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Research article

Linear-quadratic-Gaussian mean-field games driven by Poisson jumps with major and minor agents

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Abstract: This paper studies mean-field linear-quadratic-Gaussian (LQG) games with a major agent and a large number of minor agents, where each agent's state process is driven by a Poisson random measure and independent Brownian motion. The major and minor agents were coupled via both their state dynamics as well as in their individual cost functionals. By the Nash certainty equivalence (NCE) methodology, two limiting control problems were constructed and the decentralized strategies were derived through the consistency condition. The ϵ-Nash equilibrium property of the obtained decentralized strategies was shown for a finite N population system where $\epsilon = O(1/\sqrt{N})$. A numerical example was presented to illustrate the consistency of the mean-field estimation and the impact of the population's collective behavior.

Keywords: mean-field games; major and minor agents; decentralized control; jump-diffusion; ϵ-Nash equilibrium
Mathematics Subject Classification: 91A16

1. Introduction

Mean-field games of a large population system have attracted consistent and intense attention in recent years (see, e.g., [1–10]) due to their wide applicability in many fields such as finance, economics, engineering, biological science, and social science. The agents in mean-field games are individually insignificant, while their aggregated behavior has a substantial effect on each agent. This collective influence can be captured by the mean-field couplings in their individual dynamics and/or individual cost functionals. For mean-field games, it is unrealistic for a given agent to collect detailed state information of all agents due to the highly complex interactions among its peers. To tackle the dimensionality difficulty caused by the highly complex interactions among the agents in mean-field games, Huang, Caines, and Malhamé [11], Huang [12], and Nourian and Caines [13] developed a powerful approach—the Nash certainty equivalence (NCE) methodology. The key idea of this methodology is to establish a consistency relationship between the individual strategies and the mass effect (i.e., the asymptotic limit of state-average) as the population size goes to infinity. Based on this effective analytical tool, one can construct a set of decentralized strategies for each agent in the mean-field game, and verify the asymptotic Nash equilibrium property (namely, ϵ-Nash equilibrium) of the decentralized strategies where the individual optimality loss level ϵ depends on the population size N. A closely related method for solving mean-field games was independently developed by Lasry and Lions [14–16]. For a comprehensive survey of the theory of the mean-field game and its applications, one is referred to [11, 12, 14, 16–21] and the references therein.

The consideration of major and minor a agent game problems under a large population framework has been well studied in [3, 12, 13, 21, 22]. Huang [12] investigated a kind of stochastic dynamic linear-quadratic-Gaussian mean-field games model involving a major agent interacting with a large number of minor agents. The major agent has a significant influence in affecting minor agents, while the minor agents individually have negligible impact on others, but their collective behavior will impose a significant impact on all agents through mean-field coupling terms in the individual dynamics and costs. Applications of this type of mean-field game appear in many socio-economic problems such as economic and social opinion models with an influential leader (e.g., [23]), such as the charging control of plug-in electric vehicles [24]. Xu and Wu [21] studied large-population dynamic games involving a LQG system with an exponential cost functional, and the parameter in the cost functional can describe an investor's risk attitude. Moreover, in the game, there is a major agent and a population of N minor agents where N is very large. Wang and Xu [22] investigated a time-inconsistent linear-quadratic game involving a major agent as well as numerous minor agents.

Motivated by the absence of relevant theory and some practical applications, this paper studies mean-field LQG games with random jumps involving a major agent and plenty of minor agents. Specifically, we consider mean-field games with agents of the following mixed types: (i) a major agent and (ii) a large population of N minor agents where N is very large. The dynamic of each agent follows a linear stochastic differential equation driven by both Brownian motions and Poisson random measures. Moreover, the present study considers the mean-field LQG mixed games in which the diffusion term depends on the major agent's and the minor agent's states as well as the individual control strategy. Stochastic processes with random jumps can be used to model fluctuations in the financial market, both for option pricing purposes and risk management (see [20, 25–27]). As for mean-field LQG games with random jumps, Benazzoli, Campi, and Di Persio [1] studied a symmetric n-player nonzero-sum stochastic differential game with jump-diffusion dynamics and mean-field type interaction among the players, and they constructed an approximate Nash equilibrium for the n-player game with n sufficiently large. Xu and Shi [20] investigated LQG games of a stochastic large population system with jump diffusion processes. It is worth noting that in existing research on mean-field games of a stochastic large population system driven by jump-diffusion processes, all agents are comparably small and may be regarded as peers.

To obtain an asymptotic Nash equilibrium property (i.e., ϵ-Nash equilibrium) for the original mean-field game, we apply the NCE approach to establish a certain consistency relationship between all minor agents and the mass effect. First, we construct two auxiliary stochastic control problems driven by stochastic differential equations driven by Poisson jumps (SDEPs) which depict the state of the major agent and a generic minor agent, and obtain the corresponding optimal control in feedback form. Next, to devise the decentralized strategies of individual agents, we formulate a kind of fully coupled forward-backward stochastic differential equation driven by Poisson jumps which is called a consistency condition (CC) system. Then, a set of decentralized strategies are constructed by using the solution of the CC system, which are demonstrated to be the ϵ-Nash equilibrium.

The main contributions of this paper can be summarized as follows:

- A new class of LQG mean-field games involving major and minor agents is investigated. The dynamics of each agent follows a linear stochastic differential equation driven by both Brownian motions and Poisson random measures, in which the diffusion terms of the major and minor agents depend on their states and control strategy.
- The average state of all minor agents $x^{(N)}(\cdot)$ appears in the drift term and diffusion term of the state equations for both the major agent and all the minor agents, as well as in their cost functionals.
- The consistency condition system called the NCE equation is represented through a fully coupled two-point boundary value problem, and based on this equation, we design a set of decentralized feedback control strategies for the $N+1$ agents by use of two limiting control systems.
- By the approximation relationship between the closed-loop mean-field game system and the limiting systems, the set of NCE-based decentralized control strategies is shown to be an ϵ-Nash equilibrium for a finite $N+1$ population system where $\epsilon = O(1/\sqrt{N})$.

This paper is organized as follows. In Section 2, we formulate the LQG mean-field games driven by Poisson random jumps involving a major agent and many minor agents. Section 3 introduces two auxiliary optimization problems for the major agent and each minor agent, respectively, and the consistency condition system is derived. Section 4 aims to present the ϵ-Nash equilibrium property of the decentralized control strategies. A numerical example is given in Section 5. Finally, Section 6 concludes the paper.

2. Formulation of the problem

2.1. Notations

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. For a given Euclidean space, we denote by $|\cdot|$ (respectively, $\langle\cdot,\cdot\rangle$) the standard Euclidean norm (respectively, inner product). The transpose of a matrix (or vector) $\mathbb{X}$ is denoted by $\mathbb{X}^T$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability measure space for fixed time $T > 0$, and let the number N represent the population size of minor agents. Denote by $\mathcal{N}$ the index set $\{1, 2, \cdots, N\}$. Let $\mathcal{F}_t$ be the filtration generated by the following mutually independent processes:

(i) $(N+1)$ independent one-dimensional standard Brownian motions $\{W_i(t),\ \ i = 0, 1, \cdots, N\}_{0\le t\le T}$;

(ii) $(N+1)$ independent Poisson random measures $\{\widetilde{G}_i, i = 0, 1, \cdots, N\}$ on $E_i \times \mathbb{R}^+$, where $E_i \subset \mathbb{R}$ is a nonempty open set equipped with its Borel field $\mathcal{B}(E_i)$, with compensator $\widehat{G}_i(dedt) = \pi_i(de)dt$, such that $G_i(S\times[0,t]) = (\widetilde{G}_i - \widehat{G}_i)(S\times[0,t])_{t\ge 0}$ is a martingale for all $S \in \mathcal{B}(E_i)$. π_i is a σ-finite measure on $(E_i, \mathcal{B}(E_i))$ and is called the characteristic measure. Moreover, $\forall S \in \mathcal{B}(E_i)$, $\mathbb{C}_0 := \sup\limits_{0\le i\le N} \pi_i(S) < +\infty$ is a positive constant independent of the number N.

We also set

$$\mathcal{F}_t^0 := \sigma\{W_0(s), 0 \le s \le t\} \bigvee \sigma\{G_0(S_0 \times [0, s]), 0 \le s \le t, \forall S_0 \in \mathcal{B}(E_0)\},$$

$$\mathcal{F}_t^i := \sigma\{W_i(s), 0 \le s \le t\} \bigvee \sigma\{G_i(S_i \times [0,s]), 0 \le s \le t, \forall S_i \in \mathcal{B}(E_i)\},$$
$$\mathcal{F}_t^{0,i} := \sigma\{W_0, W_i(s), 0 \le s \le t\} \bigvee \sigma\{G_0(S_0 \times [0,s]), G_i(S_i \times [0,s]), 0 \le s \le t, \forall S_0 \in \mathcal{B}(E_0), S_i \in \mathcal{B}(E_i)\},$$

where $\bigvee_\alpha \mathcal{F}_\alpha := \sigma\left(\bigcup_\alpha \mathcal{F}_\alpha\right)$. Here, $\{\mathcal{F}_t^0\}_{0\le t\le T}$ represents the information of the major agent, whereas for the given $i \in \mathcal{N}$, $\{\mathcal{F}_t^i\}_{0\le t\le T}$ stands the individual information of the ith minor agent.

Denote by $\mathcal{S}^n$ the set of symmetric $n \times n$ matrices with real elements. If $M \in \mathcal{S}^n$ is positive (semi) definite, we write $M > (\ge)\, 0$. We also introduce the following spaces:

$$L^2_{\mathcal{G}}(\mathbb{R}^n) := \Big\{\zeta : \Omega \to \mathbb{R}^n | \zeta \text{ is } \mathcal{G}\text{-measurable and } \mathbb{E}\Big[|\zeta|^2\Big] < +\infty\Big\};$$
$$S^2_{\mathcal{G}}([0,T];\mathbb{R}^n) := \Big\{\phi(\cdot) : [0,T]\times\Omega \to \mathbb{R}^n | \phi(\cdot) \text{ is } \mathcal{G}_t\text{-adapted and } \mathbb{E}\Big[\sup_{0\le t\le T} |\phi(t)|^2\Big] < +\infty\Big\};$$
$$L^2_{\mathcal{G}}([0,T];\mathbb{R}^n) := \Big\{\phi(\cdot) : [0,T]\times\Omega \to \mathbb{R}^n | \phi(\cdot) \text{ is a } \mathcal{G}_t\text{-progressively measurable process and } \mathbb{E}\Big[\int_0^T |\phi(t)|^2 dt\Big] < +\infty\Big\}.$$

2.2. *Major-minor mean-field game problems*

Let us consider an LQG mean-field game involving a major agent $\mathcal{A}_0$ and a population of N minor agents $\{\mathcal{A}_i, i = 1, 2, \cdots, N\}$. For the major agent $\mathcal{A}_0$, $\mathcal{U}_{ad}^{c,0} := \{u(\cdot) | u(\cdot) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^k)\}$ denotes the centralized admissible control set, and $\mathcal{U}_{ad}^0 := \{u(\cdot) | u(\cdot) \in L^2_{\mathcal{F}^0}([0,T];\mathbb{R}^k)\}$ represents the corresponding decentralized admissible control set. For each $i \in \mathcal{N}$, we define the centralized admissible control set for the minor agent $\mathcal{A}_i$ as $\mathcal{U}_{ad}^{c,i} := \{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^k)\}$, while the corresponding decentralized admissible control set is $\mathcal{U}_{ad}^i := \{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathcal{F}^{0,i}}([0,T];\mathbb{R}^k)\}$. Note that we have $\mathcal{U}_{ad}^i \subset \mathcal{U}_{ad}^{c,i}$ for $i = 0, 1, \cdots, N$.

