Mathematics
https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 10764–10786.
DOI: 10.3934/math.2025489
Received: 24 January 2025
Revised: 25 April 2025
Accepted: 30 April 2025
Published: 12 May 2025

Research article

Some results on almost paracontact paracomplex Riemannian manifolds

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Abstract: In this study, a class of Riemannian almost product manifolds is obtained by the warped product of almost paracontact paracomplex Riemannian manifolds with $\mathbb{R}$. The curvature properties of the almost product manifolds are studied. Then, normal almost paracontact paracomplex Riemannian manifolds are considered. It is proven that the almost paracontact paracomplex Riemannian manifolds are normal if and only if the almost product manifolds obtained by the warped product are integrable. In addition, examples of almost paracont paracomplex Riemannian manifolds are given.

Keywords: almost paracontact paracomplex Riemannian manifold; Riemannian almost product manifold; Einstein manifold; warped product
Mathematics Subject Classification: 53C15, 53C25

1. Introduction

As an analogue of almost contact metric Riemannian manifolds, Sato introduced the notion of almost paracontact Riemannian manifolds in 1976 [1]. Afterwards, Sasaki defined almost paracontact manifolds of type (p, q), where p and q are the multiplicities of the eigenvalues 1 and -1 of the endomorphism φ. In the literature, these manifolds are called almost paracontact paracomplex Riemannian manifolds for the special case where

$$p = q = n.$$

Classifications of almost paracontact paracomplex Riemannian manifolds and Riemannian almost product manifolds are made by using the covariant derivative of their fundamental forms, see [2, 3], respectively. In this work, after presenting the necessary preliminary information, we obtain an almost product structure $\tilde{P}$ with a trace zero on the product of an almost paracontact paracomplex Riemannian manifold with $\mathbb{R}$, thereby using a method similar to that in [4] using a warped product. Then, we define a Riemannian metric on the product manifold, which is compatible with the almost product structure. Thus, the product manifold is a Riemannian almost product manifold of a special type. We write the covariant derivative of the Riemannian metric and the almost product structure of product manifold in terms of the covariant derivative of the metric of the almost paracontact paracomplex Riemannian manifold. We investigate the curvature properties of the product manifold and state relations between some classes of almost paracontact paracomplex Riemannian manifolds and Riemannian almost product manifolds. In addition, the almost product manifold obtained by the warped product is integrable if and only if the almost paracontact paracomplex Riemannian manifold is normal.

2. Preliminaries

An odd dimensional differentiable manifold M^{2n+1} has an almost paracontact structure (φ, ξ, η) if it admits an endomorphism φ of the tangent bundle, a vector field ξ and its dual 1-form η such that

$$\varphi^2(X) = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{2.1}$$

hold for an arbitrary vector field X. A differentiable manifold with an almost paracontact structure is called an almost paracontact manifold [1]. Equation (2.1) implies the following:

$$\eta(\varphi(X)) = 0, \quad \varphi(\xi) = 0,$$

for all vector fields X.

Almost paracontact manifolds of type (p, q) are introduced in [5]. Denote the multiplicity of the eigenvalues 1 and -1 of φ by p and q, respectively. In addition, the endomorphism φ has a simple eigenvalue 0; thus,

$$tr\varphi = p - q,$$

where $tr\varphi$ is the trace of φ.

If (M, φ, ξ, η) is an almost paracontact manifold with

$$tr\varphi = p - q,$$

then this manifold is called an almost paracontact manifold of type (p, q). If

$$p = q,$$

that is, if

$$tr\varphi = 0,$$

then M is called an almost paracontact paracomplex manifold.

An almost paracontact paracomplex manifold endowed with a Riemannian metric g such that

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y is called an almost paracontact paracomplex Riemannian manifold [4, 6].

Equations (2.1) and (2.2) yield the following:

$$\eta(X) = g(\xi, X), \quad g(\varphi(X), Y) = g(X, \varphi(Y)).$$

for all vector fields X, Y.

Note that almost paracontact paracomplex Riemannian manifolds are called almost paracontact Riemannian maniolds of type (n, n) in [2] and almost paracontact almost paracomplex Riemannian manifolds in [6]. We use the terminology in [4].

Let ∇ be the Levi-Civita connection of the Riemannian metric g. For all vector fields X, Y, Z on M, the structure tensor α of type $(0, 3)$ is defined as following:

$$\alpha(X, Y, Z) = g\left((\nabla_X \varphi)(Y), Z\right),$$

which has following properties:

$$\begin{aligned} \alpha(X, Y, Z) &= \alpha(X, Z, Y), \\ \alpha(X, \varphi(Y), \varphi(Z)) &= -\alpha(X, Y, Z) + \eta(Y)\alpha(X, \xi, Z) + \eta(Z)\alpha(X, Y, \xi). \end{aligned}$$

The following 1-forms are associated to the structure tensor α:

$$\theta(X) = g^{ij}\alpha(E_i, E_j, X), \quad \theta^*(X) = g^{ij}\alpha(E_i, \varphi(E_j), X), \quad \omega(X) = \alpha(\xi, \xi, X),$$

where $\{E_1, \cdots E_{2n}, \xi\}$ is a local frame, X is a vector field, and (g^{ij}) is the inverse matrix of (g_{ij}).

The space $\mathcal{F}$ of covariant derivatives of the endomorphism φ given by

$$\begin{aligned} \mathcal{F} = \Big\{\alpha \in \oplus_3^0 M \; : \; & \alpha(X, Y, Z) = \alpha(X, Z, Y), \\ & \alpha(X, \varphi(Y), \varphi(Z)) = -\alpha(X, Y, Z) + \eta(Y)\alpha(X, \xi, Z) + \eta(Z)\alpha(X, Y, \xi)\} \end{aligned}$$

decomposes into eleven subspaces

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{11},$$

which are orthogonal and invariant under the action of the structure group $O(n) \times O(n) \times I$ where $O(n)$ is the group of orthogonal matrices of size n, and I is the unit matrix of size one [2, 6]. The defining relations of basic classes $\mathcal{F}_i$ and projections F^i onto each subspace $\mathcal{F}_i$ is given in [2, 6]. An almost paracontact paracomplex Riemannian manifold is said to either be in the class $\mathcal{F}_i$ or a direct sum of some classes, if the structure tensor α is in $\mathcal{F}_i$ or in a direct sum of some classes,respectively.

Riemannian almost product manifolds are introduced in [1]. If a differentiable manifold L has a tensor field P (almost product structure) and a Riemannian metric h satisfies the conditions

- $P^2(X) = X$,
- $h(P(X), P(Y)) = h(X, Y)$,

for all vector fields X, Y on L, then L is called a Riemannian almost product manifold. In this study, we consider Riemannian almost product manifolds with

$$trP = 0$$

classified by [3]. In this case, L is even dimensional and the structure group of the tangent bundle reduces to the group $O(n) \times O(n)$. Note that

$$h(P(X), Y) = h(X, P(Y)).$$

The structure tensor F of type $(0,3)$ on L is defined as follows:

$$F(X,Y,Z) = h\left(\left(\nabla_X P\right)(Y), Z\right).$$

The tensor F has the following properties:

$$F(X,Y,Z) = F(X,Z,Y) = -F(X,P(Y),P(Z)),$$
$$F\left(X,Y,P(Z)\right) = -F(X,P(Y),Z).$$

In addition, for any vector field X on L, the 1-form $\tilde{\theta}$ associated with F is defined as follows:

$$\tilde{\theta}(X) = h^{ij}F(E_i,E_j,X),$$

where $\{E_1,E_2,\cdots E_{2n}\}$ is a local frame field on L, and (h^{ij}) is the inverse matrix of (h_{ij}).

