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Research article

Exploring solitonic wave dynamics in the context of nonlinear conformable Kairat-X equation via unified method

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Abstract: This research is aimed at finding exact soliton solutions of the nonlinear fractional Kairat-X equation, which describes soliton behavior in nonlinear media and has applications in quantum physics, materials science, signal processing, and telecommunications. We use a unified method that generalizes the tanh-function method to find new exact soliton solutions in trigonometric, hyperbolic, and plane wave forms. Computational simulations with fixed parameters are performed to produce two-dimensional and three-dimensional visualizations, e.g., contour and density plots, representing the physical properties of the derived solitons. The simulations result in the identification of several soliton types, namely kink wave solitons, dark solitons, bright solitons, and periodic wave solitons. Our results increase the knowledge of the solution properties of the Kairat-X equation and give a platform to interpret a variety of significant physical phenomena. The systematicity and stability of our methodology prove its usefulness as a device for solving other nonlinear partial differential equations in applied physics and mathematics, always returning different exact solutions.

Keywords: the unified method; nonlinear conformable Kairat-X equation; exact solutions; bright solitons
Mathematics Subject Classification: 35R11, 35B30, 35C07

1. Introduction

Partial differential equations (PDEs) are core mathematical instruments used to model dynamic processes over time, especially in populations where changes in size, birth, and death rates can be irregular. Scientists commonly utilize these nonlinear equations to manage dispersive effects and nonlinearity optimally while simulating the behavior of localized stationary and pulsing waves. A well-developed understanding of these equations is a prerequisite to comprehending the differences and physical meanings of the nonlinear behavior seen in dispersive waves. They are sometimes called "universal equations" because they can display wave propagation across many types of materials. Across a wide range of scientific fields, including fluid mechanics, solid-state physics, biology, geochemistry, ocean engineering, optical fibers, and plasma physics, nonlinear partial differential equations (NLPDEs) are of significant importance [1–4]. This group of equations is becoming more established as an essential field of research for modern-day scientists, since it is essential to the understanding of dynamic processes and natural phenomena. NLPDEs allow one to study the wave behaviors and extract accurate solutions that expose the intricate structures underlying the behaviors.

Researchers tend to explore unknown functions and parameters in many NLPDEs that are found in engineering, biology, physics, and chemistry, with the goal of developing and improving their research. The efficiency of mathematical calculations is an important element in modeling dynamic processes, which results in the construction of a better, more general, and more concise class of exact solutions. The research agenda in this domain is primarily governed by two general themes: Providing numerical stability and developing novel mathematical solvers to supplant standard computational techniques. Second, research into fractional differential equations has come to the fore with researchers because such equations are pivotal in dealing with real-world issues in various scientific contexts. The importance of nonlinear partial differential equations continues to be central not just in theoretical investigations but also in the translation of scientific wisdom to practical issues [5–8].

Analytical solutions of NLPDEs are crucial for the obtainment of information about the qualitative properties of these equations and for the correct interpretation of diverse phenomena. These solutions are both symbolic and graphical representations of the underlying structures that control intricate dynamics, e.g., the existence or nonexistence of steady states, the existence of multiple steady states, peak regimes, and spatial localization of transfer processes. Through extensive studies using a combination of strong and systematic methods, researchers have gained an intimate knowledge of these phenomena. The unified approach augments the current solutions by supplementing the limitations of conventional tangent function techniques while generalizing the tanh-function method family at the same time [9]. This new approach not only enhances our knowledge of the behaviors these equations demonstrate but does so without requiring large amounts of computational resources, making it very useful. The systematic and flexible nature of the merged approach makes it more valuable for a variety of applications compared to more complicated numerical approaches. One of the more prominent areas of study in this discipline is solitons, which have received considerable interest from many fields of study, ranging from ocean dynamics to optics, plasma physics, fluid dynamics, semiconductors, and engineering. Their ability to maintain both shape and velocity upon interactions makes them extremely interesting, which has driven intense mathematical study of solitary waves [10].

Solitons for variable coefficient nonlinear PDEs can be obtained by methods such as the ansatz method. The unified method unifies a number of approaches related to the tanh function technique.

Among the successful methods that have been developed in this regard are: The extended Fan sub-equation technique [11], the method of the exponential function [12], the generalized tanh method [13], the Kudryashov generalized method [14], the expanded Sinh-Gordon equation solving technique [15], the novel phi_6-model expansion method [16], the first integral method [17], the Riccati-Bernoulli sub ordinary differential equation method [18], the auxiliary equation method [19, 20], the extended simple equation method [21], the bilinear Hirota method [22], the modified Kudryshov method [23], the spectral collection method [24], the extended modified rational expansion method [25], and the binary Bell polynomial method [26], the sardar sub equation method [27], among other methods. These approaches, taken together, illustrate the strength and diversity of the combined strategy in obtaining soliton solutions and comprehending the complex behavior of nonlinear systems in many areas of science.