The dynamics of the major agent $\mathcal{A}_0$ is given as follows:

$$\begin{cases} dx_0(t) = [A_0 x_0(t) + B_0 u_0(t) + b_0 x^{(N)}(t) + f_0(t)]dt + [C_0 x_0(t) + D_0 u_0(t) \\ \qquad + l_0 x^{(N)}(t) + \sigma_0(t)]dW_0(t) + F_0 \int_{E_0} G_0(dedt), \\ x_0(0) = a_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

and the state of the minor agent $\mathcal{A}_i$ is described by

$$\begin{cases} dx_i(t) = [A x_i(t) + B u_i(t) + b_1 x^{(N)}(t) + f(t)]dt + [C x_i(t) + D u_i(t) \\ \qquad + b_2 x^{(N)}(t) + H x_0(t) + \sigma(t)]dW_i(t) + F \int_{E_i} G_i(dedt), \\ x_i(0) = a_i \in \mathbb{R}^n, \quad i = 1, \cdots, N, \end{cases} \tag{2}$$

where $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$ represents the average state of all minor agents. Here, $A_0 \in \mathbb{R}^{n\times n}, B_0 \in \mathbb{R}^{n\times k}, C_0 \in \mathbb{R}^{n\times n}, D_0 \in \mathbb{R}^{n\times k}, b_0 \in \mathbb{R}^{n\times n}, l_0 \in \mathbb{R}^{n\times n}, F_0 \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times k}, C \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{n\times k}, b_1 \in \mathbb{R}^{n\times n}, b_2 \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{n\times n}$, and $F \in \mathbb{R}^n$ are given constants, and $f_0(\cdot) \in \mathbb{R}^n, \sigma_0(\cdot) \in \mathbb{R}^n, f(\cdot) \in \mathbb{R}^n$, and $\sigma(\cdot) \in \mathbb{R}^n$ are given deterministic functions. For given admissible control u_0 and u_i, it follows that the systems (1) and (2) admit a unique solution $x_0(\cdot), x_i(\cdot) \in S^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$.

Let $u = (u_0, u_1, \ldots, u_i, \ldots, u_N)$ be the set of control strategies for all $N+1$ agents, and $u_{-i} = (u_0, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$ for $i = 0, 1 \cdots, N$. The cost functional for the major agent $\mathcal{A}_0$ is

$$J_0(u_0, u_{-0}) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q_0(x_0(t) - \beta_0 x^{(N)}(t)), (x_0(t) - \beta_0 x^{(N)}(t))\rangle + \langle R_0 u_0(t), u_0(t)\rangle\Big]dt$$

$$+ \langle M_0 x_0(T), x_0(T)\rangle\Big\}. \tag{3}$$

The cost functional for minor agent $\mathcal{A}_i$, $1 \le i \le N$, is

$$\begin{aligned} J_i(u_i, u_{-i}) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T & \Big[\langle Q(x_i(t) - \beta_1 x^{(N)}(t) - \beta_2 x_0(t)), (x_i(t) - \beta_1 x^{(N)}(t) - \beta_2 x_0(t))\rangle \\ & + \langle R u_i(t), u_i(t)\rangle\Big]dt + \langle M x_i(T), x_i(T)\rangle\Big\}. \end{aligned} \tag{4}$$

The coefficients of cost functionals satisfy that $Q_0, Q \in \mathcal{S}^n, Q_0 \ge 0, Q \ge 0, \beta_0, \beta_1, \beta_2 \in \mathbb{R}^n, R_0 > 0, R > 0, R_0, R \in \mathcal{S}^k$ and $M_0 \ge 0, M \ge 0, M_0, M \in \mathcal{S}^n$.

Parallel to (2), the cost functional (4) contains the term $\beta_2 x_0(t)$ to capture the strong influence of the major agent. Note that the state dynamics (1) and (2), and the cost functionals (3) and (4), indicate that the major agent $\mathcal{A}_0$ has a significant influence on minor agents, while each minor agent $\mathcal{A}_i, i \in \mathcal{N}$, has a negligible impact on other agents in a large N population system.

Now, we propose the following LQG mean-field games.

Problem (LP): Find an admissible strategy $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_i, \ldots, \bar{u}_N)$ where $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^{c,i}, i = 0, 1, \cdots, N$, such that

$$J_i(\bar{u}_i, \bar{u}_{-i}) = \inf_{u_i(\cdot) \in \mathcal{U}_{ad}^{c,i}} J_i(u_i, u_{-i}), \quad i = 0, 1, \cdots, N.$$

We call $\bar{u}$ a Nash equilibrium strategy for Problem (LP).

Remark 2.1. *It should be noted that this paper only addresses the existence of Nash equilibrium strategies and does not involve whether the Nash equilibrium is unique. The study of the uniqueness of Nash equilibrium strategies is also an active research topic. The variational inequality approach proposed in He and Wang [28] provides a feasible methodology for studying the uniqueness of Nash equilibrium strategies.*

3. Closed-loop behavior of the agents

In this section, we first construct two auxiliary stochastic optimal control problems, which are called limiting systems, for the major and a generic minor agent in Sections 3.1 and 3.2, respectively. Then we present the approximations between the limiting systems and the corresponding mean-field system in Section 3.3.

3.1. Optimal control of the major agent

For any $v_0(\cdot) \in \mathcal{U}_{ad}^0$, the state $y_0(\cdot)$ of agent $\mathcal{A}_0$ satisfies the following stochastic differential equation:

$$\begin{cases} dy_0(t) = [A_0 y_0(t) + B_0 v_0(t) + b_0 x^{(0)}(t) + f_0(t)]dt + [C_0 y_0(t) + D_0 v_0(t) \\ \qquad\qquad + l_0 x^{(0)}(t) + \sigma_0(t)]dW_0(t) + F_0 \int_{E_0} G_0(dedt), \\ y_0(0) = a_0, \end{cases} \tag{5}$$

where function $x^{(0)}(\cdot)$ will be given later.

The corresponding cost functional is given by

$$\widetilde{J_0}(v_0) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q_0(y_0(t)-\beta_0 x^{(0)}(t)), (y_0(t)-\beta_0 x^{(0)}(t))\rangle + \langle R_0 v_0(t), v_0(t)\rangle\Big]dt + \langle M_0 y_0(T), y_0(T)\rangle\Big\}.$$

Problem (LM1): The objective is to find $\bar{v}_0(\cdot) \in \mathcal{U}_{ad}^0$ such that

$$\widetilde{J_0}(\bar{v}_0) = \inf_{v_0 \in \mathcal{U}_{ad}^0} \widetilde{J_0}(v_0).$$

Let $P_0(\cdot)$ be the solution of the following Riccati equation:

$$\begin{cases} -\dot{P}_0(t) = P_0(t)A_0 + A_0^\top P_0(t) + C_0^\top P_0(t)C_0 + Q_0 - (B_0^\top P_0(t) + D_0^\top P_0(t)C_0)^\top \\ \qquad \times (R_0 + D_0^\top P_0(t)D_0)^{-1}(B_0^\top P_0(t) + D_0^\top P_0(t)C_0), \\ R_0 + D_0^\top P_0(t)D_0 \geq 0, \\ P_0(T) = M_0. \end{cases}$$

Let $\eta_0(\cdot)$ denote the solution of

$$\begin{cases} \dot{\eta}_0(t) = -\Big\{[A_0 - B_0(R_0 + D_0^\top P_0(t)D_0)^{-1} \times (B_0^\top P_0(t) + D_0^\top P_0(t)C_0)]^\top \eta_0(t) \\ \qquad + [C_0 - D_0(R_0 + D_0^\top P_0(t)D_0)^{-1} \times (B_0^\top P_0(t) + D_0^\top P_0(t)C_0)]^\top \\ \qquad \times P_0(t)(l_0 x^{(0)}(t) + \sigma_0(t)) + [P_0(t)(b_0 x^{(0)}(t) + f_0(t)) - \beta_0 Q_0 x^{(0)}(t)]\Big\}, \\ \eta_0(T) = 0. \end{cases}$$

The following result presents the optimal control of Problem (LM1).

Theorem 3.1. *Suppose that*

$$\begin{cases} \Lambda_0(t) := -(R_0 + D_0^\top P_0(t)D_0)^{-1} \times (B_0^\top P_0(t) + D_0^\top P_0(t)C_0), \\ \Theta_0(t) := -(R_0 + D_0^\top P_0(t)D_0)^{-1} \times [B_0^\top \eta_0(t) + D_0^\top P_0(t)(l_0 x^{(0)}(t) + \sigma_0(t))]. \end{cases}$$

Then the optimal control strategy of Problem (LM1) is

$$\bar{v}_0(t) = \Lambda_0(t)\bar{y}_0(t) + \Theta_0(t),$$

where $\bar{y}_0(\cdot)$ satisfies

$$\begin{cases} d\bar{y}_0(t) = [(A_0 + B_0\Lambda_0(t))\bar{y}_0(t) + B_0\Theta_0(t) + b_0 x^{(0)}(t) + f_0(t)]dt \\ \qquad + [(C_0 + D_0\Lambda_0(t))\bar{y}_0(t) + D_0\Theta_0(t) + l_0 x^{(0)}(t) + \sigma_0(t)]dW_0(t) \\ \qquad + F_0 \int_{E_0} G_0(dedt), \\ \bar{y}_0(0) = a_0. \end{cases} \tag{6}$$

Proof. Let $\hat{b}(t) := b_0 x^{(0)}(t) + f_0(t), \hat{\sigma}(t) := l_0 x^{(0)}(t) + \sigma_0(t)$. Then the state equation (5) can be written as

$$\begin{cases} dy_0(t) = \Big[A_0 y_0(t) + B_0 u_0(t) + \hat{b}(t)\Big]dt + \Big[C_0 y_0(t) + D_0 u_0(t) + \hat{\sigma}(t)\Big]dW_0(t) \\ \qquad + F_0 \int_{E_0} G_0(dedt), \\ y_0(0) = a_0. \end{cases}$$

For simplicity, we denote $\hat{R}_0(t) := R_0 + D_0^\top P_0(t) D_0$, $\hat{B}_0(t) := B_0^\top P_0(t) + D_0^\top P_0(t) C_0$. Applying Itô's formula to $\left(\frac{1}{2}y_0^\top(t)P_0(t)y_0(t) + y_0^\top(t)\eta_0(t)\right)$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big\{\frac{1}{2}y_0^\top(T)P_0(T)y_0(T) - \frac{1}{2}y_0^\top(0)P_0(0)y_0(0) + y_0^\top(T)\eta_0(T) - y_0^\top(0)\eta_0(0)\Big\}\\
&= \mathbb{E}\Big\{\frac{1}{2}M_0 y_0^2(T) - \frac{1}{2}y_0^\top(0)P_0(0)y_0(0) - y_0^\top(0)\eta_0(0)\Big\}\\
&= \mathbb{E}\int_0^T \Big[-\frac{1}{2}Q_0 y_0^2 - \frac{1}{2}y_0^2\hat{B}_0^2\hat{R}_0^{-1} + P_0 y_0 v_0^\top B_0^\top + P_0 C_0 y_0 v_0^\top D_0^\top + \eta_0 v_0^\top B_0^\top + \eta_0\hat{b}^\top\Big]dt\\
&\quad + \mathbb{E}\int_0^T (\frac{1}{2}P_0 D_0^2 v_0^2 + P_0 D_0 v_0\hat{\sigma} + \frac{1}{2}P_0\hat{\sigma}^2)dt + \mathbb{E}\int_0^T \Big[B_0\hat{R}_0^{-1}(B_0^\top P_0 + D_0^\top P_0 C_0)\Big]^\top \eta_0 y_0^\top dt\\
&\quad + \mathbb{E}\int_0^T \Big\{\Big[D_0\hat{R}_0^{-1}\hat{B}_0\Big]^\top P_0\hat{\sigma}y_0^\top + \beta_0 Q_0 x^{(0)}(t)y_0^\top\Big\}dt + \frac{1}{2}P_0F_0^2\int_{E_0}\int_0^T \pi_0(dedt).
\end{aligned}$$