Then, the subspace W of $\otimes_3^0 L$ is defined as follows:

$$W := \left\{F \in \otimes_3^0 L \,|F(X,Y,Z) = F(X,Z,Y) = -F(X,P(Y),P(Z))\; F(X,Y,P(Z)) = -F(X,P(Y),Z)\right\}.$$

According to the symmetries of W, this space splits into the direct sum $W = W_1 \oplus W_2 \oplus W_3$. The subspaces W_i are invariant and irreducible under the group $O(n) \times O(n)$. The defining relations for invariant subspaces are as follows:

(1) Riemannian P-manifolds:

$$F(X,Y,Z) = 0.$$

(2) Class W_1:

$$F(X,Y,Z) = \frac{1}{2n}\Big\{h(X,Y)\tilde{\theta}(Z) + h(X,Z)\tilde{\theta}(Y) -h(X,P(Y))\tilde{\theta}(P(Z)) - h(X,P(Z))\tilde{\theta}(P(Y))\Big\}. \tag{2.3}$$

(3) Class W_2:

$$F(X,Y,P(Z)) + F(Y,Z,P(X)) + F(Z,X,P(Y)) = 0 \tag{2.4}$$

and

$$\tilde{\theta} = 0,$$

or equivalently,

$$[P,P] = 0 \text{ and } \tilde{\theta} = 0.$$

(4) Class W_3:

$$F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0. \tag{2.5}$$

(5) Class $W_1 \oplus W_2$ (The class of integrable Riemannian almost product manifolds with $trP = 0$):

$$F(X,Y,P(Z)) + F(Y,Z,P(X)) + F(Z,X,P(Y)) = 0,$$

or equivalently,

$$[P,P] = 0.$$

(6) Class $W_2 \oplus W_3$:

$$\tilde{\theta} = 0.$$

(7) Class $W_1 \oplus W_3$:

$$F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = \frac{1}{n}\Big\{h(X,Y)\tilde{\theta}(Z) + h(X,Z)\tilde{\theta}(Y) + h(Y,Z)\tilde{\theta}(X) - h(X,P(Y))\tilde{\theta}(P(Z)) - h(X,P(Z))\tilde{\theta}(P(Y)) - h(Y,P(Z))\tilde{\theta}(P(X))\Big\}.$$

(8) Class $W_1 \oplus W_2 \oplus W_3$: no condition.

Note that $[P,P]$ denotes the Nijenhuis tensor of the almost paracomplex structure P and is defined by the following:

$$[P,P](X,Y) = [PX,PY] + [X,Y] - P[PX,Y] - P[X,PY] \tag{2.6}$$

for all vector fields X, Y on L, see [3].

It is well known that if (M,φ,ξ,η) is an almost paracontact paracomplex Riemannian manifold, then $M \times \mathbb{R}$ is canonically an almost product manifold with the following almost paracomplex structure:

$$P(X, a\frac{d}{dt}) = \left(\varphi(X) + \frac{a}{t}\xi, t\eta(X)\frac{d}{dt}\right), \tag{2.7}$$

where a is a function on $M \times \mathbb{R}$, and t is the coordinate on $\mathbb{R}$, see [1,6]. P has the following properties:

$$P^2 = I$$

and

$$trP = 0.$$

An almost paracontact paracomplex Riemannian manifold (M,φ,ξ,η) is called normal if the corresponding almost product structure (2.7) on the even dimensional product manifold $M \times \mathbb{R}$ is integrable. i.e., the Nijenhuis tensor $[P,P]$ of the almost product structure P is identically zero. Additionally, it is known that the vanishing of $[P,P]$ is equivalent to the vanishing of the Nijenhuis tensor N of the structure (φ,ξ,η), where

$$N = [\varphi,\varphi] - d\eta \otimes \xi \tag{2.8}$$

and the Nijenhuis torsion $[\varphi,\varphi]$ of φ is given by

$$[\varphi,\varphi](X,Y) = [\varphi X,\varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] \tag{2.9}$$

and

$$d\eta(X,Y) = -\alpha(X,\varphi Y,\xi) + \alpha(Y,\varphi X,\xi);$$

refer to [6]. In addition, if $N = 0$, then the following tensors are also zero [1]:

$$N^{(2)}(X,Y) = \left(\mathcal{L}_{\varphi(X)}\eta\right)(Y) - \left(\mathcal{L}_{\varphi(Y)}\eta\right)(X) = 0, \tag{2.10}$$

$$N^{(3)}(X) = \left(\mathcal{L}_\xi \varphi\right)(X) = 0, \tag{2.11}$$

$$N^{(4)}(X) = \left(\mathcal{L}_\xi \eta\right)(X) = 0. \tag{2.12}$$

The [6, Theorem 5.1] states that an almost paracontact paracomplex Riemannian manifold is normal if and only if the manifold is in $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ [7]; the relations of these subspaces are defined as follows:

$$\begin{aligned}
\mathcal{F}_1 : \alpha(X, Y, Z) = \frac{1}{2n} &\left\{ g(\varphi(X), \varphi(Y))\theta(\varphi^2(Z)) + g(\varphi(X), \varphi(Z))\theta(\varphi^2(Y)) \right. \\
&\left. -g(X, \varphi(Y))\theta(\varphi(Z)) - g(X, \varphi(Z))\theta(\varphi(Y)) \right\}, \quad (2.13)\\
\mathcal{F}_2 : \alpha(\xi, Y, Z) &= \alpha(X, \xi, Z) = 0, \quad \theta = 0, \quad (2.14)\\
\alpha(X, Y, \varphi(Z)) &+ \alpha(Y, Z, \varphi(X)) + \alpha(Z, X, \varphi(Y)) = 0,\\
\mathcal{F}_4 : \alpha(X, Y, Z) &= \frac{\theta(\xi)}{2n} \left\{ g(\varphi(X), \varphi(Y))\eta(Z) + g(\varphi(X), \varphi(Z))\eta(Y) \right\}, \quad (2.15)\\
\mathcal{F}_5 : \alpha(X, Y, Z) &= \frac{\theta^*(\xi)}{2n} \left\{ g(X, \varphi(Y))\eta(Z) + g(X, \varphi(Z))\eta(Y) \right\} \quad (2.16)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_6 : \alpha(X, Y, Z) &= \alpha(Z, X, \xi)\eta(Y) + \alpha(Y, X, \xi)\eta(Z) - 2\eta(Y)\eta(Z)\alpha(\xi, \xi, X)\\
&= \eta(Y)\alpha(\varphi(X), \xi, \varphi(Z)) + \eta(Z)\alpha(\varphi(X), \xi, \varphi(Y)),\\
\theta(\xi) &= \theta^*(\xi) = 0,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\alpha(X, Y, Z) &= \alpha(X, Y, \xi)\eta(Z) + \alpha(X, Z, \xi)\eta(Y),\\
\alpha(X, Y, \xi) &= \alpha(Y, X, \xi) = \alpha(\varphi(X), \varphi(Y), \xi),\\
\theta &= \theta^* = 0.
\end{aligned}$$

In Example 3.9, we show that [6, Theorem 5.1] is not an if and only if statement. If the structure belongs to $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$, then it is normal. By Example 3.9, there exist structures which are normal but do not belong to $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$.

3. Almost product manifolds from almost paracontact paracomplex Riemannian manifolds

In this section, we first define an almost paracomplex (almost product) structure $\tilde{P}$ on the product of an almost paracontact paracomplex Riemannian manifold with $\mathbb{R}$ by a warped product. Note that $\tilde{P}$ is different than the canonical almost product structure P in (2.7). The structure (2.7) depends on t, where t is the coordinate of $\mathbb{R}$. The new almost product structure $\tilde{P}$ depends on any function σ of t. Then, we write a Riemannian metric on the product manifold depending on a function σ, where

$$\sigma : \mathbb{R} \to \mathbb{R}$$

is an arbitrary function on $\mathbb{R}$; in this way, we obtain an almost product Riemannian manifold whose structure tensor $\tilde{P}$ has a trace zero and we give the relations between the covariant derivatives.

