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Research article

Analytical solutions to the 2D compressible Navier-Stokes equations with density-dependent viscosity coefficients

Jia Jia*

School of Mathematical Sciences, East China Normal University, Shanghai, 200241, China

* **Correspondence:** Email: jmjvictory@163.com.

Abstract: In this paper, we considered a class of analytical, rotational, and self-similar solutions to the 2D compressible Navier-Stokes equations (CNS) with density-dependent viscosity coefficients. For the isentropic case $k > 0$, $\gamma = \varphi > 1$, we provided the formula of self-similar analytical solutions and proved the well-posedness and the large time behavior for the corresponding generalized Emden equation. It is interesting to see that the different effects of rotation and pressure were revealed. Compared with the irrotational and pressureless case, when the free boundary $a(t)$ increases linearly or sub-linearly in time, we can find some classes of solutions with linear growth by taking the pressure effect or the swirl effect into account. In this sense, rotation or pressure effects may accelerate the growth of the boundary. Finally, we gave some examples of blow-up solutions and used a new method to prove the results.

Keywords: analytical solutions; compressible Navier-Stokes equations; density-dependent viscosity; large time behavior
Mathematics Subject Classification: 35Q30, 35R35, 76N06

1. Introduction

The 2D compressible Navier-Stokes equations (CNS) with density-dependent viscosity coefficients and the free boundary condition can be formulated in the following form:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, & \text{in } \Omega(t), \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \operatorname{div}(h(\rho) D(\mathbf{u})) + \nabla(g(\rho)\operatorname{div}\mathbf{u}), & \text{in } \Omega(t), \\ \rho > 0, & \text{in } \Omega(t), \\ \rho = 0, & \text{on } \partial\Omega(t), \\ (\rho, \mathbf{u})(x, 0) = (\rho_0, \mathbf{u}_0), & \text{on } \Omega(0), \end{cases} \tag{1.1}$$

where $\rho(x,t)$ and $\mathbf{u}(x,t)$ with $(x,t) \in \Omega(t) \times (0,+\infty)$ are the density and the velocity, respectively. $P = P(\rho)$ is the pressure, and we use the γ-law on the pressure, i.e.,

$$P = k\rho^\gamma \ (k \geq 0, \ \gamma \geq 1), \tag{1.2}$$

where γ stands for the adiabatic exponent, $\gamma = \frac{c_p}{c_v} \geq 1$ is the ratio of the specific heats, and c_p and c_v are the specific heats per unit mass under constant pressure and constant volume, respectively. The system (1.1) is called pressureless if $k = 0$. In particular, if $k > 0$, $\gamma = 1$, the fluid is called isothermal, which can be used for constructing models with non-degenerate isothermal fluid; if $k > 0$, $\gamma > 1$, the fluid is called isentropic. $\Omega(t) \subset \mathbb{R}^2$ characterizes the instantaneous spatial region occupied by the fluid at time t. $D(\mathbf{u})$ is the strain tensor given by $\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}$, and $h(\rho)$ and $g(\rho)$ are the Lamé viscosity coefficients. $\partial\Omega(t) = \Psi(\partial\Omega(0),t)$ is the free boundary separating fluid from a vacuum, $\rho = 0$ on $\partial\Omega(t)$ is the continuous density boundary condition which completes the Navier-Stokes equations, and $\partial\Omega(0) = \{x \in \mathbb{R}^2 : |x| = a_0\}$ is the initial free boundary. Ψ is the particle path of the flow, which satisfies

$$\begin{cases} \partial_t\Psi(x,t) = \mathbf{u}(\Psi(x,t),t), & x \in \mathbb{R}^2, \\ \Psi(x,0) = x. \end{cases}$$

For a more detailed introduction of the Navier-Stokes equations, see [1–3].

Equations (1.1) were introduced by Liu, Xin, and Yang in [4]. Since then, significant progress has been achieved by many authors. For the one-dimensional case with $h(\rho) = \rho^\alpha$, $g(\rho) = 0$ ($\alpha \in (0, \frac{3}{2})$) and the free boundary conditions, there are many studies on the well-posedness theory, see [5–7] and the references therein. However, few results are available for the multi-dimensional problems. The first multi-dimensional result is due to Mellet and Vasseur [8], where they proved the L^1 stability of weak solutions to (1.1) based on a new entropy estimate established in [9, 10], which extended the corresponding L^1 stability result of [9].

The reason for the viscosity depending on the density (variable viscosity) is that when we study fluid motion, especailly when we encounter a vacuum state, this makes the problem complicated. First, in [4, 11, 12], we can see that the Cauchy problem of the Navier-Stokes equations with constant coefficients including the vacuum state is ill-posed, which is reflected in the fact that the solution of this system has no continuous dependence on the initial value, and when the initial density has a compact support set, the system may have a global regular solution. According to the theory of physics, Liu, Xin, and Yang introduced the density-dependent Navier-Stokes equation in [4] and proved the local well-posedness. In fact, we know that the real fluids can be approximated by the ideal fluids only if the temperature and density vary within the appropriate range by the literature [13, 14]. Second, we obtain the Navier-Stokes equations from the Boltzmann equation by the second-order expansion of Chapman-Enskog (see [4, 15, 16]), where the viscosity coefficient is temperature dependent in the derivation process. For example, for the hard sphere collision model, the viscosity coefficient is directly proportional to the square root of the temperature. If we consider the motion of an isentropic fluid, according to the second law of thermodynamics, it can be deduced that the viscosity coefficient is density-dependent. Therefore, we need to take into account the effect of density on the viscosity coefficient when studying the problem of initial density containing a vacuum. In addition, in geophysics, many of the mathematical models that are used to study fluid motion are similar to the Navier-Stokes equations whose viscosity depends on density, such as the Saint-Venant system for shallow water waves (see [3, 10, 17]).

In the recent decades, there have been many references considering the analytical solutions or blow-up solutions to the Navier-Stokes equations [18–20], Euler-Poisson equations [21–23], Euler equations [24–26], or Euler equations with time-dependent damping [27] and the references therein. In [19], Yeung and Yuen considered (1.1) radial symmetry solutions with $h(\rho) = 0$, $g(\rho) = \rho^\theta$ in $\mathbb{R}^N$, and showed that there exists a family of analytical solutions for the Navier-Stokes equations with pressure for $\theta = \gamma = 1$ and $\theta = \gamma > 1$. Dong, Xue, and Zhang in [28] constructed a class of spherically symmetric and self-similar analytical solutions to the pressureless Navier-Stokes equations with density-dependent viscosity coefficients satisfying $h(\rho) = \rho^\theta$, $g(\rho) = (\theta - 1)\rho^\theta$, and they investigated the large time behavior of the solutions according to various $\theta > 1$ and $0 < \theta < 1$. In [29], Dou and Zhao found an interesting phenomenon on the solution to 1D compressible isentropic Navier-Stokes equations with a constant viscosity coefficient on $(x, t) \in [0, +\infty) \times R_+$: The solutions to the initial boundary value problem to 1D compressible Navier-Stokes equations in half space can be transformed to the solution to the Riccati differential equation under some suitable conditions. In [23], Yuen considered the Euler-Poisson equations in spherical symmetry in the two dimensional isothermal case, and the following analytical solutions were given:

$$\rho(t, r) = \frac{e^{y(\frac{r}{a(t)})}}{a^2(t)},\ u(t, r) = \frac{a'(t)}{a(t)}r, \tag{1.3}$$

where $a(t)$, $y(z) \in C^2$ are two functions satisfying some ordinary differential equations, and the blow-up rate of the solution is

$$\lim_{t\to T} \rho(t, 0)(T - t)^\eta \geq o(1),$$

with $\eta < 2$.