Combing the above equation with the definition of $\widetilde{J}_0(v_0)$, it follows that

$$\begin{aligned}
\widetilde{J}_0(v_0) &= \mathbb{E}\bigg\{\int_0^T (\frac{1}{2}Q_0(y_0 - \beta_0 x^{(0)}(t))^2 + \frac{1}{2}R_0 v_0^2)dt + \frac{1}{2}M_0 y_0^2(T)\bigg\}\\
&= \mathbb{E}\bigg\{\int_0^T \bigg[-\beta_0 Q_0 x^{(0)}(t)y_0 + \frac{1}{2}Q_0(\beta_0 x^{(0)}(t))^2 + \frac{1}{2}R_0 v_0^2 + \frac{1}{2}P_0 D_0^2 v_0^2 + y_0 v_0^\top P_0 B_0^\top\\
&\quad + y_0 v_0^\top P_0 C_0 D_0^\top + \beta_0 Q_0 x^{(0)}(t)y_0^\top + \frac{1}{2}y_0^2\hat{B}_0^2\hat{R}_0^{-1} + P_0 D_0 v_0\sigma_0 + \eta_0 v_0^\top B_0^\top\\
&\quad + \Big[B_0\hat{R}_0^{-1}\hat{B}_0\Big]^\top \eta_0 y_0^\top + \Big[D_0\hat{R}_0^{-1}\hat{B}_0\Big]^\top P_0\hat{\sigma}y_0^\top + \frac{1}{2}P_0\hat{\sigma}^2 + \eta_0\hat{b}^\top\Big]dt\Big\}\\
&\quad + \frac{1}{2}P_0F_0^2\int_{E_0}\int_0^T \pi_0(dedt) + \frac{1}{2}a_{i0}^2 P(0) + a_{i0}\eta(0)\\
&= \mathbb{E}\bigg\{\int_0^T \bigg[\frac{1}{2}\hat{R}_0^{-1}\Big\{\Big[\hat{R}_0 v_0 + \hat{B}_0 y_0\Big]^2 + 2(B_0^\top\eta_0 + D_0^\top P_0\hat{\sigma})(\hat{R}_0 v_0 + \hat{B}_0 y_0)\Big\}\\
&\quad + \frac{1}{2}P_0\hat{\sigma}^2 + \eta_0\hat{b}^\top\bigg]dt\bigg\} + \frac{1}{2}P_0F_0^2\int_{E_0}\int_0^T \pi_0(dedt) + \frac{1}{2}a_{i0}^2 P(0) + a_{i0}\eta(0)\\
&= \mathbb{E}\Big\{\int_0^T \Big[\frac{1}{2}\hat{R}_0^{-1}\|\hat{R}_0 v_0 + \hat{B}_0 y_0 + (B_0^\top\eta_0 + D_0^\top P_0\hat{\sigma})\|^2 - \frac{1}{2}\hat{R}_0^{-1}(B_0^\top\eta_0 + D_0^\top P_0\hat{\sigma})^2\\
&\quad + \frac{1}{2}P_0\hat{\sigma}^2 + \eta_0\hat{b}^\top\Big]dt\Big\} + \frac{1}{2}P_0F_0^2\int_{E_0}\int_0^T \pi_0(dedt) + \frac{1}{2}a_{i0}^2 P(0) + a_{i0}\eta(0).
\end{aligned}$$

Hence we obtain the optimal control

$$\begin{aligned}
\bar{v}_0(t) &= -\hat{R}_0^{-1}(t)\hat{B}_0^{-1}(t)\bar{y}_0(t) - \hat{R}_0^{-1}(t)(B_0^\top\eta_0(t) + D_0^\top P_0(t)\hat{\sigma}(t))\\
&= \Lambda_0(t)\bar{y}_0(t) + \Theta_0(t).
\end{aligned}$$

The proof is therefore complete. □

3.2. Optimal control of the minor agent

For any $i \in \mathcal{N}$, the limiting state of minor agent $\mathcal{A}_i$ is

$$\begin{cases} dy_i(t) = [Ay_i(t) + Bv_i(t) + b_1 x^{(0)}(t) + f(t)]dt + [Cy_i(t) + Dv_i(t) + b_2 x^{(0)}(t) \\ \qquad\qquad + Hy_0(t) + \sigma(t)]dW_i(t) + F \int_{E_i} G_i(dedt), \\ y_i(0) = a_i. \end{cases}$$

The limiting cost functional is given by

$$\widetilde{J}_i(v_i) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[\langle Q(y_i(t) - \beta_1 x^{(0)}(t) - \beta_2 y_0(t)), (y_i(t) - \beta_1 x^{(0)}(t) - \beta_2 y_0(t))\rangle \\ + \langle Rv_i(t), v_i(t)\rangle\Big]dt + \langle My_i(T), y_i(T)\rangle\Big\}.$$

Problem (LM2): Find a control strategy $\bar{v}_i(\cdot) \in \mathcal{U}^i_{ad}$, $1 \le i \le N$, such that

$$\widetilde{J}_i(\bar{v}_i) = \inf_{v_i \in \mathcal{U}^i_{ad}} \widetilde{J}_i(v_i).$$

Let $P_1(\cdot)$ be the solution of the following Riccati equation:

$$\begin{cases} -\dot{P}_1(t) = P_1(t)A + A^\top P_1(t) + C^\top P_1(t)C + Q - (B^\top P_1(t) + D^\top P_1(t)C)^\top \\ \qquad\qquad \times (R + D^\top P_1(t)D)^{-1}(B^\top P_1(t) + D^\top P_1(t)C), \\ R + D^\top P_1(t)D \ge 0, \\ P_1(T) = M. \end{cases}$$

$\eta_1(\cdot)$ satisfies

$$\begin{cases} \dot{\eta}_1(t) = -\Big\{[A - B(R + D^\top P_1(t)D)^{-1} \times (B^\top P_1(t) + D^\top P_1(t)C)]^\top \eta_1(t) \\ \qquad\quad + [C - D(R + D^\top P_1(t)D)^{-1} \times (B^\top P_1(t) + D^\top P_1(t)C)]^\top \times P_1(t)(b_2 x^{(0)}(t) \\ \qquad\quad + Hy_0(t) + \sigma(t)) + [P_1(t)(b_1 x^{(0)}(t) + f(t)) - \beta_1 Q x^{(0)}(t) - \beta_2 Q y_0(t)]\Big\}, \\ \eta_1(T) = 0. \end{cases}$$

Denote

$$\begin{cases} \Lambda_1(t) := -(R + D^\top P_1(t)D)^{-1} \times (B^\top P_1(t) + D^\top P_1(t)C), \\ \Theta_1(t) := -(R + D^\top P_1(t)D)^{-1} \times [B^\top \eta_1(t) + D^\top P_1(t)(b_2 x^{(0)}(t) + Hy_0(t) + \sigma(t))], \\ \bar{\Theta}_1(t) := -(R + D^\top P_1(t)D)^{-1} \times [B^\top \eta_1(t) + D^\top P_1(t)(b_2 x^{(0)}(t) + H\bar{y}_0(t) + \sigma(t))]. \end{cases}$$

Using a similar proof as in Theorem 3.1, we have the following result.

Theorem 3.2. *The optimal control strategy of Problem (LM2) is*

$$\bar{v}_i(t) = \Lambda_1(t)\bar{y}_i(t) + \bar{\Theta}_1(t),$$

where $\bar{y}_i(\cdot)$ satisfies

$$\begin{cases} d\bar{y}_i(t) = [(A + B\Lambda_1(t))\bar{y}_i(t) + B\bar{\Theta}_1(t) + b_1 x^{(0)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\bar{y}_i(t) \\ \qquad\qquad + D\bar{\Theta}_1(t) + b_2 x^{(0)}(t) + H\bar{y}_0(t) + \sigma(t)]dW_i(t) + F \int_{E_i} G_i(dedt), \\ \bar{y}_i(0) = a_i. \end{cases} \tag{7}$$

3.3. Approximation for the closed-loop system

In this subsection, we design a closed-loop mean-field system, and show the approximations between the limiting system and the corresponding closed-loop system.

Based on the feedback formulation of the optimal control for major agent $\mathcal{A}_0$ and minor agents $\mathcal{A}_i, 1 \le i \le N$, we obtain

$$\begin{cases} d\bar{x}_0(t) = [(A_0 + B_0\Lambda_0(t))\bar{x}_0(t) + B_0\Theta_0(t) + b_0\bar{x}^{(N)}(t) + f_0(t)]dt + [(C_0 + D_0\Lambda_0(t))\bar{x}_0(t) \\ \qquad + D_0\Theta_0(t) + l_0\bar{x}^{(N)}(t) + \sigma_0(t)]dW_0(t) + F_0\int_{E_0} G_0(dedt), \\ \bar{x}_0(0) = a_0, \end{cases} \tag{8}$$

and

$$\begin{cases} d\bar{x}_i(t) = [(A + B\Lambda_1(t))\bar{x}_i(t) + B\bar{\Theta}_1(t) + b_1\bar{x}^{(N)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\bar{x}_i(t) \\ \qquad + D\bar{\Theta}_1(t) + b_2\bar{x}^{(N)}(t) + H\bar{x}_0(t) + \sigma(t)]dW_i(t) + F\int_{E_i} G_i(dedt), \\ \bar{x}_i(0) = a_i. \end{cases} \tag{9}$$

By $\bar{x}^{(N)}(t) = \frac{1}{N}\sum\limits_{k=1}^{N} \bar{x}_i(t)$, the function $x^{(0)}(t)$ fulfills

$$\begin{cases} dx^{(0)}(t) = [(A + B\Lambda_1(t) + b_1)x^{(0)}(t) + B\bar{\Theta}_1(t) + f(t)]dt, \\ x^{(0)}(0) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a_j. \end{cases} \tag{10}$$

Now, we introduce the following NCE equation:

$$\begin{cases} d\bar{y}_0(t) = \Big[(A_0 + B_0\Lambda_0(t))\bar{y}_0(t) + B_0\Theta_0(t) + b_0x^{(0)}(t) + f_0(t)\Big]dt \\ \qquad + \Big[(C_0 + D_0\Lambda_0(t))\bar{y}_0(t) + D_0\Theta_0(t) + l_0x^{(0)}(t) + \sigma_0(t)\Big]dW_0(t) + F_0\int_{E_0} G_0(dedt), \\ \dot{x}^{(0)}(t) = (A + B\Lambda_1(t) + b_1)x^{(0)}(t) - B(R + D^\top P_1(t)D)^{-1} \\ \qquad \times [B^\top\eta_1(t) + D^\top P_1(t)(b_2x^{(0)}(t) + H\bar{y}_0(t) + \sigma(t))] + f(t), \\ -\dot{\eta}_1(t) = [A + B\Lambda_1(t)]^\top\eta_1(t) + [C + D\Lambda_1(t)]^\top P_1(t)[b_2x^{(0)}(t) + H\bar{y}_0(t) + \sigma(t)] \\ \qquad + P_1(t)(b_1x^{(0)}(t) + f(t)) - \beta_1Qx^{(0)}(t) - \beta_2Q\bar{y}_0(t), \\ -\dot{\eta}_0(t) = [A_0 + B_0\Lambda_0(t)]^\top\eta_0(t) + [C_0 + D_0\Lambda_0(t)]^\top P_0(t)l_0x^{(0)}(t)\sigma_0(t) \\ \qquad + P_0(t)(b_0x^{(0)}(t) + f_0(t)) - \beta_0Q_0x^{(0)}(t), \\ \bar{y}_0(0) = a_0,\ \ \eta_0(T) = \eta_1(T) = 0,\ \ x^{(0)}(0) = \frac{1}{N}\sum_{j=1}^{N} a_j, \end{cases}$$

which can be written as

$$\begin{cases} d\bar{y}_0(t) = \Big[\hat{A}_0(t)\bar{y}_0(t) + \mathbb{G}_0(t)x^{(0)}(t) - B_0\hat{R}_0^{-1}(t)B_0^\top\eta_0(t) + \hat{\mathbb{G}}_0(t)\Big]dt \\ \qquad + \Big[C_0(t)\bar{y}_0(t) + \mathbb{H}_0(t)x^{(0)}(t) - D_0\hat{R}_0^{-1}(t)B_0^\top\eta_0(t) + \hat{\mathbb{H}}_0(t)\Big]dW_0(t) + F_0\int_{E_0} G_0(dedt), \\ \dot{x}^{(0)}(t) = \mathbb{G}_1(t)x^{(0)}(t) - B\hat{R}^{-1}(t)[B^\top\eta_1(t) + D^\top P_1(t)H\bar{y}_0(t) + D^\top P_1(t)\sigma(t)] + f(t), \\ -\dot{\eta}_1(t) = \hat{A}^\top(t)\eta_1(t) + \mathbb{L}_1(t)x^{(0)}(t) + \mathbb{H}_1(t)\bar{y}_0(t) + \mathbb{K}_1(t), \\ -\dot{\eta}_0(t) = \hat{A}_0^\top\eta_0(t) + \mathbb{L}_0(t)x^{(0)}(t) + P_0(t)f_0(t) \\ \bar{y}_0(0) = a_0,\ \ \eta_0(T) = \eta_1(T) = 0,\ \ x^{(0)}(0) = \frac{1}{N}\sum_{j=1}^{N} a_j, \end{cases} \tag{11}$$