Additionally, we investigate the curvature properties of $M \times \mathbb{R}$. Then, we consider the normal almost paracontact paracomplex Riemannian manifolds and we show that a paracontact paracomplex Riemannian manifold is normal if and only if the almost product Riemannian manifold defined by our warped product is integrable.

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost paracontact paracomplex Riemannian manifold and consider the product manifold $M \times \mathbb{R}$. A vector field on the manifold $M \times \mathbb{R}$ is of the form $\left(X, a\frac{d}{dt}\right)$, where t is the coordinate of $\mathbb{R}$, and a is a smooth function on $M \times \mathbb{R}$. The almost paracomplex structure $\tilde{P}$ on $M \times \mathbb{R}$ is defined by the following:

$$\tilde{P}\left(X, a\frac{d}{dt}\right) = \left(\varphi(X) + ae^{-\sigma}\xi, e^{\sigma}\eta(X)\frac{d}{dt}\right), \tag{3.1}$$

where σ is any function of t. One can easily see that

$$\tilde{P}^2 = I.$$

In addition, we define a Riemannian metric h on $M \times \mathbb{R}$ by the following:

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := e^{2\sigma}g(X, Y) + ab.$$

Then we have

$$h\left(\tilde{P}\left(X, a\frac{d}{dt}\right), \tilde{P}\left(Y, b\frac{d}{dt}\right)\right) = h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right)$$

and

$$tr\tilde{P} = 0.$$

Hence, $(M \times \mathbb{R}, \tilde{P}, h)$ is an almost product Riemannian manifold with

$$tr\tilde{P} = 0.$$

Let ∇ denote the Levi-Civita covariant derivatives of both Riemannian metrics g on M and h on $M \times \mathbb{R}$. The Levi-Civita covariant derivative of the metric h on $M \times \mathbb{R}$ is obtained by using the Koszul formula as follows:

$$\nabla_{\left(X, a\frac{d}{dt}\right)}\left(Y, b\frac{d}{dt}\right) = \left(\nabla_X Y + \frac{d\sigma}{dt}(aY + bX), \left\{X[b] + a\frac{db}{dt} - e^{2\sigma}\frac{d\sigma}{dt}g(X, Y)\right\}\frac{d}{dt}\right).$$

Additionally, the covariant derivative of the almost product structure $\tilde{P}$ is calculated as follows:

$$\begin{aligned}\left(\nabla_{\left(X, a\frac{d}{dt}\right)}\tilde{P}\right)\left(Y, b\frac{d}{dt}\right) &= \Big((\nabla_X\varphi)(Y) + be^{-\sigma}\nabla_X\xi - b\tfrac{d\sigma}{dt}\varphi(X) + e^{\sigma}\tfrac{d\sigma}{dt}(\eta(Y)X + g(X, Y)\xi)\,a \\ &\quad \left\{e^{\sigma}(\nabla_X\eta)(Y) - e^{2\sigma}\tfrac{d\sigma}{dt}g(X, \varphi(Y)) - 2be^{\sigma}\tfrac{d\sigma}{dt}\eta(X)\right\}\tfrac{d}{dt}\Big),\end{aligned}$$

for any vector fields $\left(X, a\frac{d}{dt}\right)$, $\left(Y, b\frac{d}{dt}\right)$, and $\left(Z, c\frac{d}{dt}\right)$ on $M \times \mathbb{R}$. Unless otherwise stated, throughout the paper, we will use the notation $\tilde{X}, \tilde{Y}, \tilde{Z}, ...$ for the vector fields

$$(X, a\frac{d}{dt}), (Y, b\frac{d}{dt}), (Z, c\frac{d}{dt}), ...$$

on the product manifold, respectively. It follows that

$$\begin{aligned} F\left(\tilde{X},\tilde{Y},\tilde{Z}\right) &= h\left(\left(\nabla_{\tilde{X}}\tilde{P}\right)(\tilde{Y}),\tilde{Z}\right) \\ &= e^{2\sigma}\alpha(X,Y,Z) - 2bce^{\sigma}\frac{d\sigma}{dt}\eta(X) + e^{\sigma}\left\{bg\left(\nabla_X\xi,Z\right) + cg\left(\nabla_X\xi,Y\right)\right\} \\ &\quad + e^{3\sigma}\frac{d\sigma}{dt}\left\{\eta(Y)g(X,Z) + \eta(Z)g(X,Y)\right\} - e^{2\sigma}\frac{d\sigma}{dt}\left\{bg(X,\varphi(Z)) + cg(X,\varphi(Y)\right\}. \end{aligned} \tag{3.2}$$

By choosing

$$\tilde{X} = (\xi,0), \quad \tilde{Y} = \tilde{Z} = \left(0,\frac{d}{dt}\right),$$

we obtain

$$F(\tilde{X},\tilde{Y},\tilde{Z}) = -2e^{\sigma}\frac{d\sigma}{dt} \neq 0.$$

Therefore, the almost product Riemannian structure obtained is nontrivial for any non-constant function σ.

Since

$$\nabla_{(X,a\frac{d}{dt})}(\xi,0) = \left(\nabla_X\xi + \frac{d\sigma}{dt}a\xi, -e^{2\sigma}\frac{d\sigma}{dt}\eta(X)\frac{d}{dt}\right),$$

if ξ is parallel, then $(\xi,0)$ is not parallel for any non-constant function σ. In addition, if ξ is Killing, then $(\xi,0)$ is also Killing:

$$\begin{aligned} h\left(\nabla_{(X,a\frac{d}{dt})}(\xi,0),(Y,b\frac{d}{dt})\right) &= e^{2\sigma}g(\nabla_X\xi,Y) + ae^{2\sigma}\frac{d\sigma}{dt}\eta(Y) - be^{2\sigma}\frac{d\sigma}{dt}\eta(Y) \\ &= -h\left(\nabla_{(Y,b\frac{d}{dt})}(\xi,0),(X,a\frac{d}{dt})\right). \end{aligned}$$

Moreover, note that

$$\begin{aligned} h\left(\nabla_{(X,a\frac{d}{dt})}\left(0,\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right) &= e^{2\sigma}\frac{d\sigma}{dt}g(X,Y) \\ &= g\left(\nabla_{(Y,b\frac{d}{dt})}\left(0,\frac{d}{dt}\right),\left(X,a\frac{d}{dt}\right)\right). \end{aligned}$$

Let $\{e_1,\cdots,e_{2n},\xi\}$ be a local orthonormal frame field on M. Then, one can obtain an orthonormal frame field on $M\times\mathbb{R}$ as follows:

$$\left\{\left(e^{-\sigma}e_1,0\right),\cdots,\left(e^{-\sigma}e_{2n},0\right),\left(e^{-\sigma}\xi,0\right),\left(0,\frac{d}{dt}\right)\right\}.$$

Using this frame, the 1-form $\tilde{\theta}$, associated with F is as follows:

$$\tilde{\theta}(X,a\frac{d}{dt}) = \theta(X) - ae^{-\sigma}\theta^*(\xi) + \omega(X) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X). \tag{3.3}$$

In addition, we write the curvature tensor $\widetilde{R}$ on the product manifold $M\times\mathbb{R}$ in terms of the curvature tensor R on M as follows:

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \left(R(X,Y)Z + c\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right)(aY - bX)\right.$$

$$- e^{2\sigma}\left(\frac{d\sigma}{dt}\right)^2 (g(Y,Z)X - g(X,Z)Y)\,,\ e^{2\sigma}\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right) g(bX - aY, Z)\frac{d}{dt}\right).$$

Then, the Ricci curvature can be calculated as follows:

$$\begin{aligned}\tilde{Q}(\tilde{X}, \tilde{Y}) = Q(X, Y) - ab(2n+1)\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right)\\ - e^{2\sigma}\left(\frac{d\sigma}{dt}\right)^2 (2n+1)g(X, Y) - e^{2\sigma}\frac{d^2\sigma}{dt^2} g(X, Y).\end{aligned}$$

In addition, we can evaluate the scalar curvature as follows:

$$\tilde{s} = e^{-2\sigma} s - (2n+1)(2n+2)\left(\frac{d\sigma}{dt}\right)^2 - 2(2n+1)\frac{d^2\sigma}{dt^2}. \tag{3.4}$$

Let M be an almost paracontact paracomplex Riemannian manifold with a zero scalar curvature. Then, we can construct an almost product Riemannian manifold with a scalar curvature as follows:

$$\tilde{s} = k > 0,$$

where k is a positive real number, with the appropriate choice of the function σ. For $s = 0$, the Eq (3.4) becomes the following:

$$k = -(2n+1)(2n+2)\left(\frac{d\sigma}{dt}\right)^2 - 2(2n+1)\frac{d^2\sigma}{dt^2}. \tag{3.5}$$

The solution of the differential Eq (3.5) is as follows:

$$\sigma(t) = \frac{1}{n+1}\ln\left[\cos\left(\frac{\sqrt{k}}{\sqrt{2}}\left(\frac{\sqrt{n+1}}{\sqrt{2n+1}}t - 2\sqrt{(2n+1)(n+1)}c_1\right)\right)\right] + c_2,$$

where $c_1, c_2 \in \mathbb{R}$. For example, if k and c_1 are chosen so that

$$-\frac{\pi}{2} < \frac{\sqrt{k}}{\sqrt{2}}\left(\frac{\sqrt{n+1}}{\sqrt{2n+1}}t - 2\sqrt{(2n+1)(n+1)}c_1\right) < \frac{\pi}{2}, \tag{3.6}$$

then the product manifold $M \times (t_0, t_1)$ has the positive scalar curvature k, where

$$t_0 = \frac{\sqrt{2}}{\sqrt{k}}\left(-\frac{\pi\sqrt{2n+1}}{2\sqrt{n+1}} + 2c_1(2n+1)\right),$$

$$t_1 = \frac{\sqrt{2}}{\sqrt{k}}\left(\frac{\pi\sqrt{2n+1}}{2\sqrt{n+1}} + 2c_1(2n+1)\right).$$

Thus, from any almost paracontact paracomplex Riemannian manifold with scalar curvature $s = 0$, it is possible to construct an almost product Riemannian manifold with any constant positive scalar curvature

$$\tilde{s} = k.$$

We illustrate this result by the following example.

Example 3.1. *Consider the Lie group G of dimension* 5 *with a basis of left-invariant vector fields* $\{e_1, e_2, e_3, e_4, e_5\}$ *defined by the following non-zero brackets:*

$$[e_1, e_5] = e_2, \ \ [e_2, e_5] = -e_1, \ \ [e_3, e_5] = e_4, \ \ [e_4, e_5] = -e_3.$$

One can define an invariant almost paracontact paracomplex Riemannian structure on G as follows:

$$\begin{aligned} & g(e_i, e_i) = 1, \\ & g(e_i, e_j) = 0, \ \ i \neq j, \\ & \quad e_5 = \xi, \ \ \varphi(e_1) = e_3, \ \ \varphi(e_3) = e_1, \ \ \varphi(e_2) = e_4, \ \ \varphi(e_4) = e_2. \end{aligned}$$

The nonzero Levi-Civita covariant derivatives are as follows:

$$\nabla_{e_5} e_1 = -e_2, \ \ \nabla_{e_5} e_2 = e_1, \ \ \nabla_{e_5} e_3 = -e_4, \ \ \nabla_{e_5} e_4 = e_3.$$

This structure is cosymplectic and the scalar curvature is $s = 0$, *as shown by Example 3.10. Let us choose* $k = 1$ *and* $c_1 = 0$ *in the solution* $\sigma(t)$ *of the differential Eq (3.5). Then, the product manifold* $G \times (t_0, t_1)$ *has a positive scalar curvature* $k = 1$, *where*

$$t_0 = -\frac{\pi}{2} \frac{\sqrt{10}}{\sqrt{3}}$$

and

$$t_1 = \frac{\pi}{2} \frac{\sqrt{10}}{\sqrt{3}}$$

since Eq (3.6) holds and

$$\sigma(t) = \frac{1}{3} \ln \left(\cos \frac{\sqrt{3}}{\sqrt{10}} t \right)$$

is a solution to (3.5).

It is possible to obtain Einstein almost product Riemannian manifolds from Einstein almost paracontact paracomplex Riemannian manifolds by appropriately choosing the function σ . If the almost paracontact paracomplex Riemannian manifold M is Einstein with an Einstein constant λ, that is if

$$Q(X, Y) = \lambda g(X, Y),$$

then the almost product manifold $M \times \mathbb{R}$ is Einstein if and only if

$$-\frac{\lambda}{2n} = e^{2\sigma} \frac{d^2 \sigma}{dt^2}. \tag{3.7}$$

In this case, the Einstein constant K of the product manifold is

$$K = -(2n + 1) \left(\left(\frac{d\sigma}{dt} \right)^2 + \frac{d^2 \sigma}{dt^2} \right),$$

that is,

$$\tilde{Q}(\tilde{X}, \tilde{Y}) = K h(\tilde{X}, \tilde{Y}).$$

The differential Eq (3.7) has the following solution:

$$\sigma(t) = \ln\left(-\frac{1}{2}e^{-\sqrt{c_1}(t+c_2)}\lambda + \frac{e^{\sqrt{c_1}(t+c_2)}}{4nc_1}\right),$$

where $c_1, c_2 \in \mathbb{R}$, $c_1 > 0$, and

$$K = -(2n+1)c_1,$$

since

$$\left(\left(\frac{d\sigma}{dt}\right)^2 + \frac{d^2\sigma}{dt^2}\right) = c_1.$$

If λ is negative, then the function σ is defined for all real numbers. Hence, the product manifold $M \times \mathbb{R}$ is an Einstein manifold with a negative Einstein constant:

$$K = -(2n+1)c_1 < 0.$$

In addition, if λ is positive, then it can easily be seen that the domain of the function σ is (t_0, ∞), where

$$t_0 = \frac{\ln(2nc_1\lambda)}{2\sqrt{c_1}} - c_2,$$

and the product manifold $M \times (t_0, \infty)$ is Einstein with an Einstein constant

$$K = -(2n+1)c_1 < 0.$$

Now, we will investigate the relation between the normal almost paracontact paracomplex Riemannian manifolds $(M, \varphi, \xi, \eta, g)$ and integrable almost product Riemannian manifolds $(M \times \mathbb{R}, \tilde{P}, h)$ (that is, $[\tilde{P}, \tilde{P}] = 0$). We calculate the Nijenhuis tensor $[\tilde{P}, \tilde{P}]$ of $\tilde{P}$ on $M \times \mathbb{R}$ as follows:

$$\begin{aligned}
[\tilde{P}, \tilde{P}](\tilde{X}, \tilde{Y}) = \Big(& [\varphi, \varphi](X, Y) - (d\eta)(X, Y)\xi - be^{-\sigma}\left(\mathcal{L}_\xi\varphi\right)(X) + ae^{-\sigma}\left(\mathcal{L}_\xi\varphi\right)(Y)\,, \\
& \left\{e^{\sigma}\left(\mathcal{L}_{\varphi(X)}\eta\right)(Y) - e^{\sigma}\left(\mathcal{L}_{\varphi(Y)}\eta\right)(X) + a\left(\mathcal{L}_\xi\eta\right)(Y) - b\left(\mathcal{L}_\xi\eta\right)(Y)\right\}\frac{d}{dt}\Big),
\end{aligned} \tag{3.8}$$

where

$$\tilde{X} = \left(X, a\frac{d}{dt}\right)$$

and

$$\tilde{Y} = \left(Y, b\frac{d}{dt}\right).$$

If M is normal, then the Nijenhuis tensor N in Eq (2.9) vanishes; thus, Eqs (2.10)–(2.12) hold. This yields that

$$[\tilde{P}, \tilde{P}] = 0,$$

that is, $M \times \mathbb{R}$ is integrable.

