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Research article

Exploration of exact wave solutions for the Lord-Shulman thermo-elasticity theory with temperature dependence using advanced techniques

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Abstract: This study sought to improve the comprehension of wave propagation in thermo-elastic materials according to Lord-Shulman (L-S) theory by developing precise wave solutions for the governing equations that take into consideration temperature-dependent material features. The research utilized the improved simple equation method (ISEM) to analyze the interrelated thermal and mechanical properties of these materials, allowing the creation of analytical solutions that exactly characterize intricate wave processes. The ISEM facilitates the development of various wave shapes. These solutions, defined by configurable free parameters, offer a flexible framework for examining diverse physical circumstances in thermo-elasticity. The work includes detailed graphical representations of crucial discoveries such as temperature distributions, stress tensors, and displacement which provide amazing visual insights into the complex interactions that occur within thermo-elastic systems.

Keywords: nonlinear thermo-elasticity; temperature-dependence; analytical wave solutions
Mathematics Subject Classification: 74B20, 35C05, 35C07, 35C08

List of Nomenclature

Quantity	Definition
$\mathbf{u}$	Displacement vector
α_τ	Thermal expansion coefficient, where $\gamma = 2\mu\alpha_\tau + 3\lambda\alpha_\tau$
τ	Time
σ_{ij}	Stress tensor
T	Increment temperature
k	Thermal conductivity
θ_0	Reference temperature
c_e	Specific heat
μ and λ	Lame's constants
$s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})$	A function of $\tilde{\tau}$
(,)	Illustrates the derivative
($\prime$)	The derivative with respect to ϵ
$dot(.)$	The derivative with respect to $\tilde{\tau}$
ρ	Density

1. Introduction

The L-S theory of thermo-elasticity, created in 1967, is a significant advancement in the field of coupled thermo-elasticity that addresses the constraints of classical thermo-elastic models by introducing a finite speed of heat propagation. Classical thermo-elastic theories, based on the traditional Fourier heat conduction law, assume an infinite speed of thermal wave propagation, which leads to unrealistic physical predictions, especially in high-frequency or transient thermal scenarios [1]. To overcome this issue, Lord and Shulman [2] built a generalized thermo-elasticity model that incorporates just one relaxation time parameter, which modifies the heat conduction equation to a hyperbolic form rather than the conventional parabolic type. This modification leads to the phenomenon of thermal wave propagation with finite velocity, a critical factor in applications involving rapid thermal processes, such as laser heating, aerospace materials, and microelectronics. The theory of L-S combines the equations of motion, heat conduction, and constitutive relations, creating a more comprehensive framework that takes into account both thermal and mechanical effects simultaneously. One of the most significant advantages of L-S theory is its ability to predict the thermal wave propagation phenomenon, which has been observed experimentally in a variety of materials, notably at microscale systems and cryogenic temperatures. The governing equations of the L-S model consist of the balance of linear motion, the modified energy equation incorporating relaxation time, and the constitutive relations between strain, stress, and temperature. These equations are extremely useful for analyzing thermo-elastic problems, where the delayed response of heat flux and mechanical fields plays a crucial role. The inclusion of relaxation time introduces additional characteristic wave speeds in the system, leading to the propagation of coupled thermo-elastic waves with different attenuation and dispersion properties. Researchers have extensively studied the implications of the L-S theory in various contexts, such as layered structures [3], isotropic and anisotropic materials [4, 5], and rotating media [6]. Analytical and numerical methods, such as finite element analysis [7], the Laplace transform [8–10], the normal mode [11, 12], and perturbation techniques [13], have been widely employed to derive solutions for distinct geometric configurations and boundary conditions. The L-S theory has found significant applications in engineering fields requiring precise control over thermal and mechanical interactions, including the vibration control in thermal environments and design of heat-resistant materials. Furthermore, the theory provides a robust foundation for investigating thermo-elastic phenomena under extreme conditions, such as high-temperature gradients and dynamic loading. So, the L-S theory is considered a cornerstone of modern thermo-elasticity, offering valuable insights into the intricate interplay between thermal and mechanical fields in a wide range of practical applications.

The temperature-dependence of thermo-elasticity is critical for precisely simulating the behavior of materials subjected to thermal and mechanical loads in a variety of engineering applications. In classical thermo-elasticity, material properties such as thermal expansion coefficients, Lamé's constant, and thermal conductivity remain constant. However, experimental evidence has demonstrated that these qualities change dramatically with temperature, particularly in high-temperature environments such as microelectronics, aerospace structures, and power plants [14]. Incorporating temperature-dependent characteristics into thermo-elastic models results in nonlinear governing equations that better represent real-world material responses to thermal stresses. As the temperature increases, the materials may soften or harden, affecting wave propagation speeds, stress distributions, and energy dissipation in the system [2]. The addition of temperature dependence becomes especially significant in transient heat conduction problems, where fast temperature variations can cause localized thermal stresses [15, 16]. Advanced thermo-elastic models, such as the L-S theory and Green-Naghdi theories, have been expanded to include changeable material properties, allowing for more accurate analysis of complicated thermal effects. Numerical and analytical techniques, involving the perturbation technique and the finite element technique, are frequently employed to solve the resulting complicated equations and provide insights into stress distribution and deformation behavior under temperature-dependent conditions [17]. Studies have suggested that neglecting the temperature dependence in thermo-elastic analysis can lead to massive errors in forecasting material behavior, underscoring the importance of incorporating such effects for applications in high-performance materials and structures [18, 19]. The findings of these studies are critical for the design of materials with improved thermal stability, such as advanced composite materials used in high-temperature applications.

The ISEM is a advanced analytical technique utilized to obtain exact solutions of nonlinear partial differential equations (NLPDEs) that frequently arise in different scientific and engineering fields. NLPDEs play a crucial role in modeling complex physical phenomena, such as wave propagation in thermo-elasticity, fluid dynamics, nonlinear optics, and plasma physics [20–22]. Traditional techniques struggle with the inherent complexity of these equations, necessitating the development of efficient analytical methods like the ISEM to derive accurate and closed-form solutions [23]. The ISEM enhances the classical simple equation technique by incorporating more general solution forms and additional free parameters, allowing for greater flexibility and broader applicability. This method typically assumes a trial solution involving an elementary function, such as the trigonometric, hyperbolic, or rational function, which satisfies a reduced form of the original NLPDE [24, 25]. By systematically balancing the highest-order nonlinear and linear terms, the ISEM transforms the given NLPDE into an algebraic equation, which can be solved to obtain exact wave solutions with physical significance. Studies have demonstrated the effectiveness of the ISEM in solving various nonlinear models, involving the nonlinear Schrödinger equation [26, 27]. One of the key advantages of the ISEM is its capability to generate soliton-like, rational, and periodic solutions, which are critical for understanding wave dynamics in dispersive and nonlinear media [28, 29]. Further, the inclusion of free parameters in the solution structure provides a useful method for controlling and analyzing physical behaviors under a variety of initial and boundary conditions. However, there are limitations to the method. The ISEM assumes linear material properties and isotropic conditions, which may not apply to nonlinear or anisotropic materials. It also works best with simpler geometries and may require hybrid approaches when dealing with more complex structures.

The main objective of this research is to thoroughly explore the significant influence of temperature-dependence on thermo-elastic materials. This investigation is carried out within the framework of L-S theory and leverages the ISEM as the primary analytical tool. In the following sections, a comprehensive discussion of the ISEM will be provided, highlighting its importance and effectiveness in addressing NLPDEs. A detailed examination of each solution type will be conducted to ensure a thorough understanding of the thermo-elastic behavior under various conditions. Additionally, to facilitate better interpretation and visualization of the study's findings, 2D graphical illustrations will be incorporated. These visual representations will play a crucial role in demonstrating the key results, offering valuable insights into the intricate interactions within thermo-elastic systems, and supporting a more profound comprehension of the complex phenomena revealed in this study.

2. Basic equations

This section illustrates the governing equations for 2-dimensional thermo-elasticity within the framework of the L-S theory, with particular emphasis on examining the influence of temperature-dependence. The equations are subsequently transformed into a dimensionless form. Lastly, a moving wave transformation is applied as follows:

$$\begin{aligned} \tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) &= U(\epsilon), \qquad \tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = V(\epsilon), \\ \tilde{\theta}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) &= \theta(\epsilon), \qquad \epsilon = s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}), \end{aligned}$$

and the nonlinear PDEs are reduced to nonlinear ordinary differential equations for simplified analysis.

According to the findings reported by Othman et al. [30], the motion equation is given as follows:

$$\sigma_{ij,j} = \rho u_{i,\tau\tau}. \tag{2.1}$$

The formulation of the energy equation is now provided below, as derived and discussed in previous works referenced in [2, 31, 32]:

$$[kT_{,i}]_{,i} = \left[1 + \tau_0 \frac{\partial}{\partial \tau}\right] [\rho c_e T_{,\tau} + \gamma \theta_0 u_{r,r\tau}], \tag{2.2}$$

where $\frac{|T|}{\theta_0} << 1$.

Note that

$$u = u(x,\ y,\ \tau), \qquad v = v(x,\ y,\ \tau), \qquad w = 0, \tag{2.3}$$

and consequently, the constitutive relations are

$$\sigma_{ij} = (\lambda u_{r,r} - \gamma T)\,\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right). \tag{2.4}$$

We suppose that

$$(\lambda,\ \mu,\ \rho,\ \gamma,\ k) = (\lambda_0,\ \mu_0,\ \rho_0,\ \gamma_0,\ k_0)\,f(\theta). \tag{2.5}$$

In this context, it is presumed that the function $f(T)$ remains continuous throughout the interval $0 \leq T \leq \infty$. Additionally, the parameters μ_0, λ_0, γ_0, k_0, and ρ_0 are treated as fixed constants, which characterize the material properties and play a crucial role in defining the behavior of the system under investigation.