where

$$\hat{A}_0(t) := A_0 + B_0\Lambda_0(t), \qquad \mathbb{G}_0(t) := -B_0\hat{R}_0^{-1}(t)D_0^\top P_0(t)l_0 + b_0,$$

$$\hat{R}_0(t) := R_0 + D_0^\top P_0(t) D_0, \qquad \hat{\mathbb{G}}_0(t) := f_0(t) - B_0 \hat{R}_0^{-1}(t) D_0^\top P_0(t)\sigma_0(t),$$
$$C_0(t) := C_0 + D_0\Lambda_0(t), \qquad \mathbb{H}_0(t) := -D_0\hat{R}_0^{-1}(t) D_0^\top P_0(t) l_0 + l_0,$$
$$\hat{\mathbb{H}}_0(t) := \sigma_0(t) - D_0\hat{R}_0^{-1}(t) D_0^\top P_0(t)\sigma_0(t), \quad \mathbb{G}_1(t) := \hat{A}(t) + b_1 - B\hat{R}^{-1}(t) D^\top P_1(t) b_2,$$
$$\hat{A}(t) := A + B\Lambda_1(t), \qquad \mathbb{L}_1(t) := [C + D\Lambda_1(t)]^\top P_1(t) b_2 + P_1(t) b_1 - \beta_1 Q,$$
$$\hat{R}(t) := R + D^\top P_1(t) D, \qquad \mathbb{H}_1(t) := [C + D\Lambda_1(t)]^\top P_1(t) H - \beta_2 Q,$$
$$\mathbb{K}_1(t) := [C + D\Lambda_1(t)]^\top P_1(t)\sigma(t) + P_1(t) f(t),$$
$$\mathbb{L}_0(t) := [C_0 + D_0\Lambda_0(t)]^\top P_0(t) l_0 \sigma_0(t) + P_0(t) b_0 - \beta_0 Q_0.$$

The above NCE equation is a kind of coupled two-point boundary value problem, whose well-posedness can be found in Theorem 4.2 of Hu et al. [3] under some monotonicity assumptions. We will not repeat them here for simplicity.

Next, we establish the approximation relationship between the closed-loop mean-field game system and the limiting system.

Proposition 3.3. *The following estimates hold:*

$$(i)\ \sup_{0\le t\le T} \mathbb{E}\Big|\bar{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O(\frac{1}{N}),$$
$$(ii)\ \sup_{0\le t\le T} \mathbb{E}\Big||\bar{x}^{(N)}(t)|^2 - |x^{(0)}(t)|^2\Big| = O(\frac{1}{\sqrt{N}}),$$
$$(iii)\ \sup_{0\le t\le T} \mathbb{E}\Big|\bar{x}_0(t) - \bar{y}_0(t)\Big|^2 = O(\frac{1}{N}),$$
$$(iv)\ \sup_{0\le t\le T} \mathbb{E}\Big||\bar{x}_0(t)|^2 - |\bar{y}_0(t)|^2\Big| = O(\frac{1}{\sqrt{N}}),$$
$$(v)\ \sup_{0\le t\le T} \mathbb{E}\Big|\bar{x}_i(t) - \bar{y}_i(t)\Big|^2 = O(\frac{1}{N}), \quad 1 \le i \le N,$$
$$(vi)\ \sup_{0\le t\le T} \mathbb{E}\Big||\bar{x}_i(t)|^2 - |\bar{y}_i(t)|^2\Big| = O(\frac{1}{\sqrt{N}}), \quad 1 \le i \le N.$$

Proof. Let $\bar{z}(t) := \bar{x}^{(N)}(t) - x^{(0)}(t)$, $\bar{z}_0(t) := \bar{x}_0(t) - \bar{y}_0(t)$, $\bar{z}_i(t) := \bar{x}_i(t) - \bar{y}_i(t)$ $(1 \le i \le N)$. Combining (9) with (10), we derive

$$\begin{cases} d\bar{z}(t) = [(A + B\Lambda_1(t) + b_1)\bar{z}(t)]dt + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}[(C + D\Lambda_1(t))\bar{x}_j(t) + D\bar{\Theta}_1(t) \\ \qquad\qquad + b_2\bar{x}^{(N)}(t) + H\bar{x}_0(t) + \sigma(t)]dW_j(t) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} F\int_{E_j} G_j(dedt), \\ \bar{z}(0) = 0. \end{cases}$$

Define $\chi(t) := b_2\bar{x}^{(N)}(t) + H\bar{x}_0(t) + \sigma(t)$. Applying Itô's formula to $\bar{z}^2(t)$, we obtain

$$\mathbb{E}[\bar{z}^2(t)] = 2\int_0^t (A + B\Lambda_1(s) + b_1)\mathbb{E}[\bar{z}^2(s)]ds + \frac{1}{N^2}\sum_{j=1}^{N}\mathbb{E}\int_0^t \Big\{[(C + D\Lambda_1(s))\bar{x}_j(s)$$

$$+ D\bar{\Theta}_1(s) + \chi(s)]\Big\}^2 ds + \frac{F^2}{N^2}\sum_{j=1}^{N}\mathbb{E}\int_{E_j}\int_0^t \pi_j(deds),$$

$$\leq 2\sup_{0\leq t\leq T}(A + B\Lambda_1(t) + b_1)\times\int_0^t \mathbb{E}[\bar{z}^2(s)]ds + \frac{T}{N}\max_{0\leq j\leq N}\mathbb{E}[(C + D\Lambda_1(t))\bar{x}_j(t)$$

$$+ D\bar{\Theta}_1(t) + \chi(t)]^2 + \frac{F^2}{N}\max_{0\leq j\leq N}\mathbb{E}\int_{E_j}\int_0^t \pi_j(deds).$$

According to Gronwall's inequality, it follow that

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|\bar{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O\left(\frac{1}{N}\right). \tag{12}$$

For (ii), according to Hölder's inequality, we have

$$\mathbb{E}\Big||\bar{x}^{(N)}(t)|^2 - |x^{(0)}(t)|^2\Big| = \mathbb{E}\Big||\bar{x}^{(N)}(t) - x^{(0)}(t)|^2 + 2x^{(0)}(t)(\bar{x}^{(N)}(t) - x^{(0)}(t))\Big|$$

$$\leq \mathbb{E}\Big[|\bar{x}^{(N)}(t) - x^{(0)}(t)|^2\Big] + 2|x^{(0)}(t)|\Big(\mathbb{E}[|\bar{x}^{(N)}(t) - x^{(0)}(t)|^2]\Big)^{\frac{1}{2}}.$$

By (12) and the boundedness of $|x^{(0)}(t)|$, one has

$$\sup_{0\leq t\leq T}\mathbb{E}\Big||\bar{x}^{(N)}(t)|^2 - |x^{(0)}(t)|^2\Big| = O\left(\frac{1}{\sqrt{N}}\right).$$

We now prove (iii). According to (6) and (8), it follows that

$$\begin{cases} d\bar{z}_0(t) = [(A_0 + B_0\Lambda_0(t))\bar{z}_0(t) + b_0\bar{z}(t)]dt + [(C_0 + D_0\Lambda_0(t))\bar{z}_0(t) + l_0\bar{z}(t)]dW_0(t), \\ \bar{z}_0(0) = 0. \end{cases}$$

Applying Itô's formula to $\bar{z}_0^2(t)$, we obtain

$$\mathbb{E}[\bar{z}_0^2(t)] = 2\int_0^t \mathbb{E}\Big[(A_0 + B_0\Lambda_0(s))\bar{z}_0^2(s) + b_0\bar{z}(s)\bar{z}_0(s)\Big]ds$$

$$+ \int_0^t \mathbb{E}\Big[(C_0 + D_0\Lambda_0(s))\bar{z}_0(s) + l_0\bar{z}(s)\Big]^2 ds$$

$$\leq 2\int_0^t \Big[(A_0 + B_0\Lambda_0(s)) + (C_0 + D_0\Lambda_0(s))^2 + b_0^2\Big]\mathbb{E}\bar{z}_0^2(s)ds + \int_0^t \Big(\frac{1}{2} + 2l_0^2\Big)\mathbb{E}\bar{z}^2(s)ds.$$

By (12) and Gronwall's inequality, we have

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|\bar{x}_0(t) - \bar{y}_0(t)\Big|^2 = O(\frac{1}{N}). \tag{13}$$

Note that

$$\mathbb{E}\Big||\bar{x}_0(t)|^2 - |\bar{y}_0(t)|^2\Big| = \mathbb{E}\Big||\bar{x}_0(t) - \bar{y}_0(t)|^2 + 2\bar{y}_0(t)(\bar{x}_0(t) - \bar{y}_0(t))\Big|$$

$$\leq \mathbb{E}\Big[|\bar{x}_0(t)-\bar{y}_0(t)|^2\Big] + 2\Big(\mathbb{E}[|\bar{y}_0(t)|^2]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[|\bar{x}_0(t)-\bar{y}_0(t)|^2]\Big)^{\frac{1}{2}}.$$

According to (13) and the boundedness of $|\bar{y}_0(t)|$, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\Big||\bar{x}_0(t)|^2-|\bar{y}_0(t)|^2\Big| = O(\frac{1}{\sqrt{N}}).$$

Next, we prove (v). Combining (7) with (9), we have

$$\begin{cases} d\bar{z}_i(t) = [(A+B\Lambda_1(t))\bar{z}_i(t)+b_1\bar{z}(t)]dt + [(C+D\Lambda_1(t))\bar{z}_i(t)+b_2\bar{z}(t)+H\bar{z}_0(t)]dW_i(t),\\ \bar{z}_i(0)=0. \end{cases}$$

Applying Itô's formula to $\bar{z}_i^2(t)$, we obtain

$$\begin{aligned}\mathbb{E}[\bar{z}_i^2(t)] &= 2\int_0^t \mathbb{E}\Big[(A+B\Lambda_1(s))\bar{z}_i^2(s)+b_1\bar{z}(s)\bar{z}_i(s)\Big]ds\\ &\quad+\int_0^t \mathbb{E}\Big[(C+D\Lambda_1(s))\bar{z}_i(s)+b_2\bar{z}(s)+H\bar{z}_0(t)\Big]^2 ds\\ &\leq \int_0^t \Big[2(A+B\Lambda_1(s))+b_1^2+3(C+D\Lambda_1(s))^2\Big]\mathbb{E}\bar{z}_i^2(s)ds\\ &\quad+\int_0^t (1+3b_2^2)\mathbb{E}\bar{z}^2(s)ds + 3H^2\int_0^t \mathbb{E}\bar{z}_0^2(s)ds.\end{aligned}$$

By Gronwall's inequality, and estimates (12) and (13), we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\Big|\bar{x}_i(t)-\bar{y}_i(t)\Big|^2 = O(\frac{1}{N}). \tag{14}$$

Finally, we prove (vi). Since

$$\begin{aligned}\mathbb{E}\Big||\bar{x}_i(t)|^2-|\bar{y}_i(t)|^2\Big| &\leq \mathbb{E}\Big[|\bar{x}_i(t)-\bar{y}_i(t)|^2\Big] + 2\mathbb{E}\Big[|\bar{y}_i(t)||\bar{x}_i(t)-\bar{y}_i(t)|\Big]\\ &\leq \mathbb{E}\Big[|\bar{x}_i(t)-\bar{y}_i(t)|^2\Big] + 2\Big(\mathbb{E}[|\bar{y}_i(t)|^2]\Big)^{\frac{1}{2}}\Big(\mathbb{E}[|\bar{x}_i(t)-\bar{y}_i(t)|^2]\Big)^{\frac{1}{2}}.\end{aligned}$$

According to (14) and the boundedness of $|\bar{y}_i(t)|$, we get

$$\sup_{0\leq t\leq T}\mathbb{E}\Big||\bar{x}_i(t)|^2-|\bar{y}_i(t)|^2\Big| = O(\frac{1}{\sqrt{N}}).$$

The proof is then complete. □

Define the control strategy for the major agent as

$$\bar{u}_0(t) = \Lambda_0(t)\bar{x}_0(t) + \Theta_0(t), \tag{15}$$

and the control strategy for minor agents as

$$\bar{u}_i(t) = \Lambda_1(t)\bar{x}_i(t) + \bar{\Theta}_1(t). \tag{16}$$

Based on the approximation relationship between the closed-loop mean-field systems and the limiting system, the following approximation relationship between cost functionals can be derived.

