



Research article

Novel optical soliton solutions for the generalized integrable (2+1)-dimensional nonlinear Schrödinger system with conformable derivative

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Abstract: For the generalized integrable (2+1)-dimensional nonlinear Schrödinger system, new and creative analytical solutions were derived using a novel extended direct algebraic method incorporating conformable derivatives, which could be expressed in terms of elementary functions and yielded a variety of analytical solutions, such as single, optical periodic, and wave solitons. The analytical solutions provided key insights into the effects of conformable derivatives and temporal parameters on the behavior of optical solitons, such as their stability, propagation, and interaction. To further elucidate the dynamics of these solitons, 2D, 3D, and contour plots were created to provide a visual representation of the soliton's form, amplitude, and phase. This helps to better understand the behavior of the soliton and its potential applications in nonlinear equations. Based on the study's demonstration of the extended direct algebraic method's strength and versatility in obtaining analytical solutions for complex non linear systems, it may be a useful tool for solving a variety of nonlinear problems in science and engineering.

Keywords: extended Direct algebraic approach; conformable derivatives; nonlinear system; analytical solutions; generalized integrable nonlinear Schrödinger system

Mathematics Subject Classification: 65F10, 65H10, 90C30, 90C33

1. Introduction

The study of the exact solutions and dynamical behavior of nonlinear equations not only helps to explain the essential properties and algebraic structure of soliton theory [1–3], but also plays an important role in the rational explanation and practical application of corresponding natural phenomena [4–6]. For instance, the Biswas-Milovic equation (BME) has attracted a lot of attention because it may explain the generation of modes and nonlinear wave dynamics in a range of physical systems [7–9]. With its enlarged version that accounts for the effects of noise or random perturbations, the stochastic BME provides a more realistic description of dynamic behaviors in complex environments in real-world applications [10]. The discovery of soliton solutions in nonlinear differential equations is a compelling area of research [11, 12]. Multiple methodologies can be used to examine the soliton solutions. The translation of the NLDE into non-linear ordinary differential equations (ODE) is an innovative methodology [13–15]. Solitons have the remarkable ability to traverse extensive distances in optical fibers without undergoing distortion [16, 17]. The fiber's precise dispersion and nonlinearity equilibrium, which ensure the soliton maintains its form throughout transit, contribute to its distinctive quality [18–20]. Utilizing this phenomenon, optical solitons significantly enhance data transmission over long distances, thereby promoting global communication. Localized electromagnetic waves, termed optical solitons, sustain a uniform intensity despite the effects of nonlinearity and dispersion. This property renders them highly advantageous for this application [21, 22], Kudryashov's enhanced approach [23, 24], Kudryashov's auxiliary equation method [25], the new Kudryashov method [26–28]-Hirota bilinear method [29]. Recent developments in a number of approaches have improved soliton solutions for NLDEs. The methods used are the following: Lie symmetry analysis method and the exploration of bifurcation analysis [30, 31], Perturbation Binary Salp Swarm Algorithm [32], modified extended tanh method [33], extended transformed rational function algorithm [34] and the F-expansion scheme [35], the auxiliary equation method, nonstandard finite difference discretization method [36–38], and the general projective Riccati equations approach [39–41]. The multiple exp-function method [42, 43]- the Riccati modified extended simple equation method [44], contain a description of the integrable (2+1)-dimensional nonlinear Schrödinger (NLS) equation by Radha and Lakshmanan [45]-

$$\begin{cases} \frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial x \partial y} - pq = 0, \\ \frac{\partial q}{\partial x} - 2 \frac{\partial}{\partial x} (|p|^2) = 0. \end{cases} \quad (1)$$

According to the researchers in [46], the extended modified auxiliary equation mapping method is used to create a number of optical soliton solutions for the system. For the integrable (2+1)-dimensional NLS, Hossieni et al. investigated the exact solutions [47]. Akinyemi et al. [48] employed three novel techniques to search for analytical solutions to the integrable generalized (2+1)-dimensional NLS equations. Exploring the temporal evolution of bifurcation behavior and chaos analysis in the generalized integrable (2+1)-dimensional NLS is our primary objective of this work. Furthermore, several optical soliton solutions with a conformable derivative for the generalized integrable (2+1)-dimensional NLS were created using the novel Kudryashov approach.

$$\begin{cases} i \frac{\partial^2 p}{\partial t^2} + a_1 \frac{\partial^2 p}{\partial x \partial y} + a_2 pq = 0, \\ a_3 \frac{\partial q}{\partial x} + a_4 \frac{\partial}{\partial y} (|p|^2) = 0, \end{cases} \quad (2)$$

where a_1, a_2, a_3 , and a_4 are real constants, and α denotes the variable fractional exponent. The most recent study of this model entailed the improved modified Tanh expansion method, which led to the creation of several optical soliton solutions [49]. Fractional calculus serves as a potent instrument for elucidating many physical phenomena. Modern fractional-order models provide greater flexibility and adaptability compared to traditional integer-order models. Our main aim of this research is to examine the fundamental principles of conformable derivatives, which are essential for comprehending the dynamics of many physical processes. Various applications in physics, engineering, economics, and biology demonstrate the potential of conformable derivatives as an effective analytical instrument for complex systems [50,51].

Definition 1. Let $d : (0, \infty) \rightarrow \mathbb{R}$, α order conformable derivatives will be

$$A_\alpha(d)(x) = \lim_{l \rightarrow 0} \frac{d(x + lx^{1-\alpha}) - d(x)}{l}, \quad (3)$$

for all $x > 0$ and $l \in (0, 1]$ [52].

2. EDAM methodology

In this section, the EDAM approach is described. We examine the FPDE with the following form [53–55]:

$$P(w, \partial_t^\alpha w, \partial_{y_1}^\beta w, \partial_{y_2}^\gamma w, w^2, \dots) = 0, \quad 0 < \alpha, \beta, \gamma \leq 1, \quad (4)$$

where w is determined by $y_1, y_2, y_3, \dots, y_r$, and t . (4) is solved using the following steps:

Step 1.

Start by transforming the variables $w(y_1, y_2, y_3, \dots, y_r)$ into $W(\Pi)$, where Π is specified in a variety of ways. (4) is changed by this transformation into a nonlinear ODE of the following form:

$$Q(W, W'W, W', \dots) = 0, \quad (5)$$

where W in (5) has derivatives with respect to Π . The constant(s) of integration can be obtained by integrating (5) once or more times.

Step 2.

Next, we presume that the solution to (5) is as follows:

$$W(\Pi) = \sum_{l=-M}^M d_l (\zeta(\Pi))^l, \quad (6)$$

where d_l ($l = -M, \dots, 0, 1, 2, \dots, M$) are constants to be determined, and $\zeta(\Pi)$ is the general solution of the following ODE:

$$\zeta'(\Pi) = \ln(\mathfrak{U})(a + e\zeta(\Pi) + f(\zeta(\Pi))^2), \quad (7)$$

The constants a, e , and f are represented as $\mathfrak{U} \neq 0, 1$.

Step 3.

By establishing the homogeneous balance between the highest order derivative and the largest nonlinear term in (5), the positive integer M in (6) is obtained. More specifically, the two equations provided in [33] can be used to estimate the balance number:

$$D\left(\frac{d^k W}{d\xi^k}\right) = M + k \quad \text{and} \quad D(W^j \left(\frac{d^k W}{d\xi^k}\right)^l) = Mj + l(k + M),$$

In this case, D represents the degree of $W(\xi)$ since $D[W(\xi)] = m$ and j, k , and l are whole numbers.

Step 4.

Next, we integrate (5) to obtain (6), or the equation that results, and all the terms of $\zeta(\Pi)$ are arranged in the same order.

The system of algebraic equations for $d_l (l = -M, \dots, 0, 1, 2, \dots, M)$ and other parameters is then obtained by setting all the coefficients of the following polynomial to zero thereafter.

Step 5.

MAPLE is used to solve this collection of algebraic equations.

Step 6.

With the $\zeta(\Pi)$ (solution of Eq (7)), the unknown values are then found and inserted into (6) to obtain the analytical answers to (4). It is possible to get the following families of solutions using the generic solution of (7).

