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Research article

Constructing the soliton wave structure and stability analysis to generalized Calogero–Bogoyavlenskii–Schiff equation using improved simple equation method

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Abstract: In this work, we investigated the (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation, which models long wave propagation in shallow water and plays a significant role in fluid mechanics and plasma physics. Using the improved simple equations method, we obtained various solutions, including dark, bright, and singular solitons, and combinations of singular periodic solutions and exponential rational solutions. Additionally, we performed a linear stability analysis to examine the stability properties of these wave solutions. To further illustrate their characteristics during propagation, we provided 3D and contour plots for some opted wave solutions.

Keywords: generalized Calogero-Bogoyavlenskii-Schiff equation; improved simple equation method; nonlinear partial differential equation (NLPDE); exact solution; stability analysis
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1. Introduction

The natural world surrounding us contains incidents and phenomena that repeat themselves at precise intervals of time, and broaden the underlying order in seemingly complex systems. This periodicity can be seen frequently in daily life, such as the motion of atoms and molecules, the rhythmic behavior of ocean waves, and even biological rhythms like heartbeat. These phenomena, though diverse in scale and nature, can be consistently described and predicted using partial differential equations, providing a unifying framework to understand and analyze the dynamics of our universe.

One of the stunning phenomena observed across many fields is soliton waves. These waves have a remarkable ability to travel through dispersion mediums without losing their shape, and even after collisions, they tend to keep their structure and amplitude without distortion. This unique property makes solitons highly valuable in communication systems, fluid mechanics, and plasma physics [1–3], where stable and distortion-free signal transmission is crucial. Additionally, solitons are prominently observed in natural settings, such as in ocean waves, highlighting their relevance in both technological and environmental contexts.

In this work, we point out our attention to attaining the soliton solutions to the generalized Calogero-Bogoyavlenskii-Schiff equation [4] in (3+1) dimensional form, which is expressed as follows:

$$\begin{cases} u_t + u_{xxy} + 3uu_y + 3u_xv_y + \mu_1 u_y + \mu_2 u_x + \mu_3 v_{yy} + \mu_4 u_z + \mu_5 \left(u_{xxxxx} + 15u_xu_{xx} + 15uu_{xxx} + 45u^2u_x\right) = 0, \\ \quad v_x = u, \end{cases} \tag{1.1}$$

where $v = v(x, y, z, t)$ is the wave profile in the three dispersion axes x, y, z; the time dimension t and $u(x, y, z, t)$ represents the dispersion rate of the wave profile v in the direction of the x axis. Also, Eq (1.1) contains 5 arbitrary constants μ_i, where μ_1, μ_2, and μ_4 represent the strength of dispersion along the y-, x-, and z-axes, respectively; μ_3 denotes the coefficient of second-order dispersion in the y-direction; and μ_5 accounts for higher-order nonlinear effects. This equation has a great impact in fluid mechanics and plasma physics, providing crucial insights into the behavior of nonlinear waves in these domains.

The classical form of the Calogero–Bogoyavlenskii–Schiff equation was derived by two different approaches; both approaches were focused on modeling the propagation of long waves (usually in shallow waters) in a medium having an index of nonlinear dispersion. Schiff [5] obtained the equation by reducing the Yang-Mills self-dual equation, while Bogoyavlenskii [6] used the modified Lax pair equation alongside an inverse scattering problem to obtain it.

Various mathematical techniques have been proposed and employed to address such problems, including Hirota's bilinear method, which has been effectively used to extract soliton, breather, and lump solutions [7, 8]. It has also proven highly effective in extracting interaction solutions and multi-soliton solutions of a variable-coefficient Schrödinger equation [9, 10].

Besides the above, the modified Hirota bilinear method has made significant contributions in dealing with systems where standard bilinearization fails or becomes too complicated [11, 12]. Bilinear neural network method [13], Bäcklund transformation [14–16], and the inverse scattering method [17, 18] all have great impacts on finding exact solutions. In addition, numerous approaches based on the ansatz method have been developed, offering straightforward and effective solutions, such as the improved modified tanh function method [19, 20], the modified extended direct algebraic method [21, 22], and the improved simple equation method [23].

In [4], Yue et al. implemented the Hirota bilinear method to solve Eq (1.1) and obtained soliton solutions, N-soliton solutions, and breather solutions. They also discussed the transformation of breathers into nonlinear localized waves. In this work, we focus on the derivation and stability analysis of soliton solutions using the improved simple equation method. This approach yields a rich set of novel solutions, including various forms of bright, dark, and singular solitons, as well as singular periodic and exponential solutions—many of which were not reported in [4]. Bright solitons, corresponding to localized wave packets, are relevant in optical fibers and Bose–Einstein condensates, where nonlinearity balances dispersion. Dark solitons, characterized by localized intensity dips, arise under different boundary or phase conditions in similar physical contexts. The appearance of singular periodic solutions, marked by sharp transitions within each cycle, suggests potential applications in systems such as plasma waves and nonlinear electrical circuits. These results highlight the physical significance and flexibility of the obtained solutions in describing a wide range of nonlinear wave phenomena.

Our paper is organized as follows: Section 2 discusses the improved simple equation method. Section 3 focuses on applying this method to our problem and presenting the outcomes. Section 4 presents three different kinds of solitons in 3D and contour plots to display the obtained solutions properties. Section 5 is dedicated to discussing the linear stability analysis. Finally, the conclusions of our work are presented in Section 6.

2. Improved simple equation approach

This section is dedicated to illustrating the improved simple equation method in organized and brief steps.

First, consider the nonlinear partial differential equation (NLPDE) [23]

$$\mathcal{F}(f, f_x, f_y, f_z, f_t, f_{xx}, f_{xt}, ...) = 0, \tag{2.1}$$

where $\mathcal{F}$ is a polynomial of f and its partial derivatives, with constant coefficients. So to proceed to the improved simple equation method, we are going to track the following steps.

First step: Transform the NLPDE in Eq (2.1) to an ordinary differential equation (ODE), by using the transformation $f(x, y, z, t) = f(\zeta)$, where $\zeta = \gamma t + \alpha x + \beta y + \delta z$, $\gamma \neq 0$. Then, Eq (2.1) is transformed into an ODE in the following form:

$$\mathcal{F}(f, f', f'', f''', f^{(4)}, f^{(5)}, ...) = 0. \tag{2.2}$$

Second step: Assume the solution for Eq (2.2) in a finite series form

$$f(\zeta) = \sum_{i=-N}^{N} a_i Q^i, \tag{2.3}$$

where N is going to be calculated later by using the homogeneous balance principle method and Q satisfies the following ODE:

$$Q' = d_0 + d_1 Q + d_2 Q^2. \tag{2.4}$$

Third step: Substituting Eq (2.3) into Eq (2.2), and taking into consideration Eq (2.4), we obtain a polynomial of $Q(\zeta)$.
Fourth step: Balancing the coefficients of $Q(\zeta)^i$ to zero for every i in the polynomial, resulting in a system of equations to be dealt with by any mathematical software.
Fifth step: We are going to solve the system obtained from step four to calculate the arbitrary constants assumed in the solution for Eq (2.3) $\{a_i\}$. Using these constants, along with the general solution of Eq (2.4), we get the general exact solution to our problem.

In the following subsections, we demonstrate the solution of Eq (2.4), with different settings for the values of d_i.