Using Eq (2.5) in Eqs (2.1), (2.2) and (2.4), we realize

$$\sigma_{ij,j} = f(T)\,\rho_0 u_{i,\tau\tau}, \tag{2.6}$$

$$\left[k_0 f(T)\,T_{,i}\right]_{,i} = \left[1 + \tau_0\frac{\partial}{\partial\tau}\right]\left[\rho_0 f(T)\,c_e T_{,\tau} + \gamma_0 f(T)\,\theta_0 u_{r,r\tau}\right], \tag{2.7}$$

$$\sigma_{ij} = f(T)\left[(\lambda_0 u_{r,r} - \gamma_0 T)\,\delta_{ij} + \mu_0\left(u_{i,j} + u_{j,i}\right)\right]. \tag{2.8}$$

Employing Eqs (2.3) and (2.8) in Eq (2.6), the equations of motion can be expressed as:

$$\begin{aligned} 0 \;=\;& f(T)\left[(2\mu_0 + \lambda_0)\,u_{,xx} + \mu_0 u_{,yy} + (\mu_0 + \lambda_0)\,v_{,yx} - \rho_0 u_{,\tau\tau} - \gamma_0 T_{,x}\right] \\ &+ f'(T)\,T_{,x}\left[(2\mu_0 + \lambda_0)\,u_{,x} + \lambda_0 v_{,y} - \gamma_0 T\right] + f'(T)\,\mu_0 T_{,y}\left[u_{,y} + v_{,x}\right], \end{aligned} \tag{2.9}$$

$$\begin{aligned} 0 \;=\;& f(T)\left[(2\mu_0 + \lambda_0)\,v_{,yy} + \mu_0 v_{,xx} + (\mu_0 + \lambda_0)\,u_{,yx} - \rho_0 v_{,\tau\tau} - \gamma_0 T_{,y}\right] \\ &+ f'(T)\,T_{,y}\left[(2\mu_0 + \lambda_0)\,v_{,y} + \lambda_0 u_{,x} - \gamma_0 T\right] + f'(T)\,\mu_0 T_{,x}\left[u_{,y} + v_{,x}\right], \end{aligned} \tag{2.10}$$

where $f'(T) = \frac{df(T)}{dT}$.

From Eq (2.3), Eq (2.7) becomes

$$\begin{aligned} 0 \;=\;& f(T)\left[k_0 T_{,xx} + k_0 T_{,yy} - \rho_0 c_e T_{,\tau} - \gamma_0\theta_0\left(u_{,x\tau} + v_{,y\tau}\right) - \tau_0\rho_0 c_e T_{,\tau\tau} - \tau_0\gamma_0\theta_0\left(u_{,x\tau\tau} + v_{,y\tau\tau}\right)\right] \\ &+ f'(T)\left[k_0\,(T_{,x})^2 + k_0\left(T_{,y}\right)^2 - \tau_0\rho_0 c_e\,(T_{,\tau})^2 - \tau_0\gamma_0\theta_0 T_{,\tau}\left(u_{,x\tau} + v_{,y\tau}\right)\right]. \end{aligned} \tag{2.11}$$

To simplify the analysis, the following set of dimensionless variables is introduced:

$$(\tilde{u},\ \tilde{v},\ \tilde{x},\ \tilde{y}) = \omega c_0\,(u,\ v,\ x,\ y),\ \tilde{\tau} = c_0^2\omega\tau,\ \tilde{T} = \frac{T}{\theta_0},\ \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\lambda_0 + 2\mu_0}, \tag{2.12}$$

where $c_0^2 = \frac{2\mu_0+\lambda_0}{\rho_0}$ and $\omega = \frac{\rho_0 c_e}{k_0}$.

From Eq (2.12) in Eq (2.9), it can be inferred that

$$\begin{aligned} 0 \;=\;& f\left(\tilde{T}\right)\left[\tilde{u}_{,\tilde{x}\tilde{x}} + a_1\tilde{u}_{,\tilde{y}\tilde{y}} + a_2\tilde{v}_{,\tilde{y}\tilde{x}} - \tilde{u}_{,\tilde{\tau}\tilde{\tau}} - a_3\tilde{T}_{,\tilde{x}}\right] \\ &+ f'\left(\tilde{T}\right)\tilde{T}_{,\tilde{x}}\left[\tilde{u}_{,\tilde{x}} + a_4\tilde{v}_{,\tilde{y}} - a_3\tilde{T}\right] + a_1 f'\left(\tilde{T}\right)\tilde{T}_{,\tilde{y}}\left[\tilde{u}_{,\tilde{y}} + \tilde{v}_{,\tilde{x}}\right], \end{aligned} \tag{2.13}$$

where $a_1 = \frac{\mu_0}{2\mu_0+\lambda_0}$, $a_2 = \frac{\mu_0+\lambda_0}{2\mu_0+\lambda_0}$, $a_3 = \frac{\gamma_0\theta_0}{2\mu_0+\lambda_0}$, and $a_4 = \frac{\lambda_0}{2\mu_0+\lambda_0}$.

Utilizing Eq (2.12) in Eq (2.10), we infer

$$\begin{aligned} 0 &= f\left(\tilde{T}\right)\left[\tilde{v}_{,\tilde{y}\tilde{y}} + a_1\tilde{v}_{,\tilde{x}\tilde{x}} + a_2\tilde{u}_{,\tilde{y}\tilde{x}} - \tilde{v}_{,\tilde{\tau}\tilde{\tau}} - a_3\tilde{T}_{,\tilde{y}}\right] \\ &\quad + f'\left(\tilde{T}\right)\tilde{T}_{,\tilde{y}}\left[\tilde{v}_{,\tilde{y}} + a_4\tilde{u}_{,\tilde{x}} - a_3\tilde{T}\right] + a_1 f'\left(\tilde{T}\right)\tilde{T}_{,\tilde{x}}\left[\tilde{u}_{,\tilde{y}} + \tilde{v}_{,\tilde{x}}\right]. \end{aligned} \tag{2.14}$$

Using Eq (2.12) in Eq (2.11), one discovers

$$\begin{aligned} 0 &= f\left(\tilde{T}\right)\left[\tilde{T}_{,\tilde{x}\tilde{x}} + \tilde{T}_{,\tilde{y}\tilde{y}} - \tilde{T}_{,\tilde{\tau}} - a_5\left(\tilde{u}_{,\tilde{x}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}}\right) - \tilde{\tau}_0\tilde{T}_{,\tilde{\tau}\tilde{\tau}} - \tilde{\tau}_0 a_5\left(\tilde{u}_{,\tilde{x}\tilde{\tau}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}\tilde{\tau}}\right)\right] \\ &\quad + f'\left(\tilde{T}\right)\left[\left(\tilde{T}_{,\tilde{x}}\right)^2 + \left(\tilde{T}_{,\tilde{y}}\right)^2 - \tilde{\tau}_0\left(\tilde{T}_{,\tilde{\tau}}\right)^2 - \tilde{\tau}_0 a_5\tilde{T}_{,\tilde{\tau}}\left(\tilde{u}_{,\tilde{x}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}}\right)\right], \end{aligned} \tag{2.15}$$

where $a_5 = \frac{\beta_0}{k_0\omega}$.

Using Eq (2.12) in Eq (2.8), we see

$$\tilde{\sigma}_{xx} = f\left(\tilde{T}\right)\left[\tilde{u}_{,\tilde{x}} - a_3\tilde{T} + a_4\tilde{v}_{,\tilde{y}}\right], \tag{2.16}$$

$$\tilde{\sigma}_{yy} = f\left(\tilde{T}\right)\left[\tilde{v}_{,\tilde{y}} - a_3\tilde{T} + a_4\tilde{u}_{,\tilde{x}}\right], \tag{2.17}$$

$$\tilde{\sigma}_{zz} = f\left(\tilde{T}\right)\left[a_4\tilde{u}_{,\tilde{x}} - a_3\tilde{T} + a_4\tilde{v}_{,\tilde{y}}\right], \tag{2.18}$$

$$\tilde{\sigma}_{xy} = \tilde{\sigma}_{yx} = a_1 f\left(\tilde{T}\right)\left[\tilde{v}_{,\tilde{x}} + \tilde{u}_{,\tilde{y}}\right], \tag{2.19}$$

$$\tilde{\sigma}_{xz} = \tilde{\sigma}_{zx} = \tilde{\sigma}_{zy} = \tilde{\sigma}_{yz} = 0. \tag{2.20}$$

We consider the next function [33]:

$$f(T) = 1 - \frac{\alpha}{\theta_0}T,$$

where α is a positive constant.