Family. 1: Assuming that $f \neq 0$ and $\Lambda < 0$, then, we have

$$\begin{aligned}\zeta_1(\Pi) &= -\frac{e}{2f} + \frac{\sqrt{-\Lambda} \tan_{\mathcal{U}}\left(\frac{1}{2} \sqrt{-\Lambda} \Pi\right)}{2f}, \\ \zeta_2(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{-\Lambda} \cot_{\mathcal{U}}\left(\frac{1}{2} \sqrt{-\Lambda} \Pi\right)}{2f}, \\ \zeta_3(\Pi) &= -\frac{e}{2f} + \frac{\sqrt{-\Lambda} \left(\tan_{\mathcal{U}}\left(\sqrt{-\Lambda} \Pi\right) + \sec_{\mathcal{U}}\left(\sqrt{-\Lambda} \Pi\right) \right)}{2f}, \\ \zeta_4(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{-\Lambda} \left(\cot_{\mathcal{U}}\left(\sqrt{-\Lambda} \Pi\right) + \csc_{\mathcal{U}}\left(\sqrt{-\Lambda} \Pi\right) \right)}{2f}, \\ \text{and} \\ \zeta_5(\Pi) &= -\frac{e}{2f} + \frac{\sqrt{-\Lambda} \left(\tan_{\mathcal{U}}\left(\frac{1}{4} \sqrt{-\Lambda} \Pi\right) - \cot_{\mathcal{U}}\left(\frac{1}{4} \sqrt{-\Lambda} \Pi\right) \right)}{4f}.\end{aligned}$$

Family. 2: $f \neq 0$ and $\Lambda < 0$ are present.

$$\begin{aligned}\zeta_6(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{\Lambda} \tanh_{\mathcal{U}}\left(\frac{1}{2} \sqrt{\Lambda} \Pi\right)}{2f}, \\ \zeta_7(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{\Lambda} \coth_{\mathcal{U}}\left(\frac{1}{2} \sqrt{\Lambda} \Pi\right)}{2f}, \\ \zeta_8(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{\Lambda} \left(\tanh_{\mathcal{U}}\left(\sqrt{\Lambda} \Pi\right) + \operatorname{sech}_{\mathcal{U}}\left(\sqrt{\Lambda} \Pi\right) \right)}{2f}, \\ \zeta_9(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{\Lambda} \left(\coth_{\mathcal{U}}\left(\sqrt{\Lambda} \Pi\right) + \operatorname{csch}_{\mathcal{U}}\left(\sqrt{\Lambda} \Pi\right) \right)}{2f}, \\ \text{and} \\ \zeta_{10}(\Pi) &= -\frac{e}{2f} - \frac{\sqrt{\Lambda} \left(\tanh_{\mathcal{U}}\left(\frac{1}{4} \sqrt{\Lambda} \Pi\right) - \coth_{\mathcal{U}}\left(\frac{1}{4} \sqrt{\Lambda} \Pi\right) \right)}{4f}.\end{aligned}$$

Family. 3: For $af > 0$ and $e = 0$, then, we have

$$\zeta_{11}(\Pi) = \sqrt{\frac{a}{f}} \tan_{\mathcal{U}}(\sqrt{af}\Pi),$$

$$\mu_{12}(\Pi) = -\sqrt{\frac{a}{f}} \cot_{\mathcal{U}}(\sqrt{af}\Pi),$$

$$\zeta_{13}(\Pi) = \sqrt{\frac{a}{f}} \left(\tan_{\mathcal{U}}(2\sqrt{df}\Pi) + \sec_{\mathcal{U}}(2\sqrt{af}\Pi) \right),$$

$$\zeta_{14}(\Pi) = -\sqrt{\frac{a}{f}} \left(\cot_{\mathcal{U}}(2\sqrt{af}\Pi) + \csc_{\mathcal{U}}(2\sqrt{af}\Pi) \right),$$

and

$$\zeta_{15}(\Pi) = \frac{1}{2} \sqrt{\frac{a}{f}} \left(\tan_{\mathcal{U}}\left(\frac{1}{2}\sqrt{af}\Pi\right) - \cot_{\mathcal{U}}\left(\frac{1}{2}\sqrt{af}\Pi\right) \right).$$

Family. 4: For $af < 0$ and $e = 0$, then, we have

$$\zeta_{16}(\Pi) = -\sqrt{-\frac{a}{f}} \tanh_{\mathcal{U}}(\sqrt{-af}\Pi),$$

$$\zeta_{17}(\Pi) = -\sqrt{-\frac{a}{f}} \coth_{\mathcal{U}}(\sqrt{-af}\Pi),$$

$$\zeta_{18}(\Pi) = -\sqrt{-\frac{d}{f}} \left(\tanh_{\mathcal{U}}(2\sqrt{-af}\Pi) + \operatorname{sech}_{\mathcal{U}}(2\sqrt{-af}\Pi) \right),$$

$$\zeta_{19}(\Pi) = -\sqrt{-\frac{a}{f}} \left(\coth_{\mathcal{U}}(2\sqrt{-af}\Pi) + \operatorname{csch}_{\mathcal{U}}(2\sqrt{-df}\Pi) \right),$$

and

$$\zeta_{20}(\Pi) = -\frac{1}{2} \sqrt{-\frac{a}{f}} \left(\tanh_{\mathcal{U}}\left(\frac{1}{2}\sqrt{-af}\Pi\right) + \coth_{\mathcal{U}}\left(\frac{1}{2}\sqrt{-af}\Pi\right) \right).$$

Family. 5: For $f = a$ and $e = 0$, then, we have

$$\zeta_{21}(\Pi) = \tan_{\mathcal{U}}(a\Pi),$$

$$\zeta_{22}(\Pi) = -\cot_{\mathcal{U}}(a\Pi),$$

$$\mu_{23}(\Omega) = \tan_{\mathcal{U}}(2a\Pi) + \sec_{\mathcal{U}}(2d\Omega),$$

$$\zeta_{24}(\Pi) = -\cot_{\mathcal{U}}(2a\Pi) + \csc_{\mathcal{U}}(2a\Pi),$$

and

$$\zeta_{25}(\Pi) = \frac{1}{2} \tan_{\mathcal{U}}\left(\frac{1}{2}a\Pi\right) - \frac{1}{2} \cot_{\mathcal{U}}\left(\frac{1}{2}a\Pi\right).$$

Family. 6: $f = -a$ and $e = 0$, then, we have

$$\zeta_{26}(\Pi) = -\tanh_{\mathcal{U}}(a\Pi),$$

$$\zeta_{27}(\Pi) = -\coth_{\mathfrak{U}}(a\Pi),$$

$$\zeta_{28}(\Pi) = -\tanh_{\mathfrak{U}}(2a) + i\operatorname{sech}_{\mathfrak{U}}(2a\Pi),$$

$$\zeta_{29}(\Pi) = -\coth_{\mathfrak{U}}(2a\Pi) + \operatorname{csch}_{\mathfrak{U}}(2a\Pi),$$

and

$$\zeta_{30}(\Pi) = -\frac{1}{2}\tanh_{\mathfrak{U}}\left(\frac{1}{2}a\Pi\right) - \frac{1}{2}\coth_{\mathfrak{U}}\left(\frac{1}{2}a\Pi\right).$$

Family. 7: $\Lambda = 0$, then, we have

$$\zeta_{31}(\Pi) = -2\frac{a(e\Omega \ln \mathfrak{U} + 2)}{e^2 \ln(\mathfrak{U})\Omega}.$$

Family. 8: $f = 0$, $e = \varsigma$ and $a = n\varsigma$ (with $n \neq 0$).

$$\zeta_{32}(\Pi) = \mathfrak{U}^{\varsigma\Pi} - n.$$

Family. 9: For $e = f = 0$, then, we have

$$\zeta_{33}(\Pi) = a\Omega \ln(\mathfrak{U}).$$

Family. 10: $e = a = 0$, then, we have

$$\zeta_{34}(\Pi) = -\frac{1}{f\Pi \ln(\mathfrak{U})}.$$

Family. 11: $e \neq 0$, $f \neq 0$ and $a = 0$:

$$\zeta_{35}(\Pi) = -\frac{e}{f(\cosh_{\mathfrak{U}}(e\Pi) - \sinh_{\mathfrak{U}}(e\Pi) + 1)},$$

and

$$\zeta_{36}(\Pi) = -\frac{e(\cosh_{\mathfrak{U}}(e\Pi) + \sinh_{\mathfrak{U}}(e\Pi))}{f(\cosh_{\mathfrak{U}}(e\Pi) + \sinh_{\mathfrak{U}}(e\Pi) + 1)}.$$

Family. 12: $e = \varsigma$, $f = n\varsigma$ (with $n \neq 0$), and $a = 0$:

$$\zeta_{37}(\Pi) = \frac{\mathfrak{U}^{\varsigma\Pi}}{1 - n\mathfrak{U}^{\varsigma\Pi}}.$$