2.1. *First family:*

When $d_2 = 0$,

$$Q(\zeta) = \frac{\exp(d_1\zeta)}{d_1} - \frac{d_0}{d_1}.$$

2.2. *Second family:*

When $d_1 = 0$, we get the following [24, 25]:

$$Q(\zeta) = -\sqrt{-\frac{d_0}{d_2}}\tanh\left(\sqrt{-d_0d_2}\,\zeta\right), \quad d_0d_2 < 0, \quad Q(\zeta) = \sqrt{\frac{d_0}{d_2}}\tan\left(\sqrt{d_0d_2}\,\zeta\right), \quad d_0d_2 > 0.$$

2.3. *Third family:*

When $d_0 = 0$, we get the following:

$$Q(\zeta) = \frac{d_1\exp(d_1\zeta)}{1 - d_2\exp(d_1\zeta)}, \quad Q(\zeta) = \frac{-d_1\exp(d_1\zeta)}{1 + d_2\exp(d_1\zeta)}.$$

2.4. *Fourth family ($d_0, d_1, d_2 \neq 0$) [26]*

Case 1: When $d_1^2 > 4d_0d_2$,

$$Q(\zeta) = -\frac{\sqrt{d_1^2 - 4d_0d_2}}{2d_2}\tanh\left(\frac{\sqrt{d_1^2 - 4d_0d_2}}{2}\zeta\right) + \frac{d_1}{2d_2}, \quad Q(\zeta) = -\frac{\sqrt{d_1^2 - 4d_0d_2}}{2d_2}\coth\left(\frac{\sqrt{d_1^2 - 4d_0d_2}}{2}\zeta\right) + \frac{d_1}{2d_2},$$

$$Q(\zeta) = \frac{\sqrt{(d_1^2 - 4d_0d_2)(A^2 + B^2)} - A\sqrt{d_1^2 - 4d_0d_2}\cosh\left(\sqrt{d_1^2 - 4d_0d_2}\,\zeta\right)}{2d_2\left(A\sinh\left(\sqrt{d_1^2 - 4d_0d_2}\,\zeta\right) + B\right)} - \frac{d_1}{2d_2}, \quad A, B \neq 0.$$

$$Q(\zeta) = \frac{2d_0\cosh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right)}{\sqrt{d_1^2 - 4d_0d_2}\sinh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right) - d_1\cosh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right)},$$

$$Q(\zeta)=\frac{2d_0\sinh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right)}{\sqrt{d_1^2-4d_0d_2}\cosh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right)-d_1\sinh\left(\frac{\sqrt{d_1^2-4d_0d_2}}{2}\zeta\right)}.$$

Case 2: When $d_1^2<4d_0d_2$,

$$Q(\zeta)=\frac{\sqrt{4d_0d_2-d_1^2}}{2d_2}\tan\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)-\frac{d_1}{2d_2},\quad Q(\zeta)=-\frac{\sqrt{4d_0d_2-d_1^2}}{2d_2}\cot\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)+\frac{d_1}{2d_2},$$

$$Q(\zeta)=\frac{\sqrt{(4d_0d_2-d_1^2)(A^2-B^2)}-A\sqrt{4d_0d_2-d_1^2}\cos\left(\sqrt{4d_0d_2-d_1^2}\,\zeta\right)}{2d_2\Big(A\sin\left(\sqrt{4d_0d_2-d_1^2}\,\zeta\right)+B\Big)}-\frac{d_1}{2d_2},\quad B^2<A^2,$$

and $A,B\neq 0$.

$$Q(\zeta)=-\frac{2d_0\cos\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)}{\sqrt{4d_0d_2-d_1^2}\sin\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)+d_1\cos\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)},$$

$$Q(\zeta)=\frac{2d_0\sin\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)}{\sqrt{4d_0d_2-d_1^2}\cos\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)-d_1\sin\left(\frac{\sqrt{4d_0d_2-d_1^2}}{2}\zeta\right)}.$$

3. Outcomes

Now, we extract the waves solution for our problem Eq (1.1) by applying the improved simple equation method. To begin, we will simplify it in one single form as follows:

$$v_{xxxy}+\mu_1v_{xy}+\mu_2v_{xx}+\mu_3v_{yy}+\mu_4v_{xz}+\mu_5\left(v_{xxxxxx}+15v_{xxxx}v_x+15v_{xx}v_{xxx}+45v_{xx}(v_x)^2\right)+3v_xv_{xy}+v_{xt}+3v_{xx}v_y=0.\quad(3.1)$$

Substituting the wave-like solution $v(x,y,z,t)=v(\zeta)$ into Eq (3.1), where $\zeta=\gamma t+\alpha x+\beta y+\delta z$, with $\gamma\neq 0$; the parameters α, β, and δ represent the spatial coefficients in the x, y, and z directions, respectively; and γ characterizes the temporal evolution or wave speed, we obtain the following ordinary differential equation:

$$\alpha^6\mu_5v^{(6)}+\alpha^3v^{(4)}\left(15\alpha^2\mu_5v'+\beta\right)+v''\left(15\alpha^5\mu_5v^{(3)}+45\alpha^4\mu_5\left(v'\right)^2+6\alpha^2\beta v'+\alpha^2\mu_2+\alpha\beta\mu_1+\alpha\gamma+\alpha\delta\mu_4+\beta^2\mu_3\right)=0.\quad(3.2)$$

By equating the greatest nonlinear term $v''v^{(3)}$ with the greatest derivative $v^{(6)}$, we obtain that $2N + 5 = N + 6$, which implies that $N = 1$. Therefore, we write the wave profile v in this form:

$$v(\zeta) = a_0 + a_1 Q + a_{-1} Q^{-1}.$$

Inserting the above equation and Eq (2.4) into Eq (3.2), we get a polynomial in Q. Equating the coefficients of the polynomial to zero, we get a system of algebraic equations to be solved for each family we have discussed in the previous section.

3.1. First family:

When $d_2 = 0$, we get the following solution set:

$$\left\{ d_1 = \frac{i\sqrt{\beta}}{\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}, a_1 = 0, a_{-1} = 4\alpha d_0, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5} \right\}.$$

According to the above solution set, we obtain the following pair of exponential wave solutions:

$$\begin{cases} v(\zeta) = a_0 + \dfrac{4\sqrt{-\beta}d_0}{\sqrt{5\alpha\mu_5}\left(\exp\left(\frac{\sqrt{-\beta}\zeta}{\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}\right) - d_0\right)}, \\ u(\zeta) = \dfrac{4\beta d_0 e^{\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\zeta}}{5\alpha\mu_5\left(e^{\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\zeta} - d_0\right)^2}. \end{cases} \tag{3.3}$$

3.2. Second family:

When $d_1 = 0$, we get three different sets of solutions:

Set I

$$\left\{ a_1 = -\frac{\beta}{5\alpha^2 d_0\mu_5}, a_{-1} = 0, d_2 = \frac{\beta}{20\alpha^3 d_0\mu_5}, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5} \right\}.$$

According to the above solution set, we attain the following dark and bright soliton wave solutions and a pair of singular periodic solutions:

$$\begin{cases} v(\zeta) = a_0 + \sqrt{-\dfrac{4\beta}{5\alpha\mu_5}}\tanh\left(\sqrt{-\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right), \\ u(\zeta) = -\dfrac{\beta\,\mathrm{sech}^2\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)}{5\alpha\mu_5}. \end{cases} \tag{3.4}$$

$$\begin{cases} v(\zeta) = a_0 - \sqrt{\dfrac{4\beta}{5\alpha\mu_5}} \tan\left(\sqrt{\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right), \\ u(\zeta) = -\dfrac{\beta \sec^2\left(\dfrac{\sqrt{\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right)}{5\alpha\mu_5}. \end{cases} \tag{3.5}$$

