \in \mathfrak{X}$,

$$\epsilon < \max\left\{d(c,s), d(c,\mathfrak{T}c), d(s,\mathfrak{T}s), \frac{d(c,\mathfrak{T}s)+d(s,\mathfrak{T}c)}{2}\right\} < \epsilon+\delta, \Rightarrow d(\mathfrak{T}c,\mathfrak{T}s) \le \epsilon,$$

(ii) for every $\epsilon > 0$ and $\delta > 0$, for any $c, s \in \mathfrak{X}$,

$$d(\mathfrak{T}c,\mathfrak{T}s) < \max\left\{d(c,s), d(c,\mathfrak{T}s), d(s,\mathfrak{T}s), \frac{d(c,\mathfrak{T}s)+d(c,\mathfrak{T}c)}{2}\right\}$$

then $\mathfrak{T}$ has a unique fixed point $\zeta \in \mathfrak{X}$ and $\lim_{n\to\infty} \mathfrak{T}^n c = \zeta$ for each $c \in \mathfrak{X}$.

3. Results

In this section, we shall cover our main results. We start the section with the following theorem.

Theorem 3.1. Let $K \in \mathfrak{X}$ be a non-empty, bounded, closed, weakly uniformly convex subset of a $CAT_p(0)$ metric space, where $p \ge 2$, $\lambda_1, \lambda_2 \in [0,1]$, and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a CMJ-type mapping. Suppose that for every $\epsilon > 0$ and $c, s \in \mathfrak{X}$,

$$0 < \max\left\{d^p(\mathfrak{T}c,s), d^p(s,\mathfrak{T}s), d^p(c,s), \frac{d^p(\mathfrak{T}c,s)+d^p(c,\mathfrak{T}s)}{2}\right\} \le \epsilon \Rightarrow d^p(\mathfrak{T}c,\mathfrak{T}s) < \epsilon, \tag{3.1}$$

then for every $c \in \mathfrak{X}$, the sequence $\{\mathfrak{T}^n c\}$ converges. Furthermore, if $\mathfrak{T}$ is continuous, or if, for any $c, s \in \mathfrak{X}$, there exists μ, $0 < \mu < \epsilon$ such that

$$\left.\begin{array}{l}0 < \max\left\{d^p(\mathfrak{T}c,c), \frac{d^p(\mathfrak{T}c,s)+d^p(c,\mathfrak{T}s)}{2}\right\} \le \epsilon \\ 0 < \max\left\{d^p(c,s), d^p(s,\mathfrak{T}s)\right\} < \mu\end{array}\right\} \Rightarrow d^p(\mathfrak{T}c,\mathfrak{T}s) < \epsilon-\mu.$$

Then $\mathfrak{T}$ has a unique fixed point $z \in \mathfrak{X}$ and for each $c \in \mathfrak{X}$, $\lim_{i\to\infty} \mathfrak{T}^i c = z$.

Proof. Let K be a bounded, closed convex subset of $\mathfrak{X}$. For simplicity, let $\mathfrak{X}$ represnted $CAT_p(0)$-metric spaces. Assume $c_0 \in K$. Now, for all $i \ge 0$, $\{c_i\}$ is a Picard sequence defined by $c_i = \mathfrak{T}^i c_0$. Clearly for

$c_i = c_{i+1}$, we can have $\mathfrak{T}^i c_0$ be a fixed point of $\mathfrak{T}$. This means that for some i, $\mathfrak{T}^i c_0 = \mathfrak{T}^{i+1} c_0$. If possible, let $\mathfrak{T}^i c_0 \neq \mathfrak{T}^{i+1} c_0$. If $c = c_0$ and $s = \mathfrak{T} c_0$, then from the inequality (3.1), we have

$$\begin{aligned} 0 &\leq \max\left\{d^p(\mathfrak{T}c_0, \mathfrak{T}c_0), d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0), d(c_0, \mathfrak{T}c_0), \frac{d^p(\mathfrak{T}c_0, \mathfrak{T}c_0) + d^p(c_0, \mathfrak{T}^2c_0)}{2}\right\} \leq \epsilon \\ &\Rightarrow d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) < \epsilon. \end{aligned} \tag{3.2}$$

Now using Lemma 2.1, we can compute all metrics of the above inequality as follows.

$$\begin{aligned} d^p(\mathfrak{T}c_0, \mathfrak{T}c_0) &= d^p(\gamma(\mathfrak{T}c_0), \gamma(\lambda_2\mathfrak{T}c_0 \oplus \lambda_1\mathfrak{T}c_0)) = \lambda_1 d^p(\gamma(\mathfrak{T}c_0), \gamma(\mathfrak{T}c_0)) \\ &= \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}c_0) = 0. \\ d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) &= d^p(\gamma(\mathfrak{T}c_0), \gamma(\lambda_2\mathfrak{T}c_0 \oplus \lambda_1\mathfrak{T}^2c_0)) = \lambda_1 d^p(\gamma(\mathfrak{T}c_0), \gamma(\mathfrak{T}^2c_0)) \\ &= \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0). \\ d^p(c_0, \mathfrak{T}c_0) &= d^p(\gamma(c_0), \gamma(\lambda_2 c_0 \oplus \lambda_1\mathfrak{T}c_0)) = \lambda_1 d^p(\gamma(c_0), \gamma(\mathfrak{T}c_0)) \\ &= \lambda_1 d^p(c_0, \mathfrak{T}c_0). \\ d^p(c_0, \mathfrak{T}^2c_0) &= d^p(\gamma(c_0), \gamma(\lambda_2 c_0 \oplus \lambda_1\mathfrak{T}^2c_0)) = \lambda_1 d^p(\gamma(c_0), \gamma(\mathfrak{T}^2c_0)) \\ &= \lambda_1 d^p(c_0, \mathfrak{T}^2c_0). \end{aligned}$$

Substituting the derived metrics in (3.2), we obtain

$$\begin{aligned} 0 &< \max\Big\{\lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}c_0), \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0), \lambda_1 d^p(c_0, \mathfrak{T}c_0), \\ &\quad \frac{d^p(\mathfrak{T}c_0, \mathfrak{T}c_0) + \lambda_1 d^p(c_0, \mathfrak{T}^2c_0)}{2}\Big\} \leq \epsilon \Rightarrow \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) < \epsilon, \\ 0 &< \max\left\{0, \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0), \lambda_1 d^p(c_0, \mathfrak{T}c_0), \frac{\lambda_1 d^p(c_0, \mathfrak{T}c_0) + \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0)}{2}\right\} \leq \epsilon \\ &\Rightarrow \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) < \epsilon. \end{aligned}$$

Suppose $\frac{\lambda_1 d^p(c_0,\mathfrak{T}c_0)+\lambda_1 d^p(\mathfrak{T}c_0,\mathfrak{T}^2c_0)}{2} \leq \lambda_1 d^p(c_0, \mathfrak{T}c_0)$, and we get

$$\begin{aligned} 0 &< \max\left\{0, \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0), \lambda_1 d^p(c_0, \mathfrak{T}c_0), \lambda_1 d^p(c_0, \mathfrak{T}c_0)\right\} \leq \epsilon \Rightarrow \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) < \epsilon, \\ 0 &< \lambda_1 d^p(c_0, \mathfrak{T}c_0) \Rightarrow \lambda_1 d^p(\mathfrak{T}c_0, \mathfrak{T}^2c_0) < \epsilon. \end{aligned}$$

Next, by using mathematical induction, we have

$$0 < \lambda_1 \lim_{i\to\infty} d^p(\mathfrak{T}^n c_0, \mathfrak{T}^{n+1}c_0) \Rightarrow \lambda_1 \lim_{i\to\infty} d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1}c_0) < \epsilon.$$

Hence, we get

$$0 < \lim_{i\to\infty} d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1}c_0) < \epsilon.$$

This is a contradiction to our assumption. So the sequences $\{d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1}c_0)\}$ converge.