Proposition 3.4. *For any $i = 0, 1, \cdots, N$, we have*

$$\Big|J_i(\bar{u}_i, \bar{u}_{-i}) - \widetilde{J}_i(\bar{v}_i)\Big| = O(\frac{1}{\sqrt{N}}).$$

Proof. Based on the definitions of the cost functionals, we obtain

$$\begin{aligned}
&\Big|J_i(\bar{u}_i, \bar{u}_{-i}) - \widetilde{J}_i(\bar{v}_i)\Big| \\
=&\Bigg|\frac{1}{2}\mathbb{E}\int_0^T \Big\{\Big[Q(\bar{x}_i(t) - \beta_1\bar{x}^{(N)}(t) - \beta_2\bar{x}_0(t))^2 - Q(\bar{y}_i(t) - \beta_1 x^{(0)}(t) - \beta_2\bar{y}_0(t))^2\Big] \\
&+ \Big[R\bar{u}_i^2(t) - R\bar{v}_i^2(t)\Big]\Big\}dt + \frac{1}{2}\mathbb{E}\Big[M\bar{x}_i^2(T) - M\bar{y}_i^2(T)\Big]\Bigg| \\
=&\Bigg|\frac{1}{2}\mathbb{E}\int_0^T \Big\{Q\Big[(\bar{x}_i(t) - \beta_1\bar{x}^{(N)}(t) - \beta_2\bar{x}_0(t)) + (\bar{y}_i(t) - \beta_1 x^{(0)}(t) - \beta_2\bar{y}_0(t))\Big] \\
&\times\Big[(\bar{x}_i(t) - \beta_1\bar{x}^{(N)}(t) - \beta_2\bar{x}_0(t)) - (\bar{y}_i(t) - \beta_1 x^{(0)}(t) - \beta_2\bar{y}_0(t))\Big] \\
&+ R\Big[(\Lambda_1(t)\bar{x}_i(t) + \bar{\Theta}_1(t))^2 - (\Lambda_1(t)\bar{y}_i(t) + \bar{\Theta}_1(t))^2\Big]\Big\}dt + \frac{1}{2}\mathbb{E}\Big[M\bar{x}_i^2(T) - M\bar{y}_i^2(T)\Big]\Bigg| \\
=&\Bigg|\frac{1}{2}\mathbb{E}\int_0^T \Bigg\{Q\Big[\Big(2\bar{x}_i(t) - 2\beta_1\bar{x}^{(N)}(t) - 2\beta_2\bar{x}_0(t)\Big) - L(t)\Big]\times L(t) \\
&+ R\Big[(\Lambda_1(t))^2\Big(\bar{x}_i^2(t) - \bar{y}_i^2(t)\Big) + 2\Lambda_1(t)\bar{\Theta}_1(t)(\bar{x}_i(t) - \bar{y}_i(t))\Big]\Bigg\}dt + \frac{1}{2}\mathbb{E}\Big[M\bar{x}_i^2(T) - M\bar{y}_i^2(T)\Big] \\
\le& \frac{1}{2}\int_0^T \Bigg\{Q\mathbb{E}\Big[\Big|(2\bar{x}_i(t) - 2\beta_1\bar{x}^{(N)}(t) - 2\beta_2\bar{x}_0(t))L(t)\Big|\Big] + Q\mathbb{E}\Big[\Big|L^2(t)\Big|\Big] \\
&+ R(\Lambda_1(t))^2\mathbb{E}\Big[\Big|\bar{x}_i^2(t) - \bar{y}_i^2(t)\Big|\Big] + 2R\Lambda_1(t)\bar{\Theta}_1(t)\mathbb{E}\Big[\Big|\bar{x}_i(t) - \bar{y}_i(t)\Big|\Big]\Bigg\}dt \\
&+ \frac{1}{2}M\mathbb{E}\Big[\Big|\bar{x}_i^2(T) - \bar{y}_i^2(T)\Big|\Big] \\
\le& \frac{1}{2}QT\sup_{0\le t\le T}\mathbb{E}\Big[\Big|(2\bar{x}_i(t) - 2\beta_1\bar{x}^{(N)}(t) - 2\beta_2\bar{x}_0(t))L(t)\Big|\Big] + \frac{1}{2}QT\sup_{0\le t\le T}\mathbb{E}\Big[\Big|L^2(t)\Big|\Big] \\
&+ \frac{1}{2}RT(\Lambda_1(t))^2\sup_{0\le t\le T}\mathbb{E}\Big[\Big|\bar{x}_i^2(t) - \bar{y}_i^2(t)\Big|\Big] + RT\Lambda_1(t)\bar{\Theta}_1(t)\sup_{0\le t\le T}\mathbb{E}\Big[\Big|\bar{x}_i(t) - \bar{y}_i(t)\Big|\Big] \\
&+ \frac{1}{2}M\sup_{0\le t\le T}\mathbb{E}\Big[\Big|\bar{x}_i^2(T) - \bar{y}_i^2(T)\Big|\Big],
\end{aligned}$$

where $L(t) := \Big[(\bar{x}_i(t) - \bar{y}_i(t)) - \beta_1(\bar{x}^{(N)}(t) - x^{(0)}(t)) - \beta_2(\bar{x}_0(t) - \bar{y}_0(t))\Big]$. Obviously, according to Proposition 3.3, we have $\mathbb{E}\Big[\Big|L(t)\Big|^2\Big] = O(\frac{1}{N})$. Therefore, it follows that $\Big|J_i(\bar{u}_i, \bar{u}_{-i}) - \widetilde{J}_i(\bar{v}_i)\Big| = O(\frac{1}{\sqrt{N}})$. The proof is then complete. □

4. ε-Nash equilibrium for Problem (LP)

This section will verify the asymptotic Nash equilibrium property of the decentralized control strategies $\bar{u} = (\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_N)$ specified by (15) and (16).

4.1. Major agent's perturbation

Let the major agent $\tilde{\mathcal{A}}_0$ take an alternative control strategy u_0, and let the minor agent $\tilde{\mathcal{A}}_i$ take the control law (16). Then the state system with the major agent's perturbation is

$$\begin{cases} d\tilde{x}_0(t) = [A_0\tilde{x}_0(t) + B_0u_0(t) + b_0\tilde{x}^{(N)}(t) + f_0(t)]dt + [C_0\tilde{x}_0(t) + D_0u_0(t) \\ \qquad + l_0\tilde{x}^{(N)}(t) + \sigma_0(t)]dW_0(t) + F_0\displaystyle\int_{E_0} G_0(dedt), \\ d\tilde{x}_i(t) = [(A + B\Lambda_1(t))\tilde{x}_i(t) + B\bar{\Theta}_1(t) + b_1\tilde{x}^{(N)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\tilde{x}_i(t) \\ \qquad + D\bar{\Theta}_1(t) + b_2\tilde{x}^{(N)}(t) + H\tilde{x}_0(t) + \sigma(t)]dW_i(t) + F\displaystyle\int_{E_i} G_i(dedt), \\ \tilde{x}_0(0) = a_0, \tilde{x}_i(0) = a_i, \quad i = 1, \cdots, N, \end{cases} \tag{17}$$

where $\tilde{x}^{(N)}(t) = \frac{1}{N}\sum_{k=1}^{N}\tilde{x}_k(t)$. The cost functional for major agent $\tilde{\mathcal{A}}_0$ is

$$J_0(u_0, u_{-0}) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q_0(\tilde{x}_0(t) - \beta_0\tilde{x}^{(N)}(t)), (\tilde{x}_0(t) - \beta_0\tilde{x}^{(N)}(t))\rangle + \langle R_0u_0(t), u_0(t)\rangle\Big]dt + \langle M_0\tilde{x}_0(T), \tilde{x}_0(T)\rangle\Big\}.$$

The corresponding limiting state equation with the major agent's perturbation control is

$$\begin{cases} d\tilde{y}_0(t) = [A_0\tilde{y}_0(t) + B_0u_0(t) + b_0x^{(0)}(t) + f_0(t)]dt + [C_0\tilde{y}_0(t) + D_0u_0(t) \\ \qquad + l_0x^{(0)}(t) + \sigma_0(t)]dW_0(t) + F_0\int_{E_0} G_0(dedt) \\ d\tilde{y}_i(t) = [(A + B\Lambda_1(t))\tilde{y}_i(t) + B\bar{\Theta}_1(t) + b_1x^{(0)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\tilde{y}_i(t) \\ \qquad + D\bar{\Theta}_1(t) + b_2x^{(0)}(t) + H\tilde{y}_0(t) + \sigma(t)]dW_i(t) + F\int_{E_i} G_i(dedt) \\ \tilde{y}_0(0) = a_0, \quad \tilde{y}_i(0) = a_i, \quad i = 1, \cdots, N. \end{cases}$$

The cost functional is

$$\widetilde{J_0}(u_0) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q_0(\tilde{y}_0(t) - \beta_0x^{(0)}(t)), (\tilde{y}_0(t) - \beta_0x^{(0)}(t))\rangle + \langle R_0u_0(t), u_0(t)\rangle\Big]dt + \langle M_0\tilde{y}_0(T), \tilde{y}_0(T)\rangle\Big\}.$$

The following result presents an approximation relationship between two perturbation systems.

Proposition 4.1. *We have the following conclusion:*

$$(i)\ \sup_{0\le t\le T}\mathbb{E}\Big|\tilde{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O(\frac{1}{N}),$$

$$(ii)\ \sup_{0\le t\le T}\mathbb{E}\Big||\tilde{x}^{(N)}(t)|^2 - |x^{(0)}(t)|^2\Big| = O(\frac{1}{\sqrt{N}}),$$

$$(iii)\ \sup_{0\le t\le T}\mathbb{E}\Big|\tilde{x}_0(t) - \tilde{y}_0(t)\Big|^2 = O(\frac{1}{N}),$$

$$(iv)\ \sup_{0\le t\le T}\mathbb{E}\Big||\tilde{x}_0(t)|^2 - |\tilde{y}_0(t)|^2\Big| = O(\frac{1}{\sqrt{N}}).$$

Proof. We only need to prove the first approximation relationship, and a other three approximation relationships can be obtained by a similar proof as in Proposition 3.3.

Define $\Phi(t) := \tilde{x}^{(N)}(t) - x^{(0)}(t)$. Combining (10) with (17), we have

$$\begin{cases} d\Phi(t) = \Big[(A + B\Lambda_1(t) + b_1)\Phi(t)\Big]dt + \frac{1}{N}\sum_{k=1}^{N}\Big[(C + D\Lambda_1(t))\tilde{x}_k(t) + D\bar{\Theta}_1(t) + b_2\tilde{x}^{(N)}(t) \\ \qquad\qquad + H\tilde{x}_0(t) + \sigma(t)\Big]dW_k(t) + \frac{1}{N}\sum_{k=1}^{N} F\int_{E_k} G_k(dedt), \\ \Phi(0) = 0. \end{cases}$$

Define $L_k(t) := [(C + D\Lambda_1(t))\tilde{x}_k(t) + D\bar{\Theta}_1(t) + b_2\tilde{x}^{(N)}(t) + H\tilde{x}_0(t) + \sigma(t)]$. Therefore

$$\begin{aligned} &\mathbb{E}\int_0^t |L_k(s)|^2 ds \\ =& \mathbb{E}\int_0^t [(C + D\Lambda_1(s))\tilde{x}_k(s) + D\tilde{\Theta}_1(s) + b_2(\tilde{x}^{(N)}(s) - x^{(0)}(s)) + b_2x^{(0)}(s) + H\tilde{x}_0(s) + \sigma(s)]^2 ds \\ \le& \mathbb{C}\mathbb{E}\int_0^t [|\tilde{x}_k(s)|^2 + 1 + |\tilde{x}^{(N)}(s) - x^{(0)}(s)|^2 + |x^{(0)}(s)|^2 + |\tilde{x}_0(s)|^2 + |\sigma(s)|^2]ds \\ \le& \mathbb{C}\mathbb{E}\int_0^t |(\tilde{x}^{(N)}(s) - x^{(0)}(s))|^2 ds + \mathbb{C}_1, \end{aligned}$$

where

$$\mathbb{C} := \max\left\{\sup_{t\in[0,T]} |C + D\Lambda_1(t)|, \sup_{t\in[0,T]} |D\tilde{\Theta}_1(t)|, |b_2|, |H|, 1\right\},$$

$$\mathbb{C}_1 := \mathbb{C}\mathbb{E}\int_0^T [|\tilde{x}_k(s)|^2 + 1 + |x^{(0)}(s)|^2 + |\tilde{x}_0(s)|^2 + |\sigma(s)|^2]ds$$

are constants independent of N.