In the above solutions, $\Lambda = e^2 - 4df$.

$$\sin_{\mathfrak{U}}(\Pi) = \frac{\mathfrak{U}^{i\Pi} - \mathfrak{U}^{-i\Pi}}{2i}, \quad \cos_{\mathfrak{U}}(\Pi) = \frac{\mathfrak{U}^{i\Pi} + \mathfrak{U}^{-i\Pi}}{2},$$

$$\sec_{\mathfrak{U}}(\Pi) = \frac{1}{\cos_{\mathfrak{U}}(\Pi)}, \quad \csc_{\mathfrak{U}}(\Pi) = \frac{1}{\sin_{\mathfrak{U}}(\Pi)},$$

$$\tan_{\mathfrak{U}}(\Pi) = \frac{\sin_{\mathfrak{U}}(\Pi)}{\cos_{\mathfrak{U}}(\Pi)}, \quad \cot_{\mathfrak{U}}(\Pi) = \frac{\cos_{\mathfrak{U}}(\Pi)}{\sin_{\mathfrak{U}}(\Pi)}.$$

Similarly,

$$\sinh_{\mathfrak{U}}(\Pi) = \frac{\mathfrak{U}^{\Pi} - \mathfrak{U}^{-\Pi}}{2}, \quad \cosh_{\mathfrak{U}}(\Pi) = \frac{\mathfrak{U}^{\Pi} + \mathfrak{U}^{-\Pi}}{2},$$

$$\operatorname{sech}_{\mathfrak{U}}(\Pi) = \frac{1}{\cosh_{\mathfrak{U}}(\Pi)}, \quad \operatorname{csch}_{\mathfrak{U}}(\Pi) = \frac{1}{\sinh_{\mathfrak{U}}(\Pi)},$$

$$\tanh_{\mathfrak{U}}(\Pi) = \frac{\sinh_{\mathfrak{U}}(\Pi)}{\cosh_{\mathfrak{U}}(\Pi)}, \quad \coth_{\mathfrak{U}}(\Pi) = \frac{\cosh_{\mathfrak{U}}(\Pi)}{\sinh_{\mathfrak{U}}(\Pi)}.$$

3. Problem formulation

Using the method, we investigate several optical solutions for the generalized integrable (2+1)-dimensional nonlinear conformable Schrodinger system (NLCS). Here, we begin by transforming the following:

$$\begin{cases} p(x, y, t) = K(\eta)e^{i\zeta(x,t)}, \\ q(x, y, t) = W(\eta), \end{cases} \quad (8)$$

where $\eta = f_1x + s_1y + c_1\frac{t^\alpha}{\alpha}$ and $\zeta(x, t) = f_2x + s_2y + c_2\frac{t^\alpha}{\alpha}$ are included. Here, the wave's speed is represented by f_1 and f_2 , while the soliton's wave number and frequency are represented by s_1 and s_2 , respectively. The following real and imaginary components, respectively, are obtained by replacing the aforementioned transformations into (2):

$$(c_1 + a_1(f_1s_2 + f_2s_1))K' = 0, \quad (9)$$

$$-c_2K - a_1f_2s_2K + a_1f_1s_1K'' + a_2KW = 0, \quad (10)$$

and

$$a_3f_1W' + a_4s_1 + (K^2)' = 0. \quad (11)$$

After setting the integration constant to zero and integrating (11), we obtain the following:

$$W = \frac{a_4s_1}{a_3f_1}K^2. \quad (12)$$

From (9), we have

$$c_1 = -a_1(f_1s_2 + f_2s_1). \quad (13)$$

By substituting (12) into (10), we find

$$-(c_2 + a_1f_2s_2)K + a_1f_1s_1K'' - \frac{s_1a_2a_4}{f_1a_3}K^3 = 0. \quad (14)$$

When K and K^3 are balanced in (14), $M = 1$. Enter $M=1$ in (6) to solve the NODE in (14) produced by the fractional coupled Higgs system using mEDAM:

$$K(\eta) = \sum_{i=-1}^1 d_i(K(\eta))^i = d_{-1}(K(\eta))^{-1} + d_0 + d_1(K(\eta))^1, \quad (15)$$

The coefficients that will be determined are d_{-1} , d_0 , and d_1 .

Using (7) to put (15) into (14), we can collect terms that have the same power of $\zeta(\eta)$ and generate a polynomial in $\zeta(\eta)$. The polynomial's coefficients can be set to zero to produce a system of nonlinear algebraic equations: After solving the system with Maple, we were able to identify the following two different solution cases.

Case 1.

$$\begin{aligned} d_{-1} &= \sqrt{2} \sqrt{\frac{a_1a_3}{a_2a_4}} \rho \ln(\beta) f_1, \quad d_0 = \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1a_3}{a_2a_4}}}, \\ d_1 &= 0, \quad s_1 = -2 \frac{c_2 + a_1f_2s_2}{a_1f_1 (\ln(A))^2 \Omega}. \end{aligned} \quad (16)$$

Case 2.

$$d_{-1} = 0, d_0 = \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}}, d_1 = \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1, \quad (17)$$

$$s_1 = 2 \frac{c_2 + a_1 f_2 s_2}{a_1 f_1 (\ln(A))^2 (-\mu^2 + 4 \eta \nu)}.$$

Assuming scenario 1, we obtain the following families of soliton solutions for (14).

Family. 1: With $\Omega < 0$ and $\kappa \neq 0$, the following solitary wave solutions are obtained for (2) using (8) and (12):

$$K_1(x, y, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (18)$$

$$W_1(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (19)$$

$$K_2(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (20)$$

$$W_2(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (21)$$

$$K_3(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\tan\left(\sqrt{-\Omega} \eta\right) \pm \left(\sec\left(\sqrt{-\Omega} \eta\right) \right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (22)$$

$$W_3(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\tan(\sqrt{-\Omega} \eta) \pm (\sec(\sqrt{-\Omega} \eta)) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (23)$$

$$K_4(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\cot(\sqrt{-\Omega} \eta) \pm (\csc(\sqrt{-\Omega} \eta)) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (24)$$

$$W_4(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\cot(\sqrt{-\Omega} \eta) \pm (\csc(\sqrt{-\Omega} \eta)) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (25)$$

and

$$K_5(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{4} \frac{\sqrt{-\Omega} \left(\tan\left(\frac{1}{4} \sqrt{-\Omega} \eta\right) - \cot\left(\frac{1}{4} \sqrt{-\Omega} \eta\right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (26)$$

$$W_5(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{4} \frac{\sqrt{-\Omega} \left(\tan\left(\frac{1}{4} \sqrt{-\Omega} \eta\right) - \cot\left(\frac{1}{4} \sqrt{-\Omega} \eta\right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (27)$$

Family. 2: Using (8) and (12) the following solitary wave solutions are obtained for (2), when $\Omega > 0$, and $\kappa \neq 0$:

$$K_6(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \tanh\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (28)$$

$$W_6(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \tanh\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (29)$$

$$K_7(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \coth \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (30)$$

$$W_7(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \coth \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (31)$$

$$K_8(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} \left(\tanh \left(\sqrt{-\Omega} \eta \right) \pm \left(\operatorname{sech} \left(\sqrt{-\Omega} \eta \right) \right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (32)$$

$$W_8(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} \left(\tanh \left(\sqrt{-\Omega} \eta \right) \pm \left(\operatorname{sech} \left(\sqrt{-\Omega} \eta \right) \right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (33)$$

$$K_9(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} \left(\coth \left(\sqrt{-\Omega} \eta \right) \pm \left(\operatorname{sech} \left(\sqrt{-\Omega} \eta \right) \right) \right)}{\gamma} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (34)$$

$$W_9(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} \left(\coth \left(\sqrt{-\Omega} \eta \right) \pm \left(\operatorname{sech} \left(\sqrt{-\Omega} \eta \right) \right) \right)}{\gamma} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (35)$$

and

$$K_{10}(x, y, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{4} \frac{\sqrt{\Omega} \left(\tanh \left(\frac{1}{4} \sqrt{\Omega} \eta \right) - \coth \left(\frac{1}{4} \sqrt{\Omega} \eta \right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (36)$$

$$W_{10}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{4} \frac{\sqrt{\Omega} \left(\tanh\left(\frac{1}{4} \sqrt{\Omega} \eta\right) - \coth\left(\frac{1}{4} \sqrt{\Omega} \eta\right) \right)}{\kappa} \right)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (37)$$