Further, let $(\mathfrak{X}, d)$ be a complete $CAT_p(0)$ metric space and $\mathfrak{T}$ is closed, bounded, and uniformly convex. We need to prove $\{\mathfrak{T}^i c_0\}$ is a Cauchy sequenc. Let us construct a sequence $\{\mathfrak{T}^i c_0\}$ in K in such a way that $\mathfrak{T}c = \lim_{i\to\infty} d(c_i, c)$. Let $\gamma_0 = \inf\{\gamma(c) : c \in K\}$ and $\{s_i\}$ be a γ minimized sequence. Since K is bounded, then for any $c, s \in K$, there exists an $R > 0$ such that $d(c, s) \leq R$. Since $(\mathfrak{X}, d)$ is a $CAT_p(0)$ metric space, it is enough to show that the inequality (2.6) of Definition 2.10 is satisfied.

$$d^p(c_i, \lambda_2 s_i \oplus \lambda_1 s_j) \leq \lambda_2 d^p(c_i, s_i) + \lambda_1 d^p(s_j, c_i) - \lambda_1\lambda_2 d^p(s_i, s_j). \tag{3.3}$$

By applying inequality (2.3) from Theorem 2.2, we can derive the inequality (3.3), which yields

$$\gamma^p(c_i, \lambda_2 s_i \oplus \lambda_1 s_j) \leq \lambda_2 \gamma^p s_i + \lambda_1 \gamma^p s_j - \lambda_1\lambda_2 \gamma^p(s_i, s_j).$$

This gives a conclusion that for all $s_{i,j} \geq 1$, we have

$$\gamma_0^p \leq \lambda_2 \gamma^p s_i + \lambda_1 \gamma^p s_j - \lambda_1\lambda_2 \gamma^p(s_i, s_j).$$

As $\{s_i\}$ is a minimizing sequence for γ, we have $\lim_{i,j\to\infty} d(s_i, s_j) = 0$, implying that $\{s_i\}$ is Cauchy. Consequently, $\{s_i\}$ converges to some $z \in K$ with $\gamma_0 = \gamma(z)$.

Since γ is continuous and $\mathfrak{X}$ is complete, Lemma 2.2 gives us

$$\begin{aligned}
d^p(z, \mathfrak{T}^i z) &\leq 3d^p(\mathfrak{T}^i z, z) + d^p(z, z), \\
\limsup_{i\to\infty} d^p(z, \mathfrak{T}^i z) &\leq 3\limsup_{i\to\infty} d^p(\mathfrak{T}^i z, z) + \limsup_{i\to\infty} d^p(z, z), \\
\limsup_{i\to\infty} d^p(z, \mathfrak{T} z) &\leq \frac{1}{2}\limsup_{i\to\infty} d^p(z, z).
\end{aligned}$$

Therefore, $d^p(z, \mathfrak{T}z) = 0$, which implies that z is a fixed point of $\mathfrak{T}$.

Case 1: If $\mathfrak{T}$ is continuous, then taking limits yields:

$$z = \lim_{i\to\infty} \mathfrak{T}^i c = \lim_{i\to\infty} \mathfrak{T} c_0 = \mathfrak{T} z,$$

and hence z is a fixed point.

Case 2: Suppose instead that the second condition in the theorem holds. Assume, for contradiction, that $\mathfrak{T}z \neq z$. Then $d^p(\mathfrak{T}z, z) > 0$. Since $c_i \to z$, for sufficiently large i, we have

$$\begin{aligned}
&0 < \max\left\{d^p(\mathfrak{T}z, z), \frac{d^p(\mathfrak{T}z, c_i) + d^p(z, \mathfrak{T}c_i)}{2}\right\} \leq \epsilon, \\
&0 < \max\left\{d^p(\mathfrak{T}z, z), \frac{d^p(\mathfrak{T}z, z) + d^p(z, \mathfrak{T}z}{2}\right\} \leq \epsilon, \\
&0 < \max\{d^p(\mathfrak{T}z, z), d^p(z, \mathfrak{T}z\} \leq \epsilon, \\
&0 < d^p(z, \mathfrak{T}z) \leq \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&0 < \max\{d^p(z, c_i), d^p(c_i, \mathfrak{T}c_i)\} < \mu, \\
&0 < \max\{d^p(z, z), d^p(z, \mathfrak{T}z)\} < \mu, \\
&0 < d^p(z, \mathfrak{T}z) < \mu,
\end{aligned}$$

for some $0 < \mu < \epsilon$. Then the hypothesis implies

$$d^p(\mathfrak{T}z, \mathfrak{T}c_i) < \epsilon - \mu,$$

$$d^p(\mathfrak{T}z, \mathfrak{T}z) < \epsilon - \mu,$$

$$0 < \epsilon - \mu.$$

$$0 < \epsilon - \mu < d^p(z, \mathfrak{T}z) \leq \epsilon.$$

Letting $i \to \infty$, the right-hand side remains positive while the left-hand side tends to $d^p(\mathfrak{T}z, z)$, a contradiction. Hence, $\mathfrak{T}z = z$.

Next for uniqueness: let $\gamma(\mathfrak{T}z) \leq \gamma(z)$. According to the asymptotic center's uniqueness, $\mathfrak{T}z = z$.

Given $s_i = \zeta$ and $s_j = z$, we may use inequality (2.6) to determine the uniqueness of $\mathfrak{T}$. Since,

$$d^p(\zeta_i, \lambda_2\zeta \oplus \lambda_1 z) \quad \leq \quad \lambda_2 d^p(\zeta_i, \zeta) + \lambda_1 d^p(z, \zeta_i) - \lambda_1\lambda_2 d^p(\zeta, z),$$

by taking the limit on each side of the inequality shown above, we arrive at

$$\begin{aligned}\limsup_{i\to\infty} d^p(\zeta_i, \lambda_2\zeta \oplus \lambda_1 z) \quad \leq \quad & \limsup_{i\to\infty} \lambda_2 d^p(\zeta_i, \zeta) + \\ & \limsup_{i\to\infty} \lambda_1 d^p(z, \zeta_i) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z).\end{aligned}$$

Since, $\limsup_{i\to\infty} d^p(\zeta_i, z) \leq \limsup_{i\to\infty} d^p(\zeta_i\lambda_2\zeta \oplus \lambda_1 z)$, then we have:

$$\begin{aligned}\limsup_{i\to\infty} d^p(\zeta_i, z) \quad &\leq \quad \limsup_{i\to\infty} \lambda_2 d^p(\zeta_i, \zeta) + \\ & \quad\quad \limsup_{i\to\infty} \lambda_1 d^p(z, \zeta_i) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z), \\ \limsup_{i\to\infty} d^p(z, z) \quad &\leq \quad \limsup_{i\to\infty} \lambda_2 d^p(z, \zeta) + \\ & \quad\quad \limsup_{i\to\infty} \lambda_1 d^p(z, z) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z), \\ 0 \quad &\leq \quad \limsup_{i\to\infty} \lambda_2 d^p(z, \zeta) - \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z), \\ 0 \quad &\leq \quad \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z). \\ \limsup_{i\to\infty} \lambda_1\lambda_2 d^p(\zeta, z) \quad &\geq \quad 0, \\ \limsup_{i\to\infty} d^p(\zeta, z) \quad &\geq \quad 0,\end{aligned}$$

which is a conflict. Thus, $\zeta = z$. In conclusion, z represents a single fixed point in $\mathfrak{T}$ and Opial's condition is equal to this argument. □

The second important work extends Theorem 2.6 to consider an analog of the theorem in the context of Meir and Keeler-type uniformly convex contractive mappings in $CAT_p(0)$ space.

Theorem 3.2. Given $\epsilon > 0$, there exists $\delta > 0$ such that, given $(\mathfrak{X}, d)$, a full $CAT_p(0)$, where $p \geq 2$, $\lambda_1, \lambda_2 \in [0, 1]$, and let K be a non-empty, bounded, closed , convex subset of a $CAT_p(0)$ metric space and $\mathfrak{T} : K \to K$ is a convex contractive mapping, satisfying the condition

$$\epsilon \quad < \quad d^p(c, s) < \epsilon + \delta \implies d^p(\mathfrak{T}c, \mathfrak{T}s) \leq \epsilon. \tag{3.4}$$

Then $\mathfrak{T}$ has a unique fixed point $\zeta \in \mathfrak{X}$ and $\lim_{n\to\infty} \mathfrak{T}^i c_0 = \zeta$ for each $c \in \mathfrak{X}$.