Through the application of the dimensionless variable, the resulting expression can be derived as follows:

$$f\left(\tilde{T}\right) = 1 - \alpha\tilde{T}. \tag{2.21}$$

Employing Eq (2.21) in Eqs (2.13)–(2.15), one uncovers

$$\begin{aligned} 0 &= \tilde{u}_{,\tilde{x}\tilde{x}} + a_1\tilde{u}_{,\tilde{y}\tilde{y}} + a_2\tilde{v}_{,\tilde{y}\tilde{x}} - \tilde{u}_{,\tilde{\tau}\tilde{\tau}} - a_3\tilde{T}_{,\tilde{x}} - \alpha\tilde{T}\tilde{u}_{,\tilde{x}\tilde{x}} - \alpha a_1\tilde{T}\tilde{u}_{,\tilde{y}\tilde{y}} - \alpha a_2\tilde{T}\tilde{v}_{,\tilde{y}\tilde{x}} \\ &\quad + \alpha\tilde{T}\tilde{u}_{,\tilde{\tau}\tilde{\tau}} + 2\alpha a_3\tilde{T}\tilde{T}_{,\tilde{x}} - \alpha\tilde{T}_{,\tilde{x}}\tilde{u}_{,\tilde{x}} - \alpha a_4\tilde{T}_{,\tilde{x}}\tilde{v}_{,\tilde{y}} - \alpha a_1\tilde{T}_{,\tilde{y}}\tilde{u}_{,\tilde{y}} - \alpha a_1\tilde{T}_{,\tilde{y}}\tilde{v}_{,\tilde{x}}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 &= \tilde{v}_{,\tilde{y}\tilde{y}} + a_1\tilde{v}_{,\tilde{x}\tilde{x}} + a_2\tilde{u}_{,\tilde{y}\tilde{x}} - \tilde{v}_{,\tilde{\tau}\tilde{\tau}} - a_3\tilde{T}_{,\tilde{y}} - \alpha\tilde{T}\tilde{v}_{,\tilde{y}\tilde{y}} - \alpha a_1\tilde{T}\tilde{v}_{,\tilde{x}\tilde{x}} - \alpha a_2\tilde{T}\tilde{u}_{,\tilde{y}\tilde{x}} \\ &\quad + \alpha\tilde{T}\tilde{v}_{,\tilde{\tau}\tilde{\tau}} + 2\alpha a_3\tilde{T}\tilde{T}_{,\tilde{y}} - \alpha\tilde{T}_{,\tilde{y}}\tilde{v}_{,\tilde{y}} - \alpha a_4\tilde{T}_{,\tilde{y}}\tilde{u}_{,\tilde{x}} - \alpha a_1\tilde{T}_{,\tilde{x}}\tilde{u}_{,\tilde{y}} - \alpha a_1\tilde{T}_{,\tilde{x}}\tilde{v}_{,\tilde{x}}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tilde{T}_{,\tilde{x}\tilde{x}} + \tilde{T}_{,\tilde{y}\tilde{y}} - \tilde{T}_{,\tilde{\tau}} - a_5\left(\tilde{u}_{,\tilde{x}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}}\right) - \tilde{\tau}_0\tilde{T}_{,\tilde{\tau}\tilde{\tau}} - \tilde{\tau}_0 a_5\left(\tilde{u}_{,\tilde{x}\tilde{\tau}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}\tilde{\tau}}\right) - \alpha\tilde{T}\tilde{T}_{,\tilde{x}\tilde{x}} \\ &\quad - \alpha\tilde{T}\tilde{T}_{,\tilde{y}\tilde{y}} + \alpha\tilde{T}\tilde{T}_{,\tilde{\tau}} + \alpha a_5\tilde{T}\left(\tilde{u}_{,\tilde{x}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}}\right) + \alpha\tilde{\tau}_{0\tilde{\theta}}\tilde{T}_{,\tilde{\tau}\tilde{\tau}} + \alpha\tilde{\tau}_0 a_5\tilde{T}\left(\tilde{u}_{,\tilde{x}\tilde{\tau}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}\tilde{\tau}}\right) \\ &\quad - \alpha\left(\tilde{T}_{,\tilde{x}}\right)^2 - \alpha\left(\tilde{T}_{,\tilde{y}}\right)^2 + \alpha\tilde{\tau}_0\left(\tilde{T}_{,\tilde{\tau}}\right)^2 + \alpha\tilde{\tau}_0 a_5\tilde{T}_{,\tilde{\tau}}\left(\tilde{u}_{,\tilde{x}\tilde{\tau}} + \tilde{v}_{,\tilde{y}\tilde{\tau}}\right). \end{aligned} \tag{2.24}$$

Incorporating Eq (2.21) into Eqs (2.16)–(2.19) enables the derivation of the stress tensor, which can be expressed in the following form:

$$\tilde{\sigma}_{xx} = \left(1-\alpha\tilde{T}\right)\left[\tilde{u}_{,\tilde{x}} - a_3\tilde{T} + a_4\tilde{v}_{,\tilde{y}}\right], \tag{2.25}$$

$$\tilde{\sigma}_{yy} = \left(1-\alpha\tilde{T}\right)\left[\tilde{v}_{,\tilde{y}} - a_3\tilde{T} + a_4\tilde{u}_{,\tilde{x}}\right], \tag{2.26}$$

$$\tilde{\sigma}_{zz} = \left(1-\alpha\tilde{T}\right)\left[a_4\tilde{u}_{,\tilde{x}} - a_3\tilde{T} + a_4\tilde{v}_{,\tilde{y}}\right], \tag{2.27}$$

$$\tilde{\sigma}_{xy} = \tilde{\sigma}_{yx} = a_1\left(1-\alpha\tilde{T}\right)\left[\tilde{v}_{,\tilde{x}} + \tilde{u}_{,\tilde{y}}\right]. \tag{2.28}$$

Let us assume the following representation for the moving wave transformation:

$$\begin{aligned} \tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) &= U(\epsilon), \qquad \tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = V(\epsilon), \\ \tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) &= T(\epsilon), \qquad \epsilon = s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}). \end{aligned} \tag{2.29}$$

From Eq (2.29) in Eqs (2.22)–(2.24), one acquires

$$\begin{aligned} 0 = {} & b_1U'' + b_2V'' - b_3U' - b_4T' - b_5TU'' - b_6TV'' \\ & + b_7TU' + b_8TT' - b_9T'U' - b_{10}T'V', \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 = {} & b_{11}V'' + b_2U'' - b_3V' - b_{12}T' - b_{13}TV'' - b_6TU'' \\ & + b_7TV' + b_{14}TT' - b_{15}T'V' - b_{10}T'U', \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 = {} & b_{16}T'' - b_{17}T' - b_{18}U''' - b_{19}U'' - b_{20}U' - b_{21}V''' - b_{22}V'' - b_{23}V' \\ & - b_{24}TT'' + b_{25}TT' + b_{26}TU''' + b_{27}TU'' + b_{28}TU' + b_{29}TV''' + b_{30}TV'' \\ & + b_{31}TV' - b_{24}\left(T'\right)^2 + b_{26}T'U'' + b_{32}T'U' + b_{33}T'V' + b_{29}T'V'', \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} & b_1 = s^2 + a_1h^2 - \left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right)^2, b_2 = a_2sh, b_3 = \left(\ddot{s}\tilde{x} + \ddot{h}\tilde{y} + \ddot{\aleph}\right), b_4 = a_3s, b_5 = \alpha b_1, b_6 = \alpha b_2, b_7 = \alpha b_3, \\ & b_8 = 2\alpha b_4, b_9 = \alpha s^2 + \alpha a_1h^2, b_{10} = (a_4 + a_1)\alpha sh, b_{11} = h^2 + a_1h^2 - \left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right)^2, b_{12} = a_3h, \\ & b_{13} = \alpha b_{11}, b_{14} = 2\alpha b_{12}, b_{15} = \alpha h^2 + \alpha a_1s^2, b_{16} = s^2 + h^2 - \tilde{\tau}_0\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right)^2, \\ & b_{17} = \left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{p}\right) + \tilde{\tau}_0\left(\ddot{s}\tilde{x} + \ddot{h}\tilde{y} + \ddot{\aleph}\right), b_{18} = \tilde{\tau}_0a_5s\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right)^2, \\ & b_{19} = (a_5s + 2\tilde{\tau}_0a_5\dot{s})\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right) + \tilde{\tau}_0a_5s\left(\ddot{s}\tilde{x} + \ddot{h}\tilde{y} + \ddot{\aleph}\right), b_{20} = a_5\dot{s} + \tilde{\tau}_0a_5\ddot{s}, b_{21} = \tilde{\tau}_0a_5h\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right)^2, \\ & b_{22} = \left(a_5h + 2\tilde{\tau}_0a_5\dot{h}\right)\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right) + \tilde{\tau}_0a_5h\left(\ddot{s}\tilde{x} + \ddot{h}\tilde{y} + \ddot{\aleph}\right), b_{23} = a_5\dot{q} + \tilde{\tau}_0a_5\ddot{h}, b_{24} = \alpha b_{16}, b_{25} = \alpha b_{17}, \\ & b_{26} = \alpha b_{18}, b_{27} = \alpha b_{19}, b_{28} = \alpha b_{20}, b_{29} = \alpha b_{21}, b_{30} = \alpha b_{22}, b_{31} = \alpha b_{23}, b_{32} = \alpha\tilde{\tau}_0a_5\dot{s}\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right), \\ & b_{33} = \alpha\tilde{\tau}_0a_5\dot{h}\left(\dot{s}\tilde{x} + \dot{h}\tilde{y} + \dot{\aleph}\right). \end{aligned}$$

3. Description of the method

The ISEM that will be employed in this paper is briefly described in this section [34]. Considering that we have the following NLPDE:

$$F(T, T_{\tilde{x}}, T_{\tilde{\tau}}, T_{\tilde{x}\tilde{\tau}}, T_{\tilde{x}\tilde{x}}, ...) = 0, \tag{3.1}$$

the fundamental procedures of the ISEM can be stated below:

Step 1: Initially transform the NLPDE in Eq (3.1) to an ordinary differential equation (ODE) utilizing the following transformation:

$$T(\tilde{x}, \tilde{y}, \tilde{\tau}) = T(\epsilon), \qquad \epsilon = s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}). \tag{3.2}$$

Hence, Eq (3.1) becomes

$$H(T, T', T'', T''', ...) = 0. \tag{3.3}$$

Step 2: The solution of Eq (3.3) is supposed as follows:

$$T(\epsilon) = \sum_{j=0}^{N} s_j \psi^j(\epsilon) + \sum_{j=1}^{N} r_j \psi^{-j}(\epsilon), \tag{3.4}$$

where s_j and r_j are unknown constants to be found and $\psi(\epsilon)$ fulfills the next differential equation:

$$\psi'(\epsilon) = d_0 + d_1\psi(\epsilon) + d_2\psi^2(\epsilon). \tag{3.5}$$

Step 3: In Eq (3.3), the nonlinear term is balanced with the highest-order derivative to determine the integer N.