Furthermore,

$$\begin{aligned} \mathbb{E}\Phi^2(t) =& 2\mathbb{E}\Big\{\int_0^t \Big[(A + B\Lambda_1(s) + b_1)\Phi(s)\Big]ds\Big\}^2 + \frac{2}{N^2}\mathbb{E}\Big\{\int_0^t \sum_{j=1}^{N} L_j^2(s)ds\Big\} \\ &+ 2\mathbb{E}\Big\{\int_0^t \frac{1}{N}\sum_{k=1}^{N} F\int_{E_k} G_k(deds)\Big\}^2 \\ \le& 2\mathbb{E}\int_0^T \Big[T|(A + B\Lambda_1(s) + b_1)\Phi(s)|^2 + \frac{1}{N}\max_{1\le k\le N}|L_k(s)|^2\Big]ds \\ &+ \frac{2}{N^2}E\int_0^t\sum_{k=1}^{N}\int_{E_k} |FG_k|^2 n(de)ds. \end{aligned}$$

By Grownwall's inequality, we have

$$\sup_{0\le t\le T}\mathbb{E}\Big|\tilde{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O(\frac{1}{N}).$$

Then, the proof is complete. □

Similarly to the proof of Proposition 3.4, we can obtain the following result.

Proposition 4.2. *For any* $u_0(\cdot) \in \mathcal{U}_{ad}^{c,0}$*, we have*

$$\Big|J_0(u_0, \bar{u}_{-0}) - \widetilde{J}_0(u_0)\Big| = O(\frac{1}{\sqrt{N}}).$$

4.2. Minor agent's perturbation

Now, let us consider the following case: a given minor agent $\tilde{\mathcal{A}}_i$ takes an alternative control strategy $u_i(\cdot) \in \mathcal{U}_{ad}^{c,i}$, the major agent uses the optimal control strategy $\bar{u}_0(\cdot)$ defined by (15), while other minor agents $\tilde{\mathcal{A}}_j$ take the control strategy $\bar{u}_j(\cdot)$, $j \neq i$, $1 \leq j \leq N$, defined by (16). Then the dynamics of the agents with the given minor agent's perturbation can be written in the form

$$\begin{cases} d\hat{x}_0(t) = [(A_0 + B_0\Lambda_0(t))\hat{x}_0(t) + B_0\Theta_0(t) + b_0\hat{x}^{(N)}(t) + f_0(t)]dt + [(C_0 + D_0\Lambda_0(t))\hat{x}_0(t) \\ \qquad\qquad + D_0\Theta_0(t) + l_0\hat{x}^{(N)}(t) + \sigma_0(t)]dW_0(t) + F_0\displaystyle\int_{E_0} G_0(dedt), \\ d\hat{x}_i(t) = [A\hat{x}_i(t) + Bu_i(t) + b_1\hat{x}^{(N)}(t) + f(t)]dt + [C\hat{x}_i(t) + Du_i(t) + b_2\hat{x}^{(N)}(t) \\ \qquad\qquad + H\hat{x}_0(t) + \sigma(t)]dW_i(t) + F\displaystyle\int_{E_i} G_i(dedt), \\ d\hat{x}_j(t) = [(A + B\Lambda_1(t))\hat{x}_j(t) + B\bar{\Theta}_1(t) + b_1\hat{x}^{(N)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\hat{x}_j(t) \\ \qquad\qquad + D\bar{\Theta}_1(t) + b_2\hat{x}^{(N)}(t) + H\hat{x}_0(t) + \sigma(t)]dW_j(t) + F\displaystyle\int_{E_j} G_j(dedt), \\ \tilde{x}_0(0) = a_0,\ \tilde{x}_i(0) = a_i,\ \tilde{x}_j(0) = a_j, j = 1, 2, \cdots, N,\ j \neq i, \end{cases} \tag{18}$$

where $\hat{x}^{(N)}(t) = \frac{1}{N}\sum_{k=1}^{N} \hat{x}_k(t)$. The cost functional is

$$J_i(u_i, u_{-i}) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q(\hat{x}_i(t) - \beta_1\hat{x}^{(N)}(t) - \beta_2\hat{x}_0(t)), (\hat{x}_i(t) - \beta_1\hat{x}^{(N)}(t) - \beta_2\hat{x}_0(t))\rangle \\ + \langle Ru_i(t), u_i(t)\rangle\Big]dt + \langle M\hat{x}_i(T), \hat{x}_i(T)\rangle\Big\}.$$

The corresponding limiting system with the minor agent's perturbation strategy is

$$\begin{cases} d\hat{y}_0(t) = [(A_0 + B_0\Lambda_0(t))\hat{y}_0(t) + B_0\Theta_0(t) + b_0x^{(0)}(t) + f_0(t)]dt + [(C_0 + D_0\Lambda_0(t))\hat{y}_0(t) \\ \qquad\qquad + D_0\Theta_0(t) + l_0x^{(0)}(t) + \sigma_0(t)]dW_0(t) + F_0\displaystyle\int_{E_0} G_0(dedt), \\ d\hat{y}_i(t) = [A\hat{y}_i(t) + Bu_i(t) + b_1x^{(0)}(t) + f(t)]dt + [C\hat{y}_i(t) + Du_i(t) + b_2x^{(0)}(t) \\ \qquad\qquad + H\hat{y}_0(t) + \sigma(t)]dW_i(t) + F\displaystyle\int_{E_i} G_i(dedt), \\ d\hat{y}_j(t) = [(A + B\Lambda_1(t))\hat{y}_j(t) + B\bar{\Theta}_1(t) + b_1x^{(0)}(t) + f(t)]dt + [(C + D\Lambda_1(t))\hat{y}_j(t) \\ \qquad\qquad + D\bar{\Theta}_1(t) + b_2x^{(0)}(t) + H\hat{y}_0(t) + \sigma(t)]dW_j(t) + F\displaystyle\int_{E_j} G_j(dedt), \\ \tilde{y}_0(0) = a_0,\ \tilde{y}_i(0) = a_i,\ \tilde{y}_j(0) = a_j,\ j = 1, 2, \cdots, N,\ j \neq i. \end{cases}$$

The cost functional is

$$\widetilde{J_i}(u_i) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\langle Q(\hat{y}_i(t) - \beta_1 x^{(0)}(t) - \beta_2\hat{y}_0(t)), (\hat{y}_i(t) - \beta_1 x^{(0)}(t) - \beta_2\hat{y}_0(t))\rangle + \langle Ru_i(t), u_i(t)\rangle\Big]dt + \langle M\hat{y}_i(T), \hat{y}_i(T)\rangle\Big\}.$$

Now, we are in a position to state the following approximation results.

Proposition 4.3. *For the fixed i, we have*

$$\begin{aligned}
&(i)\ \sup_{0\le t\le T}\mathbb{E}\Big|\hat{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O(\frac{1}{N}),\\
&(ii)\ \sup_{0\le t\le T}\mathbb{E}\Big||\hat{x}^{(N)}(t)|^2 - |x^{(0)}(t)|^2\Big| = O(\frac{1}{\sqrt{N}}),\\
&(iii)\ \sup_{0\le t\le T}\mathbb{E}\Big|\hat{x}_i(t) - \hat{y}_i(t)\Big|^2 = O(\frac{1}{N}),\\
&(iv)\ \sup_{0\le t\le T}\mathbb{E}\Big||\hat{x}_i(t)|^2 - |\hat{y}_i(t)|^2\Big| = O(\frac{1}{\sqrt{N}}).
\end{aligned}$$

Proof. We prove only the first approximation relationship, and the other three approximation relationships can be obtained by a similar proof as in Proposition 3.3.

Define $\tilde{z}(t) := \hat{x}^{(N)}(t) - x^{(0)}(t)$. According to (10) and (18), we get

$$\begin{cases}
d\tilde{z}(t) = [(A + B\Lambda_1(t) + b_1)\tilde{z}(t)]dt + \mathbb{S}(t)dt + d\mathbb{L}(t) + \dfrac{1}{N}\displaystyle\sum_{k=1}^{N} F\int_{E_k} G_k(dedt),\\
\tilde{z}(0) = 0,
\end{cases}$$

where

$$\begin{aligned}
\mathbb{S}(t) &= \frac{B}{N}[u_i(t) - \Lambda_1(t)\hat{x}_i(t) - \bar{\Theta}_1(t)],\\
\mathbb{L}(t) &= \frac{1}{N}\sum_{k=1,k\neq i}^{N}\int_0^t [(C + D\Lambda_1(r))\hat{x}_k(r) + D\bar{\Theta}_1(r) + b_2\hat{x}^{(N)}(r) + H\hat{x}_0(r) + \sigma(r)]dW_k(r)\\
&\quad + \frac{1}{N}\int_0^t [C\hat{x}_i(r) + Du_i(r) + b_2\hat{x}^{(N)}(r) + H\hat{x}_0(r) + \sigma(r)]dW_i(r).
\end{aligned}$$

Since

$$\int_0^t \mathbb{E}|\mathbb{S}(r)|^2dr \le \frac{3B^2}{N^2}\Big(\int_0^t \mathbb{E}[u_i^2(r)]dr + \int_0^t \mathbb{E}[(\Lambda_1(r))^2\hat{x}_i^2(r)]dr + \int_0^t \mathbb{E}[(\bar{\Theta}_1(r))^2]dr\Big),$$

we get

$$\int_0^t \mathbb{E}|\mathbb{S}(r)|^2dr = O(\frac{1}{N^2}). \tag{19}$$

Note that

$$
\begin{aligned}
V(t) &:= \mathbb{E}\int_0^t (d\mathbb{L}(r))^2 \\
&= \frac{1}{N^2}\sum_{k=1,k\neq i}^{N}\int_0^t \mathbb{E}\Big|(C+D\Lambda_1(r))\hat{x}_k(r)+D\bar{\Theta}_1(r)+b_2\hat{x}^{(N)}(r)+H\hat{x}_0(r)+\sigma(r)\Big|^2 dr \\
&\quad + \frac{1}{N^2}\int_0^t \mathbb{E}\Big|C\hat{x}_i(r)+Du_i(r)+b_2\hat{x}^{(N)}(r)+H\hat{x}_0(r)+\sigma(r)\Big|^2 dr \\
&\leq \frac{T}{N}\sup_{0\leq t\leq T}\max_{0\leq t\leq T}\mathbb{E}\Big|(C+D\Lambda_1(t))\hat{x}_k(t)+D\bar{\Theta}_1(t)+b_2\hat{x}^{(N)}(t)+H\hat{x}_0(t)+\sigma(t)\Big|^2 \\
&\quad + \frac{T}{N^2}\sup_{0\leq t\leq T}\mathbb{E}\Big|C\hat{x}_i(t)+Du_i(t)+b_2\hat{x}^{(N)}(t)+H\hat{x}_0(t)+\sigma(t)\Big|^2.
\end{aligned}
$$

Thus

$$
V(t) = O(\frac{1}{N}). \tag{20}
$$

Applying Itô's formula to $\tilde{z}^2(t)$, we obtain

$$
\begin{aligned}
\mathbb{E}[\tilde{z}^2(t)] &= 2\int_0^t (A+B\Lambda_1(r)+b_1)\mathbb{E}[\tilde{z}^2(r)]dr + 2\int_0^t \mathbb{E}[\tilde{z}(r)\mathbb{S}(r)]dr + V(t) \\
&\quad + \frac{F^2}{N^2}\sum_{i=1}^{N}\mathbb{E}\int_{E_i}\int_0^t \pi_i(dedr) \\
&\leq \sup_{0\leq t\leq T}(|2A+2B\Lambda_1(t)+2b_1|+1)\int_0^t \mathbb{E}[\tilde{z}^2(r)]dr + \int_0^t \mathbb{E}[\mathbb{S}^2(r)]dr + V(t) \\
&\quad + \frac{F^2}{N}\max_{0\leq t\leq T}\mathbb{E}\int_{E_i}\int_0^t \pi_i(dedr).
\end{aligned}
$$

Combining (19) and (20) with Gronwall's inequality, we get

$$
\sup_{0\leq t\leq T}\mathbb{E}\Big|\hat{x}^{(N)}(t) - x^{(0)}(t)\Big|^2 = O(\frac{1}{N}).
$$

This completes the proof. □

By using similar arguments as in Proposition 3.4, we can obtain the following conclusion.