Proof. Let $c_0 \in \mathfrak{X}$. Let us construct a series $\{c_i\}$ by $c_i = \mathfrak{T}^i c_0$ for each $i \in \mathbb{N}$. Then c_i is a fixed point on $\mathfrak{T}$ if there exists i such that $c_i = c_{i+1}$. The proof is hereby finalized. Alternatively, suppose that for every $i \geq 0$, $c_i \neq c_{i+1}$. As $c = c_0$ and $s = \mathfrak{T}^i c_0$ in (3.4) implies that $\mathfrak{T}$ is contractive, therefore real sequence $d^p(c_0, \mathfrak{T}^i c_0)$ is non-growing and has a limit $\epsilon \geq 0$. Based on monotonicity, we obtain

$$\lambda_1 d^p(c_0, \mathfrak{T}^i c_0) > \epsilon, for\ i = 0, 1, 2, \ldots \tag{3.5}$$

Let us assume $\epsilon > 0$. Then, for all i such that $\delta = \delta(\epsilon) > 0$,

$$\epsilon < \lambda_1 d^p(c_0, \mathfrak{T}^i c_0) < \epsilon + \delta. \tag{3.6}$$

By (3.4),

$$\lambda_1 d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1} c_0) \leq \epsilon, \tag{3.7}$$

which is a contradiction with (3.5). Therefore $\epsilon = 0$ and

$$\begin{aligned} \lambda_1 d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1} c_0) &= 0, \\ d^p(\mathfrak{T}^i c_0, \mathfrak{T}^{i+1} c_0) &= 0. \end{aligned} \tag{3.8}$$

Let us assume $c_0 \in K$. Using the argument of induction, we will create a sequence $\{c_i\}$ in K such that, for any $i \geq 0$, the sequence $\{\mathfrak{T} c_i\}$ has a point z linked with it in Theorem 2.2. For any integer $i \geq 0$,

$$\epsilon = r_i \quad = \quad \lambda_1 \limsup_{j \to \infty} d^p(c_i, \mathfrak{T}^j c_i), \tag{3.9}$$

$$\delta = R_i \quad = \quad \lambda_1 \limsup_{j \to \infty} d^p(\mathfrak{T}^j c_i, \mathfrak{T}^{j+1} c_i). \tag{3.10}$$

By (3.9) and (3.10), we have

$$\begin{aligned} r_i \quad &< \quad \lambda_1 \limsup_{j \to \infty} d^p(c_i, \mathfrak{T}^{j+1} c_i) < \lambda_1 \limsup_{j \to \infty} d^p(c_i, \mathfrak{T}^j c_i) + \\ &\quad \lambda_1 \limsup_{j \to \infty} d^p(\mathfrak{T}^j c_i, \mathfrak{T}^{j+1} c_i), \\ r_i \quad &< \quad \lambda_1 \limsup_{j \to \infty} d^p(c_i, \mathfrak{T}^{j+1} c_i) < r_i + R_i, \\ r_i \quad &< \quad \lambda_1 \limsup_{j \to \infty} d^p(c_i, \mathfrak{T}^{j+1} c_i) < 2R_i < r_i. \end{aligned} \tag{3.11}$$

Hence by (3.4) for $i \geq 1$, the series $\sum_{i=j=1}^{\infty} \lambda_i d^p(c_i, \mathfrak{T}^{j+1} c_i)$ is also convergent and therefore $\{c_i\}$ is Cauchy. The remaining steps to demonstrate that $\{c_i\}$, the fixed point's existence and uniqueness, follow Theorem 3.1's proof in a similar manner. This concludes the proof. □

Utilizing the concepts derived from CMJ-type contractive mappings within the context of $CAT_p(0)$ space, we establish the subsequent corollary that supports Theorem 3.1.

Corollary 3.1. *Let $K \in \mathfrak{X}$ be a non-empty, bounded, closed, weakly uniformly convex subset of a $CAT_p(0)$ metric space since $p \geq 2$, $\lambda_1, \lambda_2 \in [0, 1]$ and we specify $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as a CMJ-type mapping. For all $c, s \in \mathfrak{X}$ and $\epsilon > 0$, let us assume*

$$0 < \max\left\{d^p(c, s), d^p(s, \mathfrak{T} s)\right\} < \epsilon + \mu <\Longrightarrow d^p(\mathfrak{T} c, \mathfrak{T} s) < \epsilon.$$

Then $\mathfrak{T}$ has a unique fixed point $\zeta \in \mathfrak{X}$ and $\lim_{i \to \infty} \mathfrak{T}^i c = \zeta$ for each $c \in \mathfrak{X}$.

Proof. The similar evidence for Theorem 3.1 is applied in the proofs of the remaining stages. Therefore, the proofs have been concluded. □

To support the established results mentioned above, an example is illustrated below.

Example 3.1. Let $\mathfrak{X} = [0,1]$, $K = [1, \frac{1}{3}, \ldots, \frac{1}{2n-1}] \in X$ for $n \in \mathbb{N}$, and $(\mathfrak{X}, d)$ be a complete $CAT_p(0)$ metric space, $p \geq 2$. Define a metric by $d^p(c,s) = \|c-s\|^p$, and a mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ given by

$$\mathfrak{T}c = \begin{cases} \frac{1}{2c}, & for\ n \in \mathbb{N} \\ 0, & otherwise. \end{cases}$$

We show that $\mathfrak{T}$ satisfies inequality (3.1).

First we calculate the following metrics for $c = \frac{1}{2n-1}$, $s = \frac{1}{2n+1}$.

$$\begin{aligned}
d^p(c,s) = d^p(\frac{1}{2n-1}, \frac{1}{2n+1}) &= \lambda_1 \Big\| \frac{2}{(2n-1)(2n+1)} \Big\|^p, \\
d^p(\mathfrak{T}c, s) = d^p\Big(\frac{1}{2c}, s\Big) = \lambda_1 \Big\| \frac{(2n-1)}{2} - \frac{1}{2n+1} \Big\|^p &= \lambda_1 \Big\| \frac{4n^2}{(2n+1)} \Big\|^p, \\
d^p(s, \mathfrak{T}s) = d^p\Big(s, \frac{1}{2s}\Big) = \lambda_1 \Big\| \frac{1}{2n+1} - \frac{2n+1}{2} \Big\|^p &= \lambda_1 \Big\| \frac{1-(4n^2+4n)}{2(2n+1)} \Big\|^p, \\
d^p(c, \mathfrak{T}s) = d^p\Big(c, \frac{1}{2s}\Big) = \lambda_1 \Big\| \frac{1}{2n-1} - \frac{2n+1}{2} \Big\|^p &= \lambda_1 \Big\| \frac{3-4n^2}{2(2n-1)} \Big\|^p, \\
d^p(\mathfrak{T}c, c) = d^p\Big(\frac{1}{2c}, \frac{1}{2n-1}\Big) = \lambda_1 \Big\| \frac{(2n-1)}{2} - \frac{1}{2n-1} \Big\|^p &= \lambda_1 \Big\| \frac{(2n-1)^2-2}{(2(2n-1))} \Big\|^p, \\
d^p(\mathfrak{T}c, \mathfrak{T}s) = d^p\Big(\frac{1}{2c}, \frac{1}{2s}\Big) = \lambda_1 \Big\| \frac{(2n-1)}{2} - \frac{2n+1}{2} \Big\|^p &= \lambda_1 \Big\| \frac{2n-1-2n-1}{2} \Big\|^p.
\end{aligned}$$

By applying all of the above equalities in (3.1), for $n = 2\ldots$, $\lambda_1 = \frac{1}{2}$ and $p \geq 2$, we get

$$\begin{aligned}
0 &< \max\Big\{ \lambda_1 \Big\| \frac{4n^2}{(2n+1)} \Big\|^p, \lambda_1 \Big\| \frac{1-(4n^2+4n)}{2(2n+1)} \Big\|^p, \lambda_1 \Big\| \frac{2}{(2n-1)(2n+1)} \Big\|^p, \\
&\quad \frac{\lambda_1 \Big\| \frac{4n^2}{(2n+1)} \Big\|^p + \lambda_1 \Big\| \frac{3-4n^2}{2(2n-1)} \Big\|^p}{2} \Big\} \leq \epsilon \\
&\quad \Rightarrow \lambda_1 \Big\| \frac{2n-1}{2} - \frac{2n+1}{2} \Big\|^p < \epsilon. \\
0 &< \max\Big\{ 5.12, 2.645, 0.008, \frac{5.12+2.3464}{2} \Big\} \leq \epsilon \Rightarrow 1 < \epsilon. \\
0 &< 5.12 \leq \epsilon \Rightarrow 0.5 < \epsilon.
\end{aligned}$$

Furthermore, since $\mathfrak{T}$ is continuous, it follows that

$$\left.\begin{aligned} 0 < \max\Big\{0.6805, 3.733\Big\} \leq \epsilon \\ 0 < \max\Big\{0.0088, 2.645\Big\} < \mu \end{aligned}\right\} \Rightarrow 0.5 < \epsilon - \mu.$$

Consequently

$$\left.\begin{aligned} 0 < 3.733 \leq \epsilon \\ 0 < 2.645, < \mu \end{aligned}\right\} \Rightarrow\ 0.5 < \epsilon - \mu.$$

This shows that the conditions given in Theorem 3.1 are satisfied.