Set II

$$\left\{a_1 = 0, a_{-1} = 4\alpha d_0, d_2 = \frac{\beta}{20\alpha^3 d_0\mu_5}, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5}\right\}.$$

According to the above solution set, we obtain the following pair of singular soliton solutions and a pair of singular periodic solutions:

$$\begin{cases} v(\zeta) = a_0 - \sqrt{-\dfrac{4\beta}{5\alpha\mu_5}} \coth\left(\sqrt{-\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right), \\ u(\zeta) = -\dfrac{\beta \text{csch}^2\left(\dfrac{\sqrt{-\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right)}{5\left(\alpha\mu_5\right)}. \end{cases} \tag{3.6}$$

$$\begin{cases} v(\zeta) = a_0 + \sqrt{\dfrac{4\beta}{5\alpha\mu_5}} \cot\left(\sqrt{\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right), \\ u(\zeta) = -\dfrac{\beta \csc^2\left(\dfrac{\sqrt{\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right)}{5\left(\alpha\mu_5\right)}. \end{cases} \tag{3.7}$$

Set III

$$\left\{a_1 = -\frac{\beta}{20\alpha^2 d_0\mu_5}, a_{-1} = 4\alpha d_0, d_2 = \frac{\beta}{80\alpha^3 d_0\mu_5}, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5}\right\}.$$

According to the above solution set, we obtain the following pair of singular soliton solutions and a pair of singular periodic solutions:

$$\begin{cases} v(\zeta) = a_0 - \sqrt{-\frac{\beta}{5\alpha\mu_5}} \tanh\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)\left(\coth^2\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right) + 1\right), \\ u(\zeta) = -\frac{2\beta \operatorname{csch}^2\left(\frac{\sqrt{-\frac{\beta}{\alpha^3\mu_5}}\zeta}{\sqrt{5}}\right)}{5\left(\alpha\mu_5\right)}. \end{cases} \tag{3.8}$$

$$\begin{cases} v(\zeta) = a_0 + \sqrt{\frac{\beta}{5\alpha\mu_5}} \tan\left(\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)\left(\cot^2\left(\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\zeta\right) - 1\right), \\ u(\zeta) = -\frac{2\beta \csc^2\left(\frac{\sqrt{\frac{\beta}{\alpha^3\mu_5}}\zeta}{\sqrt{5}}\right)}{5\left(\alpha\mu_5\right)}. \end{cases} \tag{3.9}$$

3.3. Third family

When $d_0 = 0$, we get the following solution set:

$$\left\{a_1 = -4\alpha d_2, a_{-1} = 0, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5}, d_1 = \frac{i\sqrt{\beta}}{\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}\right\}.$$

The above set implies the following two pairs of exponential wave solutions:

$$\begin{cases} v(\zeta) = a_0 + 4\sqrt{-\frac{\beta}{5\alpha\mu_5}}\left(\frac{1}{d_2 e^{\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\zeta} - 1} + 1\right), \\ u(\zeta) = \frac{4\left(\beta d_2\right) e^{\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}}}{5\alpha\mu_5\left(d_2 e^{\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}} - 1\right)^2}. \end{cases} \tag{3.10}$$

$$\begin{cases} v(\zeta) = a_0 + 4\sqrt{-\frac{\beta}{5\alpha\mu_5}}\left(1 - \frac{1}{d_2 e^{\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\zeta} + 1}\right), \\ u(\zeta) = -\frac{4\left(\beta d_2\right) e^{\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}}}{\alpha\mu_5 5\left(d_2 e^{\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}} + 1\right)^2}. \end{cases} \tag{3.11}$$

3.4. Fourth family

When $(d_0, d_1, d_2 \neq 0)$, we get the following sets of solutions:

Set I

$$\left\{a_1 = -4\alpha d_2, a_{-1} = 0, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5}, d_1 = \frac{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}}{\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}\right\}.$$

Considering the condition $d_1^2 > 4d_0 d_2$, the above set implies the following solutions:

$$\begin{cases} v(\zeta) = a_0 + 2\left(\sqrt{\dfrac{20\alpha^3 d_0 d_2\mu_5 - \beta}{5\alpha\mu_5}} + \sqrt{-\dfrac{\beta}{5\alpha\mu_5}}\tanh\left(\sqrt{-\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right)\right), \\ u(\zeta) = -\dfrac{\beta\operatorname{sech}^2\left(\frac{\zeta\sqrt{-\frac{\beta}{\alpha^3\mu_5}}}{2\sqrt{5}}\right)}{5\alpha\mu_5}. \end{cases} \tag{3.12}$$

$$\begin{cases} v(\zeta) = a_0 + 2\left(\dfrac{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}}{\sqrt{5\alpha\mu_5}} + \sqrt{-\dfrac{\beta}{5\alpha\mu_5}}\coth\left(\dfrac{\sqrt{-\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right)\right), \\ U(\zeta) = \dfrac{\beta\operatorname{csch}^2\left(\frac{\sqrt{-\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right)}{5\left(\alpha\mu_5\right)}. \end{cases} \tag{3.13}$$

$$\begin{cases} v(\zeta) = a_0 + 2\left(\dfrac{\sqrt{-\frac{\beta}{5\alpha\mu_5}}\left(\sqrt{A^2 + B^2} + A\cosh\left(\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}(\gamma t + \alpha x + \beta y + \delta z)\right)\right)}{A\sinh\left(\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}(\gamma t + \alpha x + \beta y + \delta z)\right) + B} + \dfrac{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}}{\sqrt{5\mu_{5\alpha}}}\right), \\ u(\zeta) = \dfrac{2A\beta\left(\sqrt{A^2 + B^2}\cosh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right) + A - B\sinh\left(\frac{\zeta\sqrt{-\frac{\beta}{\alpha^3\mu_5}}}{\sqrt{5}}\right)\right)}{(\alpha\mu_5)\left(5\left(A\sinh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right) + B\right)^2\right)}. \end{cases} \tag{3.14}$$

$$\begin{cases} v(\zeta) = a_0 + \dfrac{8d_0 d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta} - \sqrt{-\beta}\tanh\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)}, \\ u(\zeta) = -\dfrac{4\alpha^2\beta d_0 d_2\operatorname{sech}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta} - \sqrt{-\beta}\tanh\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)\right)^2}. \end{cases} \tag{3.15}$$

$$\begin{cases} v(\zeta) = a_0 + \dfrac{8d_0d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3d_0d_2\mu_5-\beta} - \sqrt{-\beta}\coth\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)}, \\ u(\zeta) = \dfrac{4\alpha^2\beta d_0d_2\operatorname{csch}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{20\alpha^3d_0d_2\mu_5-\beta} - \sqrt{-\beta}\coth\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)\right)^2}, \end{cases} \tag{3.16}$$

where Eq (3.12) represents a pair of dark and bright solitons; Eq (3.15) represent a pair of rational wave solutions; and Eq (3.13), (3.14), and (3.16) represent three pairs of singular solitons.