Solitons are unique, robust waveforms that occur in different nonlinear systems through a delicate interplay between nonlinearity and dispersion, enabling them to propagate over long distances without deformation or change in velocity. Solitons are mathematical solutions to certain nonlinear partial differential equations, like the Korteweg-de Vries equation for shallow water waves and the nonlinear Schrödinger equation, which can be used in applications such as fiber optics [28]. Among the most characteristic aspects of solitons is how they interact with other solitons by colliding and coming back to their original form afterward, a peculiarity not found in standard wave phenomena. This peculiarity is an outcome of conservation laws inherent to these equations that render solitons highly tolerant of perturbations and disruptions. Due to their fascinating characteristics, solitons are applied in a broad range of disciplines. In ocean dynamics, they are employed to describe events like rogue waves and shock waves in tides, which help us better understand complicated marine environments [29]. In fiber optics, solitons are important in preserving the integrity of data transmission over long distances since they can reverse the effects of dispersion that normally cause signal degradation. In plasma physics, solitons describe the dynamics of waves in magnetized plasmas and are essential in the understanding of energy transmission in tokamak devices. Engineering applications are also flourishing, with solitons being investigated for advances in communications systems and advanced materials, where their characteristics can guide the creation of new technologies. In addition, soliton dynamics play a role in understanding numerous biological and chemical processes, highlighting their diversity and intrinsic importance in many scientific fields [30]. Hence, the study of solitons is not a mere theoretical pursuit; it carries deep ramifications for theoretical work as well as actual applications in practical situations. N-soliton solutions can typically be derived for numerous nonlinear partial differential equations, including the Nonlinear Conformable Kairat-X Equation, by employing certain methods like the Hirota direct method, inverse scattering transform, or other analytic methods. It is important to find these N-soliton solutions in order to study soliton turbulence, as they provide researchers with knowledge about interactions of several solitons and resulting dynamics within a system. Examining how the N-soliton solutions interact may yield complicated behavior like collisions, fusion, or the development of turbulence-like interaction among solitons. Although the calculation of N-soliton solutions yields theoretical results, plotting or graphing these solutions may not always be required to derive useful conclusions. Alternatively, the equations that describe these solutions may be examined to investigate their stability, persistence, and interactions using analytical techniques. This analysis may include studying current theoretical frameworks and using numerical simulations to validate the analytical results. By concentrating on the properties and interactions of N-soliton solutions, scientists can develop a better understanding of soliton dynamics and their implications in physical systems without having to create visual representations. This kind of approach emphasizes the role of firm theoretical analysis to help us know more about the behavior of solitons and how they affect phenomena such as soliton turbulence [31].

A fractional differential equation is characterized by the occurrence of derivatives that are of fractional order. Soliton wave studies result in certain nonlinear fractional differential equations, as well as other related challenges. Solitons, with their unique properties, are now the subject of research interest in nonlinear science. Their novel solutions, especially those that are similar to solitons, are of great interest because they are inherently stable; solitons do not change shape and speed even after a collision, and they can exist as stable objects. There are a number of different types of solitons, such as "dark solitons", "periodic solitons", "special solitons", "bright solitons", "dark-bright solitons", "kink solitons", and "anti-kink solitons". Fractional order models in the field of nonlinear sciences are frequently preferred because they are better able to capture real-world complicated phenomena. Though basic calculus properties like Rolle's theorem, the chain rule, the mean value theorem, and derivative rules for the quotient and product of two functions still hold, conformable fractional operators are more readily compared to other fractional operators owing to some limitations. The conformable derivative is an easy and natural derivative, serving to clarify the importance of physical interpretations, thus being a useful tool in applications [32]. The method has real-world implications in a wide range of fields, such as biology, computer networking, chemistry, laser optics, nonlinear dynamics, optical fibers, and engineering. One of the earliest definitions of fractional derivatives is the Riemann-Liouville fractional derivative [33], which is defined by the presence of an integral limit. Another view is given by the Caputo fractional derivative, developed by Michele Caputo, which presents a revised method to the classical derivative [34]. In addition, the Grünwald-Letnikov fractional derivative [35] is a discretized version of fractional derivatives that uses finite difference methods. With the help of fractional derivatives and their various uses, scientists can access the newest research and discoveries within the growing community of fractional calculus and, thus, advance our knowledge of sophisticated systems in numerous scientific fields.

The Kairat-X equation, in its particular form and context, well explains wave propagation and the surface geometry of curves. The equation is used in many areas, such as optical fibers, optical communication, quantum mechanics, and other physical systems where nonlinearity is important. The main focus is on quantitatively investigating the Kairat-X equation, which is paramount due to the fact that solitons, owing to their ability to maintain their shape upon propagation, are important within various fields like "signal processing", "quantum physics", "materials science", and "telecommunications". The Kairat-X equation is important in investigating soliton solutions owing to their fundamental dynamics. This new model is key in finding the gauge equivalency among various models. Myrzakulova et al. have recently posted their work regarding the equations controlling the Kairat equation with explorations on numerous wave solutions [36]. There are two versions of Kairat equations that exist under the category of Kairat equations: The Kairat-X equation and the Kairat-II equation. Various authors have studied the models and presented numerous solutions. For example, to identify exact traveling wave solutions for the Kairat-II equation, Awadalla et al. (2023) employed three methods, namely the exp function method, the modified simple equation technique, and the generalized Kudryashov method [37]. Iqbal et al. (2024) obtained several solitary wave solutions for the governing model by using the extended simple equation technique [38]. Moreover, Wazwaz (2024) built multisoliton solutions of the explored model using the Hirota bilinear method [39]. Still, Tipu et al. (2024) computed photonic soliton solutions via an extended modified direct algebraic approach [40]. Not only are the results of these publications interesting and creative, but they are also enlightening from the viewpoint of different physiological phenomena, thus putting emphasis on the applicability and novelty of the framework of the Kairat equation.