Step 4: Substituting the solution of Eq (3.4) and the differential equation Eq (3.5) into Eq (3.3) yields a polynomial of $\psi(\epsilon)$.

Step 5: Setting every coefficient of ψ^j of the polynomial increased in Step 4 to null creates a system of equations that could be solved by software such as Maple or Mathematica to identify the unknown constants.

Step 6: Changing d_0, d_1, d_2 with varied values yields several general solutions for Eq (3.5) below:

First Family: $d_2 = 0$

$$\psi(\epsilon) = \frac{\exp(d_1\epsilon)}{d_1} - \frac{d_0}{d_1}.$$

Second Family: $d_1 = 0$

$$\psi(\epsilon) = -\sqrt{-\frac{d_0}{d_2}} \tanh\left(\sqrt{-d_0 d_2}\,\epsilon\right), \quad d_0 d_2 < 0.$$

Third Family: $d_0 = 0$

$$\psi(\epsilon) = \frac{d_1 \exp(d_1\xi)}{1 - d_2 \exp(d_1\epsilon)}.$$

By incorporating the determined constants r_j, s_j and the general solutions of Eq (3.5) into the suggested solution of Eq (3.4), several accurate solutions to the differential equation can be obtained.

The ISEM offers distinct advantages over traditional analytical methods, such as the homotopy perturbation method (HPM) and the extended tanh function method. Unlike these approaches, which rely on perturbative expansions or specific solution forms, the ISEM directly transforms nonlinear partial differential equations into algebraic equations by balancing nonlinear and linear terms. This results in more precise, closed-form solutions that do not require iterative corrections. Furthermore, the ISEM's flexibility in incorporating free parameters makes it especially effective for handling temperature-dependent thermo-elasticity, where material properties vary with temperature. Recent advancements, including studies in [36, 37], demonstrate the ISEM's superiority in solving complex, nonlinear models and highlight its unique ability to generate accurate, physically meaningful solutions for real-world applications.

4. Exact solutions for the suggested model

Setting

$$U = \pi_1 T, \qquad V = \pi_2 T, \;\; \pi_1 \neq 0, \;\; \pi_2 \neq 0 \tag{4.1}$$

in which π_1 and π_2 are constants.

From Eq (4.1) in Eqs (2.30)–(2.32), we obtain

$$\begin{aligned} 0 \;=\; & [\pi_1 b_1 + \pi_2 b_2] T^{''} - [\pi_1 b_3 + b_4] T^{'} - [\pi_1 b_5 + \pi_2 b_6] T T^{''} \\ & + [\pi_1 b_7 + b_8] T T^{'} - [\pi_1 b_9 + \pi_2 b_{10}] \left(T^{'}\right)^2, \end{aligned} \tag{4.2}$$

$$\begin{aligned} 0 \;=\; & [\pi_2 b_{11} + \pi_1 b_2] T^{''} - [\pi_2 b_3 + b_{12}] T^{'} - [\pi_2 b_{13} + \pi_1 b_6] T T^{''} \\ & + [\pi_2 b_7 + b_{14}] T T^{'} - [\pi_2 b_{15} + \pi_1 b_{10}] \left(T^{'}\right)^2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} 0 \;=\; & -[\pi_1 b_{18} + \pi_2 b_{21}] T^{'''} + [b_{16} - \pi_1 b_{19} - \pi_2 b_{22}] T^{''} - [b_{17} + \pi_1 b_{20} + \pi_2 b_{23}] T^{'} \\ & + [\pi_1 b_{26} + \pi_2 b_{29}] T T^{'''} + [\pi_2 b_{30} + \pi_1 b_{27} - b_{24}] T T^{''} + [b_{25} + \pi_1 b_{28} + \pi_2 b_{31}] T T^{'} \\ & + [\pi_1 b_{26} + \pi_2 b_{29}] T^{'} T^{''} + [\pi_1 b_{32} + \pi_2 b_{33} - b_{24}] \left(T^{'}\right)^2. \end{aligned} \tag{4.4}$$

Differentiating Eqs (4.2) and (4.3), we get

$$\begin{aligned} 0 \;=\; & -(\pi_1 b_5 + \pi_2 b_6) T T''' + (\pi_1 b_1 + \pi_2 b_2) T''' - (b_4 + \pi_1 b_3) T'' + (b_8 + \pi_1 b_7) T T'' + (b_8 + \pi_1 b_7) (T')^2 \\ & -(\pi_1 b_5 + 2\pi_1 b_9 + \pi_2 b_6 + 2\pi_2 b_{10}) T' T'', \end{aligned} \tag{4.5}$$

$$0 = -(\pi_1 b_6 + \pi_2 b_{13}) TT''' + (\pi_1 b_2 + \pi_2 b_{11}) T''' - (b_{12} + \pi_2 b_3) T'' + (b_{14} + \pi_2 b_7) TT'' + (b_{14} + \pi_2 b_7) (T')^2 - (\pi_1 b_6 + 2\pi_1 b_{10} + \pi_2 b_{13} + 2\pi_2 b_{15}) T'T''. \tag{4.6}$$

It is noted that Eqs (4.4)–(4.6) will have the same form under the following conditions:

$$\begin{aligned}
&\pi_1 b_1 + \pi_2 b_2 = \pi_1 b_2 + \pi_2 b_{11} = -\pi_1 b_{18} - \pi_2 b_{21},\\
&b_4 + \pi_1 b_3 = b_{12} + \pi_2 b_3 = -b_{16} + \pi_1 b_{19} + \pi_2 b_{22},\\
&\pi_1 b_5 + 2\pi_1 b_9 + \pi_2 b_6 + 2\pi_2 b_{10} = \pi_1 b_6 + 2\pi_1 b_{10} + \pi_2 b_{13} + 2\pi_2 b_{15} = -\pi_1 b_{26} - \pi_2 b_{29},\\
&b_8 + \pi_1 b_7 = b_{14} + \pi_2 b_7 = -b_{24} + \pi_1 b_{27} + \pi_2 b_{30},\\
&b_8 + \pi_1 b_7 = b_{14} + \pi_2 b_7 = -b_{24} + \pi_1 b_{32} + \pi_2 b_{33},\\
&\pi_1 b_5 + \pi_2 b_6 = \pi_1 b_6 + \pi_2 b_{13} = -\pi_1 b_{26} - \pi_2 b_{29},\\
&b_{17} + \pi_1 b_{20} + \pi_2 b_{23} = 0,\\
&b_{25} + \pi_1 b_{28} + \pi_2 b_{31} = 0.
\end{aligned} \tag{4.7}$$

Based on the previous conditions, Eqs (4.4)–(4.6) can be rewritten in the following form:

$$A_1 T''' + A_2 T'' + A_3 T'T'' + A_4 TT'' + A_4 (T')^2 + A_5 TT''' = 0, \tag{4.8}$$

where

$$\begin{aligned}
&A_1 = \pi_1 b_1 + \pi_2 b_2,\\
&A_2 = -b_4 - \pi_1 b_3,\\
&A_3 = -\pi_1 b_5 - 2\pi_1 b_9 - \pi_2 b_6 - 2\pi_2 b_{10},\\
&A_4 = b_8 + \pi_1 b_7,\\
&A_5 = -\pi_1 b_5 - \pi_2 b_6.
\end{aligned}$$

The integer value N should be determined at first to implement the improved simple equation method. This can be done by applying the balancing rule. Balancing T''' with TT'', we get $N = 1$. Therefore, the solution of the resulting ODE Eq (4.8) can be represented in the following form:

$$T(\epsilon) = s_0 + s_1 \psi(\epsilon) + \frac{r_1}{\psi(\epsilon)}. \tag{4.9}$$

Substituting Eq (4.9) along with its auxiliary equation (3.5) into Eq (4.8) yields a nonlinear algebraic equation. Collecting the coefficients of ψ^i and equating them to zero gives a system of equations that can be handled by using software packages of Mathematica. The following results are obtained:
Case 1. $d_2 = 0$:

Result 1

$$r_1 = 0,\ d_0 = -\frac{2\left(A_4 (A_3 - A_5) s_0 + A_1 (A_3 - 2A_4 + A_5)\right)}{\left(A_3^2 - A_5^2\right) s_1},\ d_1 = -\frac{2A_4}{A_3 + A_5}.$$