Considering the condition $d_1^2 < 4d_0d_2$, the above set implies the following solutions:

$$\begin{cases} v(\zeta) = a_0 + 2\left(\sqrt{\dfrac{20\alpha^3d_0d_2\mu_5-\beta}{5\alpha\mu_5}} - \sqrt{\dfrac{\beta}{5\alpha\mu_5}}\tan\left(\sqrt{\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right)\right), \\ u(\zeta) = -\dfrac{2\beta}{5\alpha\mu_5\left(\cos\left(\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\zeta\right)+1\right)}. \end{cases} \tag{3.17}$$

$$\begin{cases} v(\zeta) = a_0 + 2\left(\dfrac{\sqrt{20\alpha^3d_0d_2\mu_5-\beta}}{\sqrt{5\alpha\mu_5}} + \sqrt{\dfrac{\beta}{5\alpha\mu_5}}\cot\left(\sqrt{\dfrac{\beta}{20\alpha^3\mu_5}}\zeta\right)\right), \\ u(\zeta) = \dfrac{\beta}{5\alpha\mu_5\left(\cos\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)-1\right)}. \end{cases} \tag{3.18}$$

$$\begin{cases} v(\zeta) = a_0 + \dfrac{\sqrt{\frac{\beta}{5\alpha\mu_5}}\left(2\left(\sqrt{A^2-B^2}+A\cos\left(\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\zeta\right)\right)\right)}{A\sin\left(\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\zeta\right)+B} + 2\sqrt{\dfrac{20\alpha^3d_0d_2\mu_5-\beta}{5\alpha\mu_5}}, \\ u(\zeta) = -\dfrac{2A\beta\left(\sqrt{(A-B)(A+B)}\cos\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)+A+B\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)\right)}{5\alpha\mu_5\left(A\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)+B\right)^2}. \end{cases} \tag{3.19}$$

$$\begin{cases} v(\zeta) = a_0 + \dfrac{8d_0d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3d_0d_2\mu_5-\beta} + \sqrt{\beta}\tan\left(\sqrt{\frac{\beta}{20\alpha^3\mu_5}}(\gamma t+\alpha x+\beta y+\delta z)\right)}, \\ u(\zeta) = -\dfrac{4i\alpha^2\beta d_0d_2\sec^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{\beta}\tan\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)+\sqrt{20\alpha^3d_0d_2\mu_5-\beta}\right)^2}. \end{cases} \tag{3.20}$$

$$\begin{cases} v(\zeta) = a_0 + \dfrac{8d_0d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3d_0d_2\mu_5-\beta} - \sqrt{\beta}\cot\left(\sqrt{\frac{\beta}{20\alpha^3\mu_5}}(\gamma t+\alpha x+\beta y+\delta z)\right)}, \\ u(\zeta) = -\dfrac{4i\alpha^2\beta d_0d_2\csc^2\left(\sqrt{-\frac{\beta\zeta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{20\alpha^3d_0d_2\mu_5-\beta} - \sqrt{\beta}\cot\left(\sqrt{-\frac{\beta\zeta}{20\alpha^3\mu_5}}\right)\right)^2}. \end{cases} \tag{3.21}$$

Where, Eq (3.17) to Eq (3.21) represent singular periodic solutions.

Set II

$$\left\{a_1 = 0, a_{-1} = 4\alpha d_0, \gamma = \frac{-25\alpha^2\mu_2\mu_5 - 25\alpha\beta\mu_1\mu_5 - 25\alpha\delta\mu_4\mu_5 - 25\beta^2\mu_3\mu_5 + 4\beta^2}{25\alpha\mu_5}, d_1 = \frac{\sqrt{20\alpha^3d_0d_2\mu_5-\beta}}{\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}\right\}.$$

The above set implies the following solutions:

$$\begin{cases} v(\zeta) = a_0 - \dfrac{8\sqrt{5}d_0d_2\sqrt{\alpha^5\mu_5}}{\sqrt{20\alpha^3d_0d_2\mu_5-\beta} + \sqrt{-\beta}\tanh\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)}, \\ u(\zeta) = -\dfrac{4\alpha^2\beta d_0d_2\text{sech}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{-\beta}\tanh\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right) + \sqrt{20\alpha^3d_0d_2\mu_5-\beta}\right)^2}. \end{cases} \tag{3.22}$$

$$\begin{cases} v(\zeta) = a_0 - \dfrac{8\sqrt{5}d_0d_2\sqrt{\alpha^5\mu_5}}{\sqrt{20\alpha^3d_0d_2\mu_5-\beta} + \sqrt{-\beta}\coth\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right)}, \\ u(\zeta) = -\dfrac{-4\alpha^2\beta d_0d_2\text{csch}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{\left(\sqrt{-\beta}\coth\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right) + \sqrt{20\alpha^3d_0d_2\mu_5-\beta}\right)^2}. \end{cases} \tag{3.23}$$

$$\begin{cases} v(\zeta) = a_0 - \dfrac{8\sqrt{5}d_0d_2\sqrt{\alpha^5\mu_5}\left(A\sinh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)+B\right)}{\sqrt{-\beta}\left(\sqrt{A^2+B^2}+A\cosh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)\right) + \sqrt{20\alpha^3d_0d_2\mu_5-\beta}\left(A\sinh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)+B\right)}, \\ u(\zeta) = \dfrac{8\alpha^2A\beta d_0d_2\left(\sqrt{A^2+B^2}\cosh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)+A-B\sinh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)\right)}{\left(\sqrt{-\beta}\sqrt{A^2+B^2}+A\sqrt{-\beta}\cosh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)+A\sqrt{20\alpha^3d_0d_2\mu_5-\beta}\sinh\left(\zeta\sqrt{-\frac{\beta}{5\alpha^3\mu_5}}\right)+B\sqrt{20\alpha^3d_0d_2\mu_5-\beta}\right)^2}. \end{cases} \tag{3.24}$$

$$\begin{cases} v(\zeta) = a_0 + 2\left(\sqrt{-\frac{\beta}{5\alpha\mu_5}}\tanh\left(\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\zeta\right) - \sqrt{\frac{20\alpha^3 d_0 d_2\mu_5 - \beta}{5\alpha\mu_5}}\right), \\ u(\zeta) = -\frac{\beta\operatorname{sech}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{5\alpha\mu_5}. \end{cases} \tag{3.25}$$

$$\begin{cases} v(\zeta) = a_0 + 2\left(\sqrt{-\frac{\beta}{5\alpha\mu_5}}\coth\left(\frac{\sqrt{-\frac{\beta}{\alpha^3\mu_5}}\zeta}{2\sqrt{5}}\right) - \sqrt{\frac{20\alpha^3 d_0 d_2\mu_5 - \beta}{5\alpha\mu_5}}\right), \\ u(\zeta) = \frac{\beta\operatorname{csch}^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{5\alpha\mu_5}. \end{cases} \tag{3.26}$$

Where Eq (3.22) represents a pair of rational soliton solutions; Eq (3.25) represents a pair of dark and bright solitons; and Eq (3.23), (3.24), and (3.26) represent singular soliton wave solutions.