Conversely, if $M \times \mathbb{R}$ is integrable, then the identity (3.8) vanishes for all $\tilde{X}, \tilde{Y}$. In particular, we choose $\tilde{X}, \tilde{Y}$ so that

$$a = b = 0.$$

Then we obtain $N = 0$, which implies that

$$N^2 = N^3 = N^4 = 0,$$

and as a result, M is normal. Hence, the manifold M is normal if and only if the product manifold $M \times \mathbb{R}$ is integrable (the class $W_1 \oplus W_2$).

It is known that the classes $\mathcal{F}_1$–$\mathcal{F}_6$ are normal, that is, the Nijenhuis tensor of the almost paracontact paracomplex Riemannian manifold vanishes for these classes [6]. Thus, the product manifold is either in $W_1 \oplus W_2$ if M belongs to $\mathcal{F}_1$–$\mathcal{F}_6$ or a direct sum of these classes. Now, we prove that the product manifold $M \times \mathbb{R}$ does not belong to the subclasses W_1 or W_2 of $W_1 \oplus W_2$, except if M is in $\mathcal{F}_4$.

Theorem 3.2. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_1$, then the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$ and not in the subclasses W_1 or W_2 for any non-constant function σ.*

Proof. From the definition (2.13) of $\mathcal{F}_1$, we have $\omega = 0$ and $\theta^*(\xi) = 0$; thus,

$$\tilde{\theta}(\tilde{X}) = \theta(X) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X).$$

For

$$\tilde{X} = (\xi, 0),$$

we have

$$\tilde{\theta}(\xi, 0) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}.$$

Hence, $\tilde{\theta}$ is not equal to zero for any non-constant function σ. Therefore, the product manifold $M \times \mathbb{R}$ is not in the class W_2.

Additionally, the defining relation (2.3) of the class W_1 does not hold. Take

$$\tilde{X} = \left(0, \frac{d}{dt}\right), \quad \tilde{Y} = (\xi, 0) \quad \text{and} \quad \tilde{Z} = (Z, 0)$$

in Eq (2.3). By the Eq (3.2), the left hand side of (2.3) becomes

$$F(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0,$$

whereas the right hand side is

$$\frac{1}{(n+1)}e^{\sigma}\theta(\varphi(Z)),$$

which need not be zero. Thus, the product manifold $M \times \mathbb{R}$ is not in the class W_1. □

Theorem 3.3. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_2$, then the product manifold $M \times \mathbb{R}$ is of the class $W_1 \oplus W_2$ and not in the subclasses W_1 or W_2 for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_2$, by the defining relation (2.14), we have the following:

$$\theta(X) = 0, \quad \omega(X) = 0$$

for all vector fields X on M, and ξ is parallel [2]. Then, by (3.3),

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X) \neq 0.$$

Hence, $M \times \mathbb{R}$ is not in W_2.

Take

$$\tilde{X} = (\xi, 0), \quad \tilde{Y} = \tilde{Z} = \left(0, \frac{d}{dt}\right)$$

in Eq (2.3). The left hand side of Eq (2.3) is

$$F(\tilde{X}, \tilde{Y}, \tilde{Z}) = -2e^{\sigma}\frac{d\sigma}{dt},$$

and the right hand side of the Eq (2.3) is

$$-2e^{\sigma}\frac{d\sigma}{dt}\frac{n+1}{n},$$

which is different than $F(\tilde{X}, \tilde{Y}, \tilde{Z})$. Thus, the defining relation (2.3) of W_1 does not hold, so $M \times \mathbb{R}$ does not belong to W_1. □

Theorem 3.4. *Let $(M, \varphi, \xi, \eta, g)$ be of the class $\mathcal{F}_4$. If*

$$\sigma(t) = \ln(c_1 - \frac{\theta(\xi)t}{2(n+1)}),$$

then either the product manifold $M \times (-\infty, t_0)$ or $M \times (t_0, \infty)$ belongs to W_2, where

$$t_0 = \frac{2(n+1)c_1}{\theta(\xi)}.$$

Otherwise, $M \times \mathbb{R}$ is in $W_1 \oplus W_2$ and not in the subclasses W_1, W_2 for any non-constant function σ.

Proof. Since $\mathcal{F}_4$ is a sublass of normal manifolds [6], the product manifold is in $W_1 \oplus W_2$, and the defining relation of $W_1 \oplus W_2$ holds. By the defining relation (2.15) of $\mathcal{F}_4$, we have $\omega = 0$, $\theta^*(\xi) = 0$ (since $tr\varphi = 0$), and

$$\theta(X) = \theta(\xi)\eta(X).$$

Thus, from (3.3),

$$\tilde{\theta}(\tilde{X}) = \left(\theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\right)\eta(X).$$

If σ satisfies the differential equation

$$\theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt} = 0, \tag{3.9}$$

then $\tilde{\theta} = 0$.

Note that the differential Eq (3.9) has the solution

$$\sigma(t) = \ln(c_1 - \frac{\theta(\xi)t}{2(n+1)}), \tag{3.10}$$

and $\sigma(t)$ is defined for all t such that

$$c_1 - \frac{\theta(\xi)t}{2(n+1)} > 0.$$

If $\theta(\xi) > 0$,then the function $\sigma(t)$ is defined for all t such that

$$t < \frac{2(n+1)c_1}{\theta(\xi)}.$$

As a result, $M \times (-\infty, t_0)$, where

$$t_0 = \frac{2(n+1)c_1}{\theta(\xi)}$$

is in the class W_2.

If $\theta(\xi) < 0$, then the condition

$$c_1 - \frac{\theta(\xi)t}{2(n+1)} > 0$$

implies that

$$t > \frac{2(n+1)c_1}{\theta(\xi)}.$$

Thus, the product manifold $M \times (t_0, \infty)$, where

$$t_0 = \frac{2(n+1)c_1}{\theta(\xi)}$$

is in W_2.

To sum up, if the differential Eq (3.9) is satisfied, that is, if σ is the function given in Eq (3.10), then depending of the sign of $\theta(\xi)$, either $M \times (-\infty, t_0)$ or $M \times (t_0, \infty)$ belong to W_2, where

$$t_0 = \frac{2(n+1)c_1}{\theta(\xi)}.$$

If $\sigma(t)$ is not a solution of (3.9), then $\tilde{\theta} \neq 0$, and $M \times \mathbb{R}$ is not in W_2.