Then, we have the following solutions:

$$\tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \frac{\left(A_5^2 - A_3^2\right) s_1 e^{-\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - 2A_1(A_3 - 2A_4 + A_5)}{2A_4(A_3 - A_5)}, \tag{4.10}$$

$$\tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \pi_1 \frac{\left(A_5^2 - A_3^2\right) s_1 e^{-\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - 2A_1(A_3 - 2A_4 + A_5)}{2A_4(A_3 - A_5)}, \tag{4.11}$$

$$\tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \pi_2 \frac{\left(A_5^2 - A_3^2\right) s_1 e^{-\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - 2A_1(A_3 - 2A_4 + A_5)}{2A_4(A_3 - A_5)}, \tag{4.12}$$

$$\begin{aligned}\tilde{\sigma}_{xx} = &-\frac{e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\left(2(\alpha A_1(A_3 - 2A_4 + A_5) + A_4(A_3 - A_5))e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} + \alpha\left(A_3^2 - A_5^2\right)s_1\right)}{4A_4^2(A_3 - A_5)^2} \\ &\times\left\{2a_3A_1(-A_3 + 2A_4 - A_5)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - (A_3 - A_5)s_1(2A_4(\pi_2 a_4 h(\tilde{\tau}) + \pi_1 s(\tilde{\tau})) + a_3(A_3 + A_5))\right\},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\tilde{\sigma}_{yy} = &-\frac{e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\left(2(\alpha A_1(A_3 - 2A_4 + A_5) + A_4(A_3 - A_5))e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} + \alpha\left(A_3^2 - A_5^2\right)s_1\right)}{4A_4^2(A_3 - A_5)^2} \\ &\times\left\{2a_3A_1(-A_3 + 2A_4 - A_5)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - (A_3 - A_5)s_1(2A_4(\pi_1 a_4 s(\tilde{\tau}) + \pi_2 h(\tilde{\tau})) + a_3(A_3 + A_5))\right\},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\tilde{\sigma}_{zz} = &-\frac{e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\left(2(\alpha A_1(A_3 - 2A_4 + A_5) + A_4(A_3 - A_5))e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} + \alpha\left(A_3^2 - A_5^2\right)s_1\right)}{4A_4^2(A_3 - A_5)^2} \\ &\times\left\{2a_3A_1(-A_3 + 2A_4 - A_5)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} - (A_3 - A_5)s_1(2a_4A_4(\pi_2 h(\tilde{\tau}) + \pi_1 s(\tilde{\tau})) + a_3(A_3 + A_5))\right\},\end{aligned} \tag{4.15}$$

$$\begin{aligned}\tilde{\sigma}_{xy} = &\frac{a_1 s_1(\pi_1 h(\tilde{\tau}) + \pi_2 s(\tilde{\tau}))e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}}{2A_4(A_3 - A_5)} \\ &\times\left\{2(\alpha A_1(A_3 - 2A_4 + A_5) + A_4(A_3 - A_5))e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}} + \alpha\left(A_3^2 - A_5^2\right)s_1\right\}.\end{aligned} \tag{4.16}$$

Result 2

$$s_1 = 0,\ d_0 = \frac{r_1(2A_5A_4s_0 + A_1A_4 + A_1A_5)}{2(A_5s_0 + A_1)^2},\ d_1 = \frac{A_4s_0 + A_1}{A_5s_0 + A_1},\ A_3 = -3A_5.$$

Then, we have the following solutions:

$$\tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \frac{2s_0(A_5s_0 + A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right) + A_1r_1((A_4 + A_5)s_0 + 2A_1)}{2(A_5s_0 + A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right) - r_1(2A_4A_5s_0 + A_1(A_4 + A_5))}, \tag{4.17}$$

$$\tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau})=\pi_1\frac{2s_0(A_5s_0+A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)+A_1r_1((A_4+A_5)s_0+2A_1)}{2(A_5s_0+A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)-r_1(2A_4A_5s_0+A_1(A_4+A_5))},\tag{4.18}$$

$$\tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau})=\pi_2\frac{2s_0(A_5s_0+A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)+A_1r_1((A_4+A_5)s_0+2A_1)}{2(A_5s_0+A_1)^2\exp\left(\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)-r_1(2A_4A_5s_0+A_1(A_4+A_5))},\tag{4.19}$$

$$\tilde{\sigma}_{xx}=\frac{(\alpha s_0-1)e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}+\alpha d_1r_1+d_0(1-\alpha s_0)}{\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)^3}\times$$
$$\left\{a_3\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)\left(s_0\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)+d_1r_1\right)+d_1^2r_1(\pi_2a_4h(\tilde{\tau})+\pi_1s(\tilde{\tau}))e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\right\},\tag{4.20}$$

$$\tilde{\sigma}_{yy}=\frac{(\alpha s_0-1)e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}+\alpha d_1r_1+d_0(1-\alpha s_0)}{\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)^3}\times$$
$$\left\{a_3\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)\left(s_0\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)+d_1r_1\right)+d_1^2r_1(\pi_1a_4s(\tilde{\tau})+\pi_2h(\tilde{\tau}))e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\right\},\tag{4.21}$$

$$\tilde{\sigma}_{zz}=\frac{(\alpha s_0-1)e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}+\alpha d_1r_1+d_0(1-\alpha s_0)}{\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)^3}\times$$
$$\left\{a_3\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)\left(s_0\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)+d_1r_1\right)+a_4d_1^2r_1(\pi_2h(\tilde{\tau})+\pi_1s(\tilde{\tau}))e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\right\},\tag{4.22}$$

$$\tilde{\sigma}_{xy}=\frac{a_1d_1^2r_1(\pi_1h(\tilde{\tau})+\pi_2s(\tilde{\tau}))e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\left((\alpha s_0-1)e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}+\alpha d_1r_1+d_0(1-\alpha s_0)\right)}{\left(e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}-d_0\right)^3}.\tag{4.23}$$

Case 2. $d_1=0$:

Result 1

$$r_1=0,\ d_0=\frac{A_4s_0+A_1}{2s_1A_5},\ d_2=-\frac{A_4s_1}{2(A_5s_0+A_1)},\ A_3=-3A_5.$$

Then, we have the following solutions:

$$\tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau})=s_0-\sqrt{\frac{(A_4s_0+A_1)(A_5s_0+A_1)}{A_4A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right),\tag{4.24}$$

$$\tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau})=\pi_1s_0-\pi_1\sqrt{\frac{(A_4s_0+A_1)(A_5s_0+A_1)}{A_4A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right),\tag{4.25}$$

$$\tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \pi_2 s_0 - \pi_2\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right), \tag{4.26}$$

$$\tilde{\sigma}_{xx} = \frac{1}{2}\left(-\alpha\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + \alpha s_0 - 1\right)\times$$
$$\Bigg\{\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}(\pi_2 a_4 h(\tilde{\tau}) + \pi_1 s(\tilde{\tau}))\times$$
$$\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + 2a_3\Bigg(s_0 - \sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\times$$
$$\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)\Bigg)\Bigg\}, \tag{4.27}$$

$$\tilde{\sigma}_{yy} = \frac{1}{2}\left(-\alpha\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + \alpha s_0 - 1\right)\times$$
$$\Bigg\{\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}(\pi_1 a_4 s(\tilde{\tau}) + \pi_2 h(\tilde{\tau}))\times$$
$$\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + 2a_3\Bigg(s_0 - \sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\times$$
$$\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)\Bigg)\Bigg\}, \tag{4.28}$$

$$\tilde{\sigma}_{zz} = \frac{1}{2}\left(-\alpha\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + \alpha s_0 - 1\right)\times$$
$$\Bigg\{a_4\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}\sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}(\pi_2 h(\tilde{\tau}) + \pi_1 s(\tilde{\tau}))\times$$
$$\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right) + 2a_3\Bigg(s_0 - \sqrt{\frac{(A_4 s_0 + A_1)(A_5 s_0 + A_1)}{A_4 A_5}}\times$$
$$\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4 s_0 + A_1)}{A_5(A_5 s_0 + A_1)}}(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)\Bigg)\Bigg\}, \tag{4.29}$$

$$\tilde{\sigma}_{xy} = \frac{1}{2}a_1 \mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)\times$$
$$\Bigg\{\sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}\sqrt{\frac{(A_4s_0+A_1)(A_5s_0+A_1)}{A_4A_5}}(\pi_1h(\tilde{\tau})+\pi_2s(\tilde{\tau}))\times$$
$$\Bigg(-\alpha\sqrt{\frac{(A_4s_0+A_1)(A_5s_0+A_1)}{A_4A_5}}\tanh\left(\frac{1}{2}\sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)+\alpha s_0-1\Bigg)\Bigg\}. \tag{4.30}$$

Result 2

$$s_1 = 0,\ d_0 = \frac{A_4r_1}{2(A_5s_0+A_1)},\ d_2 = \frac{-A_4s_0-A_1}{2A_5r_1},\ A_3 = -3A_5.$$