$$\begin{cases} v(\zeta) = a_0 - \frac{8 d_0 d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta} - \sqrt{\beta}\tan\left(\frac{\sqrt{\beta}\zeta}{\sqrt{20\alpha^3\mu_5}}\right)}, \\ u(\zeta) = -\frac{4\alpha^2\beta d_0 d_2}{\left(\sqrt{\beta}\sin\left(\zeta\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\right) - \sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}\cos\left(\zeta\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\right)\right)^2}. \end{cases} \tag{3.27}$$

$$\begin{cases} v(\zeta) = a_0 - \frac{8 d_0 d_2\sqrt{5\alpha^5\mu_5}}{\sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta} + \sqrt{\beta}\cot\left(\frac{\sqrt{\beta}\zeta}{\sqrt{20\alpha^3\mu_5}}\right)}, \\ u(\zeta) = -\frac{4\alpha^2\beta d_0 d_2}{\left(\sqrt{\beta}\cos\left(\zeta\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\right) + \sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}\sin\left(\zeta\sqrt{\frac{\beta}{20\alpha^3\mu_5}}\right)\right)^2}. \end{cases} \tag{3.28}$$

$$\begin{cases} v(\zeta) = a_0 - \frac{8\sqrt{5} d_0 d_2\sqrt{\alpha^5\mu_5}\left(A\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right) + B\right)}{\sqrt{\frac{\beta\mu_5}{\mu_5}}\left(\sqrt{A^2 - B^2} + A\cos\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)\right) + \sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}\left(A\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right) + B\right)}, \\ u(\zeta) = -\frac{8\alpha^2 A\beta d_0 d_2\left(\sqrt{\beta(A^2 - B^2)}\cos\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right) + A + B\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)\right)}{\left(\left(\sqrt{\beta(A^2 - B^2)} + A\sqrt{\beta}\cos\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right)\right) + \sqrt{20\alpha^3 d_0 d_2\mu_5 - \beta}\left(A\sin\left(\zeta\sqrt{\frac{\beta}{5\alpha^3\mu_5}}\right) + B\right)\right)^2}. \end{cases} \tag{3.29}$$

$$\begin{cases} v(\zeta) = a_0 - \dfrac{2\left(\sqrt{20\alpha^3 d_0 d_2 \mu_5 - \beta} + \sqrt{\beta}\tan\left(\frac{\sqrt{\beta}\zeta}{\sqrt{20\alpha^3\mu_5}}\right)\right)}{\sqrt{5\alpha\mu_5}}, \\ u(\zeta) = -\dfrac{i\beta\sec^2\left(\zeta\sqrt{-\frac{\beta}{20\alpha^3\mu_5}}\right)}{5(\alpha\mu_5)}. \end{cases} \tag{3.30}$$

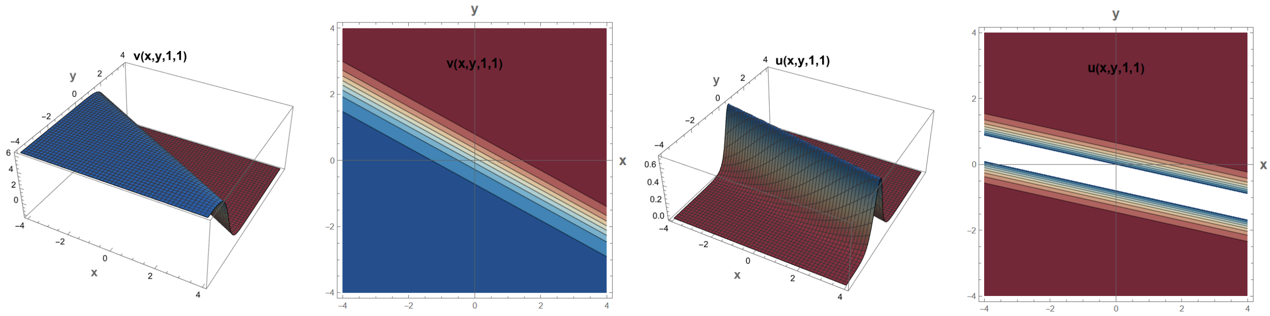
$$\begin{cases} v(\zeta) = a_0 - \dfrac{2\left(\sqrt{20\alpha^3 d_0 d_2 \mu_5 - \beta} - \sqrt{\beta}\cot\left(\frac{\sqrt{\beta}(\gamma t+\alpha x+\beta y+\delta z)}{\sqrt{20\alpha^3\mu_5}}\right)\right)}{\sqrt{5\alpha\mu_5}}, \\ u(\zeta) = \dfrac{i\beta\text{csch}^2\left(\frac{\sqrt{\beta}(\gamma t+\alpha x+\beta y+\delta z)}{2\sqrt{5}\alpha^{3/2}\sqrt{\mu_5}}\right)}{5(\alpha\mu_5)}. \end{cases} \tag{3.31}$$

Where Eq (3.27) to Eq (3.31) are singular periodic solutions.

4. Graphical illustration

In this section, we illustrate the features of selected solutions via 3D plotting and contour plotting. All the following figures are plotted at $z = 1$ and at time $t = 1$.

Figure 1 displays a dark soliton described by Eq (3.4) in 3D, and contour plotting with $\{\mu_5 = -0.09, \mu_4 = 1.08, \mu_3 = -2.05, \mu_2 = 0.04, \mu_1 = 0.46, \beta = -1.64, \alpha = -0.9, \delta = -4, a_0 = 2.265\}$, and represents a bright soliton by setting $\{\mu_5 = -1.26, \mu_4 = 0.73, \mu_3 = -0.25, \mu_2 = 0.12, \mu_1 = 0.61, \beta = -2.25, \alpha = -0.5, \delta = -4\}$.



(a) **(b)** **(c)** **(d)**

Figure 1. (a) and (b), represent dark soliton in 3D , and contour Plot at $z = t = 1$, with the parameters sets as: $\{\mu_5 = -0.09, \mu_4 = 1.08, \mu_3 = -2.05, \mu_2 = 0.04, \mu_1 = 0.46, \beta = -1.64, \alpha = -0.9, \delta = -4, a_0 = 2.265\}$; (c) and (d) represents bright solitons at $z = t = 1$, with the parameters sets as: $\{\mu_5 = -1.26, \mu_4 = 0.73, \mu_3 = -0.25, \mu_2 = 0.12, \mu_1 = 0.61, \beta = -2.25, \alpha = -0.5, \delta = -4\}$.

Figure 2 represents a pair of singular periodic solutions described by Eq (3.7) in 3D, and contour plotting with $\{\mu_5 = -1.27, \mu_4 = -1.56, \mu_3 = 1.3, \mu_2 = 1.28, \mu_1 = -0.01, \beta = -1.68, \alpha = 0.5,$

$\delta = -2.97$, $a_0 = 1\}$, and $\{\ \mu_5 = -0.4$, $\mu_4 = 0.46$, $\mu_3 = 0.51$, $\mu_2 = -0.25$, $\mu_1 = -0.72$, $\beta = -1.41$, $\alpha = 0.46$, $\delta = 0.25\}$.