Family. 3: Using (8) and (12), the following solitary wave solutions are derived for (2) for $\kappa\rho > 0$ and $\sigma = 0$:

$$K_{11}(x, y, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\tan(\sqrt{\rho\kappa}\eta))^{-1} \right), \quad (38)$$

$$W_{11}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\tan(\sqrt{\rho\kappa}\eta))^{-1} \right)^2, \quad (39)$$

$$K_{12}(x, y, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\cot(\sqrt{\rho\kappa}\eta))^{-1} \right), \quad (40)$$

$$W_{12}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\cot(\sqrt{\rho\kappa}\eta))^{-1} \right)^2, \quad (41)$$

$$K_{13}(x, y, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\tan(2\sqrt{\rho\kappa}\eta) \pm (\sec(2\sqrt{\rho\kappa}\eta)))^{-1} \right), \quad (42)$$

$$W_{13}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\tan(2\sqrt{\rho\kappa}\eta) \pm (\sec(2\sqrt{\rho\kappa}\eta)))^{-1} \right)^2, \quad (43)$$

$$K_{1,14}(x, y, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\cot(2\sqrt{\rho\kappa}\eta) \pm (\csc(2\sqrt{\rho\kappa}\eta)))^{-1} \right), \quad (44)$$

$$W_{14}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} (\cot(2\sqrt{\rho\kappa}\eta) \pm (\csc(2\sqrt{\rho\kappa}\eta)))^{-1} \right)^2, \quad (45)$$

and

$$K_{15}(x, y, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tan\left(\frac{1}{2} \sqrt{f\kappa\eta}\right) - \cot\left(\frac{1}{2} \sqrt{\rho\kappa\eta}\right) \right)^{-1} \right). \quad (46)$$

$$W_{15}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(2 \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tan\left(\frac{1}{2} \sqrt{f\kappa\eta}\right) - \cot\left(\frac{1}{2} \sqrt{\rho\kappa\eta}\right) \right)^{-1} \right)^2. \quad (47)$$

Family. 4: Using (8) and (12), the following solitary wave solutions are obtained for (2) as $\kappa\rho < 0$ and $\sigma = 0$.

$$K_{16}(x, y, t) = e^{\iota\zeta(x,t)} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tanh\left(\sqrt{-\rho\kappa\eta}\right) \right)^{-1} \right), \quad (48)$$

$$W_{16}(x, y, t) = \frac{a_4 s_1}{a_3 f_1} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tanh\left(\sqrt{-\rho\kappa\eta}\right) \right)^{-1} \right)^2, \quad (49)$$

$$K_{17}(x, y, t) = e^{\iota\zeta(x,t)} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\coth\left(\sqrt{-\rho\kappa\eta}\right) \right)^{-1} \right), \quad (50)$$

$$W_{17}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\coth\left(\sqrt{-\rho\kappa\eta}\right) \right)^{-1} \right)^2, \quad (51)$$

$$K_{18}(x, t) = e^{\iota\zeta(x,t)} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tanh\left(2 \sqrt{-\rho\kappa\eta}\right) \pm \left(\operatorname{sech}\left(2 \sqrt{-\rho\kappa\eta}\right) \right) \right)^{-1} \right), \quad (52)$$

$$W_{18}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\tanh\left(2 \sqrt{-\rho\kappa\eta}\right) \pm \left(\operatorname{sech}\left(2 \sqrt{-\rho\kappa\eta}\right) \right) \right)^{-1} \right)^2, \quad (53)$$

$$K_{19}(x, t) = e^{\iota\zeta(x,t)} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\coth\left(2 \sqrt{-\rho\kappa\eta}\right) \pm \left(\operatorname{csch}\left(2 \sqrt{-\rho\kappa\eta}\right) \right) \right)^{-1} \right), \quad (54)$$

$$W_{19}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(- \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{\frac{\rho}{\kappa}}} \left(\coth\left(2 \sqrt{-\rho\kappa\eta}\right) \pm \left(\operatorname{csch}\left(2 \sqrt{-\rho\kappa\eta}\right) \right) \right)^{-1} \right)^2, \quad (55)$$

and

$$K_{20}(x, t) = e^{\iota\zeta(x,t)} \left(-2 \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{-\frac{\rho}{\kappa}}} \left(\tanh\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) + \coth\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) \right)^{-1} \right). \quad (56)$$

$$W_{20}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-2 \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \frac{1}{\sqrt{-\frac{\rho}{\kappa}}} \left(\tanh\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) + \coth\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) \right)^{-1} \right)^2. \quad (57)$$

Family. 5: Assuming $\kappa = \rho$ and $\sigma = 0$, the following solitary wave solutions for (2) are found by applying (8) and (12):

$$K_{21}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(\rho \eta))^{-1} \right), \quad (58)$$

$$W_{21}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(\rho \eta))^{-1} \right)^2, \quad (59)$$

$$K_{22}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\cot(\rho \eta))^{-1} \right), \quad (60)$$

$$W_{22}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\cot(\rho \eta))^{-1} \right)^2, \quad (61)$$

$$K_{23}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(2\rho \eta) \pm (\sec(2\rho \eta)))^{-1} \right), \quad (62)$$

$$W_{23}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(2\rho \eta) \pm (\sec(2\rho \eta)))^{-1} \right)^2, \quad (63)$$

$$K_{24}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\cot(2\rho \eta) \pm (\csc(2\rho \eta)))^{-1} \right), \quad (64)$$

$$W_{24}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\cot(2\rho \eta) \pm (\csc(2\rho \eta)))^{-1} \right)^2, \quad (65)$$

and

$$K_{25}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(\frac{1}{2} \tan\left(\frac{1}{2} \rho \eta\right) - \frac{1}{2} \cot\left(\frac{1}{2} \rho \eta\right) \right)^{-1} \right). \quad (66)$$

$$W_{25}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(\frac{1}{2} \tan\left(\frac{1}{2} \rho \eta\right) - \frac{1}{2} \cot\left(\frac{1}{2} \rho \eta\right) \right)^{-1} \right)^2. \quad (67)$$

Family. 6: Once $\kappa = -\rho$ and $\sigma = 0$ are used, the following solitary wave solutions for (2) are found by applying (8) and (12):

$$K_{26}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tanh(\rho \eta))^{-1} \right), \quad (68)$$

$$W_{26}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tanh(\rho \eta))^{-1} \right)^2, \quad (69)$$

$$K_{27}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\coth(\rho \eta))^{-1} \right), \quad (70)$$

$$W_{27}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\coth(\rho \eta))^{-1} \right)^2, \quad (71)$$

$$K_{28}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\tanh(2\rho \eta) \pm (\operatorname{sech}(2\rho \eta)))^{-1} \right), \quad (72)$$

$$W_{28}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\tanh(2\rho \eta) \pm (\operatorname{sech}(2\rho \eta)))^{-1} \right)^2, \quad (73)$$

$$K_{29}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\coth(2\rho \eta) \pm (\operatorname{cosech}(2\rho \eta)))^{-1} \right), \quad (74)$$

$$W_{29}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\coth(2\rho \eta) \pm (\operatorname{cosech}(2\rho \eta)))^{-1} \right)^2, \quad (75)$$

and

$$K_{30}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \tanh\left(\frac{1}{2}\rho \eta\right) - \frac{1}{2} \coth\left(\frac{1}{2}\rho \eta\right) \right)^{-1} \right). \quad (76)$$

$$W_{30}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \tanh\left(\frac{1}{2}\rho \eta\right) - \frac{1}{2} \coth\left(\frac{1}{2}\rho \eta\right) \right)^{-1} \right)^2. \quad (77)$$

Family. 7: If $\Omega = 0$, then using (8) and (12) yields the following solitary wave solutions for (2):

$$K_{31}(x, t) = e^{\iota\zeta(x,t)} \left(-\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho (\ln(\beta))^2 f_1 \sigma^2 \eta (\rho (\sigma \psi \ln(\beta)) + 2)^{-1} + \right. \\ \left. 1/2 a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (78)$$

$$W_{31}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho (\ln(\beta))^2 f_1 \sigma^2 \eta (\rho (\sigma \psi \ln(\beta)) + 2)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (79)$$

Family. 8: When $\sigma = \varrho, f = \vartheta, \varrho, \kappa = 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{32}(x, t) = \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) \vartheta \varrho_1 (\beta^{\varrho \eta} - \vartheta)^{-1} + \frac{1}{2} a_1 \vartheta \varrho_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}}. \quad (80)$$

$$W_{32}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) \vartheta \varrho_1 (\beta^{\varrho \eta} - \vartheta)^{-1} + \frac{1}{2} a_1 \vartheta \varrho_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (81)$$

Family. 9: Using (8) and (12), the following solitary wave solutions are obtained for (2) for $\sigma = 0, \kappa = 0$:

$$K_{33}(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} f_1 \eta^{-1} \right). \quad (82)$$

$$W_{1,33}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} f_1 \eta^{-1} \right)^2. \quad (83)$$

Family. 10: After applying (8) and (12), the following solitary wave solutions are obtained for (2) for $\rho = 0, \sigma \neq 0, \kappa \neq 0$:

$$K_{34}(x, t) = e^{\iota \zeta(x, t)} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (84)$$

$$W_{34}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (85)$$

$$K_{35}(x, t) = e^{\iota \zeta(x, t)} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (86)$$

$$W_{35}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (87)$$

Family. 11: When $\sigma = \varrho, \kappa = \vartheta, \varrho, \rho = 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{36}(x, t) = e^{\iota \zeta(x, t)} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (88)$$

$$W_{36}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\frac{1}{2} a_1 f_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (89)$$

Considering scenario number two, we obtain the subsequent sets of soliton solutions for Eq (14).