Then, we have the following solutions:

$$\tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = s_0 - \frac{\sqrt{(A_4s_0+A_1)(A_5s_0+A_1)}\coth\left(\frac{1}{2}\sqrt{-\frac{A_4(-A_4s_0-A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)}{\sqrt{A_4A_5}}, \tag{4.31}$$

$$\tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \pi_1s_0 - \frac{\pi_1\sqrt{(A_4s_0+A_1)(A_5s_0+A_1)}\coth\left(\frac{1}{2}\sqrt{-\frac{A_4(-A_4s_0-A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)}{\sqrt{A_4A_5}}, \tag{4.32}$$

$$\tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \pi_2s_0 - \frac{\pi_2\sqrt{(A_4s_0+A_1)(A_5s_0+A_1)}\coth\left(\frac{1}{2}\sqrt{-\frac{A_4(-A_4s_0-A_1)}{A_5(A_5s_0+A_1)}}(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)}{\sqrt{A_4A_5}}, \tag{4.33}$$

$$\tilde{\sigma}_{xx} = -\frac{\mathrm{csch}^2\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)}{2A_4A_5}\times\Bigg\{\Bigg(\alpha\sqrt{A_4A_5}s_0-\sqrt{A_4A_5}-$$
$$\alpha\Omega\coth\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)\Bigg)\times\Bigg(a_3\Big(\Omega\sinh(\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau})))-$$
$$\sqrt{A_4A_5}s_0(\cosh(\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau})))-1)\Big)+\zeta\Omega(\pi_2a_4h(\tilde{\tau})+\pi_1s(\tilde{\tau}))\Bigg)\Bigg\}, \tag{4.34}$$

$$\tilde{\sigma}_{yy} = -\frac{\mathrm{csch}^2\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)}{2A_4A_5}\times\Bigg\{\Bigg(\alpha\sqrt{A_4A_5}s_0-\sqrt{A_4A_5}-$$
$$\alpha\Omega\coth\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))\right)\Bigg)\times\Bigg(a_3\Big(\Omega\sinh(\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau})))-$$
$$\sqrt{A_4A_5}s_0(\cosh(\zeta(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau})))-1)\Big)+\zeta\Omega(\pi_1a_4s(\tilde{\tau})+\pi_2h(\tilde{\tau}))\Bigg)\Bigg\}, \tag{4.35}$$

$$\tilde{\sigma}_{zz} = -\frac{\operatorname{csch}^2\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)}{2A_4A_5} \times \Bigg\{\Bigg(\alpha\sqrt{A_4A_5}s_0 - \sqrt{A_4A_5} - \alpha\Omega\coth\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)\Bigg) \times \Bigg(a_3\Big(\Omega\sinh(\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))) - \sqrt{A_4A_5}s_0(\cosh(\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))) - 1)\Big) + a_4\zeta\Omega(\pi_2h(\tilde{\tau}) + \pi_1s(\tilde{\tau}))\Bigg)\Bigg\}, \tag{4.36}$$

$$\tilde{\sigma}_{xy} = -\frac{a_1\zeta\Omega(\pi_1h(\tilde{\tau}) + \pi_2s(\tilde{\tau}))\operatorname{csch}^2\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)}{2A_4A_5} \times \left\{\alpha\sqrt{A_4A_5}s_0 - \sqrt{A_4A_5} - \alpha\Omega\coth\left(\frac{1}{2}\zeta(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau}))\right)\right\}, \tag{4.37}$$

where $\zeta = \sqrt{\frac{A_4(A_4s_0+A_1)}{A_5(A_5s_0+A_1)}}$ and $\Omega = \sqrt{(A_4s_0 + A_1)(A_5s_0 + A_1)}$.

Case 3. $d_0 = 0$:

Result 1

$$r_1 = 0,\ A_3 = -3A_5,\ d_1 = \frac{-A_4s_0 - A_1}{A_5s_0 + A_1},\ d_2 = -\frac{s_1(2A_5A_4s_0 + A_1A_4 + A_1A_5)}{2(A_5s_0 + A_1)^2}.$$

Then, we have the following solutions:

$$\tilde{T}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \frac{s_1(-A_4s_0 - A_1)\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{(A_5s_0 + A_1)\left(\frac{s_1(2A_4A_5s_0+A_1(A_4+A_5))\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{2(A_5s_0+A_1)^2} + 1\right)} + s_0, \tag{4.38}$$

$$\tilde{u}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \frac{\pi_1s_1(-A_4s_0 - A_1)\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{(A_5s_0 + A_1)\left(\frac{s_1(2A_4A_5s_0+A_1(A_4+A_5))\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{2(A_5s_0+A_1)^2} + 1\right)} + \pi_1s_0, \tag{4.39}$$

$$\tilde{v}(\tilde{x},\ \tilde{y},\ \tilde{\tau}) = \frac{\pi_2s_1(-A_4s_0 - A_1)\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{(A_5s_0 + A_1)\left(\frac{s_1(2A_4A_5s_0+A_1(A_4+A_5))\exp\left(-\frac{(A_4s_0+A_1)(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_5s_0+A_1}\right)}{2(A_5s_0+A_1)^2} + 1\right)} + \pi_2s_0, \tag{4.40}$$

$$\tilde{\sigma}_{xx} = \frac{d_2(\alpha s_0 - 1)e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))} - \alpha d_1s_1e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))} - \alpha s_0 + 1}{(d_2e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))} - 1)^3} \times \Bigg\{a_3\left(d_2e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))} - 1\right)\left(s_0\left(d_2e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))} - 1\right) - d_1s_1e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\right) - d_1^2s_1(\pi_2a_4h(\tilde{\tau}) + \pi_1s(\tilde{\tau}))e^{d_1(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}\Bigg\}, \tag{4.41}$$

$$\tilde{\sigma}_{yy} = \frac{\left(d_2\left(\alpha s_0 - 1\right) - \alpha d_1 s_1\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - \alpha s_0 + 1}{\left(d_2 e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - 1\right)^3} \times \Big\{ a_3 \left(d_2 e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - 1\right)$$
$$\left(\left(d_2 s_0 - d_1 s_1\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - s_0\right) - d_1^2 s_1 \left(\pi_1 a_4 s\left(\tilde{\tau}\right) + \pi_2 h\left(\tilde{\tau}\right)\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} \Big\}, \tag{4.42}$$

$$\tilde{\sigma}_{zz} = \frac{\left(d_2\left(\alpha s_0 - 1\right) - \alpha d_1 s_1\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - \alpha s_0 + 1}{\left(d_2 e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - 1\right)^3} \times \Big\{ a_3 \left(d_2 e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - 1\right)$$
$$\left(\left(d_2 s_0 - d_1 s_1\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - s_0\right) - a_4 d_1^2 s_1 \left(\pi_2 h\left(\tilde{\tau}\right) + \pi_1 s\left(\tilde{\tau}\right)\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} \Big\}, \tag{4.43}$$

$$\tilde{\sigma}_{xy} = \frac{a_1 d_1^2 s_1 \left(\pi_1 h\left(\tilde{\tau}\right) + \pi_2 s\left(\tilde{\tau}\right)\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} \left(\left(d_2\left(1 - \alpha s_0\right) + \alpha d_1 s_1\right) e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} + \alpha s_0 - 1\right)}{\left(d_2 e^{d_1\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)} - 1\right)^3}. \tag{4.44}$$

Result 2

$$s_1 = 0,\ d_1 = \frac{2A_4}{A_3 + A_5},\ d_2 = \frac{2\left(A_4 A_3 s_0 - A_4 A_5 s_0 + A_1 A_3 - 2A_1 A_4 + A_1 A_5\right)}{\left(A_3^2 - A_5^2\right) r_1}.$$

Then, we have the following solutions:

$$\tilde{T}\left(\tilde{x},\ \tilde{y},\ \tilde{\tau}\right) = \frac{\left(A_3^2 - A_5^2\right) r_1 e^{-\frac{2A_4(Hy+P+Sx)}{A_3+A_5}} - 2A_1\left(A_3 - 2A_4 + A_5\right)}{2A_4\left(A_3 - A_5\right)}, \tag{4.45}$$

$$\tilde{u}\left(\tilde{x},\ \tilde{y},\ \tilde{\tau}\right) = \frac{\pi_1 \left(A_3^2 - A_5^2\right) r_1 e^{-\frac{2A_4(Hy+P+Sx)}{A_3+A_5}} - 2pi_1 A_1\left(A_3 - 2A_4 + A_5\right)}{2A_4\left(A_3 - A_5\right)}, \tag{4.46}$$

$$\tilde{v}\left(\tilde{x},\ \tilde{y},\ \tilde{\tau}\right) = \frac{\pi_2 \left(A_3^2 - A_5^2\right) r_1 e^{-\frac{2A_4(Hy+P+Sx)}{A_3+A_5}} - 2pi_2 A_1\left(A_3 - 2A_4 + A_5\right)}{2A_4\left(A_3 - A_5\right)}, \tag{4.47}$$

$$\tilde{\sigma}_{xx} = -\frac{e^{-\frac{4A_4\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)}{A_3+A_5}} \left(2\left(\alpha A_1\left(A_3 - 2A_4 + A_5\right) + A_4\left(A_3 - A_5\right)\right) e^{\frac{2A_4\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)}{A_3+A_5}} + \alpha\left(A_5^2 - A_3^2\right) r_1\right)}{4A_4^2\left(A_3 - A_5\right)^2}$$
$$\times \Big\{ 2a_3 A_1\left(-A_3 + 2A_4 - A_5\right) e^{\frac{2A_4\left(s(\tilde{\tau})\tilde{x} + h(\tilde{\tau})\tilde{y} + \aleph(\tilde{\tau})\right)}{A_3+A_5}} + \left(A_3 - A_5\right) r_1 \left(2A_4\left(\pi_2 a_4 h\left(\tilde{\tau}\right) + \pi_1 s\left(\tilde{\tau}\right)\right) + a_3\left(A_3 + A_5\right)\right) \Big\}, \tag{4.48}$$

$$\tilde{\sigma}_{yy}=-\frac{e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\left(2\left(\alpha A_1\left(A_3-2A_4+A_5\right)+A_4\left(A_3-A_5\right)\right)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}+\alpha\left(A_5^2-A_3^2\right)r_1\right)}{4A_4^2\left(A_3-A_5\right)^2}$$
$$\times\left\{2a_3A_1\left(-A_3+2A_4-A_5\right)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}+\left(A_3-A_5\right)r_1\left(2A_4\left(\pi_1a_4s\left(\tilde{\tau}\right)+\pi_2h\left(\tilde{\tau}\right)\right)+a_3\left(A_3+A_5\right)\right)\right\}, \tag{4.49}$$

$$\tilde{\sigma}_{zz}=-\frac{e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\left(2\left(\alpha A_1\left(A_3-2A_4+A_5\right)+A_4\left(A_3-A_5\right)\right)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}+\alpha\left(A_5^2-A_3^2\right)r_1\right)}{4A_4^2\left(A_3-A_5\right)^2}$$
$$\times\left\{2a_3A_1\left(-A_3+2A_4-A_5\right)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}+\left(A_3-A_5\right)r_1\left(2a_4A_4\left(\pi_2h\left(\tilde{\tau}\right)+\pi_1s\left(\tilde{\tau}\right)\right)+a_3\left(A_3+A_5\right)\right)\right\}, \tag{4.50}$$

$$\tilde{\sigma}_{xy}=\frac{a_1r_1\left(\pi_1h\left(\tilde{\tau}\right)+\pi_2s\left(\tilde{\tau}\right)\right)e^{-\frac{4A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}}{2A_4\left(A_3-A_5\right)}\times$$
$$\left\{\alpha\left(A_3^2-A_5^2\right)r_1-2\left(\alpha A_1\left(A_3-2A_4+A_5\right)+A_4\left(A_3-A_5\right)\right)e^{\frac{2A_4(s(\tilde{\tau})\tilde{x}+h(\tilde{\tau})\tilde{y}+\aleph(\tilde{\tau}))}{A_3+A_5}}\right\}. \tag{4.51}$$