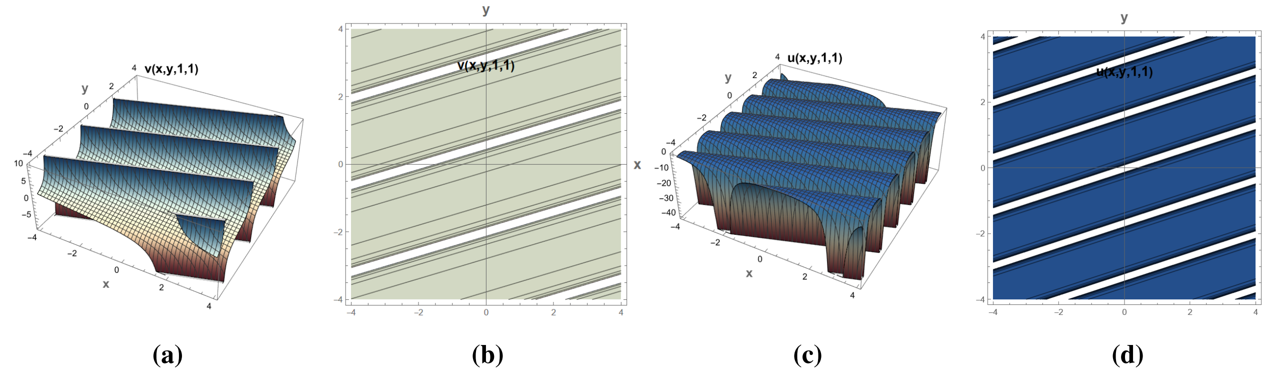


Figure 2. (a) and (b) represent singular periodic solution in 3D, and contour plot at $z = t = 1$, by the following settings$\{\ \mu_5 = -1.27$, $\mu_4 = -1.56$, $\mu_3 = 1.3$, $\mu_2 = 1.28$, $\mu_1 = -0.01$, $\beta = -1.68$, $\alpha = 0.5$, $\delta = -2.97$, $a_0 = 1\}$; (c) and (d) represents singular periodic solution in 3D, and contour plot at $z = t = 1$, by setting $\{\ \mu_5 = -0.4$, $\mu_4 = 0.46$, $\mu_3 = 0.51$, $\mu_2 = -0.25$, $\mu_1 = -0.72$, $\beta = -1.41$, $\alpha = 0.46$, $\delta = 0.25\}$.

Figure 3 illustrates a pair of singular soliton solutions represented by Eq (3.6) in 3D, and contour plot using the parameters $\{\ \mu_5 = 0.26$, $\mu_4 = 0.29$, $\mu_3 = -0.18$, $\mu_2 = 0.17$, $\mu_1 = -0.01$, $\beta = 0.07$, $\alpha = -4$, $\delta = -1.14$, $a_0 = 1\}$ and $\{\ \mu_5 = -0.6$, $\mu_4 = -0.13$, $\mu_3 = 0.56$, $\mu_2 = 0.54$, $\mu_1 = 0.73$, $\beta = -3.68$, $\alpha = -3.34$, $\delta = -3.95\}$.

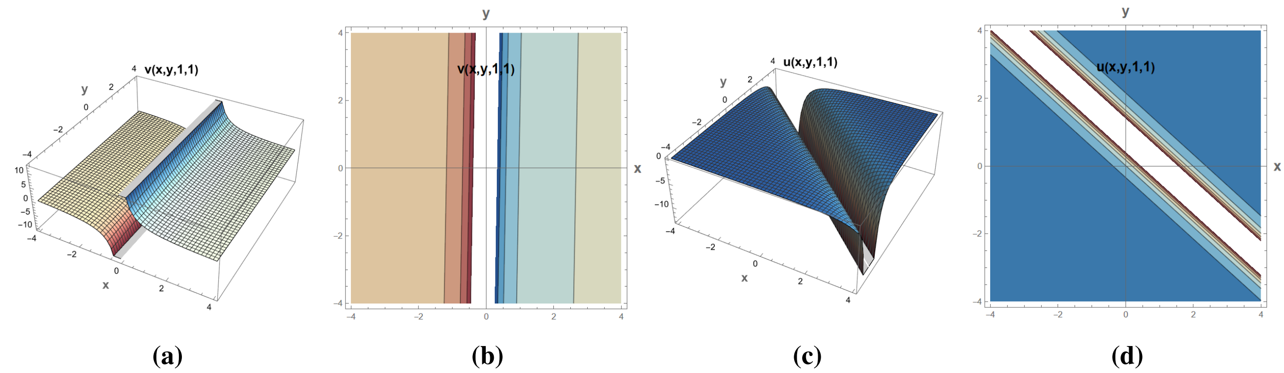


Figure 3. (a) and (b) represent singular soliton in 3D, and contour plot at $z = t = 1$, by setting $\{\ \mu_5 = 0.26$, $\mu_4 = 0.29$, $\mu_3 = -0.18$, $\mu_2 = 0.17$, $\mu_1 = -0.01$, $\beta = 0.07$, $\alpha = -4$, $\delta = -1.14$, $a_0 = 1\}$; (c) and (d) representing singular soliton in 3D, and contour plot at $z = t = 1$, by setting $\{\ \mu_5 = -0.6$, $\mu_4 = -0.13$, $\mu_3 = 0.56$, $\mu_2 = 0.54$, $\mu_1 = 0.73$, $\beta = -3.68$, $\alpha = -3.34$, $\delta = -3.95\}$.

5. Stability analysis

In this section, we apply the linear stability analysis to Eq (3.1) [27–29]. By assuming the perturbed solution as follows:

$$v(x, y, z, t) = q_0 + \lambda U(x, y, z, t). \tag{5.1}$$

Here, q_0 is the steady-state solution and λ is the perturbed parameter, usually taken to be very small. Then, we substitute the above perturbed solution into Eq (3.1) and obtain the following:

$$\begin{aligned}&\lambda\mu_3 U_{yy} + \lambda U_{tx} + \lambda\mu_4 U_{xz} + \lambda\mu_1 U_{xy} + 3\lambda^2 U_x U_{xy} + \lambda\mu_2 U_{xx} + 3\lambda^2 U_y U_{xx} + \lambda U_{xxxy}\\ &+ \mu_5\left(45\lambda^3 U_x^2 U_{xx} + 15\lambda^2 U_{xx} U_{xxx} + 15\lambda^2 U_x U_{xxxx} + \lambda U_{xxxxxx}\right) = 0.\end{aligned} \tag{5.2}$$

By ignoring the nonlinear terms in Eq (5.2), we obtain the following:

$$\mu_5 U_{xxxxxx} + \mu_1 U_{xy} + \mu_2 U_{xx} + \mu_3 U_{yy} + \mu_4 U_{xz} + U_{xxxy} + U_{tx} = 0. \tag{5.3}$$

Then, for the Eq (5.3), we solve it by substituting the following solution in Eq (5.3).

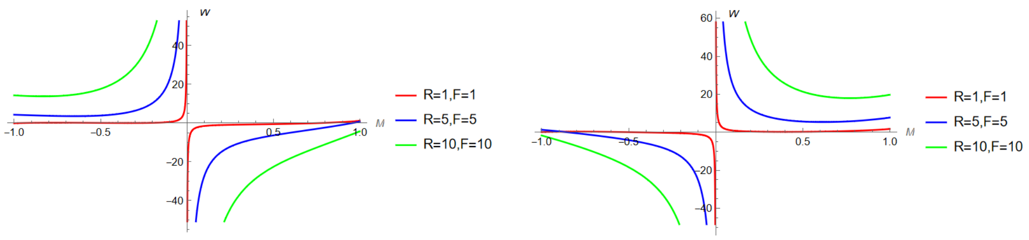
$$U(x, y, z, t) = \rho e^{i(Wt+Mx+Ry+Fz)}, \tag{5.4}$$

where M, R, and F are the wave numbers for each axis, and W is the wave frequency. The following equation are obtained:

$$F\mu_4 M + \mu_5 M^6 - M^3 R + \mu_2 M^2 + \mu_1 MR + MW + \mu_3 R^2 = 0.$$

Then, we get the relation between the wave numbers and frequency,

$$W = \frac{-F\mu_4 M - \mu_5 M^6 + M^3 R - \mu_2 M^2 - \mu_1 MR - \mu_3 R^2}{M}. \tag{5.5}$$

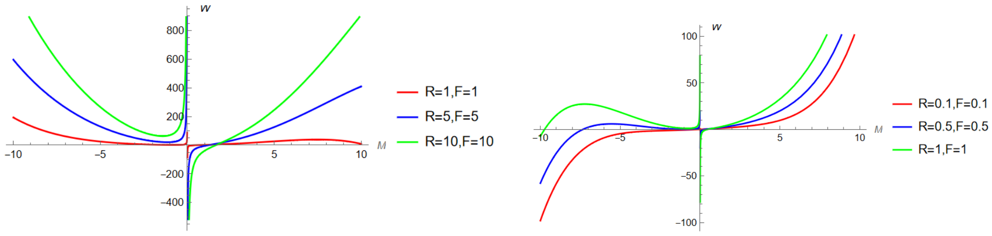


(a) **(b)**

Figure 4. The relation between the frequency $W(M, R, F)$ and the number M, by setting $\mu_1 = 0.2, \mu_2 = 0.3, \mu_3 = 0.1, \mu_4 = 0.3, \mu_5 = -1$ in (a), and $\mu_1 = -0.2, \mu_2 = 0.3, \mu_3 = -0.1, \mu_4 = 0.3, \mu_5 = -1$ in(b).