Now, we show that the product manifold is not in W_1. Assume that the defining relation of W_1 holds for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$. In particular, choose

$$\tilde{X} = (\xi, 0), \quad \tilde{Y} = (Y, 0) \quad \text{and} \quad \tilde{Z} = (Z, 0).$$

From (3.2), the left hand side of the defining relation (2.3) of W_1 is

$$F(\tilde{X}, \tilde{Y}, \tilde{Z}) = e^{2\sigma}\alpha(\xi, Y, Z) + e^{3\sigma}\frac{d\sigma}{dt}2\eta(Y)\eta(Z) \tag{3.11}$$

and the right hand side is

$$\begin{aligned} &\frac{1}{2(n+1)}\left\{h(X,Y)\tilde{\theta}(Z) + h(X,Z)\tilde{\theta}(Y) - h(X,P(Y))\tilde{\theta}(P(Z)) - h(X,P(Z))\tilde{\theta}(P(Y))\right\} \\ &= \frac{1}{(n+1)}\left\{e^{2\sigma}\eta(Y)\eta(Z)\left(\theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\right)\right\}. \end{aligned} \tag{3.12}$$

Comparing (3.11) and (3.12), and noting also that

$$\alpha(\xi, Y, Z) = 0$$

in $\mathcal{F}_4$, we obtain

$$0 = \frac{\theta(\xi)}{n+1}\eta(Y)\eta(Z),$$

which is a contradiction since $\theta(\xi) \neq 0$ in $\mathcal{F}_4$. As a result $M \times \mathbb{R}$ is not in W_1. □

Theorem 3.5. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_5$, then the product manifold $M\times\mathbb{R}$ is of the class $W_1 \oplus W_2$ and not in the subclasses W_1 or W_2 for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_5$, from (2.16), we have the following:

$$\theta(X) = 0, \quad \theta^*(X) = \theta^*(\xi)\eta(X), \quad \omega(X) = 0$$

for all vector fields X on M. Therefore, by (3.3),

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = -e^{-\sigma}a\theta^*(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X) \neq 0,$$

since

$$\tilde{\theta}\left(0, \frac{d}{dt}\right) = -e^{-\sigma}\theta^*(\xi).$$

Thus, $M \times \mathbb{R}$ is not in W_2. Taking

$$\tilde{X} = \left(0, \frac{d}{dt}\right), \quad \tilde{Y} = (\xi, 0) \quad \text{and} \quad \tilde{Z} = (\xi, 0)$$

in the defining relation (2.3) of the class W_1, we have

$$\theta^*(\xi) = 0.$$

This is a contradiction since

$$\theta^*(\xi) \neq 0$$

for a nontrivial structure in the class $\mathcal{F}_5$ for any non-constant function σ. Hence, the product manifold $M \times \mathbb{R}$ is not in the class W_1. □

Theorem 3.6. *If $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_6$, then the product manifold $M\times\mathbb{R}$ is of the class $W_1 \oplus W_2$ and not in the subclasses W_1 or W_2 for all non-constant σ functions.*

Proof. Since $(M, \varphi, \xi, \eta, g)$ is of the class $\mathcal{F}_6$, by the defining relation of this class, we have the following:

$$\theta(\xi) = 0, \quad \theta^*(\xi) = 0, \quad \omega(X) = 0$$

for all vector fields X on M [7]. Then, we have

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = \theta(X) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X) \neq 0$$

since, for instance,

$$\tilde{\theta}(\xi, 0) = 2(n+1)e^{\sigma}\frac{d\sigma}{dt}$$

is not equal to zero for any non-constant function σ. Thus, the product manifold is not in W_2. Take

$$\tilde{X} = \left(0, \frac{d}{dt}\right), \quad \tilde{Y} = (Y, 0) \quad \text{and} \quad \tilde{Z} = \left(0, \frac{d}{dt}\right)$$

in the defining relation (2.3) of the class W_1, we have the following:

$$e^{\sigma}\frac{d\sigma}{dt}\eta(Y) = 0.$$

This is a contradiction for a non-constant function σ. □

Now, we investigate the class of the almost paracontact paracomplex Riemannian structure if the class of the almost product Riemannian structure on $M \times \mathbb{R}$ is given.

Theorem 3.7. *If the product manifold $M \times \mathbb{R}$ is of the class W_1, then the almost paracontact almost paracomplex manifold is cosymplectic.*

Proof. If the product manifold $M \times \mathbb{R}$ is of the class W_1, the defining relation (2.3) is satisfied for all vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$. In Eq (2.3), taking

$$\tilde{X} = \left(0, \frac{d}{dt}\right)$$

and

$$\tilde{Y} = \tilde{Z} = (\xi, 0),$$

we obtain

$$\theta^*(\xi) = 0.$$

For

$$\tilde{X} = \left(0, \frac{d}{dt}\right), \quad \tilde{Y} = (\xi, 0) \quad \text{and} \quad \tilde{Z} = (Z, 0),$$

we have

$$\theta(\varphi(Z)) = -\omega(\varphi(Z));$$

hence,

$$\theta(Z) + \omega(Z) = \eta(Z)\theta(\xi).$$

Then, by choosing

$$\tilde{X} = (X, 0), \quad \tilde{Y} = (Y, 0) \quad \text{and} \quad \tilde{Z} = (Z, 0),$$

we obtain

$$\alpha(X, Y, Z) = \frac{1}{2(n+1)}\theta(\xi)\{\eta(Y)g(X, Z) + \eta(Z)g(X, Y)\}. \tag{3.13}$$

Now, since

$$\alpha(X, \xi, \xi) = 0$$

for any almost paracontact paracomplex structure, by Eq (3.13), we have the following:

$$\alpha(X,\xi,\xi) = \frac{1}{(n+1)}\theta(\xi)\eta(X) = 0,$$

which implies $\theta(\xi) = 0$. Thus, $\alpha = 0$. □

Theorem 3.8. *If the product manifold $M \times \mathbb{R}$ is of the class W_2, then the almost paracontact almost paracomplex manifold is of the class $\mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_4 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$.*

Proof. Since the product manifold $M \times \mathbb{R}$ is of the class W_2,

$$\tilde{\theta}(X, a\frac{d}{dt}) = 0$$

for all vector fields $(X, a\frac{d}{dt})$. From the Eq (3.3),

$$\tilde{\theta}\left(0, \frac{d}{dt}\right) = -e^{-\sigma}\theta^*(\xi) = 0,$$

which implies

$$\theta^*(\xi) = 0$$

and

$$\tilde{\theta}(\xi, 0) = \theta(\xi) + 2(n+1)e^{\sigma}\frac{d\sigma}{dt} = 0.$$

By choosing

$$\tilde{X} = \left(0, \frac{d}{dt}\right), \quad \tilde{Y} = (Y, 0) \quad \text{and} \quad \tilde{Z} = (Z, 0)$$

in Eq (2.4) and using (3.2), we have the following:

$$\alpha(Y, Z, \xi) = \alpha(Z, Y, \xi). \tag{3.14}$$

Then,

$$0 = \alpha(X,\xi,\xi) = \alpha(\xi, X, \xi) = \alpha(\xi,\xi,X) = \omega(X);$$

thus

$$\nabla_\xi \xi = 0.$$

Additionally, we have

$$0 = \tilde{\theta}\left(X, \frac{d}{dt}\right) = \theta(X) + (2n+1)e^{\sigma}\frac{d\sigma}{dt}\eta(X),$$

and

$$\theta(\varphi(X)) = 0.$$

Moreover, by taking

$$\tilde{X} = (X, 0), \quad \tilde{Y} = (Y, 0), \quad \tilde{Z} = (Z, 0)$$

in Eq (2.4), from (3.2), we obtain the following:

$$\mathfrak{S}_{XYZ}\{\alpha(X, Y, \varphi(Z) - \alpha(X, \varphi(Y), \xi)\eta(Z)\} = 0.$$

Now, we evaluate the projections α^i to determine the class of the almost paracontact paracomplex structure.