5. Visualization and interpretation of some solutions

This section provides graphical representations in two dimensions for a selection of solutions. Copper is chosen as the thermo-elastic material for the analysis, with specific values assigned to various physical constants, as outlined below [35]:

$$\begin{aligned} c_e &= 3.831\times10^2\ \mathrm{J\cdot kg^{-1}\cdot K^{-1}}, \quad \theta_0=2.93\times10^2\ K, \quad k_0=368\ \mathrm{W\cdot m^{-1}\cdot K^{-1}},\\ \mu_0 &= 38.6\times10^9\ \mathrm{N\cdot m^{-2}}, \quad \rho_0=89.54\times10^2\ \mathrm{kg\cdot m^{-3}}, \quad \lambda_0=77.6\times10^9\ \mathrm{N\cdot m^{-2}},\\ \alpha_\tau &= 1.78\times10^{-5}\ K^{-1}. \end{aligned}$$

Figure 1 depicts the solutions for Eqs (4.10)–(4.13) and (4.16) with the specified parameters $s(\tilde{\tau})=h(\tilde{\tau})=\aleph(\tilde{\tau})=\tilde{\tau}$, $\pi_1=0.6$, $\pi_2=-1.28$, $\tilde{y}=1$, $\tilde{\tau}=1$, $s_1=-0.29$, $\alpha=3$, 4 and 5. Figure 2 depicts the solutions for Eqs (4.24)–(4.27) and (4.30) with the specified parameters $s(\tilde{\tau})=h(\tilde{\tau})=\aleph(\tilde{\tau})=\tilde{\tau}$, $\pi_1=-1.68$, $\pi_2=2.62$, $\tilde{y}=1$, $\tilde{\tau}=1$, $s_0=3.4$, $\alpha=3$, 4 and 5. Figure 3 depicts the solutions for Eqs (4.45)–(4.48) and (4.51) with the specified parameters $s(\tilde{\tau})=h(\tilde{\tau})=\aleph(\tilde{\tau})=\tilde{\tau}$, $\pi_1=1.66$, $\pi_2=-3.8$, $\tilde{y}=1$, $\tilde{\tau}=1$, $r_1=2$, $\alpha=3$, 4 and 5. Varying parameters in a system can significantly influence its behavior and performance, making sensitivity analysis a crucial tool for understanding the underlying dynamics. By systematically altering key parameters, such as α, and observing the corresponding changes in the system's output, one can assess how sensitive the system is to these variations. This type of analysis helps identify regions where the system exhibits high responsiveness, as well as areas where it remains relatively unaffected by parameter changes.

Understanding the relationship between parameters and outputs is essential for optimization, control, and decision-making processes, as it provides insight into which parameters most significantly impact the system's behavior. Additionally, recognizing points of high sensitivity can guide the design of more robust systems that perform consistently across a range of operating conditions, which frequently aligns with [17].

Figures 1(a), 2(b), and 3(b) introduce the effect of changing α on $\tilde{T}$ for Eqs (4.10), (4.24), and (4.45). In addition, the effect of changing α on $\tilde{u}$, $\tilde{v}$, $\tilde{\sigma}xx$, and $\tilde{\sigma}xy$ is introduced in Figures 1(b, c, d, e), 2(b, c, d, e), and 3(b, c, d, e). For small values of α, $\tilde{T}$ may change rapidly, indicating a high sensitivity of the system to α. In contrast, for larger values of α, the rate of change of $\tilde{T}$ could become less pronounced, suggesting that the system becomes less sensitive to further changes in α. This nonlinear relationship highlights how the effect of α on $\tilde{T}$ is not constant and can vary depending on the magnitude of α. Critical points on the plot, where $\tilde{T}$ shows a sharp increase or decrease, can be interpreted as zones where small adjustments in α lead to significant changes in $\tilde{T}$, reflecting heightened sensitivity in those ranges. Understanding this behavior helps in optimizing parameters or identifying conditions where the system is most responsive to α.

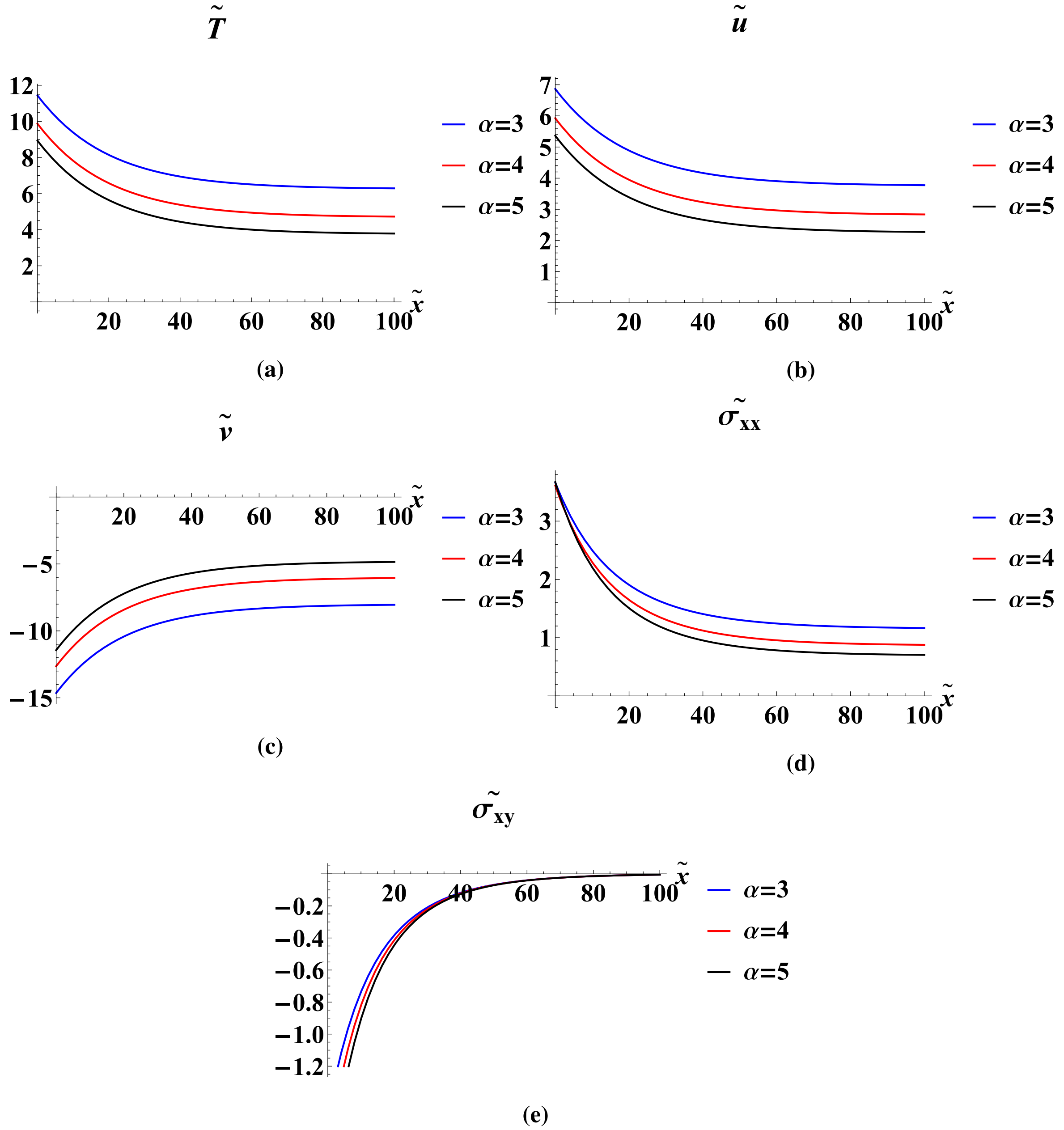

Figure 1. Graphical simulations of Eqs (4.10)–(4.13) and (4.16).