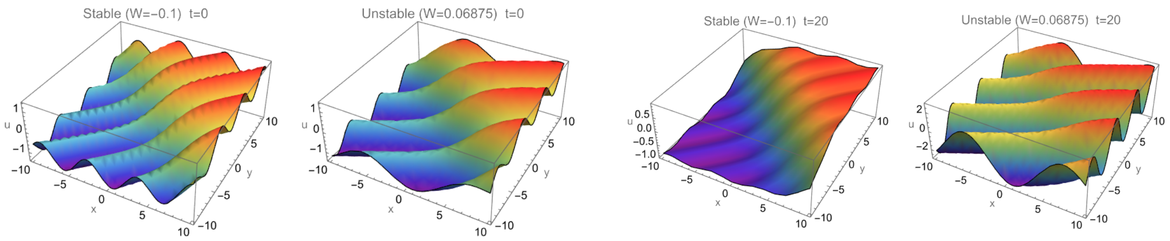
In Figures 4 and 5, we plotted the relation between the frequency W and the wave number M, under different settings of $\{\mu_i\}$ and different settings for $\{R, F\}$, which indicates how the solution will decay or converge to the steady state q_0. As W takes a negative sign, the solution will converge to q_0. Conversely, when the sign of W is positive, the solution will diverge, which means it will be an unstable solution, and if it takes 0 it will be called marginally stable. Additionally, we have perturbed and plotted the dark soliton (at $z = \alpha = \beta = 1$, and $\delta = a_0 = 0$) described by Eq (3.4) under two distinct parameter regimes. For the stable case, the parameters are set as: $\mu_1 = 0.2$, $\mu_2 = 0.3$, $\mu_3 = 0.1$, $\mu_4 = 0.3$, $\mu_5 = -1$, $M = -1$, $R = 1$, $F = 1$, yielding $W = -0.1$ according to Eq (5.5). For the unstable case, the parameters are set as : $\mu_1 = 0.2$, $\mu_2 = 0.3$, $\mu_3 = 0.1$, $\mu_4 = 0.3$, $\mu_5 = -1$, $M = -0.5$, $R = 1$, $F = 1$, resulting in

$W = 0.06875$ and the perturbed parameter $\lambda = 0.5$, $\rho = 1$ for the two cases. The time evolution of the perturbed solution, illustrated in figure 6 through a series of snapshots, which corroborates our linear stability analysis. The solution remains stable as time progresses when $W = -0.1$, while it becomes unstable for $W = 0.06875$.

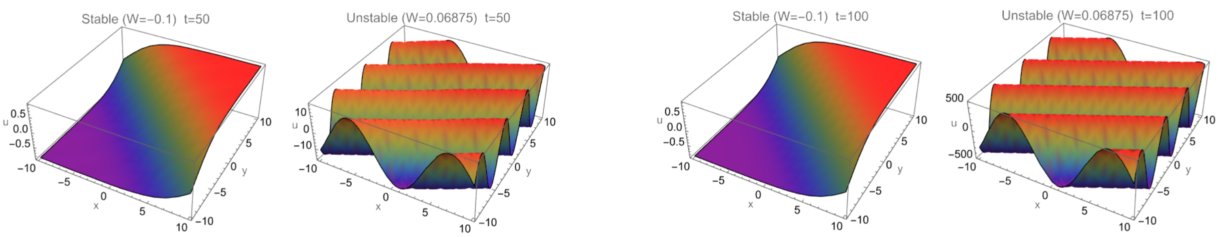


(a) **(b)**

Figure 5. The relation between the frequency $W(M, R, F)$ and the number M, by setting $\mu_1 = -0.5, \mu_2 = -0.8, \mu_3 = 0.5, \mu_4 = -0.001, \mu_5 = 0.001$in (a), and $\mu_1 = -0.5, \mu_2 = -0.8, \mu_3 = 0.5, \mu_4 = 0.001, \mu_5 = -0.001$ in (b).



(a) Represents a pair of perturbed dark soliton at $t = 0$ **(b)** Represents a pair of perturbed dark soliton at $t = 20$



(c) Represents a pair of perturbed dark soliton at $t = 50$ **(d)** Represents a pair of perturbed dark soliton at $t = 100$

Figure 6. The evolution of the perturbed dark soliton described by Eq (3.4) (at $z = \alpha = \beta = 1$, and $\delta = a_0 = 0$) is shown at four time snapshots: $t = 0$, 20, 50, and 100, under two parameter sets: (i) convergent case with $\mu_1 = 0.2$, $\mu_2 = 0.3$, $\mu_3 = 0.1$, $\mu_4 = 0.3$, $\mu_5 = -1$, $M = -1$, $R = 1$, $F = 1$, yielding $W = -0.1$; and (ii) divergent case with $\mu_1 = 0.2$, $\mu_2 = 0.3$, $\mu_3 = 0.1$, $\mu_4 = 0.3$, $\mu_5 = -1$, $M = -0.5$, $R = 1$, $F = 1$, yielding $W = 0.06875$.

6. Conclusions

Using the improved simple equation method, we derived several types of solutions to the (3+1) generalized Calogero-Bogoyavlenskii-Schif equation, which models the propagation of long waves in nonlinear mediums and has many contributions to fluid mechanics and plasma physics. These include dark, bright, and singular solitons; singular periodic solutions; and exponential solutions. Furthermore, to highlight the properties of these solutions, we have supported our findings with 3D visualizations and contour plots. Also, we have discussed the linear stability analysis for our equation and provided a 2D plot for the relation between the wave number and the frequency.