Similarly, assume that $\mathfrak{T}$ does not satisfy the CN_p inequality:

$$d^p(\lambda_2 c \oplus \lambda_1 s, z) \leq \lambda_2 d^p(c, z) + \lambda_1 d^p(s, z) - \lambda_1\lambda_2 d^p(c, s), \tag{3.12}$$

where

$$d^p(\lambda_2 c \oplus \lambda_1 s, z) \leq \lambda_2 d^p(c, z) + \lambda_1 d^p(s, z). \tag{3.13}$$

Using (3.13) in (3.12), we get

$$\begin{aligned}
\lambda_2 d^p(\kappa, z) + \lambda_1 d^p(s, z) &\leq \lambda_2 d^p(c, z) + \lambda_1 d^p(s, z) - \lambda_1\lambda_2 d^p(c, s),\\
0 &\leq -\lambda_1\lambda_2 d^p(c, s),\\
0 &\leq \lambda_1\lambda_2 \|c - s\|^p,\\
0 &\leq \lambda_1\lambda_2 \left\|\frac{2}{(2n-1)(2n+1)}\right\|^p,\\
0 &\leq \frac{1}{2}.\frac{1}{2}\left\|\frac{2}{15}\right\|^p,\\
0 &\leq \frac{1}{4}\left\|\frac{2}{15}\right\|^2,\\
0 &\leq \frac{1}{225}.
\end{aligned}$$

Thus, $\mathfrak{T}$ satisfies the CN_p inequality and possesses a unique fixed point at $c = \frac{\sqrt{2}}{2}$, which leads to a contradiction.

For CAT(0) space, we show that our results are a real extension by showing that the example given above does not satisfy the CN inequality, which is an example of a CAT(0) space. Let $c = 1, s_1 = \frac{1}{3}, s_2 = \frac{1}{5}$ be the three points in $\mathfrak{X}$ and $z = \frac{s_1 \oplus s_2}{2} = \frac{8}{15}$:

$$\begin{aligned}
d(c, z)^2 &\leq \frac{1}{2}d(c, s_1)^2 + \frac{1}{2}d(c, s_2)^2 - \frac{1}{4}d(s_1, s_2)^2,\\
0.54 &\leq 0.222 + 0.32 - 0.0088,\\
0.54 &\leq 0.53,
\end{aligned}$$

which is a contradiction when you compare with a $CAT(0)_P$ space.

4. Comparison with previous research in different spaces

Theorem 3.1 extends the scope of fixed point theory by considering CMJ-type mappings in $CAT_p(0)$ metric space with $p \geq 2$ and weak uniform convexity. We now compare this result with existing literature across various settings:

(1) Banach spaces

In normed linear spaces, especially Banach spaces, the foundational result is the Banach contraction principle [36], which requires:

$$d(\mathfrak{T}c, \mathfrak{T}s) \leq \lambda d(c, s), \quad \text{for } \lambda \in [0, 1).$$

By taking $c = 1$, $s = \frac{1}{3}$ with $\lambda = \frac{1}{2}$, we have

$$0.5 \leq 0.066,$$

which is a contradiction.

The Mankowski-type conditions in Theorem 2.7 serve as a generalization of the Banach contraction principle, particularly in cases where Example 3.1 fails to satisfy the Banach conditions. This highlights that the results established in the theorem are indeed a genuine extension.

Theorem 3.1, on the other hand:

- Uses d^p instead of d, extending the metric to include nonlinear or weighted behaviors.
- Requires only weak uniform convexity, thus covering a broader class of metric spaces.
- Applies to $CAT_p(0)$ spaces, which generalize Hilbert spaces to nonlinear, non-positive curvature settings.

(2) CAT(0) and CAT(κ) spaces

Previous works by Kirk [9], Gromov [8], and others studied fixed points of nonexpansive mappings in CAT(0) spaces. These studies typically rely on:

- Convexity of the metric (midpoint inequality).
- Nonexpansive or asymptotically nonexpansive conditions.

Theorem 3.1 strengthens such results by:

- Introducing CMJ-type conditions with d^p terms.
- Applying to $CAT_p(0)$ spaces, which accommodate generalized distances and stronger nonlinear structures.
- Allowing weakly uniformly convex subsets instead of requiring global convexity.

We present a comparison of fixed point results in various metric and geometric settings in the following table (see Table 1).

Table 1. Comparison of fixed point results in various metric and geometric settings.

Reference / Approach	Space Considered	Type of Mapping	Conditions Imposed	Limitation / Novelty
Banach (1922) [36]	Metric space	Contraction mapping	Constant Lipschitz condition $k < 1$	Limited to linear contraction in complete metric spaces
Meir-Keeler (1969) [1]	Metric space	Generalized contraction	Convergence-type condition on distance functions	Fails in geometric or non-Euclidean spaces
CMJ Mappings	Metric / b-metric space	Cyclic or multi-valued mappings	Use of cyclical and intermediate contractive conditions	Not always extendable to non-linear geometric spaces
Results in CAT(0) Spaces Kirk [9]	CAT(0) (non-positively curved) spaces	Nonexpansive mappings	Convexity + geodesic properties	Limited to single-valued and nonexpansive cases
Results in $\text{CAT}(0)_P$ Spaces Khamsi and Shukri [13].	Generalized geodesic spaces with projection properties	Generalized mappings under convexity constraints	Geometric assumptions can be too restrictive for hybrid mappings	
Present Work (This Paper)	$\text{CAT}(0)_P$ spaces / generalized metric spaces	CMJ-type + Meir–Keeler + hybrid/self/non-self mappings	Weak cyclic contraction, convexity, and non-linear projection structure	Unifies multiple mapping types; applicable in abstract geometric settings with greater flexibility

5. Existence of the solution of matrix equation in $CAT_p(0)$ metric space

In this section, we explore the existence of solutions to the non-linear matrix equation in the context of the $CAT_p(0)$ metric space, which is intended to demonstrate Theorem 3.1. Consider the following a non-linear matrix equation:

$$\mathfrak{X} + \sum_{1\leq i\leq d} A_i \mathfrak{X}^{-1} D_i \;\; = \;\; C. \tag{5.1}$$

The solution pertaining to a tree-like stochastic process, which acts as a generalization of the quasi-birth-and-death (QBD) process, is fundamental to this equation. The expression for S is regarded as minimal, specifically $S = \mathfrak{T} - I$, where $\mathfrak{T}$ is identified as a sub-stochastic matrix, as demonstrated by Bini et al. [37].

Lemma 5.1. *[37] The matrix $\mathfrak{T} = S + I$ is the minimum non-negative solution of the equation*

$$\mathfrak{X} \;=\; B + \sum_{1\le i\le d} A_i(A - I)^{-1}D_i, \tag{5.2}$$

where

(i) $C = B - I$, B is sub-stochastic,
(ii) A_i and D_i have non-negative entries,
(iii) the matrices $I + C + D_i + A_i + \cdots + A_d$, $i = 1 \ldots d$, are stochastic.

The matrices G_i are defined as $G_i = (-S)^{-1}D_i$, where $(G_i)_{k,k'}$ denotes the probability of the tree-like process starting at state (i, k) in N_i and eventually reaching the root node, with k' being the first state visited in this process. If the process exhibits positive recruitment, then G_i is stochastic for every i. Furthermore, Eq 5.1 can be expressed as a system of coupled equations.

$$S \;=\; C + \sum_{1\le i\le d} A_iG_i, \tag{5.3}$$

where $G_i = (-S)^{-1}D_i$, for $1 \le i \le d$, from which S is obtained by fixed point iteration. Then $(\mathfrak{X}, d)$ is a complete $CAT_p(0)$ metric space. We hereby introduce the subsequent theorem.