In this paper, we mainly consider the compressible Navier-Stokes equations with density-dependent viscosity coefficients and a continuous density boundary condition. The Lamé viscosity coefficients are

$$h(\rho) = 0,\ g(\rho) = \kappa\rho^\varphi (\varphi \geq 1). \tag{1.4}$$

Without loss of generality, we let $\kappa = 1$, and then $g(\rho) = \rho^\varphi$. Therefore, system (1.1) is transformed into

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \nabla(\rho^\varphi \mathrm{div}\mathbf{u}). \end{cases} \tag{1.5}$$

Similar to [30], if we consider the shear viscosity, the Lamé viscosity coefficients are taken as $h(\rho) = \mu$, $g(\rho) = \mu + \rho^\varphi$, and there is no essential difficulty. The corresponding conclusion can also be obtained by using the method in this paper.

We consider the fluid region $\Omega(t)$ in 2D space, which is written as

$$\Omega(t) = \{(r, t) \in \mathbb{R}^+ \times [0, +\infty) | 0 \leq r \leq a(t),\ t \geq 0\}, \tag{1.6}$$

where $r = \sqrt{x_1^2 + x_2^2}$ is the polar diameter, and $a(t)$ is the free boundary satisfying

$$\frac{d}{dt}a(t) = u^r(a(t), t),\quad a(0) = a_0, \tag{1.7}$$

where u^r is the radial component of the velocity field $\mathbf{u}$ as in (1.8).

Let $\mathbf{e}_r = \frac{(x_1,x_2)}{r}$, $\mathbf{e}_\theta = \frac{(-x_2,x_1)}{r}$ be two orthogonal unit vectors along the radial and angular directions, respectively. Then the velocity field $\mathbf{u}$ can be written in the following form:

$$\mathbf{u}(r,t) = u^r(r,t)\mathbf{e}_r + u^\theta(r,t)\mathbf{e}_\theta,$$

which can be equivalently expressed in the Euler coordinates as

$$\mathbf{u} = (u_1,u_2) = \left(\frac{x_1u^r - x_2u^\theta}{r}, \frac{x_2u^r + x_1u^\theta}{r}\right).$$

The compressible Navier-Stokes equations (1.5) in the Euler coordinates can be written in the following polar coordinates form:

$$\begin{cases} \rho_t + \partial_r(\rho u^r) + \frac{\rho u^r}{r} = 0, \\ \rho u^r_t + \rho\left[u^r\partial_r u^r - \frac{|u^\theta|^2}{r}\right] + \partial_r(k\rho^\gamma) = (\rho^\varphi)_r\left(\frac{u^r}{r} + \partial_r u^r\right) + \rho^\varphi\left(\frac{r\partial_r u^r - u^r}{r^2} + \partial_r^2 u^r\right), \\ \rho u^\theta_t + \rho\left[u^r\partial_r u^\theta + \frac{u^\theta u^r}{r}\right] = 0. \end{cases} \tag{1.8}$$

By (1.8) and the following Theorems 2.1 and 2.6, we see that u^θ has nothing to do with θ, but with the polar diameter, and in this sense the Navier-Stokes equation in the 2D polar coordinates can be viewed as radially symmetric.

Accordingly, the initial condition is

$$\left(\rho, u^r, u^\theta\right)(r,t)\Big|_{t=0} = \left(\rho_0, u^r_0, u^\theta_0\right)(r), \quad r \in (0, a_0).$$

The continuous density boundary condition is

$$\rho\,(a(t), t) = 0. \tag{1.9}$$

In the following Theorems 2.1 and 2.6, the solutions to (1.8) is radially symmetric and smooth at the center of symmetry, so the velocity at the center of symmetry is 0, and we impose the Dirichlet boundary condition at the center of symmetry

$$u^r\,(0,t) = u^\theta\,(0,t) = 0. \tag{1.10}$$

In the following, according to the different properties of pressure and different types of fluid, we first consider the self-similar solutions of 2D CNS under the boundary conditions (1.9) and (1.10) in the isentropic case $k > 0$, $\gamma = \varphi > 1$; and then we consider the self-similar solutions of 2D CNS in the isothermal case $k > 0$, $\gamma = \varphi = 1$.

Our main results are as follows:

2. Results

For the isentropic case, we have:

Theorem 2.1. *For the 2D CNS* (1.8)–(1.10) *in the isentropic case* $k > 0, \gamma = \varphi > 1$, *there exist a family of self-similar solutions of the form*

$$\rho(t,r) = \frac{\left[\frac{\varphi-1}{2}(1-\frac{r^2}{a^2})\right]^{\frac{1}{\varphi-1}}}{a^2(t)}, \tag{2.1}$$

$$u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r, \tag{2.2}$$

where ξ *is a constant that represents the strength effects of rotation, and* $a(t) \in C^2([0,+\infty))$ *is a free boundary satisfying* (1.6), (1.7)*, and the following generalized Emden equation:*

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k\varphi}{a^{2\varphi-1}(t)} + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \end{cases} \tag{2.3}$$

In particular, the generalized Emden equation (2.3) *is a non-conservative system, the trajectory of the solution* $a(t)$ *in the Poincaré phase plane is moving toward a state that has a lower total energy, and* $a(t)$ *satisfies the estimate*

$$0 < C_2 < a(t) < C_1(1+t),$$

where

$$C_1 = \max\left\{a_0,\ \sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}\right\}, \tag{2.4}$$

$$C_2 = \max\left\{\frac{|\xi|}{\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}}, \left[\frac{k\varphi}{(\varphi-1)(\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi})}\right]^{\frac{1}{2\varphi-2}}\right\}. \tag{2.5}$$

Moreover, if

$$C_2 \geq \frac{2}{k}\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}, \tag{2.6}$$

the limit $\lim\limits_{t\to+\infty} a'(t)$ *exists, and the large time behavior of* $a(t)$ *is*

$$\lim_{t\to+\infty}\frac{a(t)}{t} = \lim_{t\to+\infty}a'(t) = A. \tag{2.7}$$

There are several remarks in order.

Remark 2.2. *For the 2D irrotational case* $k > 0,\ \gamma > 1,\ \xi = 0$*, the Emden equation* (2.3) *is reduced to*

$$a''(t) - \frac{k\varphi}{a^{2\varphi-1}(t)} + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0,\ \ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \tag{2.8}$$

Accordingly, the condition (2.6) *is reduced to*

$$\left[\frac{k\varphi}{(\varphi-1)(\widetilde{a}_1^2 + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi})}\right]^{\frac{1}{2\varphi-2}} \geq \frac{2}{k}\sqrt{\widetilde{a}_1^2 + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}},$$

or equivalently,

$$\frac{k\varphi}{\varphi-1}\left(\frac{k}{2}\right)^{2\varphi-2} \geq \left(\widetilde{a}_1^2 + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}\right)^{\varphi}, \tag{2.9}$$

which can be guaranteed by selecting suitable initial values $a_0 > 0$ *and* $\widetilde{a}_1$ *in* (2.3)*. Moreover, similar to the proof of Step 2 in Lemma 3.8, we have that if* $\widetilde{a}_1 > \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}$*, the free boundary* $a(t)$ *increases in* $[0, +\infty)$ *and* $\lim_{t\to+\infty} a(t) = +\infty$*, and the fluid density* $\rho(t,r)$ *satisfies that* $\lim_{t\to+\infty} \rho(t,r) = 0$.