Family. 12: When $\Omega < 0$ $\kappa \neq 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{90}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (90)$$

$$W_{37}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \tan\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (91)$$

$$K_{38}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (92)$$

$$W_{38}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{-\Omega} \cot\left(\frac{1}{2} \sqrt{-\Omega} \eta\right)}{\kappa} \right) + \right. \\ \left. 1/2 a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (93)$$

$$K_{39}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\tan\left(\sqrt{-\Omega} \eta\right) \pm \left(\sec\left(\sqrt{-\Omega} \eta\right) \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (94)$$

$$W_{39}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\tan\left(\sqrt{-\Omega} \eta\right) \pm \left(\sec\left(\sqrt{-\Omega} \eta\right) \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (95)$$

$$K_{40}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\cot\left(\sqrt{-\Omega} \eta\right) \pm \left(\csc\left(\sqrt{-\Omega} \eta\right) \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (96)$$

$$W_{40}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{-\Omega} \left(\cot \left(\sqrt{-\Omega} \eta \right) \pm \left(\csc \left(\sqrt{-\Omega} \eta \right) \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (97)$$

and

$$K_{41}(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{4} \frac{\sqrt{-\Omega} \left(\tan \left(\frac{1}{4} \sqrt{-\Omega} \eta \right) - \cot \left(\frac{1}{4} \sqrt{-\Omega} \eta \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (98)$$

$$W_{41}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{4} \frac{\sqrt{-\Omega} \left(\tan \left(\frac{1}{4} \sqrt{-\Omega} \eta \right) - \cot \left(\frac{1}{4} \sqrt{-\Omega} \eta \right) \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (99)$$

Family.13: When $\Omega > 0$, $\kappa \neq 0$,

$$K_{42}(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \tanh \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right) + \right. \\ \left. 1/2 a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (100)$$

$$W_{42}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \tanh \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (101)$$

$$K_{43}(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \coth \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (102)$$

$$W_{43}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sqrt{\Omega} \coth \left(\frac{1}{2} \sqrt{-\Omega} \eta \right)}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (103)$$

$$K_{44}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} (\tanh(\sqrt{-\Omega}\eta) \pm (\operatorname{sech}(\sqrt{-\Omega}\eta)))}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (104)$$

$$W_{44}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} (\tanh(\sqrt{-\Omega}\eta) \pm (\operatorname{sech}(\sqrt{-\Omega}\eta)))}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (105)$$

$$K_{45}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} (\coth(\sqrt{-\Omega}\eta) \pm (\operatorname{sech}(\sqrt{-\Omega}\eta)))}{\gamma} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right), \quad (106)$$

$$W_{45}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} + \frac{1}{2} \frac{\sqrt{\Omega} (\coth(\sqrt{-\Omega}\eta) \pm (\operatorname{sech}(\sqrt{-\Omega}\eta)))}{\gamma} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2, \quad (107)$$

and

$$K_{46}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{4} \frac{\sqrt{\Omega} (\tanh(\frac{1}{4} \sqrt{\Omega}\eta) - \coth(\frac{1}{4} \sqrt{\Omega}\eta))}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (108)$$

$$W_{46}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \left(-\frac{1}{2} \frac{\sigma}{\kappa} - \frac{1}{4} \frac{\sqrt{\Omega} (\tanh(\frac{1}{4} \sqrt{\Omega}\eta) - \coth(\frac{1}{4} \sqrt{\Omega}\eta))}{\kappa} \right) + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (109)$$

Family. 14: When $\kappa\rho > 0$ and $\sigma = 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{47}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \tan(\sqrt{\rho\kappa}\eta) \right), \quad (110)$$

$$W_{47}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \tan(\sqrt{\rho\kappa}\eta) \right)^2, \quad (111)$$

$$K_{48}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \cot(\sqrt{\rho \kappa} \eta) \right), \quad (112)$$

$$W_{48}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \cot(\sqrt{\rho \kappa} \eta) \right)^2, \quad (113)$$

$$K_{49}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} (\tan(2 \sqrt{\rho \kappa} \eta) \pm (\sec(2 \sqrt{\rho \kappa} \eta))) \right), \quad (114)$$

$$W_{49}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} (\tan(2 \sqrt{\rho \kappa} \eta) \pm (\sec(2 \sqrt{\rho \kappa} \eta))) \right)^2, \quad (115)$$

$$K_{50}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} (\cot(2 \sqrt{\rho \kappa} \eta) \pm (\csc(2 \sqrt{\rho \kappa} \eta))) \right), \quad (116)$$

$$W_{50}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} (\cot(2 \sqrt{\rho \kappa} \eta) \pm (\csc(2 \sqrt{\rho \kappa} \eta))) \right)^2, \quad (117)$$

and

$$K_{51}(x, t) = e^{\iota\zeta(x,t)} \left(\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \left(\tan\left(\frac{1}{2} \sqrt{f \kappa} \eta\right) - \cot\left(\frac{1}{2} \sqrt{\rho \kappa} \eta\right) \right) \right). \quad (118)$$

$$W_{51}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{\frac{\rho}{\kappa}} \left(\tan\left(\frac{1}{2} \sqrt{f \kappa} \eta\right) - \cot\left(\frac{1}{2} \sqrt{\rho \kappa} \eta\right) \right) \right)^2. \quad (119)$$

Family. 15: When $\kappa\rho < 0$ and $\sigma = 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{52}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \tanh(\sqrt{-\rho \kappa} \eta) \right), \quad (120)$$

$$W_{52}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \tanh(\sqrt{-\rho \kappa} \eta) \right)^2, \quad (121)$$

$$K_{53}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \coth(\sqrt{-\rho \kappa} \eta) \right), \quad (122)$$

$$W_{53}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \coth(\sqrt{-\rho \kappa} \eta) \right)^2, \quad (123)$$

$$K_{54}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} (\tanh(2 \sqrt{-\rho \kappa} \eta) \pm (\operatorname{sech}(2 \sqrt{-\rho \kappa} \eta))) \right), \quad (124)$$

$$W_{54}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} (\tanh(2 \sqrt{-\rho \kappa} \eta) \pm (\operatorname{sech}(2 \sqrt{-\rho \kappa} \eta))) \right)^2, \quad (125)$$

$$K_{55}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} (\coth(2 \sqrt{-\rho \kappa} \eta) \pm (\operatorname{csch}(2 \sqrt{-\rho \kappa} \eta))) \right), \quad (126)$$

$$W_{55}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} (\coth(2 \sqrt{-\rho \kappa} \eta) \pm (\operatorname{csch}(2 \sqrt{-\rho \kappa} \eta))) \right)^2, \quad (127)$$

and

$$K_{56}(x, t) = e^{\iota\zeta(x,t)} \left(-\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \left(\tanh\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) + \coth\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) \right) \right). \quad (128)$$

$$W_{56}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\frac{1}{2} \sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa \ln(\beta) f_1 \sqrt{-\frac{\rho}{\kappa}} \left(\tanh\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) + \coth\left(\frac{1}{2} \sqrt{-\rho \kappa} \eta\right) \right) \right)^2. \quad (129)$$