1.1. Conformable derivative

The conformable derivative is an advanced development of the concept of the classical derivative, specifically built to provide greater flexibility for the calculation of rates of change for functions. Classical derivatives demand functions be differentiable as per normal criteria, while conformable derivatives facilitate differentiation for those functions that need not adhere to these parameters. By including a scaling factor, it allows for fractional or non-integer differentiation, which is especially beneficial in many applications. One of the main uses of the conformable derivative is in fractional differential equations, allowing the creation and solving of equations that have fractional orders of differentiation. This is very useful in physics and engineering, where most systems have behaviors described best by fractional calculus. Further, the conformable derivative is also an effective means of simulating intricate systems that are typified by non-local phenomena or memory effects, thereby providing better representations of their dynamics. In control theory, system design is further enhanced through the conformable derivative in terms of accurately representing processes that do not fit into integer-order dynamics, eventually yielding improved system stability and response. The financial industry is also advantaged by this derivative, as it enables improved stock price movement modeling that displays anomalous behaviors and irregular fluctuations. In biology, conformable derivatives are applied in modeling population dynamics and disease propagation, capturing sophisticated interactions and lags that basic models might insufficiently address. Also, in signal processing, conformable derivatives open new avenues for filtering and analyzing signals, especially when the conventional procedures fail. When it comes to wave propagation, they allow models of wave actions in situations ranging from quantum mechanics to material science with more accuracy, where complicated non-linear patterns are the norms. On the whole, the conformable derivative is a strong and versatile instrument, having far-reaching effects on theoretical studies and real-world applications in a broad range of fields, from science and engineering to finance and others [41]. The impact of different fractional orders on the solution can be vast, leading to a variety of mathematical behaviors and characteristics. Depending on the system's fractional orders, system stability, and oscillating tendencies, as well as rates of convergence, might be greatly varied. By scrutinizing these differentiations, a better understanding of underlying dynamics is acquired, resulting in more meaningful visual representations that echo the nuances initiated by different fractional orders. This emphasis not only makes the results clearer but also enhances the knowledge of the relevance of fractional calculus in the context of the research.

The nonlinear conformable Kairat-X equation is as follows [42]:

$$\mathbb{D}^{2\beta,\vartheta}_{N,\mathrm{tt}}u - 3(\mathbb{D}^{2\beta,\vartheta}_{N,\mathrm{xx}}u \cdot \mathbb{D}^{\beta,\vartheta}_{N,t}u + \mathbb{D}^{\beta,\vartheta}_{N,x}u \cdot \mathbb{D}^{2\beta,\vartheta}_{N,\mathrm{xt}}u) + \mathbb{D}^{4\beta,\vartheta}_{N,\mathrm{xxxt}}u = 0, \tag{1.1}$$

where, the variables $x > 0$ and $t > 0$. The Mittag-Leffler function is defined as elaborated in [43]. A review of existing literature reveals that the unified method has not yet been utilized in this context, nor have soliton solutions been derived through this mathematical approach. It should be noted that the unified approach is capable of yielding solutions that are more general than those produced by the recently developed methods, such as the hyperbolic tangent function approach or the $\frac{G'}{G}$-expansion methods. This observation underscores a significant gap in the current body of research, which this study aims to address. The provision of precise solutions enables the prediction of system behavior across various scenarios without necessitating substantial computational resources. Specifically, "exact soliton solutions", or simply "exact solutions", refer to analytical solutions obtained directly through rigorous mathematical methodologies, rather than through approximations or numerical simulations. In the context of the nonlinear conformable Kairat-X equation, these exact soliton solutions represent specific waveforms that not only perfectly satisfy the equation but also exhibit essential solitonic characteristics. Such exact solutions facilitate the understanding of system dynamics under different conditions, thereby enhancing predictive capabilities without the reliance on extensive computational power. This aligns with the overarching goal of our study: To contribute to the field by supplying valuable insights through the derivation of exact solutions that fill the identified research void.

The main reason for choosing the nonlinear fractional Kairat-X equation as the model for this study lies in its relevance and applicability across diverse fields such as quantum physics, materials science, and telecommunications. It effectively captures the intricate dynamics of solitons in nonlinear media, making it a valuable tool for examining complex phenomena and interactions that occur in these applications. The gap in the literature that this study focuses on includes the limited exploration of exact soliton solutions for the nonlinear fractional Kairat-X equation. While previous research may have addressed general properties or numerical simulations of the equation, there is often a lack of comprehensive studies that derive and classify various exact soliton solutions, such as kink wave solitons, dark solitons, and bright solitons. Additionally, the existing literature may not fully utilize advanced mathematical techniques, such as the generalized tanh-function method, to systematically obtain a wide variety of soliton solutions. By addressing these gaps, this study contributes to the growing body of knowledge surrounding the Kairat-X equation, providing a more detailed understanding of its soliton solutions and their physical implications. The results not only enhance theoretical insights but also pave the way for potential applications in related fields, thereby bridging existing gaps in the literature.