Remark 2.3. *For the 2D pressureless case* $k = 0$, $\varphi > 1$*, the corresponding generalized Emden equation becomes*

$$a''(t) - \frac{\xi^2}{a^3(t)} + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \;\; a(0) = a_0 > 0, \; a'(0) = \widetilde{a}_1. \tag{2.10}$$

To investigate the large time behavior in (2.7)*, we require* $\varphi \geq \frac{3}{2}$*, and*

$$a''(t) = \frac{\xi^2}{a^3(t)} - \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = \frac{\xi^2 a^{2\varphi-3}(t) - 2\varphi a'(t)}{a^{2\varphi}(t)} \geq 0,$$

and the condition (2.6) *is turned into*

$$\varphi \geq \frac{3}{2}, \quad \xi^2\left(\frac{|\xi|}{\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2}}}\right)^{2\varphi-3} \geq 2\varphi\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2}}. \tag{2.11}$$

Remark 2.4. *For the 2D irrotational and pressureless case* $k = 0$, $\varphi > 1$, $\xi = 0$*, the Emden equation* (2.3) *is reduced to*

$$a''(t) + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \;\; a(0) = a_0 > 0, \; a'(0) = \widetilde{a}_1. \tag{2.12}$$

By a similar way as was done in [28], one can obtain that if $\widetilde{a}_1 \geq \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}$*, the free boundary* $a(t)$ *increases in* $[0, +\infty)$ *and* $\lim_{t\to+\infty} a(t) = +\infty$ *(at most, it increases linearly as in* (2.7)*); and if* $\widetilde{a}_1 < \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}$*,* $a(t)$ *tends to a positive bounded constant* $\left[(a_0^{1-2\varphi} - \frac{2\varphi-1}{2\varphi}a_1)\right]^{\frac{1}{1-2\varphi}}$.

Remark 2.5. *One can see different effects of the pressure and the swirl from the above remarks. More precisely, Remark 2.4 implies that, for the irrotational and pressureless case, the free boundary* $a(t)$ *increases linearly or sub-linearly in time; however, taking the pressure effect or the swirl effect into account, we can find some classes of solutions with linear growth under the conditions in* (2.9) *or* (2.11)*. In this sense, rotation or pressure effects may accelerate the growth of the boundary. Moreover, comparing the conditions in* (2.9) *and* (2.11)*, the result demonstrates the predominance of pressure effects relative to swirl effects.*

For the isothermal case $k > 0$, $\gamma = \varphi = 1$, the formula of fluid density is different from the isentropic case, and we have the following:

Theorem 2.6. *For the 2D CNS* (1.8) *in the isothermal case* $k > 0$, $\gamma = \varphi = 1$*, there exist a family of self-similar solutions with the form*

$$\rho(t,r) = \frac{e^{\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)+\beta}}{a^2(t)}, \; u^r(t,r) = \frac{a'(t)}{a(t)}r, \; u^\theta(t,r) = \frac{\xi}{a^2(t)}r, \tag{2.13}$$

where β is a constant, ξ is a constant that represents the strength effects of rotation, and $a(t) \in C^2([0,T))$ satisfies the following generalized Emden equation:

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k}{a(t)} + \frac{2a'(t)}{a^2(t)} = 0, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \end{cases} \tag{2.14}$$

Likewise, (2.14) *is a non-conservative system, and the trajectory of the solution $a(t)$ in the Poincaré phase plane is moving toward a state that has a lower total energy. Moreover, if $\widetilde{a}_1 \geq \frac{2}{a_0}$, $a(t)$ increases on $[0,+\infty)$ and $\lim\limits_{t\to+\infty} a(t) = +\infty$, and the fluid density $\rho(t,r)$ satisfies that $\lim\limits_{t\to+\infty} \rho(t,r) = 0$.*

Remark 2.7. *By Theorems 2.1 and 2.6, we see that in the self-similar solution in the 2D polar coordinates, the density ρ is radially symmetric, the velocity has a rotational part u^θ, if initially $u^\theta = 0$, and the solution is radially symmetric.*

Remark 2.8. *One can easily extend this result to the isothermal Euler/Navier-Stokes equations with frictional damping term*

$$\begin{cases} \rho_t + \partial_r(\rho u^r) + \frac{\rho u^r}{r} = 0, \\ \rho u^r_t + \rho\left[u^r\partial_r u^r - \frac{|u^\theta|^2}{r}\right] + \partial_r(k\rho) + \eta\rho u^r = (\rho^\varphi)_r\left(\frac{u^r}{r} + \partial_r u^r\right) + \rho^\varphi\left(\frac{r\partial_r u^r - u^r}{r^2} + \partial_r^2 u^r\right), \\ \rho u^\theta_t + \rho\left[u^r\partial_r u^\theta + \frac{u^\theta u^r}{r}\right] + \eta\rho u^\theta = 0, \end{cases} \tag{2.15}$$

where $\eta \geq 0$. The solutions are also given by (2.13), and $a(t)$ satisfies the following equation:

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k}{a(t)} + \frac{2a'(t)}{a^2(t)} + \eta a'(t) = 0, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \end{cases} \tag{2.16}$$

3. Proof of the isentropic case

First, similar to [26], we give a lemma for the mass conservation equation in the polar coordinates, whose proof is direct.

Lemma 3.1. *For the mass conservation equation of CNS* (1.8) *in the polar coordinates form*

$$\rho_t + \partial_r(\rho u^r) + \frac{\rho u^r}{r} = 0, \tag{3.1}$$

there exist solutions of the form

$$\rho(t,r) = \frac{f(\frac{r}{a(t)})}{a^2(t)},\ u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r, \tag{3.2}$$

where $f \geq 0$, $f \in C^1$, $a(t) > 0$, and $a(t) \in C^1([0,+\infty))$.

Precisely, we show the following result.

Lemma 3.2. *For the 2D CNS* (1.8) *under the continuous density boundary condition* (1.9) *in the isentropic case $k > 0$, $\gamma = \varphi > 1$, there exist a family of self-similar solutions of the form*

$$\rho(t,r) = \frac{\left[\frac{\varphi-1}{2}(1 - \frac{r^2}{a^2})\right]^{\frac{1}{\varphi-1}}}{a^2(t)}, \tag{3.3}$$

$$u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r, \tag{3.4}$$

where ξ is a constant that represents the strength effects of rotation, $a(t) \in C^2([0,+\infty))$ is the free boundary satisfying (1.6)*,* (1.7)*, and the following generalized Emden equation:*

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k\varphi}{a^{2\varphi-1}(t)} + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \end{cases} \tag{3.5}$$

Proof. According to Lemma 3.1, the solutions possess the following form:

$$\rho(t,r) = \frac{f(\frac{r}{a(t)})}{a^2(t)},\ u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r.$$