Since
$$\theta(\varphi(X)) = 0, \quad \theta^*(\xi) = 0 \quad \text{and} \quad \omega(X) = 0,$$
projections α^1, α^5, and α^{11} vanish, respectively. From (3.14), we have the following:
$$\alpha^7 = \alpha^9 = 0.$$

Note that, the remaining projections need not vanish. For example, by using
$$\omega(X) = 0,$$
we have
$$\alpha^{10}(X, Y, Z) = \eta(X)\alpha(\xi, \varphi^2(Y), \varphi^2(Z)) = \eta(X)\alpha(\xi, Y, Z),$$
which is zero if and only if
$$\nabla_\xi Y = 0.$$

□

We finish by giving examples of almost paracontact paracomplex Riemannian structures in certain classes. Example 3.9 shows that a normal almost paracontact paracomplex Riemannian manifold is not necessarily in $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$, contrary to [6, Theorem 5.1].

Example 3.9. *Consider the five dimensional Lie algebra with basis elements* $\{e_1, e_2, \ldots, e_5\}$ *whose nonzero brackets are*
$$[e_1, e_2] = e_5, \quad [e_3, e_4] = e_5.$$
Let g be the Riemannian metric such that the basis elements are orthonormal. Nonzero covariant derivatives are evaluated by Koszul's formula in [8]:
$$\nabla_{e_1} e_2 = \frac{1}{2} e_5, \quad \nabla_{e_1} e_5 = -\frac{1}{2} e_2,$$
$$\nabla_{e_2} e_1 = -\frac{1}{2} e_5, \quad \nabla_{e_2} e_5 = \frac{1}{2} e_1,$$
$$\nabla_{e_3} e_4 = \frac{1}{2} e_5, \quad \nabla_{e_3} e_5 = -\frac{1}{2} e_4,$$
$$\nabla_{e_4} e_3 = -\frac{1}{2} e_5, \quad \nabla_{e_4} e_5 = \frac{1}{2} e_3,$$
$$\nabla_{e_5} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_5} e_2 = \frac{1}{2} e_1, \quad \nabla_{e_5} e_3 = -\frac{1}{2} e_4, \quad \nabla_{e_5} e_4 = \frac{1}{2} e_3.$$

Let
$$\xi = e_5,$$
η *be the 1-form metric dual to* e_5*, and the endomorphism* φ *be defined by*
$$\varphi(e_1) = e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = e_4, \quad \varphi(e_4) = e_3, \quad \varphi(e_5) = 0.$$

Then, (φ, ξ, η, g) *is an almost paracontact paracomplex structure with nonzero structure constants:*

$$\alpha(e_1, e_1, e_5) = g((\nabla_{e_1}\varphi)(e_1), e_5) = \frac{1}{2} = \alpha(e_1, e_5, e_1),$$

$$\alpha(e_2, e_2, e_5) = -\frac{1}{2} = \alpha(e_2, e_5, e_2),$$

$$\alpha(e_3, e_3, e_5) = \frac{1}{2} = \alpha(e_3, e_5, e_3),$$

$$\alpha(e_4, e_4, e_5) = -\frac{1}{2} = \alpha(e_4, e_5, e_4),$$

$$\alpha(e_5, e_1, e_1) = -\alpha(e_5, e_2, e_2) = \alpha(e_5, e_3, e_3) = -\alpha(e_5, e_4, e_4) = 1.$$

First, we determine the class of this structure by evaluating projections α^i *introduced in [6]. The only nonzero projections are*

$$\begin{aligned}\alpha^8(x, y, z) = &\frac{1}{2}\{x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4\}z_5 \\ &+ \frac{1}{2}\{x_1z_1 - x_2z_2 + x_3z_3 - x_4z_4\}y_5\end{aligned}$$

and

$$\alpha^{10}(x, y, z) = \{y_1z_1 - y_2z_2 + y_3z_3 - y_4z_4\}x_5,$$

where

$$x = x_1e_1 \dots x_5e_5, \quad y = y_1e_1 \dots y_5e_5, \quad z = z_1e_1 \dots z_5e_5.$$

Thus, the structure is in the class $\mathcal{F}_8 \oplus \mathcal{F}_{10}$ *according to the classification in [6].*

Now, by a direct calculation, we get that

$$[\varphi, \varphi](x, y) = (-x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3) = d\eta(x, y)\xi$$

which implies that the Nijenhuis tensor

$$N(x, y) = [\varphi, \varphi](x, y) - d\eta(x, y)\xi$$

vanishes; thus, the structure is normal. As a result, a normal structure need not be in $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ *given in [6].*

In the following example, we write almost paracontact paracomplex Riemannian structures of classes $\mathcal{F}_9$, $\mathcal{F}_6 \oplus \mathcal{F}_{10}$ and $\mathcal{F}_6 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10}$, and we show that these structures are normal if and only if the structure is cosymplectic.

Example 3.10. *Consider the Lie group G of dimension* 5 *with a basis of left-invariant vector fields* $\{e_1, e_2, e_3, e_4, e_5\}$ *defined by the following non-zero brackets:*

$$\begin{aligned}[e_1, e_5] &= \lambda_1e_1 + \lambda_2e_2 + \lambda_1e_3 + \lambda_3e_4, \\ [e_2, e_5] &= -\lambda_2e_1 - \lambda_1e_2 - \lambda_3e_3 - \lambda_1e_4, \\ [e_3, e_5] &= -\lambda_1e_1 - \lambda_3e_2 + \lambda_1e_3 + \lambda_2e_4,\end{aligned}$$

$$[e_4, e_5] = \lambda_3 e_1 + \lambda_1 e_2 - \lambda_2 e_3 - \lambda_1 e_4,$$

where λ_i*,* $i = 1, 2, 3$ *are arbitrary real numbers. One can define an invariant almost paracontact paracomplex Riemannian structure on G as follows:*

$$\begin{gathered} g(e_i, e_i) = 1, \\ g(e_i, e_j) = 0, \quad i \neq j, \\ e_5 = \xi, \quad \varphi(e_1) = e_3, \quad \varphi(e_3) = e_1, \quad \varphi(e_2) = e_4, \quad \varphi(e_4) = e_2. \end{gathered}$$

By Koszul's formula, the nonzero Levi-Civita covariant derivatives are

$$\begin{aligned} &\nabla_{e_1} e_1 = -\lambda_1 e_5, \;\; \nabla_{e_1} e_4 = -\lambda_3 e_5, \;\; \nabla_{e_1} e_5 = \lambda_1 e_1 + \lambda_3 e_4, \\ &\nabla_{e_2} e_2 = \lambda_1 e_5, \;\; \nabla_{e_2} e_3 = \lambda_3 e_5, \;\; \nabla_{e_2} e_5 = -\lambda_1 e_2 - \lambda_3 e_3, \\ &\nabla_{e_3} e_2 = \lambda_3 e_5, \;\; \nabla_{e_3} e_3 = -\lambda_1 e_5, \;\; \nabla_{e_3} e_5 = -\lambda_3 e_2 + \lambda_1 e_3, \\ &\nabla_{e_4} e_1 = -\lambda_3 e_5, \;\; \nabla_{e_4} e_4 = \lambda_1 e_5, \;\; \nabla_{e_4} e_5 = \lambda_3 e_1 - \lambda_1 e_4, \\ &\nabla_{e_5} e_1 = -\lambda_2 e_2 - \lambda_1 e_3, \;\; \nabla_{e_5} e_2 = \lambda_2 e_1 + \lambda_1 e_4, \;\; \nabla_{e_5} e_3 = \lambda_1 e_1 - \lambda_2 e_4, \\ &\nabla_{e_5} e_4 = -\lambda_1 e_2 + \lambda_2 e_3, \end{aligned}$$