Solutions of the nonlinear conformable Kairat-X Equation must be interpreted physically with caution, since their applicability hinges upon the context of the equation as well as the very character of the solutions. Trivial solutions that represent simple equilibrium or constant states tend to lack dynamical relevance. onversely, non-trivial solutions with more involved behavior, such as wave propagation or solitonic patterns, can capture physical processes of interest to the systems they describe. To determine their physical feasibility, it is crucial to analyze how these solutions fit into basic principles, such as conservation laws and stability of perturbations. Further, probing parameter dependencies and comparing these solutions with known outcomes in the literature can shed light on their relevance and validity. This cautious scrutiny is critical to determine if the solutions add to underlying physical phenomena or are simple mathematical artifacts. The force equation, written as $F = ma$, is fundamental in describing the interplay between mass, acceleration, and the forces on a system. Its integration into the theory of nonlinear dynamics and soliton solutions explains how external forces influence the development and stability of solitons [44]. By using force equation concepts, scientists can study how changing conditions of force affect systems controlled by nonlinear partial differential equations, for example, the Kairat-X equation. This knowledge provides better predictions for soliton behavior under changing external conditions and consolidates the theory needed for use in practical fields such as fluid dynamics, optics, and material science. In addition, studying the interaction between the force equation and soliton theory can result in new approaches to solving intricate nonlinear systems, ultimately enhancing our capacity to model real-world phenomena.

Section 2 introduces the unified approach, providing an overview of its principles and methods. In Section 3, we discuss the application of the nonlinear conformable Kairat-X equation and its relevance to our research objectives. Section 4 employs various visual representations, including contour plots, density graphs, and both 2D and 3D visualizations, to illustrate the multiple solutions identified in our analysis. Finally, Section 5 concludes the paper by summarizing the main contributions and suggesting directions for future research. This structured approach allows for a comprehensive examination of the nonlinear fractional Kairat-X equation and its solutions.

2. Summary of unified method

Consider the NLPDE in the form of the function $u(x,t)$, where t represents the time variable and x denotes the spatial variable.

$$p(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}........) = 0. \tag{2.1}$$

Let P be a polynomial in $u = u(x,t)$ and its various partial derivatives, which includes nonlinear terms as well as the highest-order derivative. Here, $u(x,t)$ is an unknown function of both the spatial variable x and the temporal variable t.

The gamma function, denoted as $\Gamma(z)$, is a fundamental mathematical function that generalizes the concept of factorials to non-integer values. It is defined for complex numbers and plays a crucial role in various areas such as calculus, complex analysis, and probability theory. The gamma function is mathematically defined for complex numbers z with a positive real part ($\mathrm{Re}(z) > 0$) by the following integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{2.2}$$

This integral converges for all z where the real part is positive. The factor e^{-t} ensures that the integral converges as t approaches infinity, while t^{z-1} allows for the extension to non-integer values.

Step 1. Consider a fractional wave transformation, which involves substituting the original variables in a wave equation with fractional derivatives.

$$u(x,t) = U(\chi), \quad \chi = \frac{\Gamma(\alpha+1)}{\beta}\left(\lambda t^\beta + \xi x^\beta\right) + \phi. \tag{2.3}$$

Step 2. By substituting Eq (2.3) into Eq (2.1), we derive a NLODE, which can be expressed as follows.

$$P(U, U', U'', U''', U'''', ...) = 0. \tag{2.4}$$

Step 3. The integration process introduces constants. When every term of the differential equation includes derivatives that can be integrated, the integration should be concluded. For the specific solution we are looking for, we assume that the constant arising from the integration is set to zero.

Step 4. Let us consider that the assumed solution to the NLPDE can be represented by the following ansatz [45]:

$$U(\chi) = a_0 + \sum_{i=1}^{M} \left[a_i F(\chi)^i + b_i F(\chi)^{-i} \right], \tag{2.5}$$

where $a_M^2 + b_M^2 \neq 0$, and a_0, a_i, b_i (for $1 \leq i \leq M$) are unknown arbitrary parameters, the explicit invariant solution $F(\chi)$ satisfies the following equation:

$$F'(\chi) = F(\chi)^2 + \Omega. \tag{2.6}$$

The general solution is as follows for Eq (2.1):

(a) Hyperbolic function solutions (If $\Omega < 0$ then)

$$F(\chi) = \frac{\mp \sqrt{\Omega\left(-\left(d^2+e^2\right)\right)} - d\sqrt{-\Omega}\cosh\left(2\sqrt{-\Omega}(f+\chi)\right)}{d\sinh\left(2\sqrt{-\Omega}(f+\chi)\right)+e},$$

$$F(\chi) = \mp\sqrt{-\Omega} + \frac{\pm 2d\sqrt{-\Omega}}{d+\cosh\left(2\sqrt{-\Omega}(f+\chi)\right)-\sinh\left(2\sqrt{-\Omega}(f+\chi)\right)}.$$