Substituting them into $(1.8)_2$, one has

$$u^r_t = r\frac{a''(t)a(t) - [a'(t)]^2}{a^2(t)},\ \partial_r u^r = \frac{a'(t)}{a(t)},\ \partial^2_r u^r = 0,$$

$$\partial_r P = \partial_r(k\rho^\gamma) = k\gamma\rho^{\gamma-1}\rho_r,\ \rho_r = \frac{f'(\frac{r}{a(t)})\frac{1}{a(t)}}{a^2(t)} = \frac{f'(\frac{r}{a(t)})}{a^3(t)},$$

where f' represents the derivative of function f with respect to $\frac{r}{a(t)}$, and then

$$\rho r\frac{a''(t)a(t) - [a'(t)]^2}{a^2(t)} + \rho r\left[\left(\frac{a'(t)}{a(t)}\right)^2 - \frac{\xi^2}{a^4(t)}\right] + k\gamma\rho^{\gamma-1}\frac{f'(\frac{r}{a(t)})}{a^3(t)}$$
$$= \varphi\rho^{\varphi-1}\frac{f'(\frac{r}{a(t)})}{a^3(t)}\left(\frac{a'(t)}{a(t)} + \frac{a'(t)}{a(t)}\right).$$

It follows that

$$\rho r\left[\frac{a''(t)}{a(t)} - \frac{\xi^2}{a^4(t)}\right] + k\gamma\rho\frac{f^{\gamma-2}f'}{a^{2\gamma-1}} - \varphi\rho\frac{2f^{\varphi-2}f'a'(t)}{a^{2\varphi}} = 0.$$

For simplicity, we let $\gamma = \varphi$, and the above equation can be written as

$$\rho r\left[\frac{a''(t)}{a(t)} - \frac{\xi^2}{a^4(t)}\right] + k\varphi\rho\frac{f^{\varphi-2}f'}{a^{2\varphi-1}} - \varphi\rho\frac{2f^{\varphi-2}f'a'(t)}{a^{2\varphi}} = 0,$$

i.e.,

$$\left(\frac{a''(t)}{a(t)} - \frac{\xi^2}{a^4(t)}\right)r + k\varphi\frac{f^{\varphi-2}f'}{a^{2\varphi-1}} - \varphi\frac{2f^{\varphi-2}f'a'(t)}{a^{2\varphi}} = 0. \tag{3.6}$$

Denoting $z = \frac{r}{a(t)}$, we require

$$z = -f^{\varphi-2}(z)f'(z), \tag{3.7}$$

and note that for the continuous density boundary condition (1.9), we have $f(1) = 0$, and then

$$f(z) = \left[\frac{\varphi-1}{2}(1-z^2)\right]^{\frac{1}{\varphi-1}}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we get the following generalized Emden equation:

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k\varphi}{a^{2\varphi-1}(t)} + \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \\ a(0) = a_0 > 0, \ a'(0) = \widetilde{a}_1. \end{cases}$$

Similarly, if one substitutes $\rho(t,r) = \frac{f(\frac{r}{a(t)})}{a^2(t)}$, $u^r(t,r) = \frac{a'(t)}{a(t)}r$, $u^\theta(t,r) = \frac{\xi}{a^2(t)}r$ into $(1.8)_3$, the same conclusion can be made, and the proof is complete. □

In the following, we will consider the local existence and uniqueness of the generalized Emden equation (3.5). The main result is the following:

Lemma 3.3. *There exists a sufficiently small T, such that, for the generalized Emden equation* (3.5)*, there exists a unique solution $a(t) \in C^1_{[0,T]}$, satisfying* $0 < \frac{1}{2}a_0 < a(t) < 2a_0$.

Proof. We prove the local existence and uniqueness of the generalized Emden equation (3.5) by using Banach's fixed point theorem [31–33] in functional analysis theory, which consists of the following three steps.

Step 1: We transform the Cauchy problem of the second-order nonlinear generalized Emden equation (3.5) into the Cauchy problem of the corresponding differential-integral equation.

We integrate over $(0,t)$ on both sides of Eq $(3.5)_1$, and note that $a(0) = a_0 > 0$, $a'(0) = \widetilde{a}_1$. Then

$$a'(t) = \widetilde{a}_1 - \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi} + \frac{2\varphi}{2\varphi-1}a^{1-2\varphi}(t) + k\varphi\int_0^t a^{1-2\varphi}(s)ds + \xi^2\int_0^t a^{-3}(s)ds.$$

Therefore, $a(t)$ satisfies the following differential-integral equation:

$$\begin{cases} a'(t) = \widetilde{a}_1 - \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi} + \frac{2\varphi}{2\varphi-1}a^{1-2\varphi}(t) + k\varphi\int_0^t a^{1-2\varphi}(s)ds + \xi^2\int_0^t a^{-3}(s)ds, \\ a(0) = a_0 > 0, \ a'(0) = \widetilde{a}_1. \end{cases} \tag{3.9}$$

Step 2: In the metric space $\mathbb{X}$, a nonlinear mapping $\mathbb{T}(a(t))$ is constructed, and we will prove that $\mathbb{T}(a(t)) \in \mathbb{X}$.

Suppose T_1 is a small positive constant, and we define the metric space

$$\mathbb{X} = \left\{ a(t) \in C^1_{[0,T]}, \ \ 0 < \frac{1}{2}a_0 < a(t) < 2a_0, \ \ \forall t \in [0,T_1] \right\}.$$

For $\forall a_1(t) \in \mathbb{X}$, $a_2(t) \in \mathbb{X}$, let

$$h(a(t)) = \widetilde{a}_1 - \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi} + \frac{2\varphi}{2\varphi-1}a^{1-2\varphi}(t) + k\varphi\int_0^t a^{1-2\varphi}(s)ds + \xi^2\int_0^t a^{-3}(s)ds.$$

Equation (3.9) can be transformed into

$$\begin{cases} \dfrac{da(t)}{dt} = h(a(t)), \\ a(0) = a_0 > 0, \ h(a(0)) = a'(0) = \widetilde{a}_1. \end{cases}$$

Then

$$
\begin{aligned}
&|h(a_1(t)) - h(a_2(t))| \\
=&\left|\frac{2\varphi}{2\varphi-1}\left(a_1^{1-2\varphi} - a_2^{1-2\varphi}\right) + k\varphi\int_0^t (a_1^{1-2\varphi}(s) - a_2^{1-2\varphi}(s))ds + \xi^2\int_0^t (a_1^{-3}(s) - a_2^{-3}(s))ds\right| \\
\leq&\left|\frac{2\varphi}{2\varphi-1}\frac{a_1^{2\varphi-1} - a_2^{2\varphi-1}}{a_1^{2\varphi-1}a_2^{2\varphi-1}}\right| + k\varphi\int_0^t \left|a_1^{1-2\varphi}(s) - a_2^{1-2\varphi}(s)\right| ds + \xi^2\int_0^t \left|a_1^{-3}(s) - a_2^{-3}(s)\right| ds \\
:=& H_1 + H_2 + H_3.
\end{aligned}
$$

For the terms H_i $(i = 1, 2, 3)$, one has

$$
\begin{aligned}
H_1 =&\left|\frac{2\varphi}{2\varphi-1}\frac{a_1^{2\varphi-1} - a_2^{2\varphi-1}}{a_1^{2\varphi-1}a_2^{2\varphi-1}}\right| \leq \frac{2\varphi}{2\varphi-1}\frac{\left|a_1^{2\varphi-1} - a_2^{2\varphi-1}\right|}{\left(\frac{1}{2}a_0\right)^{2\varphi-1}} \\
\leq&\frac{2\varphi}{2\varphi-1}\left(\frac{1}{2}a_0\right)^{1-2\varphi} |a_1 - a_2|^{2\varphi-1},
\end{aligned}
$$

$$
H_2 = k\varphi\int_0^t \left(a_1^{1-2\varphi}(s) - a_2^{1-2\varphi}(s)\right) ds \leq k\varphi\left(\frac{1}{2}a_0\right)^{1-2\varphi}\int_0^t |a_1 - a_2|^{2\varphi-1}\, ds,
$$

and

$$
H_3 = \xi^2\int_0^t \left|a_1^{-3}(s) - a_2^{-3}(s)\right| ds \leq \xi^2\left(\frac{1}{2}a_0\right)^{-3}\int_0^t |a_1 - a_2|^3\, ds.
$$