Family. 16: When $\rho = \kappa$ and $\sigma = 0$,

$$K_{57}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \tan(\rho \eta) \right), \quad (130)$$

$$W_{57}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \tan(\rho \eta) \right)^2, \quad (131)$$

$$K_{58}(x, t) = e^{\iota\zeta(x,t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \cot(\rho \eta) \right), \quad (132)$$

$$W_{58}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \cot(\rho \eta) \right)^2, \quad (133)$$

$$K_{59}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(2\rho \eta) \pm (\sec(2\rho \eta))) \right), \quad (134)$$

$$W_{59}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (\tan(2\rho \eta) \pm (\sec(2\rho \eta))) \right)^2, \quad (135)$$

$$K_{60}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\cot(2\rho \eta) \pm (\csc(2\rho \eta))) \right), \quad (136)$$

$$W_{60}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\cot(2\rho \eta) \pm (\csc(2\rho \eta))) \right)^2, \quad (137)$$

and

$$K_{61}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(\frac{1}{2} \tan\left(\frac{1}{2} \rho \eta\right) - \frac{1}{2} \cot\left(\frac{1}{2} \rho \eta\right) \right) \right). \quad (138)$$

$$W_{61}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(\frac{1}{2} \tan\left(\frac{1}{2} \rho \eta\right) - \frac{1}{2} \cot\left(\frac{1}{2} \rho \eta\right) \right) \right)^2. \quad (139)$$

Family. 17: When $\kappa = -\rho$ and $\sigma = 0$,

$$K_{62}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \tanh(\rho \eta) \right), \quad (140)$$

$$W_{62}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \tanh(\rho \eta) \right)^2, \quad (141)$$

$$K_{63}(x, t) = e^{\iota\zeta(x,t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \coth(\rho \eta) \right), \quad (142)$$

$$W_{63}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \coth(\rho \eta) \right)^2, \quad (143)$$

$$K_{64}(x, t) = e^{\iota \zeta(x, t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\tanh(2\rho \eta) \pm (\operatorname{isech}(2\rho \eta))) \right), \quad (144)$$

$$W_{64}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\tanh(2\rho \eta) \pm (\operatorname{isech}(2\rho \eta))) \right)^2, \quad (145)$$

$$K_{65}(x, t) = e^{\iota \zeta(x, t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 (-\coth(2\rho \eta) \pm (\operatorname{cech}(2\rho \eta))) \right), \quad (146)$$

and

$$W_{65}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \rho \ln(\beta) f_1 \left(-\frac{1}{2} \tanh\left(\frac{1}{2}\rho \eta\right) - \frac{1}{2} \coth\left(\frac{1}{2}\rho \eta\right) \right) \right)^2. \quad (147)$$

Family. 18: When $\Omega = 0$, as a result of utilizing (8) and (12), the subsequent solitary wave solutions are obtained for (2):

$$K_{66}(x, t) = e^{\iota \zeta(x, t)} \left(-2\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa f_1(\rho(\sigma \psi \ln(\beta)) + 2)\sigma^{-2}\eta^{-1} + \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (148)$$

$$W_{66}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-2\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \kappa f_1(\rho(\sigma \psi \ln(\beta)) + 2)\sigma^{-2}\eta^{-1} + \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (149)$$

Family. 19: When $\sigma = \varrho$, $f = \vartheta$, ϱ , $\kappa = 0$,

$$K_{67}(x, t) = e^{\iota \zeta(x, t)} \left(\frac{1}{2} a_1 \vartheta \varrho_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (150)$$

$$W_{67}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\frac{1}{2} a_1 \vartheta \varrho_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (151)$$

Family. 20: When $\sigma = 0$, $\rho = 0$,

$$K_{68}(x, t) = e^{\iota \zeta(x, t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} f_1 \eta^{-1} \right). \quad (152)$$

$$W_{68}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} f_1 \eta^{-1} \right)^2, \quad (153)$$

Family. 21: When $\rho = 0$, $\sigma \neq 0$, $\kappa \neq 0$,

$$K_{69}(x, t) = e^{\iota \zeta(x, t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \ln(\beta) f_1 \sigma (\cosh(\sigma \eta) - \sinh(\sigma \eta) + 1)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} - \frac{\ln(\Pi) \Omega(l^2 - n^2)}{q \sqrt{2n^2 - 2l^2}} \right). \quad (154)$$

$$W_{69}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \ln(\beta) f_1 \sigma (\cosh(\sigma \eta) - \sinh(\sigma \eta) + 1)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} - \frac{\ln(\Pi) \Omega(l^2 - n^2)}{q \sqrt{2n^2 - 2l^2}} \right)^2. \quad (155)$$

$$K_{70}(x, t) = e^{\iota \zeta(x, t)} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \ln(\beta) f_1 \sigma (\cosh(\sigma \eta) + \sinh(\sigma \eta)) (\cosh(\sigma \eta) + \sinh(\sigma \eta) + 1)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (156)$$

$$W_{70}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(-\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \ln(\beta) f_1 \sigma (\cosh(\sigma \eta) + \sinh(\sigma \eta)) (\cosh(\sigma \eta) + \sinh(\sigma \eta) + 1)^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \sigma a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (157)$$

Family. 22: When $\sigma = \varrho$; $\kappa = \vartheta$, $\varrho, \rho = 0$,

$$K_{71}(x, t) = e^{\iota \zeta(x, t)} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \vartheta \varrho \ln(\beta) f_1 \beta^{\varrho \eta} (1 - \vartheta \beta^{\varrho \eta})^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right). \quad (158)$$

$$W_{71}(x, t) = \frac{a_4 s_1}{a_3 f_1} \left(\sqrt{2} \sqrt{\frac{a_1 a_3}{a_2 a_4}} \vartheta \varrho \ln(\beta) f_1 \beta^{\varrho \eta} (1 - \vartheta \beta^{\varrho \eta})^{-1} + \right. \\ \left. \frac{1}{2} a_1 f_1 \ln(\beta) \varrho a_3 \sqrt{2} a_2^{-1} a_4^{-1} \frac{1}{\sqrt{\frac{a_1 a_3}{a_2 a_4}}} \right)^2. \quad (159)$$

4. Discussion & graphs

Figure 1(a) is a 3D surface plot, where the wave function is represented in three dimensions, with the x-axis and y-axis showing space and the z-axis representing the amplitude (strength) of the wave at

different points. The sharp peaks and valleys in the plot indicate wave-like behavior, with alternating high and low intensities. This type of plot is useful for visualizing the dynamics of waves in a spatial domain over time. The pattern indicates periodicity and may suggest the behavior of solitons or nonlinear waves that maintain their form over time.

Figure 1(b) is a contour plot, where the intensity of the wave is represented with colored regions and contour lines. The contours indicate areas with the same intensity, showing how the wave's energy distribution changes across space. The regions with closely packed contour lines show areas where the wave's energy is concentrated. This plot highlights the focus points of the wave, helping to identify where the wave has higher or lower intensity. The pattern suggests a localized wave that may be affected by nonlinear interactions.

Figure 1(c) is a line plot that shows the wave's intensity along the x-axis at a specific y-position. The wave intensity is plotted for different values of x, showing how the wave's amplitude changes over space at a particular moment in time. Physical meaning: The sharp spikes in the plot suggest discontinuities or singularities in the wave, which are typical of solitons or shock waves in nonlinear systems. The wave shows strong localized changes in intensity, which might indicate interactions between different parts of the wave or wave collisions.

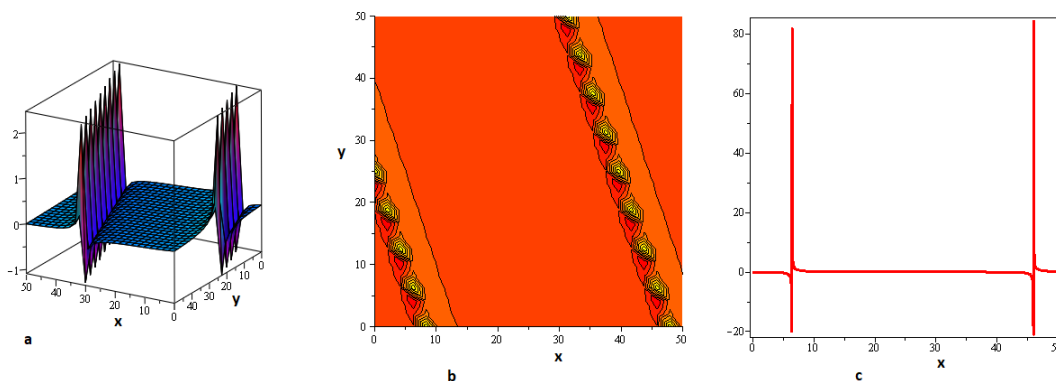


Figure 1. The two-dimensional, three-dimensional and contour soliton solution stated in (18) are graphed for $\sigma = 4$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = .5$, $f_1 = 0.3$, $a_2 = .3$, $a_3 = .9$, $a_4 = 1$, $\alpha = 1$, and $t = 1$.