Theorem 5.1. Suppose the following hypotheses hold:

(i) The sequences $\{S_n : n \ge 0\}$ and $\{G_{i,n} : n \ge 0\}$, $1 \le i \le d$, defined by $S_n = C + \sum_{1\le i\le d} A_iG_{i,n}$, where $G_{i,n+1} = (-S_n)^{-1}D_i$ $\forall$ $1 \le i \le d, n \ge 0$ with $G_{i,0} = \cdots = G_{d,0} = 0$, monotonically converge to S and G_i, respectively. S is a non-singular matrix.
(ii) There exists a matrix G such that

$$\|\sum_{1\le i\le d} A_iG_{i,n} - \sum_{1\le i\le d} A_iG_{i,n+1}\| \le \|c - s\|^p,$$

where

$$\|c - s\|^p = \max\left\{d^p(\mathfrak{T}c, s), d^p(s, \mathfrak{T}s), d^p(c, s), \frac{d^p(\mathfrak{T}c, s) + d^p(c, \mathfrak{T}s)}{2}\right\} \le \epsilon \Rightarrow d^p(\mathfrak{T}c, \mathfrak{T}s) < \epsilon.$$

Then, the stochastic matrix equation (5.1) has a minimum unique at c.

Proof. It is known that the stability of the positive steady state of the stochastic matrix equation can be ascertained from conditions (i) and (ii). We now demonstrate that a minimal unique positive point can be reached by the stochastic matrix equation (5.1), and that point is c.

Define a mapping $\mathfrak{T} : C^1([0, T], \mathfrak{X}) \to C^1([0, T], \mathfrak{X})$ by $\mathfrak{T}S_n = C + \sum_{1\le i\le d} A_iG_{i,n}$. By the continuity property, for $0 \le t \le T$, we have $\mathfrak{T}c, \mathfrak{T}s \in \mathfrak{X}$ and $\mathfrak{T}c \ne \mathfrak{T}s$. Let $c, s \in C^1([0, T], \mathfrak{X})$ for $s \le c$ and $c = S_n$, $s = S_{n+1}$, and we get

$$\|\mathfrak{T}S_n - \mathfrak{T}S_{n+1}\|^p \;=\; \left\|C + \sum_{1\le i\le d} A_iG_{i,n} - \left(C + \sum_{1\le i\le d} A_iG_{i,n+1}\right)\right\|^p,$$

$$
\begin{aligned}
\|\mathfrak{T}S_n - \mathfrak{T}S_{n+1}\|^p &\leq \Big\| \sum_{1\leq i\leq d} A_i G_{i,n} - \sum_{1\leq i\leq d} A_i G_{i,n+1} \Big\|^p, \\
\|\mathfrak{T}S_n - \mathfrak{T}S_{n+1}\|^p &\leq \Big\| S_n - S_{n+1} \Big\|^p, \\
\|\mathfrak{T}c - \mathfrak{T}s\|^p &\leq \|c - s\|^p,
\end{aligned}
$$

which implies

$$
\begin{aligned}
d^p(\mathfrak{T}c, \mathfrak{T}s) &\leq d^p(c, s), \\
d^p(\mathfrak{T}c, \mathfrak{T}s) &\leq \max\Big\{d^p(\mathfrak{T}c, s), d^p(s, \mathfrak{T}s), d^p(c, s), \frac{d^p(\mathfrak{T}c, s) + d^p(c, \mathfrak{T}s)}{2}\Big\}, \\
&\leq \epsilon \Rightarrow d^p(\mathfrak{T}c, \mathfrak{T}s) < \epsilon,
\end{aligned}
$$

which is a contradiction. As a result, there is a minimal unique positive solution for the stochastic matrix Equation (5.1) at c. □

6. Uniqueness solution for two-scale fractal fractional hybrid differential equations in $CAT_p(0)$ metric spaces

In this section, we delve into the results associated with two-scale fractal fractional hybrid fractional differential equations (HFDEs) in $CAT_p(0)$ metric space, which leads to Theorem 3.1. One can see [38] and the references therein for several hybrid differential equations. In addition to the commonly used Caputo and Riemann-Liouville definitions, several alternative formulations of fractional derivatives have been introduced to address specific modeling needs. For instance, the He [39] fractional derivative, based on an invariant transformation method, is particularly useful in solving nonlinear problems analytically. Similarly, the Atangana-Baleanu [40] fractional derivative, defined with a non-singular and non-local kernel based on the Mittag-Leffler function, provides advantages in modeling processes with fading memory. These definitions, alongside the Caputo-Fabrizio and other generalized operators, offer diverse tools for fractional-order modeling in various disciplines, including physics, engineering, and biology.

Atangana and Baleanu [40] gave the following definition on fractional derivatives:

Definition 6.1. [40] The Atangana-Baleanu fractional differential equation is defined as follows with the left and right in Caputo sense:

$$
\begin{aligned}
{}^{ABC}_{\ a}D^{\sigma} f(t) &= \frac{M(\sigma)}{1-\sigma} \int_0^t f'(s) E_{\sigma}\Big(\frac{-\sigma}{1-\sigma}(t-s)^{\sigma}\Big) ds, t \in I = [0, T], \\
{}^{ABC}_{\ b}D^{\sigma} f(t) &= -\frac{M(\sigma)}{1-\sigma} \int_t^b f'(s) E_{\sigma}\Big(\frac{-\sigma}{1-\sigma}(t-s)^{\sigma}\Big) ds, \ t \in I = [0, T].
\end{aligned}
$$

Definition 6.2. [40] The Atangana-Baleanu fractional differential equation is defined as follows, with the left and right in Riemann-Lioville sense:

$$
{}^{ABR}_{\ a}D^{\sigma} f(t) = \frac{M(\sigma)}{1-\sigma} \frac{d}{dt} \int_0^t f'(s) E_{\sigma}\Big(\frac{-\sigma}{1-\sigma}(t-s)^{\sigma}\Big) ds,
$$

$$ {}^{ABR}_{b}D^{\sigma}f(t) = -\frac{M(\sigma)}{1-\sigma}\frac{d}{dt}\int_{t}^{b} f'(s)E_{\sigma}\Big(\frac{-\sigma}{1-\sigma}(t-s)^{\sigma}\Big)ds, $$

where $f \in H'(a,b)$, $a < b$, $\sigma \in [0,1]$, and $M(\sigma)$ is a normalization function with $M(0) = M(1) = 1$.

According to He [39,41], the fractional derivative is defined in the following form.

Definition 6.3. [39]

$$ D_t^{\sigma}f(t) = \frac{1}{\Gamma(n-\sigma)}\frac{d^n}{dt^n}\int_{t_0}^{t}(s-t)^{n-\sigma-1}\Big[\sum_{i=0}^{n-1}\frac{1}{i!}(s-t)^i f^i(t_0) - f(s)\Big]ds, $$

for the continuous and differentiable case.

$$ D_t^{\sigma}f(t) = \frac{1}{\Gamma(n-\sigma)}\frac{d^n}{dt^n}\int_{t_0}^{t}(s-t)^{n-\sigma-1}\Big[\sum_{i=0}^{n-1}\frac{(s-t_0)^i}{\Gamma(1+i\sigma)}f^{i\sigma}(t_0) - f(s)\Big]ds, $$

for continuous and not differentiable case.

Shafiullah and his team [42] have recently addressed this topic, employing fixed point analysis alongside numerical approaches to derive significant theoretical and computational insights regarding a class of fractal HFDEs that utilize a power law kernel. The following challenge discussed by Shafiullah et al. [42] is presented as follows:

$$ \begin{cases} {}^{FABC}_{0}D_{\vartheta}^{\sigma,\varepsilon}[\Psi(\vartheta) - K(\vartheta, c\theta)] = F(\vartheta, \Psi(\lambda\theta)),\ \vartheta \in I = (0,T), \\ \Psi(0) = \Psi_0 + H(\vartheta, \Psi(\vartheta)), \end{cases} \tag{6.1} $$

where ${}^{FABC}_{0}D_{\theta}^{\sigma,\varepsilon}$ denotes the fractal fractional Atangana-Baleanu-Caputo (ABC) derivative fractal order $\lambda \in (0,1)$, $K, F \in C[I \times \mathbb{R}, \mathbb{R}]$ is a given function, and $\psi \in C$ and λ are the proportional delay terms.

Differential equations featuring proportional delay terms constitute the pantograph equations. These equations are applicable in both pure and applied mathematics, encompassing fields such as control systems, probability, and electrodynamics. In the case where $\lambda = 1$, Equation 6.1 yields the standard HFDE fractals. However, when $\lambda > 1$, Equation 6.1 is classified as ill-posed. The fractal dimension is denoted by the symbol ε, and a value of $\varepsilon = 1$ indicates the consideration of the commonly employed fractional differential operator.