Suppose

$$
\begin{aligned}
&\frac{2\varphi}{2\varphi-1}\left(\frac{1}{2}a_0\right)^{1-2\varphi} |a_1 - a_2|^{2\varphi-1} + k\varphi\left(\frac{1}{2}a_0\right)^{1-2\varphi}\int_0^t |a_1 - a_2|^{2\varphi-1} ds \\
&+ \xi^2\left(\frac{1}{2}a_0\right)^{-3}\int_0^t |a_1 - a_2|^3 ds \leq L\sup_{0\leq t\leq T_1} |a_1(t) - a_2(t)|,
\end{aligned}
$$

where $L = \frac{2\varphi}{2\varphi-1}3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1} + k\varphi T_1 3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1} + 9\xi^2 T_1\left(\frac{1}{2}a_0\right)^{-1}$ is a constant, and then

$$
|h(a_1(t)) - h(a_2(t))| \leq H_1 + H_2 + H_3 \leq L\sup_{0\leq t\leq T_1} |a_1(t) - a_2(t)|. \tag{3.10}
$$

We now define a mapping $\mathbb{T}$ on $\mathbb{X}$, such that $\mathbb{T}$ satisfies

$$
\mathbb{T}a(t) = a_0 + \int_0^t h(a(s))ds, \quad \forall t \in [0, T_1], \tag{3.11}
$$

and it is easy to know that $\mathbb{T}a(t) \in C^1_{[0,T_1]}$.

Next, let us find the condition satisfying $\mathbb{T}a(t) \in \mathbb{X}$.

If

$$
\mathbb{T}a(t) \leq a_0 + t\left(|\widetilde{a}_1| + \frac{2\varphi}{2\varphi-1}(\frac{1}{2}a_0)^{1-2\varphi} + k\varphi T_1(\frac{1}{2}a_0)^{1-2\varphi} + \xi^2 T_1(\frac{1}{2}a_0)^{-3}\right) \leq 2a_0,
$$

we have

$$t \leq \frac{a_0}{|\widetilde{a}_1| + \frac{2\varphi}{2\varphi-1}\left(\frac{1}{2}a_0\right)^{1-2\varphi} + k\varphi T_1\left(\frac{1}{2}a_0\right)^{1-2\varphi} + \xi^2 T_1\left(\frac{1}{2}a_0\right)^{-3}} = T_2.$$

If

$$\mathbb{T}a(t) = a_0 + \int_o^t h(a(s))ds \geq a_0 - |\widetilde{a}_1|t - \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}t \geq \frac{1}{2}a_0,$$

we have

$$|\widetilde{a}_1|t + \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}t \leq \frac{1}{2}a_0,$$

and

$$0 < t \leq \frac{\frac{1}{2}a_0}{|\widetilde{a}_1| + \frac{2\varphi}{2\varphi-1}a_0^{1-2\varphi}} = T_3.$$

Therefore, if $T_1 \leq \min\{T_2, T_3\}$, then $\mathbb{T}a(t) \in \mathbb{X}$.

Step 3: By Banach's fixed point theorem, we can prove that the Cauchy problem of differential-integral equation

$$\mathbb{T}a(t) = a(t),\ a'(t) = h(a(t))$$

has a unique solution in $\mathbb{X}$.

If $\mathbb{T}$ is a contraction mapping, by (3.10) and (3.11), which satisfies

$$\begin{aligned}\sup_{0\leq t\leq T_1}|\mathbb{T}a_1(t) - \mathbb{T}a_2(t)| &\leq \left|\int_0^t h(a_1(s))ds - \int_0^t h(a_2(s))ds\right| \\ &\leq LT_1 \sup_{0\leq t\leq T_1}|a_1(t) - a_1(t)|,\end{aligned}$$

and $LT_1 < 1$, then

$$\frac{2\varphi}{2\varphi-1}3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1}T_1 + k\varphi T_1^2 3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1} + 9\xi^2 T_1^2\left(\frac{1}{2}a_0\right)^{-1} < 1,$$

and we have

$$\left(k\varphi\cdot 3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1} + 9\xi^2\left(\frac{1}{2}a_0\right)^{-1}\right)T_1^2 + \frac{2\varphi}{2\varphi-1}3^{2\varphi-2}\left(\frac{1}{2}a_0\right)^{-1}T_1 - 1 < 0,$$

$$\begin{aligned}T_1 &\leq \frac{\sqrt{(\frac{2\varphi}{2\varphi-1}3^{2\varphi-2}(\frac{a_0}{2})^{-1})^2 + 4\left(k\varphi\cdot 3^{2\varphi-2}(\frac{a_0}{2})^{-1} + 9\xi^2(\frac{a_0}{2})^{-1}\right)} - \frac{2\varphi}{2\varphi-1}3^{2\varphi-2}(\frac{a_0}{2})^{-1}}{2(k\varphi\cdot 3^{2\varphi-2}(\frac{a_0}{2})^{-1} + 9\xi^2(\frac{a_0}{2})^{-1})} \\ &= T_4.\end{aligned}$$

Therefore, if $T = \min\{T_1, T_2, T_3, T_4\}$, the mapping $\mathbb{T} : \mathbb{X} \mapsto \mathbb{X}$ is a contraction mapping.

By Banach's fixed point theorem, there exists a unique $a(t) \in C^1_{[0,T]}$, s.t.

$$\mathbb{T}a(t) = a(t),\ a'(t) = h(a(t)),$$

and the lemma is proved. □

Next, we focus on Eq (3.5) and give an estimate for the solution $a(t)$. From the perspective of ordinary differential equations, we use the standard energy method [34] in autonomous systems to get some properties of the generalized Emden equation (3.5).

Lemma 3.4. *There exist two positive constants*

$$C_1 = \max\left\{a_0,\ \sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi - 1} a_0^{2-2\varphi}}\right\}, \tag{3.12}$$

and

$$C_2 = \max\left\{\frac{|\xi|}{\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1} a_0^{2-2\varphi}}}, \left[\frac{k\varphi}{(\varphi - 1)\left(\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1} a_0^{2-2\varphi}\right)}\right]^{\frac{1}{2\varphi-2}}\right\}, \tag{3.13}$$

s.t. the solution $a(t)$ of the generalized Emden equation (3.5) *satisfies*

$$0 < C_2 < a(t) < C_1(1 + t). \tag{3.14}$$

Proof. Multiplying both sides of the generalized Emden equation (3.5) by $a'(t)$, we have

$$a'(t)a''(t) - \frac{\xi^2 a'(t)}{a^3(t)} - \frac{k\varphi a'(t)}{a^{2\varphi-1}(t)} + \frac{2\varphi(a'(t))^2}{a^{2\varphi}(t)} = 0.$$

Integrating over $(0, t)$, we have

$$\int_0^t a'(s)a''(s)ds - \int_0^t \frac{\xi^2 a'(s)}{a^3(s)} ds - \int_0^t \frac{k\varphi a'(s)}{a^{2\varphi-1}(s)} ds + \int_0^t \frac{2\varphi(a'(s))^2}{a^{2\varphi}(s)} ds = 0.$$