(b) Trigonometric function solutions (If $\Omega > 0$ then)

$$F(\chi) = \frac{\mp \sqrt{\Omega\left(d^2+e^2\right)} - d\sqrt{\Omega}\cos\left(2\sqrt{\Omega}(f+\chi)\right)}{d\sin\left(2\sqrt{\Omega}(f+\chi)\right)+e},$$

$$F(\chi) = \mp i\sqrt{\Omega} + \frac{2id\sqrt{\Omega}}{d \pm i\sin\left(2\sqrt{\Omega}(f+\chi)\right)+\cos\left(2\sqrt{\Omega}(f+\chi)\right)}.$$

(c) Rational function solutions (If $\Omega = 0$ then)

$$F(\chi) = -\frac{1}{f+\chi}.$$

3. Application of the nonlinear conformable Kairat-X equation

By applying a fractional wave transformation to Eq (2.3) in relation to Eq (1.1), we derive an NLODE.

$$\lambda\xi^3 U^{(4)} + \lambda^2 U'' - 6\lambda\xi^2 U' U'' = 0. \tag{3.1}$$

After integrating once with regard to χ, we modify Eq (3.1) as

$$\lambda\xi^3 U^{(3)} + \lambda^2 U' - 3\lambda\xi^2 \left(U'\right)^2 = 0. \tag{3.2}$$

Equation (3.2) can be solved as follows based on M = 1, which is obtained by applying the homogeneous balancing principle.

$$U(\chi) = a_1 F(\chi) + a_0 + \frac{b_1}{F(\chi)}. \tag{3.3}$$

By substituting Eq (3.3) into Eq (3.2) and incorporating the details outlined in Eq (2.6), we formulate an algebraic system. This is achieved by setting each coefficient from the resulting expression to zero. This approach ensures that we identify the requisite conditions for the equations to be valid, ultimately resulting in a solvable system of equations based on the coefficients involved.

$$B_0 + B_1F(\chi)^2 + B_2F(\chi)^4 + B_3F(\chi)^6 + B_4F(\chi)^8 = 0,$$

where

$$B_0 = -6b_1\lambda\xi^3\Omega^3 - 3b_1^2\lambda\xi^2\Omega^2,$$

$$B_1 = 6a_1b_1\lambda\xi^2\Omega^2 - b_1\lambda^2\Omega - 8b_1\lambda\xi^3\Omega^2 - 6b_1^2\lambda\xi^2\Omega,$$

$$B_2 = 12a_1b_1\lambda\xi^2\Omega + a_1\lambda^2\Omega + 2a_1\lambda\xi^3\Omega^2 - 3a_1^2\lambda\xi^2\Omega^2 - b_1\lambda^2 - 2b_1\lambda\xi^3\Omega - 3b_1^2\lambda\xi^2,$$

$$B_3 = 6a_1b_1\lambda\xi^2 + a_1\lambda^2 + 8a_1\lambda\xi^3\Omega - 6a_1^2\lambda\xi^2\Omega,$$

$$B_4 = 6a_1\lambda\xi^3 - 3a_1^2\lambda\xi^2.$$

By setting all coefficients equal to zero

$$\begin{cases} B_0 = 0,\ B_1 = 0, \\ B_2 = 0,\ B_3 = 0,\ B_4 = 0. \end{cases} \tag{3.4}$$

Upon solving these algebraic equations, we obtain

$$a_1 = 2\xi, b_1 = -2\xi\Omega, \lambda = 16\xi^3\Omega.$$

(a) Hyperbolic function solutions (If $\Omega < 0$ then)

$$\begin{aligned} u_1(x,t) = a_0 + &\frac{2\xi\left(\sqrt{-(\Omega(d^2+e^2))} - d\sqrt{-\Omega}\cosh\left(2\sqrt{-\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)\right)}{d\sinh\left(2\sqrt{-\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) + e} \\ &- \frac{2\xi\Omega\left(d\sinh\left(2\sqrt{-\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) + e\right)}{\sqrt{-(\Omega(d^2+e^2))} - d\sqrt{-\Omega}\cosh\left(2\sqrt{-\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_2(x,t) = a_0 + 2\xi\sqrt{-\Omega}&\left(\frac{2d}{d - \sinh\left(2\sqrt{-\Omega}(f+\chi)\right) + \cosh\left(2\sqrt{-\Omega}(f+\chi)\right)}\right. \\ &\left. + \frac{1}{\frac{2d}{d - \sinh\left(2\sqrt{-\Omega}(f+\chi)\right) + \cosh\left(2\sqrt{-\Omega}(f+\chi)\right)} - 1} - 1\right). \end{aligned} \tag{3.6}$$

(b) Trigonometric function solutions (If $\Omega > 0$ then)

$$u_3(x,t) = a_0 - \frac{2\xi\left(\sqrt{\Omega\left(d^2+e^2\right)} + d\sqrt{\Omega}\cos\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)\right)}{d\sin\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) + e} + \frac{2\xi\Omega\left(d\sin\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) + e\right)}{\sqrt{\Omega\left(d^2+e^2\right)} + d\sqrt{\Omega}\cos\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)}, \tag{3.7}$$

$$u_4(x,t) = a_0 - 2i\xi\sqrt{\Omega}\left(-\frac{2d}{d + \cos\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) - i\sin\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)} + \frac{1}{1 - \frac{2d}{d + \cos\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right) - i\sin\left(2\sqrt{\Omega}\left(f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right)}} + 1\right). \tag{3.8}$$