Some examples of fractional derivatives are as follows: The symbol for Riemann Liouville is RL. CP-is the derivative of Caputo. CFD-is the derivative of Caputo Fabrizio. The exponential kernel in the CFD was replaced with a Mittag-Leffler by Atagana and Baleanu [40], and the resulting definition was dubbed ABC. It has been applied to the Klein-Gordan equation using the HIV-1 model and the Atangana-Baleanu-Caputo (ABC) differential operator in [42,45].

The following definitions and lemma are taken from [39,40,43,46].

Definition 6.4. The following is the definition of the RL fractional integral of order $0 < \sigma \leq 1$:

$$ {}^{RL}I^{\sigma}\Psi(\vartheta) = \int_0^{\theta}(\vartheta - r)^{\sigma-1}\Psi(r)\frac{1}{\Gamma(\sigma)}dr,\ \vartheta \in I = [0,T]. \tag{6.2} $$

Definition 6.5. The following is the definition of an ABC-type arbitrary-order integral with order $\sigma > 0$:

$$^{ABC}I_0^{\sigma} c(\vartheta) = \frac{1-\sigma}{N(\sigma)}\Psi(\vartheta) + \frac{\sigma}{\Gamma(\sigma)N(\sigma)}\int_0^{\vartheta}(\vartheta - r)^{\sigma-1}\Psi(r)d\tau \tag{6.3}$$

such that the right side exists.

Additionally, the definition of Mittag-Leffler is as follows:

Definition 6.6. The definition of the arbitrary-order derivative of ABC-type with order $0 < \sigma \le 1$ is as follows:

$$^{ABC}D_0^{\sigma} v(\vartheta) = \frac{N(\sigma)}{1-\sigma}\Psi(\vartheta)\int_0^{\vartheta} E_{\sigma}[-\frac{\sigma}{1-\sigma}(\vartheta - r)^{\sigma-1}]\Psi'(r)d\tau. \tag{6.4}$$

In the definition, E_{σ} is the Mittag-Leffler function, and the function $N(\sigma)$, which is called normalization, obeys $N(0) = N(1) = 1$.

Moreover, the definitions of the fractal fractional ABC integral and derivatives are as follows:

Definition 6.7. The following is the definition of the fractal fractional ABC integral with fractional order $\sigma \in (0, 1)$ and fractals $\varepsilon \in (0, 1)$:

$$^{FABC}I_{0^+}^{\sigma,\varepsilon} v(\vartheta) = \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}\Psi(\vartheta) + \frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^{\vartheta}(\vartheta - r)^{\sigma-1}r^{\varepsilon-1}\Psi(r)dr. \tag{6.5}$$

Definition 6.8. The following is the definition of the fractal fractional ABC derivative with fractional order $\sigma \in (0, 1)$ and fractals $\varepsilon \in (0, 1)$:

$$^{FABC}D_0^{\sigma,\varepsilon}\Psi(\vartheta) = \frac{N(\sigma)}{1-\sigma}\Psi(\vartheta)\int_0^{\vartheta} E_{\sigma}[-\frac{\sigma}{1-\sigma}(\vartheta - r)^{\sigma-1}]\frac{du(r)}{dr^{\varepsilon}}d(r). \tag{6.6}$$

Lemma 6.1. *If $c \in L[0, T]$ and $c(0) = 0$, the solution of*

$$^{FABC}D_0^{\sigma,\varepsilon}\Psi(\vartheta) = c(\vartheta), \textit{with } \sigma \in (0, 1],$$
$$\Psi(0) = \Psi_0,$$

is described as follows:

$$\Psi(\vartheta) = \Psi_0 + \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}c(\vartheta) + \frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^{\vartheta}(\vartheta - r)^{\sigma-1}r^{\varepsilon-1}c(r)dr. \tag{6.7}$$

The continuous function defined on I has a space $I = \mathfrak{X}$. Assume that all continuous functions from I into $\mathfrak{X}$ have a norm of $\|c\| := \sup|c(t)|$, $\vartheta \in I$ for $c \in C(I, \mathfrak{X})$. Let $(\mathfrak{X}, \|\|)$ be a Banach space.

This space defines the path metric by

$$d^p(c, s) = \sup_{\vartheta\in I}\{|c(\vartheta) - s(\vartheta)|^p\} \tag{6.8}$$

where $\forall c, s \in \mathfrak{X}$. This is a complete $CAT_p(0)$ metric space.

It is possible to define problem (6.1) as a fixed point problem by using Lemma 6.1.

$$\begin{aligned}\Psi(\vartheta) &= \Psi_0 + H(\vartheta,\Psi(\vartheta)) + K(\vartheta,\Psi(\vartheta)) + \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}c(\vartheta) + \\ &\quad \frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^{\vartheta}(\vartheta-r)^{\sigma-1}r^{\varepsilon-1}c(r)dr, \qquad (6.9)\end{aligned}$$

where

$$c(\vartheta) = F(\vartheta,\Psi(\lambda\vartheta))$$

and

$$c(r) = F(r,\Psi(\lambda r)).$$

Referencing [44], the following image in Figure 2 depicts the fractal fractional behavior of Eq (6.1), which is identical to Eq (6.9).

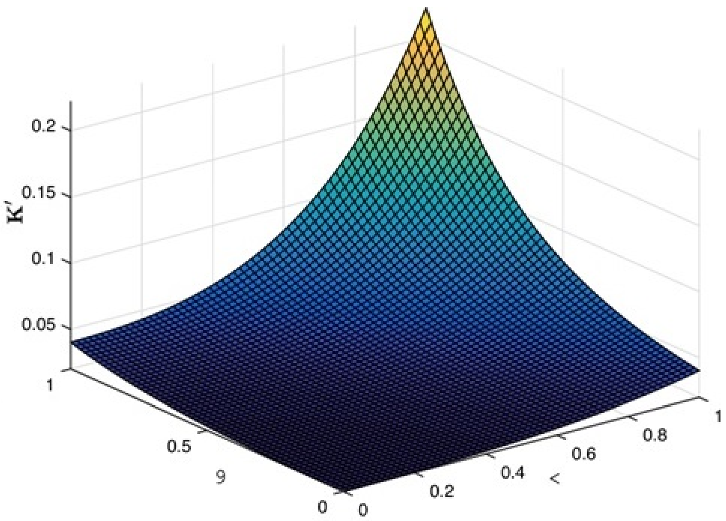

Figure 2. FABC- $CAT_p(0)$ metric spaces.

From the above information, we prove the following theorem.

Theorem 6.1. For this investigation, the following presumptions are required.

(i) There exists a constant $\zeta > 0$, such that

$$|H(\vartheta,\Psi(\vartheta)) - H(\vartheta,\bar{\Psi}(\vartheta))| \leq \zeta|\Psi(\vartheta)-\bar{\Psi}(\vartheta)| = \zeta|c(\vartheta)-s(\vartheta)|,$$

(ii) a constant $\eta > 0$, such that

$$|K(\vartheta,\Psi(\vartheta)) - K(\vartheta,\bar{\Psi}(\vartheta))| \leq \eta|\Psi(\vartheta)-\bar{\Psi}(\vartheta)| = \eta|c(\vartheta)-s(\vartheta)|,$$

(iii) a constant $\gamma > 0$ given by

$$|F(\vartheta,\Psi(\vartheta)) - F(\lambda\vartheta,\bar{\Psi}(\lambda\vartheta))| \leq \gamma|\Psi(\vartheta)-\bar{\Psi}(\vartheta)| = \gamma|c(\vartheta)-s(\vartheta)|,$$

where $\gamma = \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}$,

(iv) a constant $\rho > 0$ is given by

$$|F(\vartheta, \Psi(\lambda r)) - F(\vartheta, \bar{\Psi}(\lambda r))| \leq \rho|\Psi(\vartheta) - \bar{\Psi}(\vartheta)| = \rho|c(\vartheta) - s(\vartheta)|,$$

imply that $\rho = \frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\vartheta^{\sigma+\varepsilon-1}B(\sigma, \varepsilon)$, and

(iv) $k' = \zeta + \eta + \gamma + \rho < \epsilon < 1$, where

$$k' < \max\left\{d^p(\mathfrak{T}c, s), d^p(s, \mathfrak{T}s), d^p(c, s), \frac{d^p(\mathfrak{T}c, s) + d^p(c, \mathfrak{T}s)}{2}\right\} \leq \epsilon \Rightarrow d^p(\mathfrak{T}c, \mathfrak{T}s) < \epsilon.$$

Then Eq (6.9) has a unique solution which is also a solution to problem (6.1).