Proposition 4.4. *For any $u_i(\cdot)\in\mathcal{U}_{ad}^{c,i}$, $1\leq i\leq N$, one has*

$$
\Big|J_i(u_i,\bar{u}_{-i}) - \widetilde{J}_i(u_i)\Big| = O(\frac{1}{\sqrt{N}}). \tag{21}
$$

4.3. ϵ-Nash equilibrium

In this subsection, we will verify the ϵ-Nash equilibrium property of the decentralized control strategies (15) and (16).

Before presenting the main result, we give the definition of ϵ-Nash equilibrium in the following manner.

Definition 4.5. *A set of control strategies $\bar{u} = (\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_N)$ where $\bar{u}_i(\cdot) \in \mathcal{U}^{c,i}_{a,d}$, $i = 0, 1 \cdots, N$, is called an ϵ-Nash equilibrium with respect to costs $J_i, i = 0, 1 \cdots, N$, if there exists an $\epsilon \geq 0$, such that for any $i = 0, 1 \cdots, N$, we have*

$$J_i(\bar{u}_i, \bar{u}_{-i}) \leq J_i(u_i, \bar{u}_{-i}) + \epsilon, \tag{22}$$

when any alternative strategy $u_i(\cdot) \in \mathcal{U}^{c,i}_{a,d}$ is applied by agent $\mathcal{A}_i$.

Based on the above results, we obtain the following main result.

Theorem 4.6. *Suppose that $\bar{x}_i(\cdot), i = 0, 1, \cdots, N$, is the solution to the equation systems (8) and (9). Then the set of control strategy profiles $\bar{u} = (\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_N)$ defined by (15) and (16) is an ϵ-Nash equilibrium of* Problem (LP), where $\epsilon = O(\frac{1}{\sqrt{N}}) \to 0$ as $N \to +\infty$.

Proof. Combining Propositions 3.4 and 4.2 with Proposition 4.4, we obtain

$$\begin{aligned} J_i(\bar{u}_i, \bar{u}_{-i}) &= \widetilde{J}_i(\bar{v}_i) + O(\frac{1}{\sqrt{N}}) \\ &\leq \widetilde{J}_i(u_i) + O(\frac{1}{\sqrt{N}}) \\ &\leq J_i(u_i, \bar{u}_{-i}) + O(\frac{1}{\sqrt{N}}), \quad i = 0, 1, \cdots, N. \end{aligned}$$

Therefore, the conclusion holds with $\epsilon = O(\frac{1}{\sqrt{N}})$. □

5. Numerical examples

This section demonstrates the consistency of mean-field estimation as well as the influence of the population's collective behavior $\bar{x}^{(N)}(\cdot)$ on the state trajectories of the agents through a numerical example.

Consider a mean-field game system with one major agent and $N = 500$ minor agents. For any $u_j \in \mathcal{U}^{c,j}_{ad}$, $j = 0, 1, \cdots, N$, the dynamics of the major agent and minor agents are given by

$$\begin{cases} dx_0(t) = (\frac{1}{2}x_0(t) + u_0(t) + x^{(N)}(t))dt + \big(x_0(t) + u_0(t) + x^{(N)}(t)\big)dW_0(t) \\ \qquad\qquad + 2\int_{E_0} G_0(dedt), \\ dx_i(t) = (3x_i(t) + 5u_i(t) + x^{(N)}(t))dt + \big(2x_i(t) + u_i(t) + x^{(N)}(t) + x_0(t)\big)dW_i(t) \\ \qquad\qquad + \int_{E_i} G_i(dedt), \\ x_0(0) = 5,\ x_i(0) = a_i,\ i = 1, \cdots, N, \end{cases} \tag{23}$$

where $t \in [0, T]$ with $T = 1$. Let the initial states of the agents $\{a_i, i = 1, \cdots, N\}$ be independent and identically distributed random variables with the normal distribution $N(-5, 1)$.

The cost functional of the major agent $\mathcal{A}_0$ is

$$J_0(u_0, u_{-0}) = \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[3(x_0(t) - x^{(500)}(t))^2 + u_0^2(t)\Big]dt + 3x_0^2(1)\Big\}, \tag{24}$$

and the cost functional of the minor agent $\mathcal{A}_i, i = 1, \cdots, 500$, is

$$J_i(u_i, u_{-i}) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[2(x_i(t) - x^{(500)}(t) - x_0(t))^2 + u_i^2(t)\Big]dt + x_i^2(1)\Big\}. \tag{25}$$

It is easy to check that $\{P_0(t) \equiv 3, \forall t \in [0, 1]\}$ is a unique solution of the following Riccati equation:

$$\begin{cases} \dot{P}_0(t) + 2P_0(t) - 4P_0^2(t)(1 + P_0(t))^{-1} + 3 = 0, \\ P_0(1) = 3. \end{cases}$$

Suppose that $P_1(\cdot)$ fulfills

$$\begin{cases} \dot{P}_1(t) + 10P_1(t) - 49P_1^2(t)(1 + P_1(t))^{-1} + 2 = 0, \\ P_1(1) = 1. \end{cases}$$

Then the NCE Eq (11) turns out to be

$$\begin{cases} d\bar{y}_0(t) = \Big[- \bar{y}_0(t) + \frac{1}{4}x^{(0)}(t)\Big]dt + \Big[- \frac{1}{2}\bar{y}_0(t) + \frac{1}{4}x^{(0)}(t)\Big]dW_0(t) + 2\int_{E_0} G_0(dedt), \\ \dot{x}^{(0)}(t) = \Big(4 + 40\mathbb{P}(t)\Big)x^{(0)}(t) - 25(1 + P_1(t))^{-1}\eta_1(t) + 5\mathbb{P}(t)\bar{y}_0(t), \\ -\dot{\eta}_1(t) = \Big(3 + 35\mathbb{P}(t)\Big)\eta_1(t) + P_1(t)x^{(0)}(t) + \Big(2P_1(t) + 7\mathbb{P}(t)P_1(t) - 2\Big)\Big(\bar{y}_0(t) + x^{(0)}(t)\Big), \\ \bar{y}_0(0) = a_0, \ \ \eta_1(T) = 0, \ \ x^{(0)}(0) = \frac{1}{N}\sum_{j=1}^{N} a_j, \ \ \eta_0(t) \equiv 0, \quad t \in [0, T], \end{cases} \tag{26}$$

where $\mathbb{P}(t) = -(1 + P_1(t))^{-1}P_1(t)$.

According to Theorem 4.6, the set of control strategies $\bar{u} = (\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_N)$ defined by

$$\begin{aligned} \bar{u}_0(t) &= -\frac{3}{2}\bar{x}_0(t) - \frac{3}{4}x^{(0)}(t), \\ \bar{u}_i(t) &= \mathbb{P}(t)\Big(7\bar{x}_i(t) + x^{(0)}(t) + \bar{y}_0(t)\Big) - 5(1 + P_1(t))^{-1}\eta_1(t), \ i = 1, 2, \cdots, N, \end{aligned}$$

is an ϵ-Nash equilibrium of the mean-field systems (24) and (25), where $\bar{x}_0(\cdot)$ and $\bar{x}_i(\cdot)$ satisfy

$$\begin{cases} d\bar{x}_0(t) = (- \bar{x}_0(t) - \frac{3}{4}x^{(0)}(t) + \bar{x}^{(500)}(t))dt + (- \frac{1}{2}\bar{x}_0(t) - \frac{3}{4}x^{(0)}(t) + \bar{x}^{(500)}(t))dW_0(t) \\ \qquad\qquad + 2\int_{E_0} G_0(dedt), \\ d\bar{x}_i(t) = \Big[(3 + 35\mathbb{P}(t))\bar{x}_i(t) + 5\mathbb{P}(t)x^{(0)}(t) + 5\mathbb{P}(t)\bar{y}_0(t) + \bar{x}^{(N)}(t) \\ \qquad\qquad - 25(1 + P_1(t))^{-1}\eta_1(t)\Big]dt + \Big[(2 + 7\mathbb{P}(t))\bar{x}_i(t) + \mathbb{P}(t)x^{(0)}(t) + \mathbb{P}(t)\bar{y}_0(t) \\ \qquad\qquad + \bar{x}^{(N)}(t) + \bar{x}_0(t) - 5(1 + P_1(t))^{-1}\eta_1(t)\Big]dW_i(t) + \int_{E_i} G_i(dedt), \\ \bar{x}_0(0) = a_0, \ \bar{x}_i(0) = a_i, \ i = 1, \cdots, N, \end{cases} \tag{27}$$

where $\bar{x}^{(500)}(t) = \frac{1}{500}\sum_{j=1}^{500} \bar{x}_j(t)$.

In this article, Merton's jump model (see Merton [29], as well as Platen and Bruti-Liberati [30, pg. 37] is applied to describe the jump-diffusion process. Assume that $\int_{E_0} G_0(dedt) = \mathbb{Q}_0(\mu_0, \sigma_0)d\Pi_0(\lambda_0)$. $\mathbb{Q}_0(\mu_0, \sigma_0)$ is the jump size with a normally distributed mean $\mu_0 \sim N(2, 1)$ and a standard deviation $\sigma_0 = 0.1$. The Poisson process $\Pi_0(\lambda_0)$ has a jump intensity of $\lambda_0 = 2$. For agent $\mathcal{A}_i, i = 1, \cdots, 500$, let $\int_{E_i} G_i(dedt) = \mathbb{Q}_i(\mu_1, \sigma_1)d\Pi_i(\lambda)$. $\mathbb{Q}_i(\mu_1, \sigma_1)$ is the jump size with a normally distributed mean

$\mu_1 \sim N(1, 1)$ and a standard deviation $\sigma_1 = 0.05$. The Poisson process $\Pi_i(\lambda)$ has a jump intensity of $\lambda = 5$.

Figure 1 shows the consistency of mean-field estimation, and the interactive influence between mean-field term $\bar{x}^{(500)}(\cdot)$, and the major state $\bar{x}_0(\cdot)$. When the number of minor agents $N = 500$, as shown in Figure 1, the curves of $\bar{x}^{(500)}(\cdot)$ and $x^{(0)}(\cdot)$ coincide well, which illustrates the consistency of the mean-field estimation indicated by Proposition 3.3.

Figure 2 illustrates the state trajectories of the major agent and all the minor agents. As shown in Figure 2, for each fixed i, the trajectory $\bar{x}_i(\cdot)$ of $\mathcal{A}_i$, in addition to being influenced by its own initial values and parameters, is also affected by the major agent and the collective behavior of all the minor agents.

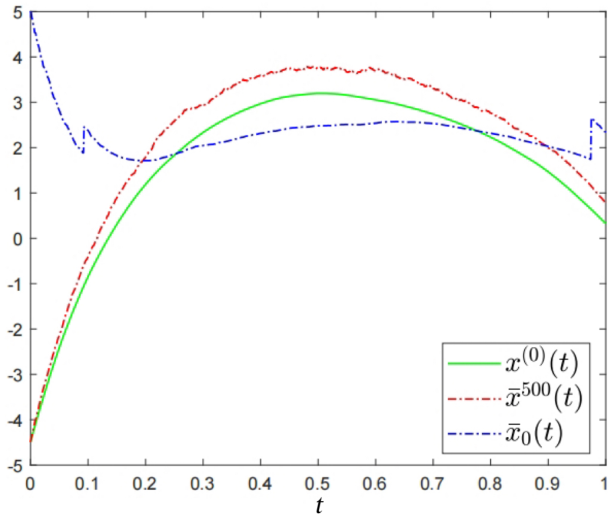

Figure 1. Consistency of mean-field estimation for $a_i \sim N(-5, 1), i = 1, \cdots, 500$, $x_0(0) = 5$.