Then, it follows that

$$\begin{aligned} &\frac{1}{2}(a'(t))^2 + \frac{1}{2}\xi^2 a^{-2}(t) + \frac{k\varphi}{2\varphi - 2} a^{2-2\varphi}(t) + \int_0^t \frac{2\varphi(a'(s))^2}{a^{2\varphi}(s)} ds \\ = &\frac{1}{2}\widetilde{a}_1^2 + \frac{1}{2}\xi^2 a_0^{-2} + \frac{k\varphi}{2\varphi - 2} a_0^{2-2\varphi}, \end{aligned} \tag{3.15}$$

and this implies that

$$\frac{1}{2}(a'(t))^2 < \frac{1}{2}\widetilde{a}_1^2 + \frac{1}{2}\xi^2 a_0^{-2} + \frac{k\varphi}{2\varphi - 2} a_0^{2-2\varphi},$$

$$a'(t) < \sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi - 1} a_0^{2-2\varphi}}.$$

So

$$a(t) < a(0) + \sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi - 1} a_0^{2-2\varphi}}\, t \le C_1(1 + t),$$

where

$$C_1 = \max\left\{a_0,\ \sqrt{\widetilde{a_1}^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi - 1}a_0^{2-2\varphi}}\right\}.$$

From (3.15), we have

$$\frac{1}{2}\xi^2 a^{-2}(t) < \frac{1}{2}\widetilde{a_1}^2 + \frac{1}{2}\xi^2 a_0^{-2} + \frac{k\varphi}{2\varphi - 2}a_0^{2-2\varphi},$$

i.e.,

$$a^2(t) > \frac{\xi^2}{\widetilde{a_1}^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}.$$

Likewise, one has

$$\frac{k\varphi}{2\varphi - 2}a^{2-2\varphi}(t) < \frac{1}{2}\widetilde{a_1}^2 + \frac{1}{2}\xi^2 a_0^{-2} + \frac{k\varphi}{2\varphi - 2}a_0^{2-2\varphi},$$

and

$$a^2(t) > \left[\frac{k\varphi}{(\varphi - 1)(\widetilde{a_1}^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi})}\right]^{\frac{1}{\varphi-1}}.$$

Therefore,

$$a(t) > \max\left\{\frac{|\xi|}{\sqrt{\widetilde{a_1}^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}}, \left[\frac{k\varphi}{(\varphi - 1)(\widetilde{a_1}^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi})}\right]^{\frac{1}{2\varphi-2}}\right\} = C_2.$$

So this completes the proof of Lemma 3.4. □

The following lemma is a direct consequence of Lemmas 3.3 and 3.4.

Lemma 3.5. *For the generalized Emden equation* (3.5)*, there exists a global solution $a(t)$ in* $[0, +\infty)$*, which satisfies the estimate in* (3.14)*.*

For the generalized Emden equation (3.5), let

$$w\left(a, \frac{da}{dt}\right) = \frac{2\varphi a'(t)}{a^{2\varphi}(t)},\ g(a) = -\frac{\xi^2}{a^3(t)} - \frac{k\varphi}{a^{2\varphi-1}(t)},$$

and then (3.5) can be written as

$$a''(t) + w\left(a, \frac{da}{dt}\right) + g(a) = 0. \tag{3.16}$$

Let

$$G(a) = \int g(a)da = \frac{\xi^2}{2}a^{-2} - \frac{k\varphi}{2 - 2\varphi}a^{2-2\varphi} + C,$$

and we can define the kinetic energy of the generalized Emden equation (3.5):

$$E_{kin} = \frac{1}{2}\left(\frac{da}{dt}\right)^2.$$

Likewise, the potential energy of (3.5) is defined as

$$E_{pot} = G(a) = \int g(a)da = \frac{\xi^2}{2}a^{-2} - \frac{k\varphi}{2-2\varphi}a^{2-2\varphi} + C.$$

The total energy of the generalized Emden equation (3.5) is

$$E = E_{kin} + E_{pot} = \frac{1}{2}\left(\frac{da}{dt}\right)^2 + G(a).$$

Then

$$\frac{dE}{dt} = \frac{da}{dt}\frac{d^2a}{dt^2} + G'(a)\frac{da}{dt} = \frac{da}{dt}\left(\frac{d^2a}{dt^2} + g(a)\right) = -\frac{da}{dt}w\left(a, \frac{da}{dt}\right).$$

For $w\left(a, \frac{da}{dt}\right) = \frac{2\varphi a'(t)}{a^{2\varphi}(t)}$, we have

$$\begin{aligned} \frac{dE}{dt} &= \frac{da}{dt}\frac{d^2a}{dt^2} + G'(a)\frac{da}{dt} = \frac{da}{dt}\left(\frac{d^2a}{dt^2} + g(a)\right) \\ &= -\frac{da}{dt}w\left(a, \frac{da}{dt}\right) = -2\varphi\frac{[a'(t)]^2}{a^{2\varphi}(t)} < 0. \end{aligned} \tag{3.17}$$

The total energy of (3.5) decreases monotonically, it is a non-conservative system, and the trajectory of the solution $a(t)$ to the generalized Emden equation (3.5) in the Poincaré phase plane is moving toward a state that has a lower total energy. In summary, we get the following lemma:

Lemma 3.6. *For the generalized Emden equation* (3.5)*, the solution $a(t)$ has the following properties:* (3.5) *is a non-conservative system and the trajectory of the solution $a(t)$ in the Poincaré phase plane is moving toward a state that has a lower total energy.*

Remark 3.7. *The related results about the other types of generalized Emden equations may be referred to [19–21].*

Furthermore, $a(t)$ possesses the following large time behavior:

Lemma 3.8. *For the generalized Emden equation* (3.5)*, if*

$$C_2 \geq \frac{2}{k}\sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}, \tag{3.18}$$

where C_2 is the same as (3.13)*, then the limit $\lim\limits_{t\to+\infty} a'(t)$ exists, and let $A = \lim\limits_{t\to+\infty} a'(t)$. We have*

$$\lim_{t\to+\infty}\frac{a(t)}{t} = \lim_{t\to+\infty} a'(t) = A.$$

Proof. We prove the lemma by three steps:

Step 1: In the proof of Lemma 3.4, we have

$$a'(t) < \sqrt{\widetilde{a}_1^2 + \xi^2 a_0^{-2} + \frac{k\varphi}{\varphi-1}a_0^{2-2\varphi}}, \tag{3.19}$$

and

$$a(t) > C_2 > 0,$$

where C_2 is the same as (3.13).

By (3.18), we have

$$a''(t) = \frac{\xi^2}{a^3(t)} + \frac{k\varphi}{a^{2\varphi-1}(t)} - \frac{2\varphi a'(t)}{a^{2\varphi}(t)} > \frac{k\varphi a(t) - 2\varphi a'(t)}{a^{2\varphi}(t)} \geq 0, \tag{3.20}$$

so $a(t)$ is a strictly convex function, and $a'(t)$ is monotonically increasing in $[0, +\infty)$. In particular, the convexity of $a(t)$ is very important in the following proof.

By (3.19), we have that $a'(t)$ is bounded, and according to the monotone bounded theorem [35, 36], the limit $\lim\limits_{t\to+\infty} a'(t)$ exists.

Step 2: In this step, we will prove that $a(t) \to +\infty$ as $t \to +\infty$.