Proof. We define an operator $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ by

$$\begin{aligned}\mathfrak{T}\Psi(\vartheta) &= \Psi_0 + H(\vartheta, \Psi(\vartheta)) + K(\vartheta, \Psi(\vartheta)) + \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}c(\vartheta) + \\ &\frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^\vartheta (\vartheta - r)^{\sigma-1}r^{\varepsilon-1}c(r)dr.\end{aligned}$$

The following uses the aforementioned conditions $(i)-(v)$ to demonstrate the existence of a fixed point:

$$\begin{aligned}\|\mathfrak{T}\Psi(\vartheta) - \mathfrak{T}\bar{\Psi}(\vartheta)\|^p &= \Bigg\|\Bigg(\Psi_0 + H(\vartheta, \Psi(\vartheta)) + K(\vartheta, \Psi(\vartheta)) + \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}c(\vartheta) \\ &+\frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^\vartheta (\vartheta - r)^{\sigma-1}r^{\varepsilon-1}c(r)dr\Bigg) - \\ &\Bigg(\Psi_0 + H(\vartheta, \bar{\Psi}(\vartheta)) + K(\vartheta, \bar{\Psi}(\vartheta)) + \frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}s(\vartheta) \\ &+\frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^\vartheta (\vartheta - r)^{\sigma-1}r^{\varepsilon-1}s(r)dr\Bigg)\Bigg\|^p, \\ &\leq \|H(\vartheta, \Psi(\vartheta)) - H(\vartheta, \bar{\Psi}(\vartheta))\|^p + \\ &\|K(\vartheta, \Psi(\vartheta)) - K(\vartheta, \bar{\Psi}(\vartheta))\|^p + \\ &\frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}\|c(\vartheta) - s(\vartheta)\|^p + \\ &\frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\int_0^\vartheta (\vartheta - r)^{\sigma-1}r^{\varepsilon-1}\|c(r) - s(r)\|^p dr,\end{aligned}$$

$$\begin{aligned}&\leq \zeta\|\Psi(\vartheta) - \bar{\Psi}(\vartheta)\|^p + \eta\|\Psi(\vartheta) - \bar{\Psi}(\vartheta)\|^p + \\ &\frac{\varepsilon(1-\sigma)\vartheta^{\varepsilon-1}}{N(\sigma)}\|c(\vartheta) - s(\vartheta)\|^p + \\ &\frac{\sigma\varepsilon}{\Gamma(\sigma)N(\sigma)}\vartheta^{\sigma+\varepsilon-1}B(\sigma, \varepsilon)\|c(r) - s(r)\|^p, \\ &\leq \zeta\|c(\vartheta) - s(\vartheta)\|^p + \eta\|c(\vartheta) - s(\vartheta)\|^p + \\ &\gamma\|c(\vartheta) - s(\vartheta)\|^p + \rho\|c(r) - s(r)\|^p,\end{aligned}$$

$$\begin{aligned}
\|\mathfrak{T}\Psi(\vartheta) - \mathfrak{T}\bar{\Psi}(\vartheta)\|^p &\leq (\zeta + \eta + \gamma + \rho)\|c(\vartheta) - s(\vartheta)\|^p, \\
d^p(\mathfrak{T}c(\vartheta), \mathfrak{T}s(\vartheta)) &\leq (\zeta + \eta + \gamma + \rho)d^p(c(\vartheta), s(\vartheta)), \\
d^p(\mathfrak{T}c(\vartheta), \mathfrak{T}s(\vartheta)) &\leq k' d^p(c(\vartheta), s(\vartheta)).
\end{aligned}$$

This suggests that

$$k' d^p(c(\vartheta), s(\vartheta)) < \max\left\{d^p(\mathfrak{T}c, s), d^p(s, \mathfrak{T}s), d^p(c, s), \frac{d^p(\mathfrak{T}c, s) + d^p(c, \mathfrak{T}s)}{2}\right\} \leq \epsilon \Rightarrow d^p(\mathfrak{T}c, \mathfrak{T}s) < \epsilon.$$

Consequently, the fractal fractional hybrid differential Eq (6.1) has a unique solution, which is also the solution to Eq (6.9). Thus, our evidence is finished. □

7. Discussion

This model captures the fractal time evolution of MHD flow velocity, reflecting the memory and multiscale effects inherent in porous or heterogeneous conducting media. Such models have relevance in:

- Geophysical flows (e.g., lava or saline water),
- Plasma confinement and
- Blood flow in biological tissues with magnetic therapy.

The two-scale derivative introduces control over scale effects ($\alpha(s)$) and temporal memory ($\beta(s)$), while VIM ensures analytical tractability and rapid convergence.

The current study demonstrates that the fractal derivative model can be used to describe the anomalous diffusion process as captured by diffusion-weighted MRI in a fixed mouse brain. In the model, the fractal dimension of the diffusion trajectory is directly expressed in terms of the Hausdorff fractal dimension, and the spectral entropy is used to measure the uncertainty in the heterogeneous and multi-scale system of biological tissues. Interestingly, the fractal derivative order influences the diffusion behavior significantly, especially in systems with anomalous transport properties.

8. Conclusions

In this manuscript, we establish and validate fundamental fixed point theorems for CMJ-type mappings, highlighting their broad applicability and improvements over existing results in the literature. Our investigation addresses the existence of solutions to a nonlinear matrix equation within the framework of a $CAT_p(0)$ metric space, thereby reinforcing the claims presented in Theorem 3.1.

Additionally, we examine the behavior of an iterative matrix process governed by a recursive scheme, which guarantees convergence. This approach is particularly relevant in the context of solving stochastic matrix equations that frequently arise in image reconstruction problems, such as those encountered in tomography and signal processing—most notably in X-ray computed tomography (CT). In these applications, the goal is to reconstruct internal images, such as anatomical cross-sections, from projection data—often requiring the resolution of large systems of equations affected by noise and uncertainty.

Our study further extends to hybrid fractional differential equations (HFDEs) within the $CAT_p(0)$ setting, leading to the formulation and conclusion of Theorem 3.1. We demonstrate that the fractal derivative framework in $CAT_p(0)$ spaces effectively models anomalous diffusion phenomena, as observed in diffusion-weighted MRI scans of fixed mouse brains. In this model, the fractal nature of the diffusion pathway is directly quantified via the Hausdorff fractal dimension, while spectral entropy serves as a metric for the uncertainty inherent in the complex, multi-scale biological tissue environment. Notably, the order of the fractal derivative is intricately linked to the underlying diffusion characteristics.

A promising direction for future work involves establishing fixed point theorems for extended (p, q)-F-interpolative mappings within $CAT(0)$ spaces. To support the theoretical development, one may construct concrete examples that validate the proposed results. These fixed point formulations can further be applied to models in economic dynamics such as economic growth, market equilibrium, and bifurcation scenarios in energy capital accumulation by incorporating fractional differential equations to demonstrate their practical utility. Additionally, the structure of $CAT(0)$ spaces can be integrated with generalized two-scale fractional derivatives within the framework of the variational iteration method (VIM), particularly for modeling heat transfer in fractal porous media and magnetohydrodynamic (MHD) flow in fractal environments.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no conflicts of interest in this paper.

References

1. A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, **28** (1969), 326–329. https://doi.org/10.1016/0022-247X(69)90031-6
2. L. B. Ćirić, A new fixed-point theorem for contractive mapping, *Publ. I. Math.*, **30** (1981), 25–27.