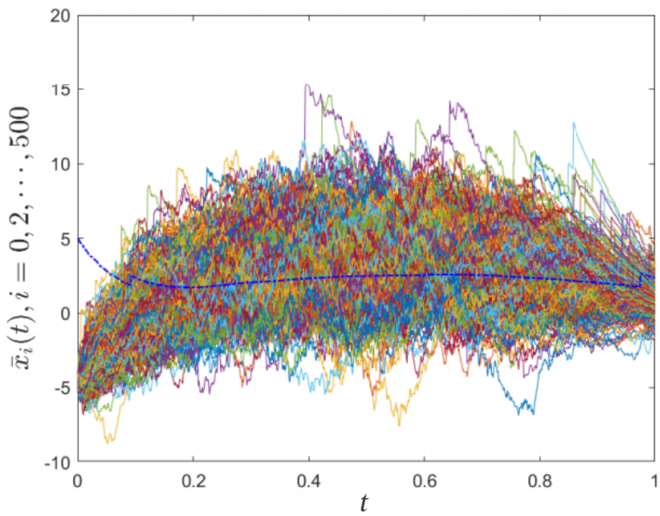

Figure 2. Curves of $\bar{x}_i, i = 0, 1, 2, \cdots, 500$, for $a_i \sim N(-5, 1), i = 1, \cdots, 500$, $x_0(0) = 5$.

To illustrate how the key parameters in the control strategies of Eqs (15) and (16) influence the system's dynamic behavior, we set another set of initial values for $N+1$ agents with $x_0(0) = -5$ and the independent and identically distributed random variables $\{a_i \sim N(5,1),\ i = 1, \cdots, 500\}$. Figures 3 and 4 are shown to elaborate the consistency of mean-field estimation and the curves of $\bar{x}_i, i = 0, 1, 2, \cdots, 500$.

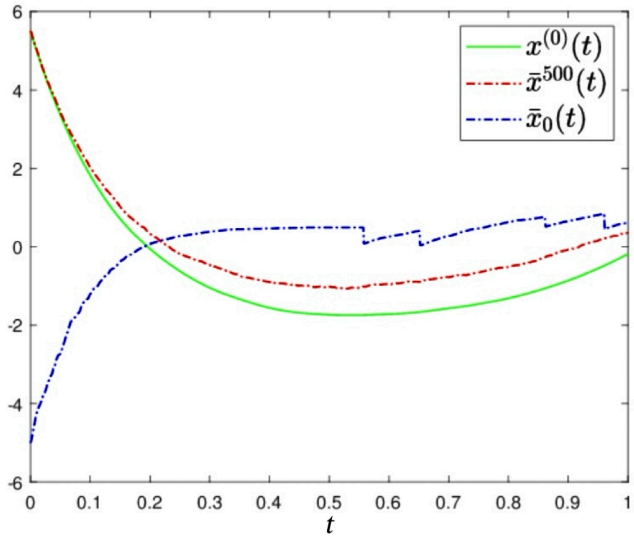

Figure 3. Consistency of mean-field estimation for $a_i \sim N(5,1), i = 1, \cdots, 500$, $x_0(0) = -5$.

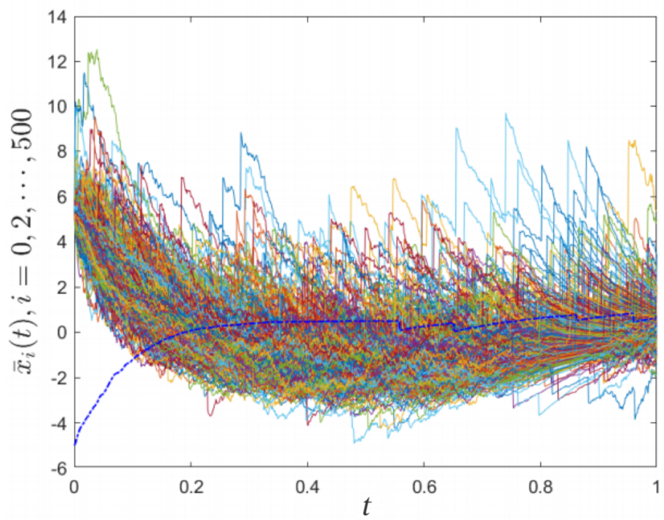

Figure 4. Consistency of mean-field estimation for $a_i \sim N(5,1), i = 1, \cdots, 500$, $x_0(0) = -5$.

6. Conclusions

Motivated by the lack of theory and some practical applications, this paper in concerned with linear-quadratic-Gaussian mean-field games involving mixed agents of a stochastic large population system with random jumps. There are two mixed types of agents: (i) a major agent and (ii) a population of N minor agents where N is very large. The coupling of the major and minor agents exists in both their state dynamics and their individual cost functions. To deal with the dimensionality difficulty and obtain decentralized strategies, the NCE methodology is applied to yield a set of decentralized strategies which is verified to be the ϵ-Nash equilibrium. We provide numerical examples to illustrate both the consistency of the mean-field estimation and the impact of the population's collective behavior. In the future, an interesting research direction is to extend the modeling and analysis to the social optima case, which may involve more applications in practice and generate more challenges in theory. Another potential direction is to study the uniqueness of the equilibrium strategy, which may be more valuable and challenging.

Author contributions

Conceptualization and methodology, R. X.; writing-original draft, review and editing, K. D., J. Z. and Y. Z.; supervision, R. X. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest.

References

1. C. Benazzoli, L. Campi, L. Di Persio, ϵ-Nash equilibrium in stochastic differential games with mean-field interaction and controlled jumps, *Stat. Probabil. Lett.*, **154** (2019), 108522. https://doi.org/10.1016/j.spl.2019.05.021
2. R. Carmona, F. Delarue, Probabilistic analysis of mean-field games, *SIAM J. Control Optim.*, **51** (2013), 2705–2734. https://doi.org/10.1137/120883499

3. Y. Hu, J. Huang, T. Nie, Linear-quadratic-gaussian mixed mean-field games with heterogeneous input constraints, *SIAM J. Control Optim.*, **56** (2018), 2835–2877. https://doi.org/10.1137/17M1151420
4. M. Huang, P. E. Caines, R. P. Malhame, Social optima in mean field LQG control: centralized and decentralized strategies, *IEEE T. Automat. Contr.*, **57** (2012), 1736–1751. https://doi.org/10.1109/TAC.2012.2183439
5. J. Huang, Z. Qiu, S. Wang, Z. Wu, A unified relation analysis of linear-quadratic mean-field game, team, and control, *IEEE T. Automat. Contr.*, **69** (2024), 3325–3332. https://doi.org/10.1109/TAC.2023.3323576
6. B. Wang, J. Zhang, Social optima in mean field linear-quadratic-gaussian models with Markov jump parameters, *SIAM J. Control Optim.*, **55** (2017), 429–456. https://doi.org/10.1137/15M104178X
7. H. Wang, R. Xu, Time-inconsistent LQ games for large-population systems and applications, *J. Optim. Theory Appl.*, **197** (2023), 1249–1268. https://doi.org/10.1007/s10957-023-02223-2
8. R. Xu, F. Zhang, ϵ-Nash mean-field games for general linear-quadratic systems with applications, *Automatica*, **114** (2020), 108835. https://doi.org/10.1016/j.automatica.2020.108835
9. H. Yuan, Q. Zhu, The well-posedness and stabilities of mean-field stochastic differential equations driven by g-Brownian motion, *SIAM J. Control Optim.*, **63** (2025), 596–624. https://doi.org/10.1137/23M1593681
10. H. Yuan, Q. Zhu, The stabilities of delay stochastic McKean-Vlasov equations in the g-framework, *Sci. China Inf. Sci.*, **68** (2025), 112203. https://doi.org/10.1007/s11432-024-4075-2
11. M. Huang, P. E. Caines, R. P. Malhame, Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ-Nash equilibria, *IEEE T. Automat. Contr.*, **52** (2007), 1560–1571. https://doi.org/10.1109/TAC.2007.904450
12. M. Huang, Large-population LQG games involving a major player: the Nash certainty equivalence principle, *SIAM J. Control Optim.*, **48** (2010), 3318–3353. https://doi.org/10.1137/080735370
13. M. Nourian, P. E. Caines, ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents, *SIAM J. Control Optim.*, **51** (2013), 3302–3331. https://doi.org/10.1137/120889496
14. J. M. Lasry, P. L. Lions, Jeux à Champ Moyen. I–Le cas stationnaire, *C. R. Math.*, **343** (2006), 619–625. https://doi.org/10.1016/j.crma.2006.09.019
15. J. M. Lasry, P. L. Lions, Jeux à Champ Moyen. II–Horizon fini et contrôle optimal, *C. R. Math.*, **343** (2006), 679–684. https://doi.org/10.1016/j.crma.2006.09.018
16. J. M. Lasry, P. L. Lions, Mean field games, *Jpn. J. Math.*, **2** (2007), 229–260. https://doi.org/10.1007/s11537-007-0657-8
17. A. Bensoussan, J. Frehse, P. Yam, *Mean field games and mean field type control theory*, New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-8508-7
18. J. Huang, S. Wang, Z. Wu, Backward mean-field linear-quadratic-gaussian (LQG) games: full and partial information, *IEEE T. Automat. Contr.*, **61** (2016), 3784–3796. https://doi.org/10.1109/TAC.2016.2519501

19. T. Nie, S. Wang, Z. Wu, Linear-quadratic delayed mean-field social optimization, *Appl. Math. Optim.*, **89** (2024), 4. https://doi.org/10.1007/s00245-023-10067-5

20. R. Xu, J. Shi, ϵ-Nash mean-field games for linear-quadratic systems with random jumps and applications, *Int. J. Control*, **94** (2021), 1415–1425. https://doi.org/10.1080/00207179.2019.1651940

21. R. Xu, T. Wu, Risk-sensitive large-population linear-quadratic-gaussian games with major and minor agents, *Asian J. Control*, **25** (2023), 4391–4403. https://doi.org/10.1002/asjc.3106

22. H. Wang, R. Xu, Time-inconsistent large-population linear-quadratic games with major and minor agents, *Int. J. Control*, 2025. https://doi.org/10.1080/00207179.2025.2491823

23. B. Düring, P. Markowich, J. F. Pietschmann, M. T. Wolfram, Boltzmann and Fokker-Planck equations modelling opinion formation in the presence of strong leaders, *P. Roy. Soc. A Math. Phy.*, **465** (2009), 3687–3708. https://doi.org/10.1098/rspa.2009.0239

24. Z. Ma, D. S. Callaway, I. A. Hiskens, Decentralized charging control of large populations of plug-in electric vehicles, *IEEE T. Control Syst. Technol.*, **21** (2011), 67–78. https://doi.org/10.1109/TCST.2011.2174059

25. J. Shi, Z. Wu, Maximum principle for forward-backward stochastic control system with random jumps and applications to finance, *J. Syst. Sci. Complex.*, **23** (2010), 219–231. https://doi.org/10.1007/s11424-010-7224-8

26. J. Shi, Z. Wu, A risk-sensitive stochastic maximum principle for optimal control of jump diffusions and its applications, *Acta Math. Sci.*, **31** (2011), 419–433. https://doi.org/10.1016/S0252-9602(11)60242-7

27. R. Cont, P. Tankov, *Financial modelling with jump processes*, Chapman and Hall/CRC, 2003.

28. W. He, Y. Wang, Distributed optimal variational GNE seeking in merely monotone games, *IEEE/CAA J. Automatic.*, **11** (2024), 1621–1630. https://doi.org/10.1109/JAS.2024.124284

29. R. Merton, Option pricing when underlying stock returns are discontinuous, *J. Financ. Econ.*, **3** (1976), 125–144. https://doi.org/10.1016/0304-405X(76)90022-2

30. E. Platen, N. Bruti-Liberati, *Numerical solution of stochastic differential equations with jumps in finance*, Berlin: Springer-Verlag, 2010. https://doi.org/10.1007/978-3-642-13694-8