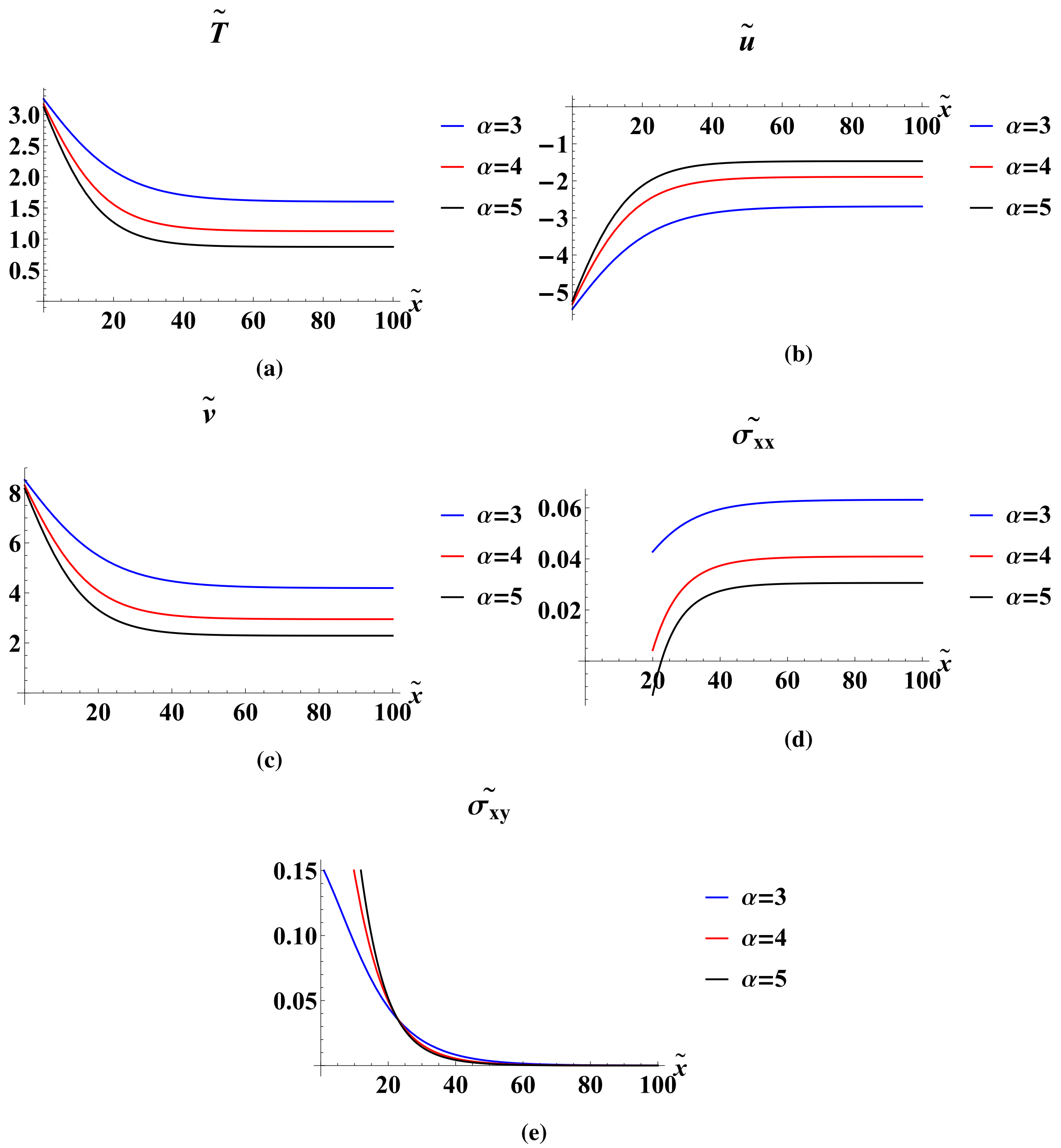

Figure 2. Graphical simulations of Eqs (4.24)–(4.27) and (4.30).

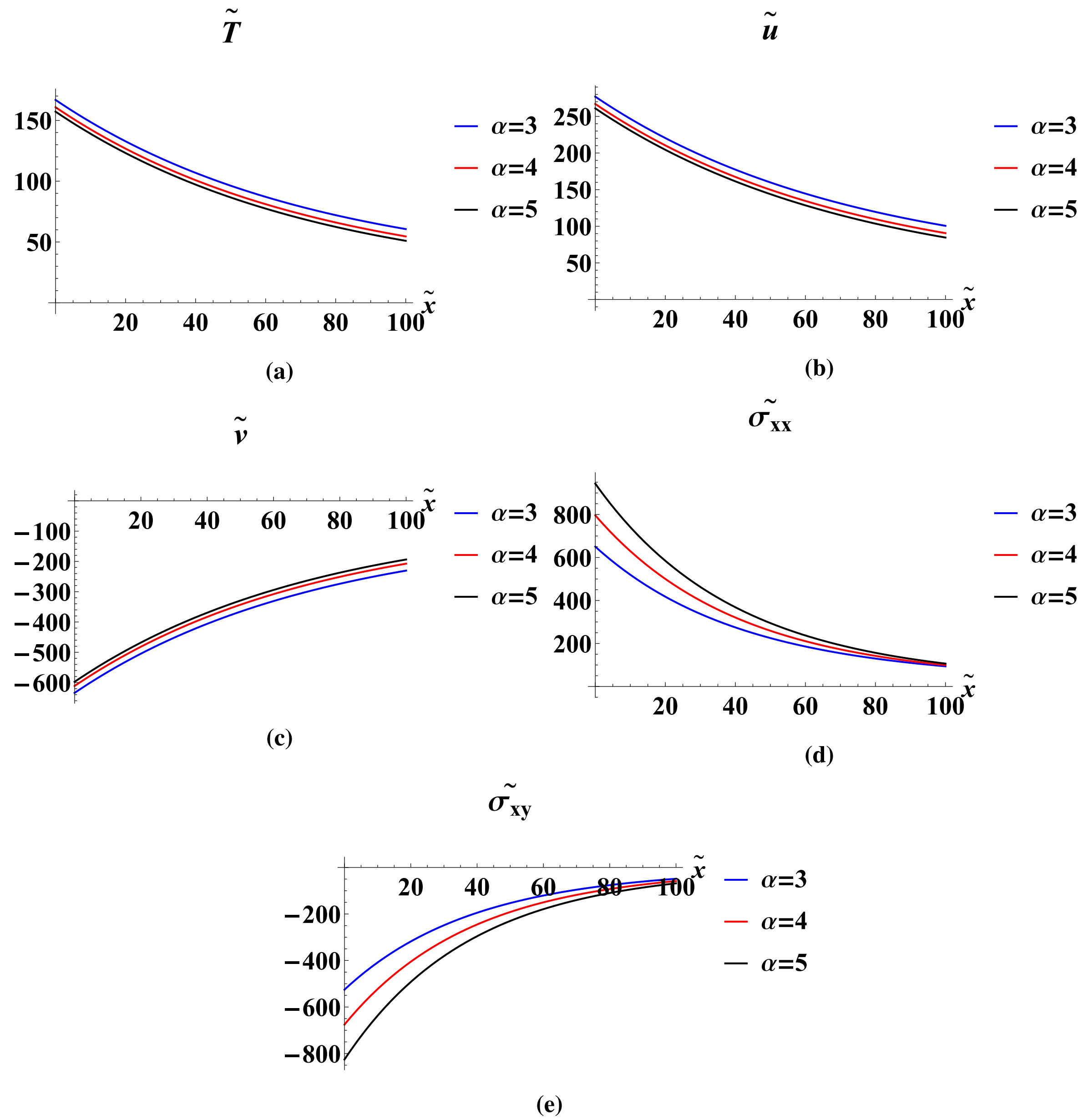

Figure 3. Graphical simulations of Eqs (4.45)–(4.48) and (4.51).

6. Conclusions

The ISEM has established itself as a powerful and flexible analytical technique for investigating temperature-dependent behavior in thermo-elastic materials, particularly within the framework of L-S theory. Its capacity to derive a diverse array of exact solutions underscores its reliability and adaptability across a broad spectrum of analytical applications. The method's effectiveness is further reinforced by the use of graphical representations, which depict key parameters such as displacement components, temperature distributions, and stress tensor components. These visualizations serve as essential tools for both researchers and engineers, offering critical insights into the complex responses of thermo-elastic materials subjected to varying environmental and operational conditions. In addition to enhancing the understanding of thermo-elasticity, the ISEM provides a solid foundation for future studies and potential real-world applications in multiple scientific and engineering disciplines. By unveiling the intricate dynamics governing these materials, the method opens new avenues for exploration, fostering innovation and contributing to the advancement of related technologies and analytical methodologies. As a result, the ISEM is not only a valuable tool for current research but also a driving force in the evolution of future developments in thermo-elastic material analysis.

Aerospace Engineering: The accurate modeling of thermo-elastic wave propagation in temperature-dependent materials is crucial for predicting thermal stresses in spacecraft structures during rapid thermal loading, such as re-entry or orbital transitions. The derived solutions can assist in the design of thermally resilient components by providing precise stress and displacement profiles.

Microelectronics: In micro-scale devices, where thermal effects significantly influence mechanical behavior, the presented model can contribute to better heat management and structural reliability analysis. Our findings can support the development of predictive tools for thermal fatigue and failure prevention in integrated circuits.

Materials Science: The study offers insights into the behavior of advanced materials under coupled thermo-mechanical loads, particularly those exhibiting temperature-dependent properties. This can guide the design and testing of smart materials or composites used in sensors and actuators.

Author contributions

Mohamed F. Ismail: Formal analysis, software; Hamdy M. Ahmed: Validation, methodology; Alaa A. El-Bary: Resources, writing–review and editing; Hamdy M. Youssef: Writing–review and editing, investigation; Islam Samir: Software, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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