3. J. Matkowski, Fixed point theorems for contractive mappings in metric spaces, *Časopis Pro Pěstování Matematiky*, **105** (1980), 341–344. https://doi.org/10.21136/CPM.1980.108246

4. D. W. Boyd, J. S. W. Wong, On nonlinear contraction, *Proc. Amer. Math. Soc.*, **20** (1969), 458–464. https://doi.org/10.2307/2035677

5. S. Reich, Fixed points of contractive functions, *Boll. UMI*, **5** (1972), 26–42.

6. C. S. Wong, Generalized contraction and fixed point theorems, *Proc. Amer. Math. Soc.*, **42** (1974), 409–417. https://doi.org/10.1090/S0002-9939-1974-0331358-4

7. j. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, *J. Math. Anal. Appl.*, **194** (1995), 293–303. https://doi.org/10.1006/jmaa.1995.1299

8. M. Gromov, *Metric structure for Riemannian and Non-Riemannian spaces*, Boston: Birkhauser, 1984. https://doi.org/10.1007/978-0-8176-4583-0

9. W. A. Kirk, Fixed point theorems in CAT(0) spaces and R-trees, *Fixed Point Theory Appl.*, **2004** (2004), 738084. https://doi.org/10.1155/S1687182004406081

10. K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and non-expansive mappings*, 1984.

11. S. Reich, I. Shafrir, Non-expansive iterations in hyperbolic spaces, *Nonlinear Anal. Theor.*, **15** (1990), 537–558. https://doi.org/10.1016/0362-546X(90)90058-O

12. S. Dhompongsa, B. Panyanak, On △-convergence theorems in CAT (0) spaces, *Comput. Math. Appl.*, **56** (2008), 2572–2579. https://doi.org/10.1016/j.camwa.2008.05.036

13. M. A. Khamsi, S. Shukri, Generalized CAT(0) spaces, *Bull. Belg. Math. Soc. Simon Stevin*, **24** (2017), 417–426. https://doi.org/10.36045/bbms/1506477690

14. S. Shukri, On monotone nonexpansive mappings in $CAT_p(0)$ spaces, *Fixed Point Theory Appl.*, **8** (2020). https://doi.org/10.1186/s13663-020-00675-z

15. A. A. Darweesh, S. Shukri, Fixed points of Suzuki-generalized nonexpansive mappings in $CAT_p(0)$ metric spaces, *Arab. J. Math.*, **13** (2024), 227–236. https://doi.org/10.1007/s40065-024-00455-2

16. Y. Sun, R. P. Agarwal, Existence of fixed point for nonlinear operator in partially ordered metric space, *Adv. Differ. Equ. Contr.*, **30** (2023), 97–116. https://doi.org/10.17654/0974324323007

17. L. Xie, J. Zhou, H. Deng, Y. He, Existence and stability of solution for multi-order nonlinear fractional differential equations, *AIMS Math.*, **7** (2022), 16440–16448. https://doi.org/10.3934/math.2022899

18. J. L. Zhou, Y. B. He, S. Q. Zhang, H. Y. Deng, X. Y. Lin, Existence and stability results for nonlinear fractional integrodifferential coupled systems, *Bound. Value Probl.*, **2023** (2023), 10. https://doi.org/10.1186/s13661-023-01698-2

19. M. A. Imran, Application of fractal fractional derivative of power law kernel ${}_0^{FFP}D_x^{\alpha,\beta}$ to MHD viscous fluid flow between two plates, *Chaos Soliton. Fract.*, **134** (2020), 10969. https://doi.org/10.1016/j.chaos.2020.109691

20. C. H. He, H. W. Liu, C. Liu, A fractal-based approach to the mechanical properties of recycled aggregate concretes, *Fact. Univ. Ser. Mech.*, **22** (2024), 329–342. https://doi.org/10.22190/FUME240605035H

21. D. Bini, B. Meini, On the solution of a nonlinear matrix equation arising in queueing problems, *SIAM J. Matrix Anal. A.*, **17** (1996), 906–926. https://doi.org/10.1137/S0895479895284804
22. Y. Lim, Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} M_i X^{\delta_i} M_i^*$ via a contraction principle, *Linear Algebra Appl.*, **430** (2009), 1380–1383. https://doi.org/10.1016/j.laa.2008.10.034
23. W. A. Kirk, A fixed point theorem for mappings which do not increases distances, *Amer. Math. Mon.*, **72** (1965), 1004–1006. https://doi.org/10.2307/2313345
24. B. Nanjaras, B. Panyanaka, W. Phuengrattanab, Fixed point theorems and convergence theorems for Suzuki-generalized non-expansive mappings in C AT (0) spaces, *Nonlinear Anal. Hybri.*, **4** (2010), 25–31. https://doi.org/10.1016/j.nahs.2009.07.003
25. M. R. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Berlin: Springer, 1999. https://doi.org/10.1007/978-3-662-12494-9
26. F. Bruhat, J. Tits, Groupes réductifs sur un corps local: I. Données radicielles valuées, In: *Publications Mathématiques de l'IHÉS*, **41** (1972), 5–251.
27. R. P. Agarwal, J. Mohamed, B. Samet, *Fixed point theory in metric spaces*, Singapore: Springer, 2018. https://doi.org/10.1007/978-981-13-2913-5
28. K. Goebel, W. A. Kirk, Iteration processes for non-expansive mappings, *Contemp. Math.*, **21** (1983), 115–123.
29. A. Papadopoulos, *Metric spaces, convexity and nonpositive curvature*, 2004.
30. U. Kohlenbach, L. Leustean, Mann iterates of directionally non-expansive mappings in hyperbolic spaces, *Abstr. Appl. Anal.*, **8** (2003), 449–477. https://doi.org/10.1155/S1085337503212021
31. Z. Opial, Weak convergence of the sequence of successive approximations for non-expansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597.
32. F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Bull. Amer. Math. Soc.*, **73** (1967), 867–874. https://doi.org/10.1090/S0002-9904-1967-11820-2
33. A. Naor, L. Silberman Poincare inequalities, embeddings, and wild groups, *Compos. Math.*, **147** (2011), 1546–1572. https://doi.org/10.1112/S0010437X11005343
34. T. Suzuki, Fixed point theorems and convergence theorems for some generalized non-expansive mapping, *J. Math. Anal. Appl.*, **340** (2008), 1088–1095.
35. T. Suzuki, Generalized metric spaces do not have the compatible topology, *Abstr. Appl. Anal.*, **2014** (2014), 458098. https://doi.org/10.1155/2014/458098
36. S. Banach, On operations in abstract sets and their application to integral equations, *Fund. Math.*, **3** (1922), 133–181.
37. D. A. Bini, G. Latouche, B. Meini, Solving nonlinear matrix equations arising in tree-like stochastic processes, *Linear Algebra Appl.*, **366** (2003), 39–64. https://doi.org/10.1016/S0024-3795(02)00593-1
38. M. L. Maheswari, K. S. Keerthana Shri, M. Sajid, Analysis on existence of system of coupled multi fractional nonlinear hybrid differential equations with coupled boundary conditions, *AIMS Math.*, **9** (2024), 13642–13658. https://doi.org/10.3934/math.2024666

39. J. H. He, Fractal calculus and its geometrical explanation, *Res. Phys.*, **10** (2018), 272–276. https://doi.org/10.1016/j.rinp.2018.06.011

40. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, 2016. https://doi.org/10.48550/arXiv.1602.03408

41. J. H. He, A tutorial review on fractal space time and fractional calculus, *Int. J. Theor. Phys.*, **53** (2014), 3698–3718. https://doi.org/10.1007/s10773-014-2123-8

42. S. K. Shafiullah, M. Sarwar, T. Abdeljawad, On theoretical and numerical analysis of fractal-fractional non-linear hybrid differential equations, *Nonlinear Eng.*, **13** (2024), 20220372. https://doi.org/10.1515/nleng-2022-0372

43. S. Qureshi, A. Atangana, Fractal-fractional diferentiation for the modeling and mathematical analysis of nonlinear diarrhea transmission dynamics under the use of real data, *Chaos Soliton. Fract.*, **136** (2020), 109812. https://doi.org/10.1016/j.chaos.2020.109812

44. T. Abdeljawad, M. Sher, K. Shah, M. Sarwar, I. Amacha, M. Alqudah, et al., Analysis of a class of fractal hybrid fractional differential equation with application to a biological model, *Sci. Rep.*, **14** (2024), 18937. https://doi.org/10.1038/s41598-024-67158-8

45. C. Xu, Z. Liu, Y. Pang, S. Saifullah, M. Inc, Oscillatory, crossover behavior and chaos analysis of HIV-1 infection model using piece-wise Atangana-Baleanu fractional operator: Real data approach, *Chaos Soliton. Fract.*, **164** (2022), 112662. https://doi.org/10.1016/j.chaos.2022.112662

46. G. S. Teodoro, J. T. Machado, E. C. De Oliveira, A review of defnitions of fractional derivatives and other operators, *J. Comput. Phys.*, **388** (2019), 195–208. https://doi.org/10.1016/j.jcp.2019.03.008