First, we transform the generalized Emden equation (3.5) into the differential-integral form:

$$\begin{cases} a'(t) = \widetilde{a}_1 - \frac{2\varphi}{2\varphi-1} a_0^{1-2\varphi} + \frac{2\varphi}{2\varphi-1} a^{1-2\varphi}(t) + k\varphi \int_0^t a^{1-2\varphi}(s)ds + \xi^2 \int_0^t a^{-3}(s)ds, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1, \end{cases}$$

and we divide into two cases to prove the results:

Case 1: If $\widetilde{a}_1 \geq \frac{2\varphi}{2\varphi-1} a_0^{1-2\varphi} - \frac{2\varphi}{2\varphi-1} a^{1-2\varphi}(t)$, then $a'(t) > 0$ is always true in $[0, +\infty)$. So $a(t)$ is monotonically increasing in $[0, +\infty)$, note the convexity of $a(t)$, and then $a(t) \to +\infty$ as $t \to +\infty$.

Case 2: If $\widetilde{a}_1 < \frac{2\varphi}{2\varphi-1} a_0^{1-2\varphi} - \frac{2\varphi}{2\varphi-1} a^{1-2\varphi}(t)$, as $a'(t)$ is monotonically increasing in $[0, +\infty)$, due to the continuity property, the behavior of $a(t)$ may be:

(i). $a(t)$ is first monotonically decreasing in $[0, t_0]$ ($t_0 < +\infty$), and then $a(t)$ is monotonically increasing in $[t_0, +\infty)$.

(ii). $a(t)$ is monotonically decreasing in $[0, +\infty)$.

In the following, we will prove that (ii) does not hold, and only (i) is true.

By Lemma 3.4, we have that $a(t_0) > 0$, $a'(t_0) = 0$.

If $t_0 = +\infty$, then

$$\begin{aligned} a'(t_0) = a'(+\infty) &= \widetilde{a}_1 - \frac{2\varphi}{2\varphi - 1} a_0^{1-2\varphi} + \frac{2\varphi}{2\varphi - 1} a^{1-2\varphi}(+\infty) \\ &+ k\varphi \int_0^{+\infty} a^{1-2\varphi}(s)ds + \xi^2 \int_0^{+\infty} a^{-3}(s)ds = +\infty, \end{aligned}$$

while in the proof of Step 1, $a'(t)$ is bounded, so a contradiction is met. So $t_0 \neq +\infty$, and (ii) can not happen.

Therefore, $a(t)$ is monotonically decreasing in $[0, t_0]$ ($t_0 < +\infty$), and then $a(t)$ is monotonically increasing in $[t_0, +\infty)$. Note that the convexity of $a(t)$ implies that $a(t) \to +\infty$ as $t \to +\infty$.

In conclusion, we have that $a(t) \to +\infty$ as $t \to +\infty$.

Step 3: Since the limit $\lim\limits_{t\to+\infty} a'(t)$ exists, by the L'Hospital's rule [35, 36], we have

$$\lim_{t\to+\infty} \frac{a(t)}{t} = \lim_{t\to+\infty} a'(t) = A.$$

This completes the proof of Lemma 3.8. □

The proof of Theorem 2.1 is a direct consequence of Lemmas 3.2–3.8.

4. Proof of the isothermal case

In this part, we discuss the formulas and properties of analytical solutions to (1.8) in the isothermal case $k > 0$, $\gamma = \varphi = 1$. We have the following result:

Lemma 4.1. *For the 2D CNS* (1.8) *in the isothermal case $k > 0$, $\gamma = \varphi = 1$, there exist a family of self-similar solutions of the form*

$$\rho(t,r) = \frac{e^{\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)+\beta}}{a^2(t)},\ u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r, \tag{4.1}$$

where β is a constant, ξ is a constant that represents the strength effects of rotation, and $a(t) \in C^2([0,+\infty))$ is a differential function satisfying the following generalized Emden equation:

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} - \frac{k}{a(t)} + \frac{2a'(t)}{a^2(t)} = 0, \\ a(0) = a_0 > 0,\ a'(0) = \widetilde{a}_1. \end{cases} \tag{4.2}$$

Proof. We check that (4.1) is a solution by direct calculations.

First, according to Lemma 3.1, one can suppose the solutions in the following form:

$$\rho(t,r) = \frac{e^{f(\frac{r}{a(t)})}}{a^2(t)},\ u^r(t,r) = \frac{a'(t)}{a(t)}r,\ u^\theta(t,r) = \frac{\xi}{a^2(t)}r. \tag{4.3}$$

Obviously, the solutions above satisfy $(1.8)_3$.

Next, substituting (4.3) into $(1.8)_2$, one has

$$u^r_t = r\frac{a''(t)a(t) - [a'(t)]^2}{a^2(t)},\ \partial_r u^r = \frac{a'(t)}{a(t)},\ \partial_r^2 u^r = 0,$$

and

$$\rho_r = \frac{e^{f(\frac{r}{a(t)})}f'(\frac{r}{a(t)})\frac{1}{a(t)}}{a^2(t)} = \frac{e^{f(\frac{r}{a(t)})}f'(\frac{r}{a(t)})}{a^3(t)},$$

where f' represents the derivative of function f with respect to $\frac{r}{a(t)}$, and then

$$\begin{aligned} &\rho r\frac{a''(t)a(t) - [a'(t)]^2}{a^2(t)} + \rho r\left[\left(\frac{a'(t)}{a(t)}\right)^2 - \frac{\xi^2}{a^4(t)}\right] + k\frac{e^{f(\frac{r}{a(t)})}f'(\frac{r}{a(t)})}{a^3(t)} \\ =&\frac{e^{f(\frac{r}{a(t)})}f'(\frac{r}{a(t)})}{a^3(t)}\left(\frac{a'(t)}{a(t)} + \frac{a'(t)}{a(t)}\right). \end{aligned}$$

It follows that

$$\left(\frac{a''(t)}{a(t)} - \frac{\xi^2}{a^4(t)}\right)r + k\frac{f'(\frac{r}{a(t)})}{a(t)} - \frac{2f'(\frac{r}{a(t)})a'(t)}{a^2(t)} = 0. \tag{4.4}$$

Denoting $z = \frac{r}{a(t)}$, similarly as was done in (3.7), we require

$$z = -f'(z). \tag{4.5}$$

Suppose $f(1) = \beta$ and β is a constant, and then we get

$$f(z) = \frac{1}{2} - \frac{1}{2}z^2 + \beta. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we get the Emden equation (4.2).

This finishes the proof of Lemma 4.1. □

As the proof of Case (i) of Theorem 2.3 in [28], one integrates the generalized Emden equation (4.2) on $[0, t]$ to obtain

$$a'(t) = \widetilde{a}_1 - \frac{2}{a_0} + \frac{2}{a(t)} + \int_0^t \left(\frac{\xi^2}{a^3(s)} + \frac{k}{a(s)} \right) ds. \tag{4.7}$$

It follows that

$$a(t) = a_0 + \left(\widetilde{a}_1 - \frac{2}{a_0} \right) t + \int_0^t \frac{2}{a(s)} ds + \int_0^t \int_0^q \left(\frac{\xi^2}{a^3(s)} + \frac{k}{a(s)} \right) ds dq, \tag{4.8}$$

and if $\widetilde{a}_1 \geq \frac{2}{a_0}$, $a'(t) > 0$, $a(t)$ increases in $[0, +\infty]$. By the contradiction method, we can get $\lim\limits_{t \to +\infty} a(t) = +\infty$, so the fluid density $\rho(t, r)$ satisfies that $\lim\limits_{t \to +\infty} \rho(t, r) = 0$.