(c) Rational function solutions (If $\Omega = 0$ then)

$$u_5(x,t) = a_0 + 2\xi\left(-\frac{1}{f + \frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi} + f\Omega + \Omega\left(\frac{\Gamma(\alpha+1)\left(\lambda t^\beta + \xi x^\beta\right)}{\beta} + \phi\right)\right). \tag{3.9}$$

4. Graphical representation

The unified method has emerged as a far better improvement over classical methods for obtaining exact soliton solutions for the nonlinear fractional Kairat-X equation. The approach presents a generalized framework that encompasses different techniques so that a wider family of solutions such as trigonometric, hyperbolic, and plane waves, can be obtained under a single systematic protocol. Such a concerted effort not only streamlines the process of solution derivation, rendering it more amenable to researchers, but is also shown to be robust in that it produces a varied set of exact solutions consistently, thus inviting greater theoretical investigation and practical usage. But there are some drawbacks to the unified approach, such as its complexity that could impose a learning curve on those not familiar with the underlying math and its requirement for more computational resources than usual methods. Additionally, this holistic approach could risk missing special properties particular to specific equations that standard methods could treat better. Consequently, while the unified method represents a significant step forward, researchers should carefully weigh its advantages against these complexities to ensure its suitability for the context of their work.

In the present section, we discuss different parametric values being used in contour graphs and compare them thoroughly in both 2D and 3D visual forms. Using Mathematica as our calculating software, we created Figures 1–5 to demonstrate these results. Figure 1 illustrates the hyperbolic soliton, which is a stable, localized packet of waves that maintains its form while moving at a steady speed. These solitons are important in many physical systems, where they take the form of rogue waves or tsunamis in shallow water waves and signal optimization in optical fibers by avoiding dispersion. In Figure 2, we show the dark soliton, an interesting nonlinear wave characteristic of a localized reduction in wave amplitude in the background of a continuous structure. Dark solitons occur in various physical situations, significantly in Bose-Einstein condensates, where they are used as a means of manipulation of quantum states, and in nonlinear optics, where they find application in pulse manipulation within fiber lasers and soliton lasers. Figures 3 and 4 show periodic soliton solutions, which can be interpreted as repeating waves that can describe different phenomena in different media. The solutions appear in plasma physics, where periodic structures occur in confined plasmas, and in nonlinear lattice systems, including optical lattices, which find applications in the control of light propagation at the nanoscale. Finally, Figure 5 shows the kink wave solution, which is important in field theory because it can be used to model phase transitions in physical systems, e.g., magnetic materials experiencing phase transitions. Kink solitons are a good example of one phase being transformed into another and have implications for processes such as domain wall motion in ferromagnets and field configurations in scalar field theories. In conclusion, the various soliton solutions obtained from the nonlinear Kairat-X equation demonstrate a rich variety of nonlinear phenomena with extensive applications in various fields of fluid dynamics, optical communications, condensed matter physics, and materials science, emphasizing the relevance of the equation in modeling intricate real-world situations.

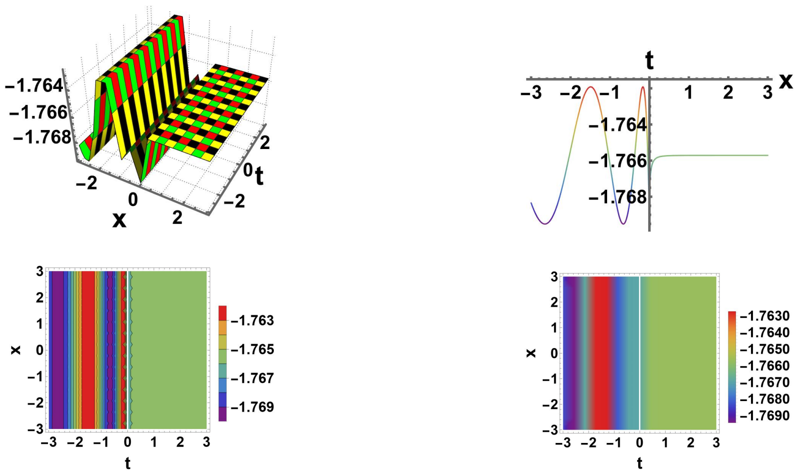

Figure 1. Visual representation based on the parameter values $\Omega = -2, d = 0.78, e = 0.65, f = 1, \phi = 2.04, a = 0.05, \beta = 0.95, \xi = 0.04, a_0 = 1.01, \alpha = 0.85, \lambda = 0.75$ as defined in Eq (3.5).