Figure 2(a) is a 3D plot that displays the wave's amplitude (height) over x and y space. The sharp peaks and valleys suggest a wave with strong intensity variations at different points.

Figure 2(b) is a contour plot showing lines of equal wave strength. The closely spaced contours indicate areas with higher energy, while the wider spaced contours show regions of lower energy.

Figure 2(c) is a simple line plot that shows how the wave's strength (amplitude) changes along the x-axis at a fixed y-value. The spikes suggest sharp changes or discontinuities in the wave's behavior.

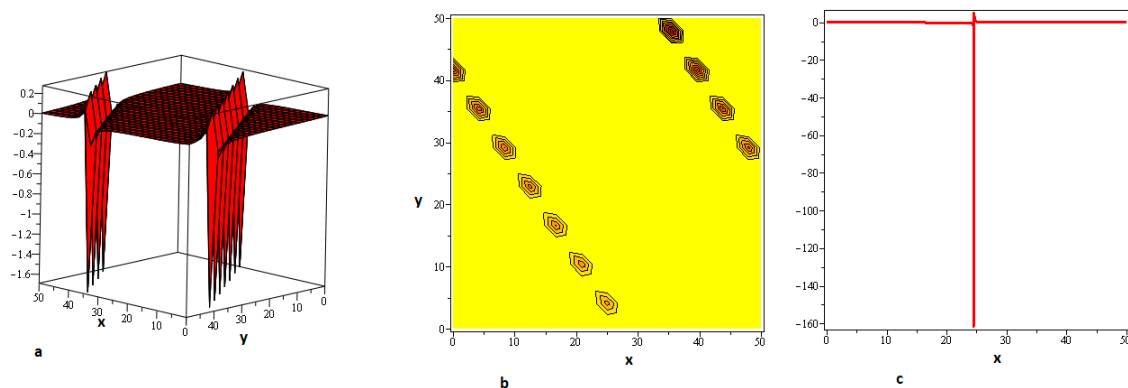


Figure 2. The two-dimensional, three-dimensional and contour soliton solution stated in (20) are graphed for $\sigma = 4$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = .05$, $f_1 = 0.03$, $a_2 = .3$, $a_3 = .9$, $a_4 = 11$, $\alpha = 1$, and $t = 1$.

Figure 3(a) is a 3D surface plot with waves oscillating in both x and y directions. The waves have alternating peaks and valleys. This represents a complex wave pattern with high-frequency oscillations, which could be related to a nonlinear wave or a wave in a medium with periodic behavior.

Figure 3(b) is a contour plot that shows areas of equal intensity, with most of the plot being red, indicating high intensity, and a small part having contours representing lower intensities. The wave has a concentrated energy region in the red area, with the intensity rapidly changing. This suggests that the wave's energy is localized or has sharp transitions in space.

Figure 3(c) is a line plot that shows the wave's intensity along the x -axis, with two curves showing the positive and negative parts of the wave. The wave has sharp changes in intensity along the x -axis, indicating a nonlinear response, where the intensity quickly increases and decreases, similar to a shock wave or soliton behavior.

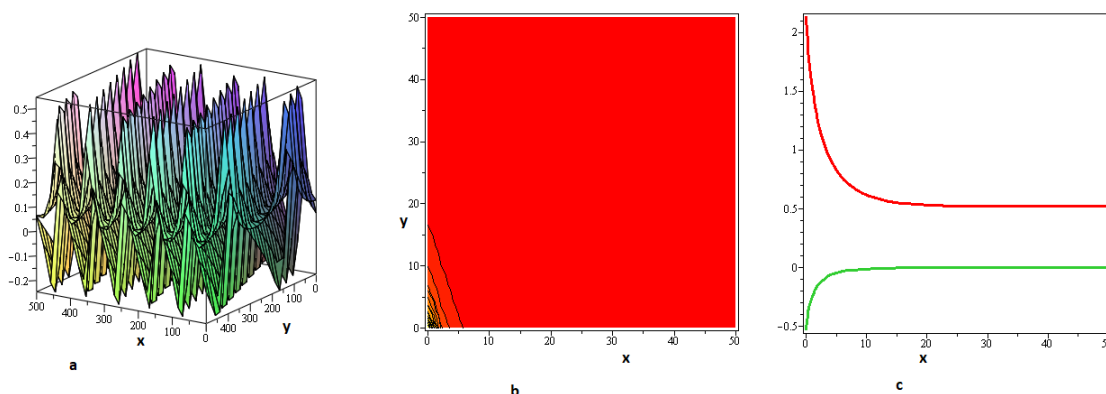


Figure 3. The two-dimensional, three-dimensional and contour soliton solution stated in (28) are graphed for $\sigma = .4$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = 5$, $f_1 = 0.03$, $a_2 = .03$, and $a_3 = .9$.

Figure 4(a) is a 3D surface plot with two wave components that intersect and create a pattern of alternating positive and negative values. The plot represents two oscillating waves that interfere with each other. The red and green regions indicate different wave intensities, which suggest constructive and destructive interferences where the waves combine.

Figure 4(b) is a contour plot that shows areas of equal intensity (contours) with a gradient of colors from yellow to red, indicating changes in wave intensity. This plot shows how the wave's energy is distributed across space, with stronger energy represented by red areas and weaker energy in the yellow areas. This helps visualize the wave's variation in space.

Figure 4(c) is a line plot showing the wave's intensity along the x-axis at a fixed y-value, with two curves (one positive and one negative). The two curves represent the oscillation of the wave at a specific location in space (y) over time or position along the x-axis. The green curve is the wave's positive intensity, and the red curve represents the negative part of the wave.

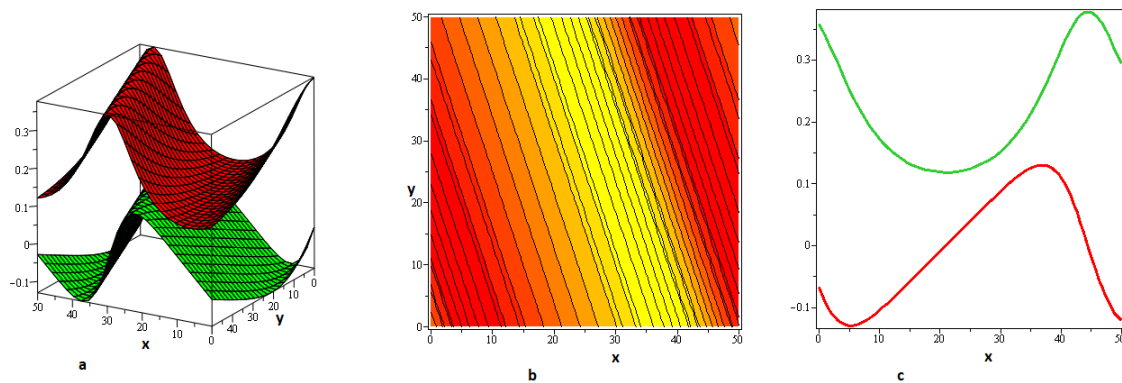


Figure 4. The two-dimensional, three-dimensional, and contour soliton solution stated in (30) are graphed for $\sigma = 8$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = .75$, $f_1 = 0.03$, $a_2 = .03$, $a_3 = .79$, $a_4 = 4$, $\alpha = .1$, and $t = 1$.

Figure 5 shows a nonlinear wave system where the wave has localized energy regions (such as solitons or shock waves). The 3D plot shows the wave's overall structure, the contour plot illustrates the concentration of energy, and the line plot reveals sharp intensity changes. This kind of behavior is typical in systems where the wave interacts nonlinearly, leading to energy localization and discontinuous wave behavior.

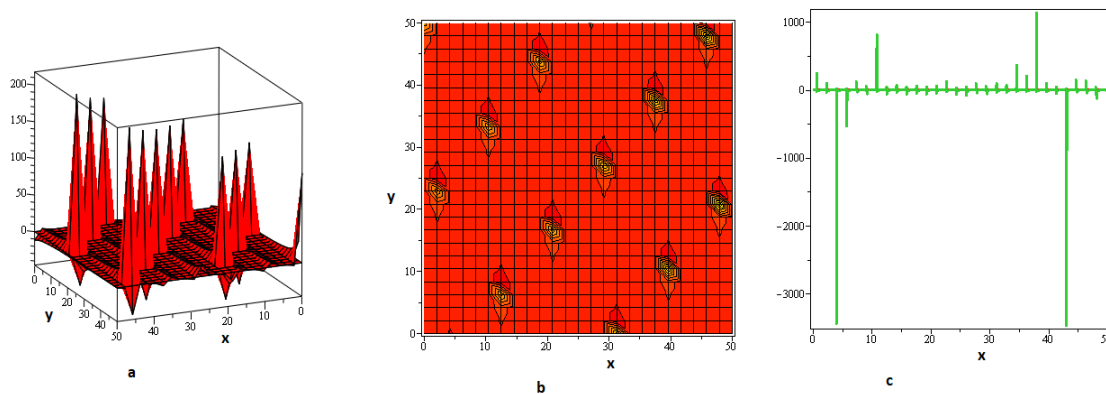


Figure 5. The two-dimensional, two-dimensional, three-dimensional, and contour soliton solution stated in (90) are graphed for $\sigma = 4$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = .75$, $f_1 = 0.7$, $a_2 = .3$, $a_3 = .3$, $a_4 = 4$, and $t = 11$.