Author contributions

Mina M. Fahim: formal analysis, software, writing—original draft; Hamdy M. Ahmed: methodology, validation, writing—review & editing; K. A. Dib: supervision, visualization; M. Elsaid Ramadan: investigation, writing—original draft; Islam Samir: formal analysis, writing—review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. G. Deng, Y. Gao, C. Ding, J. Su, Solitons and breather waves for the generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt system in fluid mechanics, ocean dynamics and plasma physics, *Chaos Soliton. Fract.*, **140** (2020), 110085. https://doi.org/10.1016/j.chaos.2020.110085
2. S. Kumar, B. Mohan, R. Kumar, Lump, soliton, and interaction solutions to a generalized two-mode higher-order nonlinear evolution equation in plasma physics, *Nonlinear Dyn.*, **110** (2022), 693–704. https://doi.org/10.1007/s11071-022-07647-5
3. S. A. AlQahtani, M. E. M. Alngar, R. M. A. Shohib, A. M. Alawwad, Enhancing the performance and efficiency of optical communications through soliton solutions in birefringent fibers, *J. Opt.*, **53** (2024), 3581–3591. https://doi.org/10.1007/s12596-023-01490-6
4. J. Yue, Z. Zhao, A. Wazwaz, Solitons, nonlinear wave transitions and characteristics of quasi-periodic waves for a (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation in fluid mechanics and plasma physics, *Chinese J. Phys.*, **89** (2024), 896–929. https://doi.org/10.1016/j.cjph.2024.03.039
5. J. Schiff, Integrability of Chern-Simons-Higgs vortex equations and a reduction of the Self-Dual Yang-Mills equations to three dimensions, In: *Painlevé Transcendents*, Boston, MA: Springer, 1992. https://doi.org/10.1007/978-1-4899-1158-2_26
6. Y. Li, H. An, Y. Zhang, Abundant fission and fusion solutions in the (2+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation, *Nonlinear Dyn.*, **108** (2022), 2489–2503. https://doi.org/10.1007/s11071-022-07306-9

7. Z. Zhao, L. He, M-lump and hybrid solutions of a generalized (2+1)-dimensional Hirota–Satsuma–Ito equation, *Appl. Math. Lett.*, **111** (2021), 106612. https://doi.org/10.1016/j.aml.2020.106612

8. H. Sheng, G. Yu, Y. Zhong, A special two-dimensional lattice by Blaszak and Szum: Solitons, breathers, lump solutions, and their interactions and dynamics, *J. Math. Anal. Appl.*, **526** (2023), 127248. 10.1016/j.jmaa.2023.127248

9. X. Lü, S. Chen, Interaction solutions to nonlinear partial differential equations via Hirota bilinear forms: one-lump-multi-stripe and one-lump-multi-soliton types, *Nonlinear Dyn.*, **103** (2021), 947–977. https://doi.org/10.1007/s11071-020-06068-6

10. L. Xu, X. Yin, N. Cao, S. Bai, Multi-soliton solutions of a variable coefficient Schrödinger equation derived from vorticity equation, nonlinear dynamics, *Nonlinear Dyn.*, **112** (2024), 2197–2208. https://doi.org/10.1007/s11071-023-09158-3

11. T. Yin, Z. Xing, J. Pang, Modified Hirota bilinear method to (3+1)-D variable coefficients generalized shallow water wave equation, *Nonlinear Dyn.*, **111** (2023), 9741–9752. https://doi.org/10.1007/s11071-023-08356-3

12. L. Li, C. Duan, F. Yu, An improved Hirota bilinear method and new application for a nonlocal integrable complex modified Korteweg-de Vries (MKdV) equation, *Phys. Lett. A*, **383** (2019), 1578–1582. https://doi.org/10.1016/j.physleta.2019.02.031

13. R. Zhang, S. Bilige, Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation, *Nonlinear Dyn.*, **95** (2019), 3041–3048. https://doi.org/10.1007/s11071-018-04739-z

14. P. Kumari, R. K. Gupta , S. Kumar, Non-auto-Bäcklund transformation and novel abundant explicit exact solutions of the variable coefficients Burger equation, *Chaos Soliton. Fract.*, **145** (2021), 110775. https://doi.org/10.1016/j.chaos.2021.110775

15. Z. Zhao, Bäcklund transformations, nonlocal symmetry and exact solutions of a generalized (2+1)-dimensional Korteweg–de Vries equation, *Chinese J. Phys.*, **73** (2021), 695–705. https://doi.org/10.1016/j.cjph.2021.07.026

16. Z. Lan, N-soliton solutions, Bäcklund transformation and Lax Pair for a generalized variable-coefficient cylindrical Kadomtsev–Petviashvili equation, *Appl. Math. Lett.*, **158** (2024), 109239. https://doi.org/10.1016/j.aml.2024.109239

17. Y. Li, B. Hu, L. Zhang, J. Li, The exact solutions for the nonlocal Kundu-NLS equation by the inverse scattering transform, *Chaos Soliton. Fract.*, **180** (2024), 114603. https://doi.org/10.1016/j.chaos.2024.114603

18. F. Yu, L. Li, Soliton robustness, interaction and stability for a variable coefficients Schrödinger (VCNLS) equation with inverse scattering transformation, *Chaos Soliton. Fract.*, **185** (2024), 115185. https://doi.org/10.1016/j.chaos.2024.115185

19. I. Samir, H. M. Ahmed, A. Darwish, H. H. Hussein, Dynamical behaviors of solitons for NLSE with Kudryashov's sextic power-law of nonlinear refractive index using improved modified extended tanh-function method, *Ain Shams Eng. J.*, **15** (2024), 102267. https://doi.org/10.1016/j.asej.2023.102267

20. O. El-shamy, R. El-barkoki, H. M. Ahmed, W. Abbas, I. Samir, Exploration of new solitons in optical medium with higher-order dispersive and nonlinear effects via improved modified extended tanh function method, *Alex. Eng. J.*, **68** (2023), 611–618. https://doi.org/10.1016/j.aej.2023.01.053

21. W. B. Rabie, H. M. Ahmed, Optical solitons for multiple-core couplers with polynomial law of nonlinearity using the modified extended direct algebraic method, *Optik*, **258** (2022), 168848. https://doi.org/10.1016/j.ijleo.2022.168848

22. F. Ashraf, R. Ashraf, A. Akgül, Traveling waves solutions of Hirota–Ramani equation by modified extended direct algebraic method and new extended direct algebraic method, *Int. J. Mod. Phys. B*, **38** (2024), 2450329.

23. I. Samir, N. Badra, H. M. Ahmed, A. H. Arnous, Solitary wave solutions for generalized Boiti–Leon–Manna–Pempinelli equation by using improved simple equation method, *Int. J. Appl. Comput. Math.*, **8** (2022), 102. https://doi.org/10.1007/s40819-022-01308-2

24. A. Bekir, A. C. Cevikel, The tanh–coth method combined with the Riccati equation for solving nonlinear coupled equation in mathematical physics, *J. King Saud Univ.-Sci.*, **23** (2011), 127–132. https://doi.org/10.1016/j.jksus.2010.06.020

25. H. Chen, H. Zhang, New multiple soliton solutions to the general Burgers–Fisher equation and the Kuramoto–Sivashinsky equation, *Chaos Soliton. Fract.*, **19** (2004), 71–76. https://doi.org/10.1016/S0960-0779(03)00081-X

26. X. Fu-Ding, C. Jing, L. Zhuo-Sheng, Using symbolic computation to exactly solve the integrable Broer–Kaup equations in (2+1)-dimensional spaces, *Commun. Theor. Phys.*, **43** (2005), 585. https://doi.org/10.1088/0253-6102/43/4/003

27. J. Manafian, O. A. Ilhan, H. F. Ismael, S. A. Mohammed, S. Mazanova, Periodic wave solutions and stability analysis for the (3+1)-D potential-YTSF equation arising in fluid mechanics, *Int. J. Comput. Math.*, **98** (2021), 1594–1616. https://doi.org/10.1080/00207160.2020.1836358

28. Y. Zhao, T. Mathanaranjan, H. Rezazadeh, L. Akinyemi, M. Inc, New solitary wave solutions and stability analysis for the generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles, *Results Phys.*, **43** (2022), 106083. https://doi.org/10.1016/j.rinp.2022.106083

29. J. Manafian, O. A. Ilhan, A. Alizadeh, Periodic wave solutions and stability analysis for the KP-BBM equation with abundant novel interaction solutions, *Phys. Scripta*, **95** (2020), 065203. http://doi.org/10.1088/1402-4896/ab68be