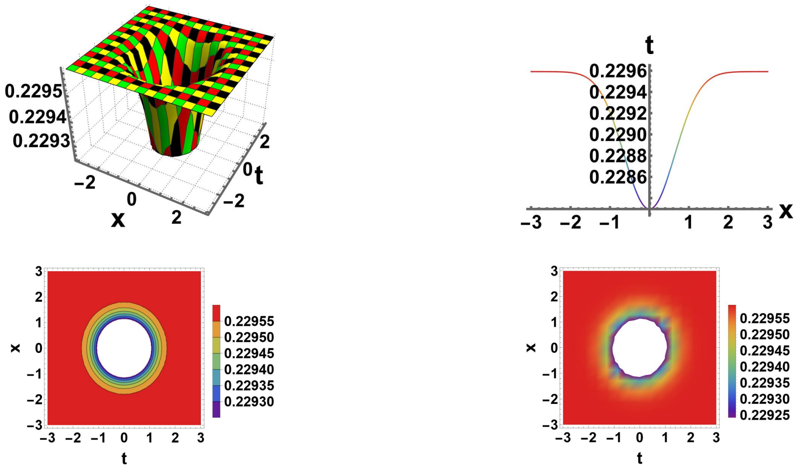

Figure 2. Visual representation based on the parameter values $\Omega = -0.2, d = 0.07, e = 0.4, f = 1.02, \phi = 0.7, a = 1.03, \beta = 2.02, \xi = 1.04, a_0 = 2.09, \alpha = 1.5, \lambda = 0.9$ as defined in Eq (3.6).

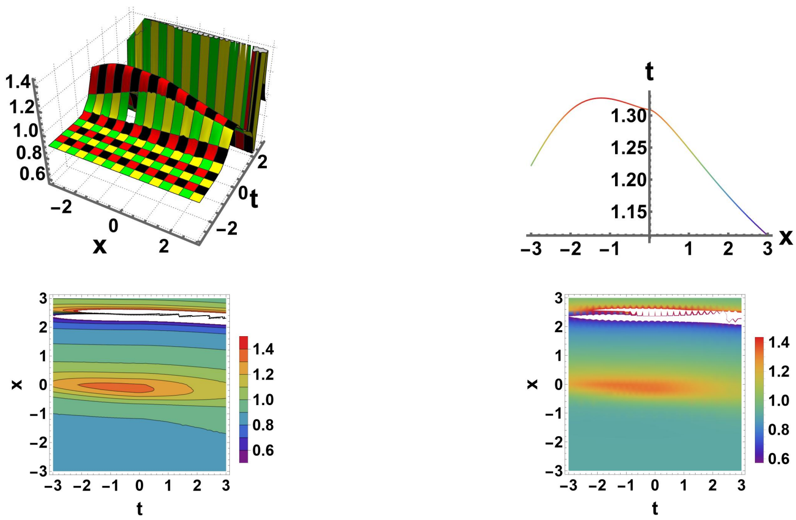

Figure 3. Visual representation based on the parameter values $\Omega = 2, d = 1, e = 2, f = 0.03, \phi = 1.3, a = 1.5, \beta = 1.4, \xi = 0.05, a_0 = 0.9, \alpha = 0.5, \lambda = 0.9$ as defined in Eq (3.7).

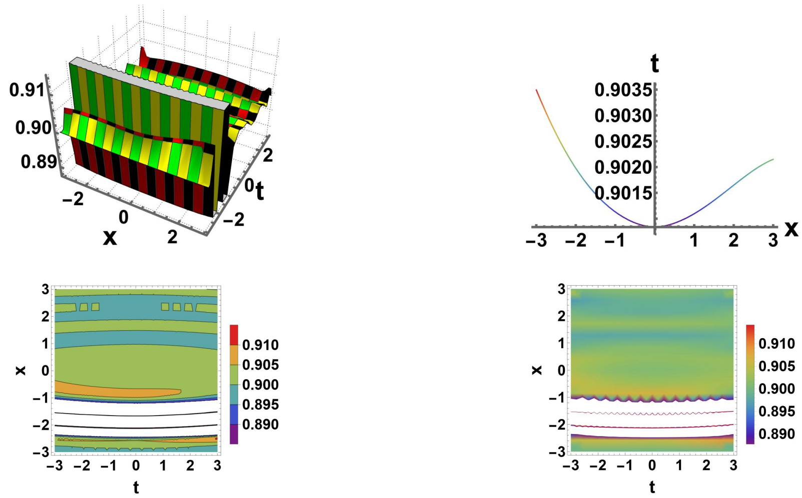

Figure 4. Visual representation based on the parameter values $\Omega = 2, d = 0.07, e = 0.01, \lambda = 0.9, f = 1.7, \phi = 1.7, a = 1.3, \beta = 1.8, \xi = 0.04, a_0 = 0.9, \alpha = 1.05$ as defined in Eq (3.8).