Figure 6 shows a nonlinear wave system that exhibits localized energy structures, sharp intensity variations, and high-energy regions. The 3D plot indicates the overall shape and behavior of the wave, the contour plot reveals the concentration of energy, and the line plot highlights the abrupt intensity changes. These characteristics are typical of systems involving solitons, shock waves, or other nonlinear wave phenomena where the wave retains its structure over time, despite the sharp intensity variations.

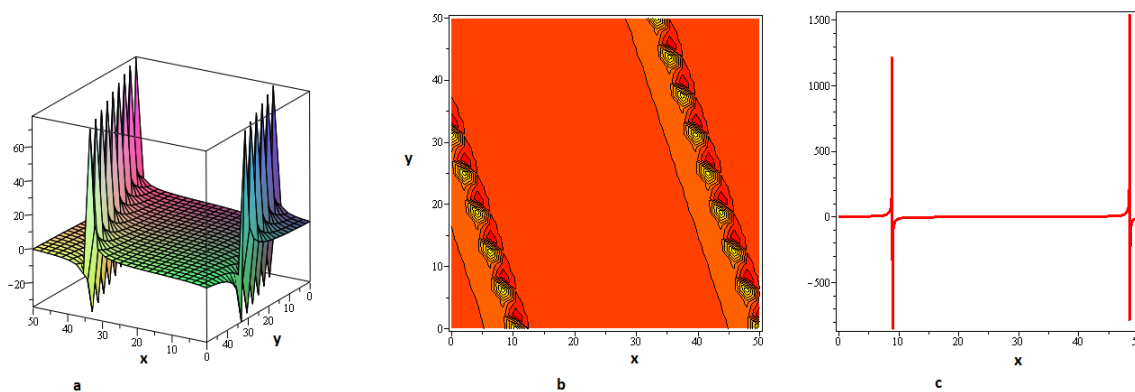


Figure 6. The two-dimensional, three-dimensional and contour soliton solution stated in (92) are graphed for $\sigma = 4$, $\kappa = 1$, $\rho = 11$, $\beta = e$, $a_1 = .5$, $f_1 = 0.03$, $a_2 = .03$, $a_3 = .9$, $a_4 = .11$, and $t = .7$.

Figure 7 illustrates a nonlinear wave system where energy is localized in specific regions, as shown by the sharp peaks in the 3D plot and the contour plot. The line plot indicates discontinuities or sudden changes in the wave's behavior, which are characteristic of solitons or shock waves. These types of

waves maintain their structure over time, even as they exhibit localized intensity variations.

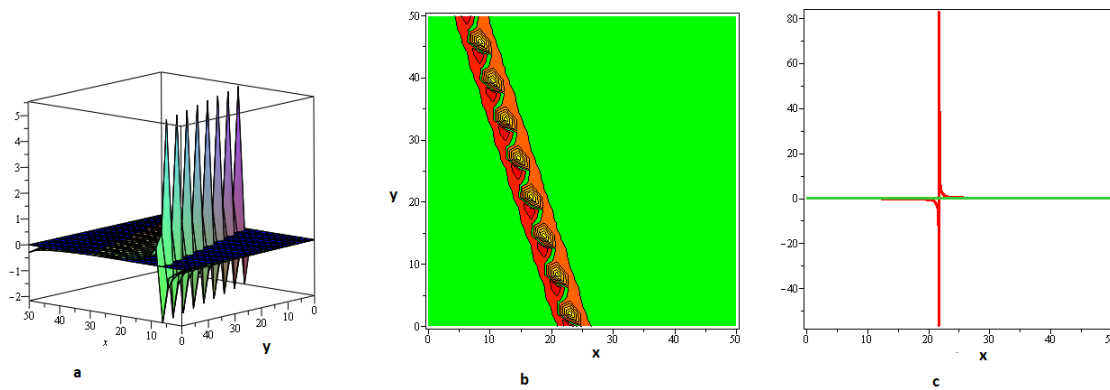


Figure 7. The two-dimensional, three-dimensional and contour soliton solution stated in (100) are graphed for $\sigma = 8$, $\kappa = 11$, $\rho = 1$, $\beta = e$, $a_1 = .5$, $f_1 = 0.03$, $a_2 = .93$, $a_3 = .99$, $a_4 = .911$, and $t = .5$.

Figure 8 represents a nonlinear wave system where the wave exhibits localized high-amplitude regions (likely solitons or other stable wave forms). The 3D plot shows the overall oscillating nature of the wave with sharp intensity spikes, the contour plot illustrates the concentration of energy in specific regions, and the line plot highlights the discontinuous intensity changes in the wave. This is indicative of a wave system that supports nonlinear phenomena, such as solitons or shock waves, where energy can become localized, and the wave exhibits sharp transitions in intensity.

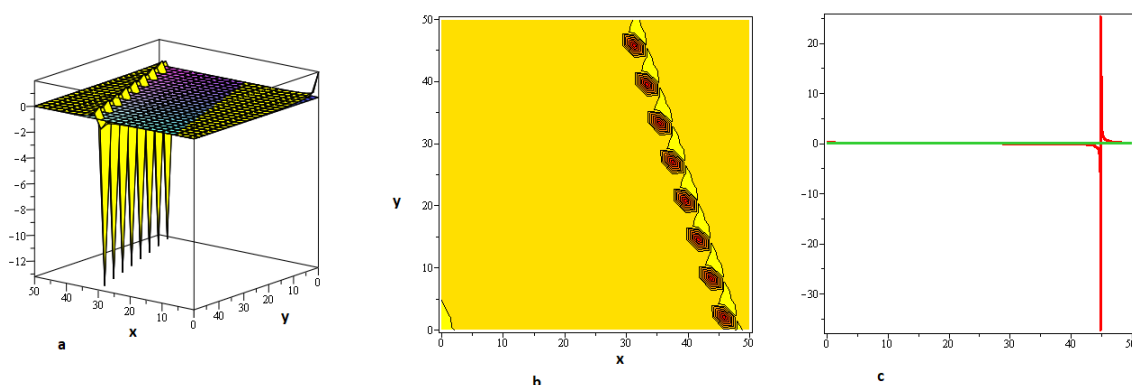


Figure 8. The two-dimensional, three-dimensional and contour soliton solution stated in (102) are graphed for $\sigma = 8$, $\kappa = 11$, $\rho = 1$, $\beta = e$, $a_1 = .5$, $f_1 = 0.03$, $a_2 = .3$, $a_3 = .9$, $a_4 = 11$, and $t = .1$.

5. Conclusions

Using a conformable derivative and the extended direct algebraic method, we successfully derived novel optical soliton solutions for the generalized integrable (2+1)-dimensional nonlinear Schrödinger system. Wave and single solitons make up the resulting solutions. Our thorough analysis of the impact of the conformable derivative and temporal parameters on optical solutions yields important new information. However, it is essential to acknowledge the limitations and potential shortcomings of the method and results. Furthermore, the results obtained in this study are based on a specific model, and the applicability of these results to other nonlinear optical systems remains to be explored. Additionally, we focus primarily on the theoretical aspects of optical soliton solutions, and experimental verification of these results is necessary to confirm their validity. Furthermore it is unknown whether the findings from this study can be applied to other nonlinear optical systems because they are based on a particular model. Furthermore, we concentrate on the theoretical features of optical soliton solutions, and to validate these findings, experimental verification is required.

Author contributions

Muhammad Bilal, Javed Iqbal, Ikram Ullah, Aditi Sharma, Hasim Khan and Sunil Kumar Sharma: Conceptualization, methodology, resources, writing-original draft, formal analysis, investigation, validation, visualization, software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article

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Conflicts of interest

The authors declare they have no conflict of interest.

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