and the nonzero structure constants

$$\alpha(e_i, e_j, e_k) = g((\nabla_{e_i} \varphi)(e_j), e_k)$$

are

$$\begin{aligned} \alpha(e_1, e_2, e_5) &= -\lambda_3 = \alpha(e_1, e_5, e_2), \\ \alpha(e_1, e_3, e_5) &= -\lambda_1 = \alpha(e_1, e_5, e_3), \\ \alpha(e_2, e_1, e_5) &= \lambda_3 = \alpha(e_2, e_5, e_1), \\ \alpha(e_2, e_4, e_5) &= \lambda_1 = \alpha(e_2, e_5, e_4), \\ \alpha(e_3, e_1, e_5) &= -\lambda_1 = \alpha(e_3, e_5, e_1), \\ \alpha(e_3, e_4, e_5) &= \lambda_3 = \alpha(e_3, e_5, e_4), \\ \alpha(e_4, e_2, e_5) &= \lambda_1 = \alpha(e_4, e_5, e_2), \\ \alpha(e_4, e_3, e_5) &= -\lambda_3 = \alpha(e_4, e_5, e_3), \\ \alpha(e_5, e_1, e_1) &= 2\lambda_1 = -\alpha(e_5, e_2, e_2), \\ -\alpha(e_5, e_3, e_3) &= 2\lambda_1 = \alpha(e_5, e_4, e_4). \end{aligned}$$

We write $N(x, y)$ *by a direct calculation. We see that* $d\eta(x, y)\xi = 0$ *and*

$$\begin{aligned} N(x, y) &= [\varphi, \varphi](x, y) \\ &= 2\{-\lambda_1 x_3 y_5 + \lambda_3 x_4 y_5 + \lambda_1 x_5 y_3 - \lambda_3 x_5 y_4\} e_1 \\ &\quad + 2\{-\lambda_3 x_3 y_5 + \lambda_1 x_4 y_5 + \lambda_3 x_5 y_3 - \lambda_1 x_5 y_4\} e_2 \\ &\quad + 2\{\lambda_1 x_1 y_5 - \lambda_3 x_2 y_5 - \lambda_1 x_5 y_1 + \lambda_3 x_5 y_2\} e_3 \\ &\quad + 2\{\lambda_3 x_1 y_5 - \lambda_1 x_2 y_5 - \lambda_3 x_5 y_1 + \lambda_1 x_5 y_2\} e_4. \end{aligned}$$

This structure is normal if and only if $N = 0$*, that is, if and only if* $\lambda_1 = \lambda_3 = 0$*. In this case, the structure is cosymplectic, and if* $\lambda_2 = 0$*, then all Lie brackets vanish and the Lie algebra is Abelian. Otherwise, if* $\lambda_2 \neq 0$*, then the algebra is not Abelian.*

We determine the class of this structure by evaluating projections α^i *given in [6]. We have* $\alpha^{11} = 0$ *since* $\omega = 0$*. Since*

$$\theta(\xi) = \theta^*(\xi) = 0,$$

we obtain

$$\alpha^4 = \alpha^5 = 0.$$

Let

$$x = \sum x_i e_i, \quad y = \sum y_i e_i \quad \textit{and} \quad z = \sum z_i e_i.$$

Then

$$F^{10}(x, y, z) = 2\lambda_1 x_5 \{y_1 z_1 - y_2 z_2 - y_3 z_3 + y_4 z_4\};$$

thus, $F^{10} = 0$ *if and only if* $\lambda_1 = 0$*. In addition,*

$$\begin{aligned}\alpha(\varphi^2 x, \varphi^2 y, \xi) &= -\lambda_3 x_1 y_2 - \lambda_1 x_1 y_3 + \lambda_3 x_2 y_1 + \lambda_1 x_2 y_4 - \lambda_1 x_3 y_1 + \lambda_3 x_3 y_4 + \lambda_1 x_4 y_2 - \lambda_3 x_4 y_3 \\ &= \alpha(\varphi y, \varphi x, \xi)\end{aligned}$$

and

$$\begin{aligned}\alpha(\varphi^2 y, \varphi^2 x, \xi) &= \lambda_3 x_1 y_2 - \lambda_1 x_1 y_3 - \lambda_3 x_2 y_1 + \lambda_1 x_2 y_4 - \lambda_1 x_3 y_1 - \lambda_3 x_3 y_4 + \lambda_1 x_4 y_2 + \lambda_3 x_4 y_3 \\ &= \alpha(\varphi x, \varphi y, \xi).\end{aligned}$$

Thus,

$$\alpha^8 = \alpha^7 = 0$$

and

$$\alpha^9(x, y, z) = \lambda_3 \{-x_1 y_2 z_5 + x_2 y_1 z_5 + x_3 y_4 z_5 - x_4 y_3 z_5 - x_1 y_5 z_2 + x_2 y_5 z_1 + x_3 y_5 z_4 - x_4 y_5 z_3\}.$$

Thus, $\alpha^9 = 0$ *if and only if* $\lambda_3 = 0$*. Similarly,* $\alpha^6 = 0$ *if and only if* $\lambda_1 = 0$*.*

Note that

$$\theta(x) = 0$$

for any vector x; thus, $\alpha^1 = 0$*. By a direct calculation,*

$$\alpha^2 = \alpha^3 = 0.$$

We can summarize these points as

(1) If $\lambda_1 = 0$ *and* $\lambda_3 = 0$*, then the structure is cosymplectic.*
(2) If $\lambda_1 = 0$ *and* $\lambda_3 \neq 0$*, then this structure is in* $\mathcal{F}_9$*.*
(3) If $\lambda_1 \neq 0$ *and* $\lambda_3 = 0$*, then this structure is in* $\mathcal{F}_6 \oplus \mathcal{F}_{10}$*.*
(4) If $\lambda_1 \neq 0$ *and* $\lambda_3 \neq 0$*, then this structure is in* $\mathcal{F}_6 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10}$*.*

The class of the product manifold can be determined by (3.2). For example, if $\lambda_1 = 0$ *and* $\lambda_3 \neq 0$*, then the structure is in* $\mathcal{F}_9$*. If* σ *is constant, then the product structure satisfies the defining relation of* $W_2 \oplus W_3$*. If* σ *is not constant, then*

$$\tilde{\theta}(\tilde{X}) \neq 0;$$

in this case, the product manifold is in the widest class.

This structure has the scalar curvature $-4(\lambda_1^2 + \lambda_3^2)$*. For instance, if* $\lambda_1 = 0$*,* $\lambda_3 \neq 0$*, and* σ *is constant, then, from (3.4), the product manifold also has a negative scalar curvature.*

4. Conclusions

In this paper, we systematically defined an almost product structure $\tilde{P}$, on the product of an almost paracontact paracomplex Riemannian manifold with $\mathbb{R}$ by a warped product. Here, the new structure $\tilde{P}$ depends on any function σ of the coordinate t of $\mathbb{R}$. Then, we considered the classifications of these structures and gave certain relations between them, that may depend on the function σ. In this way, it is possible to obtain integrable almost product Riemannian manifolds. Additionally, we studied the curvature properties of the almost product structures and gave explicit examples. We showed that the almost product Riemannian manifolds of any positive scalar curvature can be obtained from almost paracontact paracomplex Riemannian manifold with a scalar curvature of zero. In addition, we obtained Einstein almost product Riemannian manifolds from Einstein almost paracontact paracomplex Riemannian manifolds.

Author contributions

Nülifer Özdemir: supervision, conceptualization, writing-original draft, writing-review and editing, methodology, validation; Şirin Aktay: writing-review and editing, writing-original draft, validation; Mehmet Solgun: methodology, validation, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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