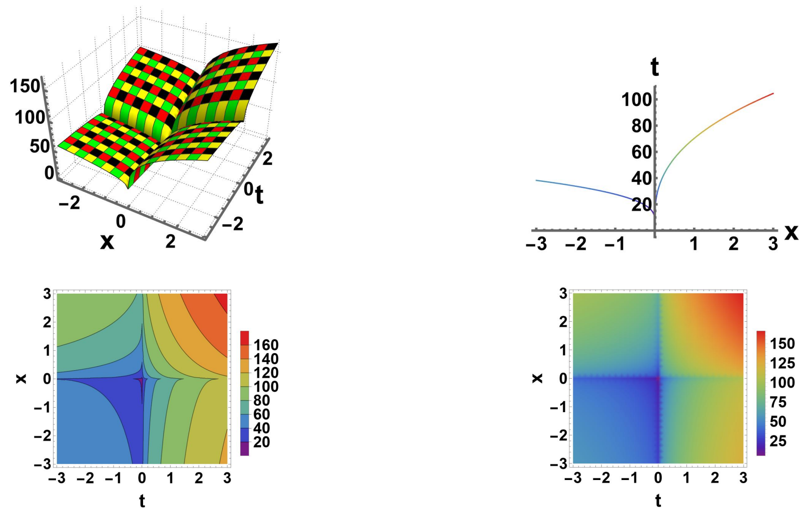

Figure 5. Visual representation based on the parameter values $\Omega = 0, d = 1.09, e = 1.6, f = 1.01, \phi = 0.07, a = 0.03, \beta = 0.4, \xi = 1.4, a_0 = 2.9, \alpha = 2.5, \lambda = 0.9$ as defined in Eq (3.9).

5. Conclusions

This study employed a unified method to investigate various types of solitary wave solutions to the nonlinear fractional Kairat-X equation with conformable derivative, which encompasses newly identified kink wave solitons, as well as dark, bright, and periodic wave solitons. The use of the conformable derivative allows for a more nuanced exploration of fractional dynamics and facilitates the handling of complex nonlinear phenomena across various applications. These solutions were given in terms of rational, hyperbolic, and trigonometric functions. In order to graphically depict the physical properties of the secured solitons, we performed computational simulations with constant parameters, and we obtained contour, two-dimensional, and three-dimensional plots. The results obtained from this study have great potential for furthering our knowledge of nonlinear phenomena in various scientific fields. The uses of these solutions are especially significant in fields like fiber optics, soliton wave theory, nonlinear dynamics, geophysics, nonlinear optics, ferromagnetic material dynamics, and many engineering applications. By taking advantage of the properties of the obtained solitons, we can further investigate their applications in practical situations, improving the design and optimization of systems described by nonlinear equations. In the future, further work may involve the extension of this research to consider the investigation of more complex boundary conditions and initial value problems, perhaps culminating in a better understanding of the stability and interactions of these soliton solutions. In addition, it would be useful to further research the solutions with higher-dimensional systems or with various kinds of fractional derivatives so as to expand applicability and range of knowledge gleaned from the nonlinear fractional Kairat-X equation. In total, this research not only adds worthy new solutions to current literature but also emphasizes the success of our strategy in tackling nonlinear complex problems and opening up areas for further innovations in this developing field.

The investigation of different soliton solutions obtained from the nonlinear Kairat-X equation has important implications in various disciplines, proving the relevance of these mathematical objects to the description and utilization of wave phenomena in actual applications. Perhaps the most significant context for the use of kink solitons is condensed matter physics, where they represent stable structures in materials that are experiencing phase transitions. For example, in liquid crystals, the kinks can enable the control of light, resulting in the advancement of display and optical devices. In superconductors, too, kink solitons could be involved in vortex motion and, as such, have an impact on the efficiency of power transmission as well as on the building of quantum computing components. Dark solitons, or localized amplitude decreases, have seen important use in the field of nonlinear optics, specifically in optical fiber and Bose-Einstein condensate applications. In telecommunications, the capability of dark solitons to preserve their shape under high-speed travel can be utilized to design more efficient data transmission systems with increased bandwidth and decreased signal degradation. In addition, their special characteristics can facilitate the creation of sophisticated pulse generation methods, which are very important in laser technology. Periodic soliton solutions carry essential implications in fluid dynamics, especially in shallow water wave systems. The understanding derived from the exploration of such waves can be used to further expand the knowledge of wave behaviors across a range of conditions, from oceanographic research modeling storm surges and tsunami dynamics to engineering use in coastal barrier development and wave energy collection.

Through the optimization of the shape of structures to reduce the effect of wave forces, coastal communities can be safeguarded and sustainable energy systems can be encouraged. In addition, plane wave solutions form a basic foundation for many physical phenomena and are the building blocks for more complicated wave interactions. Plane wave solutions are essential in acoustics and electromagnetic theory and are used to guide the design of devices like antennas and sensors that are based on wave propagation principles. Generally, the broad knowledge of soliton behaviors paves the way for innovation in applications like materials science, where soliton-like behavior can be engineered into materials so that their performance is improved for particular purposes. Moreover, new technologies can be developed with advances in soliton theory in quantum computing, where solitons can be used for transferring and processing information in quantum systems. In summary, the discoveries involving various soliton solutions not only add richness to the theoretical field of nonlinear dynamics but also have the potential to influence a wide array of real-world applications. With continued development of this field of study, the incorporation of soliton solutions into technological innovations has the potential to revolutionize the advancement of multiple fields, providing the framework for the engineering, communications, and material design of the future.

Author contributions

Jamshad Ahmad: Resources, acquisition, supervision, writing-review and editing, visualization, validation; Zulaikha Mustafa: Writing-original draft, conceptualization, methodology, software, validation, investigation, software; Mehjabeen Anwar: Methodology, writing-review and editing, formal analysis; Marouan Kouki: Writing-review and editing, formal analysis, software; Nehad Ali Shah: Writing-review and editing, formal analysis, validation, investigation. All authors have read and agreed to the published version of the manuscript. Jamshad Ahmad and Nehad Ali Shah contributed equally to this work and are co-first authors.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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