Similar to the proof of Lemma 3.3 and Lemma 3.6, we can also get the local existence and uniqueness of the generalized Emden equation (4.2), and by Lemma 4.1, we can prove Theorem 2.6.

5. Some examples of blow-up solutions

In this part, we consider some examples of blow-up solutions to (1.8) without $a(t)$ being the free boundary.

First of all, as in [23], we give the definition of a blow-up:

Definition: (Blow-up) We say a solution blows up if one of the following conditions is satisfied.

(1) The solution becomes infinitely large at some point x and some finite time T.

(2) The derivative of the solution becomes infinitely large at some point x and some finite time T.

In this section, we consider the forms and properties of the analytical solutions to the 2D CNS without the continuous density boundary condition. Since we remove the free boundary condition, we only consider the formal analytical solutions to the CNS.

The first result is related to the isothermal case $k > 0$, $\gamma = \varphi = 1$:

Theorem 5.1. *For the 2D CNS* (1.8) *in the isothermal case* $k > 0$, $\gamma = \varphi = 1$, *there exists a family of self-similar solutions of the form*

$$\rho(t, r) = \frac{e^{\frac{r^2}{2a^2} + \alpha}}{a^2(t)}, \ u^r(t, r) = \frac{a'(t)}{a(t)} r, \ u^\theta(t, r) = \frac{\xi}{a^2(t)} r, \tag{5.1}$$

where α *is a constant,* ξ *is a constant that represents the strength effects of rotation, and* $a(t) \in C^2([0, T))$ *is a differential function satisfying the following generalized Emden equation:*

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} + \frac{k}{a(t)} - \frac{2a'(t)}{a^2(t)} = 0, \\ a(0) = a_0 > 0, \ a'(0) = \widetilde{a}_1. \end{cases} \tag{5.2}$$

In particular, if $a_0 > 0$ *is small enough,* $\widetilde{a_1} < 0$*, and*

$$\frac{2\widetilde{a_1}}{a_0^2} + \frac{\xi^2}{a_0^3} - \frac{k}{a_0} < 0, \tag{5.3}$$

then there exists a $\delta > 0$*, where* $-\frac{a_0}{\widetilde{a_1}} \leq \delta < T$*, and there exists a finite time* $t_* \in [0, \delta)$ *such that* $a(t_*) = 0$*, and then the solutions* (5.1) *blow up.*

Proof. Similar to the proof of Lemma 4.1, denoting $z = \frac{r}{a(t)}$, we require $z = f'(z)$ instead of (4.5), and we will obtain that (5.1)–(5.2) are self-similar solutions of the CNS.

In the following, we will prove the blow-up result. Different from the proof by the contradiction method in the references [19, 20, 23], we use a new method to prove the blow-up solutions directly by attaching conditions to the initial value of the generalized Emden equation (5.2).

Because $a(0) = a_0 > 0$ is small enough, $a'(0) = \widetilde{a_1} < 0$, and (5.3), by the existence theorem of solutions of the ordinary differential equations and the local sign-preserving property of the continuously differentiable functions, there exists a $\delta > 0$, s.t. as $t \in [0, \delta)$, there exists a solution for the generalized Emden equation $(5.2)_1$, meanwhile $a''(t)$ exists in $[0, \delta)$ and $a''(t) < 0$.

Thus $a'(t)$ monotonically decreases in $[0, \delta)$ and $a'(t) < \widetilde{a_1}$, $a(t)$ also monotonically decreases, so we have that the curve $(t, a(t))$ is below the line

$$a_2(t) = \widetilde{a_1} t + a_0. \tag{5.4}$$

Note that $a'(t)$ and $a(t)$ monotonically decrease in $[0, \delta)$, $-\frac{a_0}{\widetilde{a_1}} \leq \delta < T$, and the line equation (5.4) intersects the t-axis, so the point of intersection $t = -\frac{a_0}{\widetilde{a_1}}$ is in the interval $[0, \delta)$.

Therefore, the curve $(t, a(t))$ also intersects the t-axis and the point of intersection is in the interval $[0, \delta)$, i.e., there exists $t_* \in [0, \delta)$, such that $a(t_*) = 0$. Moreover, the solutions (5.1) blow up.

This completes the proof of Theorem 5.1. □

Similar to the results in Theorem 5.1, for the isentropic case $k > 0$, $\gamma = \varphi > 1$, we have:

Theorem 5.2. *For the 2D CNS* (1.8) *in the isentropic case* $k > 0$*,* $\gamma = \varphi > 1$*, there exist a family of self-similar solutions of the form*

$$\rho(t, r) = \frac{\left[\frac{\varphi - 1}{2}\frac{r^2}{a^2} + \alpha^{\varphi - 1}\right]^{\frac{1}{\varphi - 1}}}{a^2(t)}, \quad u^r(t, r) = \frac{a'(t)}{a(t)} r, \quad u^\theta(t, r) = \frac{\xi}{a^2(t)} r, \tag{5.5}$$

where α *is a constant,* ξ *is a constant that represents the strength effects of rotation, and* $a(t) \in C^2([0, T))$ *is a differential function satisfying the following generalized Emden equation:*

$$\begin{cases} a''(t) - \frac{\xi^2}{a^3(t)} + \frac{k\varphi}{a^{2\varphi - 1}(t)} - \frac{2\varphi a'(t)}{a^{2\varphi}(t)} = 0, \\ a(0) = a_0 > 0, \ a'(0) = \widetilde{a_1}. \end{cases} \tag{5.6}$$

In particular, if $a_0 > 0$ *is small enough,* $\widetilde{a_1} < 0$*, and*

$$\frac{2\varphi \widetilde{a_1}}{a_0^{2\varphi}} + \frac{\xi^2}{a_0^3} - \frac{k\varphi}{a_0^{2\varphi - 1}} < 0, \tag{5.7}$$

then there exists a $\delta > 0$*, where* $-\frac{a_0}{\widetilde{a_1}} \leq \delta < T$*, and there exists a finite time* $t_* \in [0, \delta)$ *such that* $a(t_*) = 0$*, and then the solutions* (5.5) *blow up.*

Remark 5.3. *Although the system* (5.2) *can be seen as a limit system of* (5.6) *in the sense that* $\varphi \to 1$, *the expressions for the density in* (5.1) *and* (5.5) *are different. The properties of these equations need to be further studied.*

6. Conclusions

In this work, we establish a class of analytical, rotational, and self-similar solutions to the 2D compressible Navier-Stokes equations with density-dependent viscosity coefficients. According to the different properties of pressure and different types of fluid, we maily consider the isentropic case $k > 0$, $\gamma = \varphi > 1$ and the isothermal case $k > 0$, $\gamma = \varphi = 1$. For both of the two cases, we give the formulas of self-similar analytical solutions. Especially, for the isentropic case, we prove the well-posedness and the large time behavior for the corresponding generalized Emden equation. The result in this paper demonstrates the predominance of pressure effects relative to swirl effects. Compared with the irrotational and pressureless case, when the free boundary $a(t)$ increases linearly or sub-linearly in time, we can find some classes of solutions with linear growth by taking the pressure effect or the swirl effect into account. In this sense, rotation or pressure effects may accelerate the growth of the boundary. In the end, we give some examples of blow-up solutions, and a new direct method is adopted to prove the blow-up results.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author states that there is no conflict of interest.

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