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Research article

Optimizing decision precision with linguistic Pythagorean fuzzy Dombi models

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Abstract: Last-mile distribution is a subject that has drawn significant attention from both academic and industry researchers. There are several reasons for the adoption of drone delivery technology, including the growing number of customers who want more flexible and faster delivery options. Currently, there is a large selection of these models on the market. Therefore, there is a need to develop efficient methods to select the most appropriate drone delivery service. This research employs Dombi aggregation operators (AOs) within the context of linguistic Pythagorean fuzzy sets (LPFS) to tackle issues in drone delivery operations. The incorporation of linguistic concepts within the Pythagorean fuzzy framework improves the precision and dependability of delivery data analysis by providing a more thorough representation of uncertainty, consistent with human intuition and qualitative assessments. The present study presents two novel aggregation operators: the linguistic Pythagorean fuzzy Dombi weighted averaging (LPFDWA) and the linguistic Pythagorean fuzzy Dombi weighted geometric (LPFDWG) operators. Essential structural characteristics of these operators are demonstrated, and important particular cases are described. Furthermore, we developed a systematic approach for handling multi-attribute decision-making issues that incorporate LPF data through the use of the suggested operators. In order to showcase the effectiveness of the developed approaches, we provide a numerical illustration that identifies the top drone delivery service. Finally, we execute an in-depth comparative assessment to evaluate the efficacy of the proposed methods in relation to several established procedures.

Keywords: linguistic Pythagorean fuzzy sets; Dombi aggregation operators; multi-attribute decision-making; drone delivery optimization
Mathematics Subject Classification: 03E72, 94D05

1. Introduction

1.1. Abbreviations and symbols

The descriptions of the abbreviations and symbols used in this article are provided in Tables 1 and 2, respectively.

Table 1. List of abbreviations of the current study.

Abbreviations	Description
MADM	Multi-attribute decision-making
MAGDM	Multiple-attribute group decision making
AO	Aggregation operator
MD	Membership degree
NMD	Non-membership degree
IFS	Intuitionistic fuzzy set
LIFS	Linguistic intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
LPFS	Linguistic Pythagorean fuzzy set
LPFN	Linguistic Pythagorean fuzzy number
LPFDWA	Linguistic Pythagorean fuzzy Dombi weighted averaging
LPFDWG	Linguistic Pythagorean fuzzy Dombi weighted geometric

Table 2. List of symbols of the current study.

Symbols	Description
μ	Membership degree
ϑ	Non-membership degree
π	Indeterminacy degree
u_μ	Linguistic membership degree
u_ϑ	Linguistic non-membership degree
t	Cardinality of the linguistic term set
$\oplus$	To aggregate a collection of given LPFNs under certain s-norms
$\otimes$	To aggregate a collection of given LPFNs under certain t-norms
λ	A positive real number
Ψ	Operational parameter
U	Linguistic term set
$\overline{U}$	Continuous linguistic term set

1.2. Background

Previous studies and various decision-making methodologies [1−4] show that efficient multi-attribute decision-making (MADM) requires comparing a range of options and selecting the finest while considering a broad spectrum of criteria and features. Prior to a final decision being made, multiple competing issues are taken into consideration by decision-makers in solving the real-world situation. MADM's systematic approach is useful in dealing with the difficulties of intricate decision-making processes. Several groups, including those in public policy, healthcare, engineering, business, and environmental management, can profit from it. Whenever there are multiple competing issues to consider, MADM provides a straightforward and equitable method for selecting one option. MADM has become more and more important due to its capability to undertake challenging problems. Aggregation operators are essential components of the aggregation procedure, as they combine various values to select one. Since the AOs are adaptable, they can be applied in various contexts to solve difficult problems [5−7]. The methodologies proposed in [8,9] demonstrate the application of advanced decision-making models in the financial and defense sectors, respectively.

Zadeh [10] introduced fuzzy sets in 1965, indicating that they could be applied to address several problems as people were confronted with difficult situations using only exact numerical data. Fuzzy sets are particularly effective in dealing with imperfect data, as is common in human assessments. In 1975, Kahne [11] introduced a decision-making framework that considers many features with varied levels of importance. Jain [12] developed an alternative method for making decisions that finds a problematic replacement. In 1978, Dubois and Prade [13] reviewed several procedures involving a set of fuzzy variables. Yager [14] devised a number of aggregation operations based on fuzzy sets. In order to address the absence of a clear representation of hesitancy in fuzzy sets, Atanassov [15] extended upon Zadeh's research by developing the concept of IFS. This extension involved assigning MD and NMD to the elements, satisfying the condition with the hesitancy part. In 1994, Chen and Tan [16] suggested a score function for IFS with the aim of addressing MADM issues. In 1996, Szmidt and Kaepryzk [17] formulated an approach for solving MAGDM challenges within the context of the IFS. Li [18] introduced MADM models and methods using IFS in 2005. In 2006, Xu and Yager [19] developed some geometric AOs on IFS. In 2007, Xu [20] proposed arithmetic aggregation procedures for IFS. Zhao et al. [21] presented generalized AOs on IFS as a solution to the MADM issue. In [22], the authors proposed induced generalized AOs for intuitionistic fuzzy group decision-making. The researchers have placed greater emphasis on the study of FS and IFS. These studies have the potential to resolve quantitatively defined unpredictability. However, in real-life situations, many decision-making issues require a qualitative representation of uncertainty and imprecise information. When decision-makers evaluate someone, they tend to prefer using terms such as "very high", "high", "medium", "low", or "very low" to describe their level of intellect. Within this context, decision-makers can articulate their perspective on the item through the utilization of linguistic factors. Wang [23] defined the LIFS, which originates from the foundational principles of IFS. The LIFS employs an intuitionistic fuzzy number to precisely indicate the MDs and NMDs of a linguistic variable. Zhang introduced a method for MAGDM utilizing linguistic intuitionistic fuzzy numbers [24]. Ju et al. [25] proposed the MADM method by implementing linguistic intuitionistic Maclaurin symmetric mean aggregation techniques. Liu et al. [26] developed the linguistic intuitionistic geometric aggregation operator in their research. Liu presented the linguistic intuitionistic weighted Bonferroni mean operator [27]. In addition, Liu and Wang [28] devised and executed an improved LIF aggregation operator to solve MADM challenges.

The IFS theory shows its efficiency when making decisions and performing other operations.

However, the sum of the MD and NMD can be higher than one. For example, authorities have demonstrated interest in an alternative to the criterion with an MD of 0.8 and an NMD of 0.5. However, the equation 0.8+0.5≠1 does not meet the conditions of IFS. The concept of PFS, introduced in [29] as an extension of IFS theory, satisfies the condition $0 \leq (\mu(x))^2 + (\vartheta(x))^2 \leq 1$. It clearly shows that $0.8^2 + 0.5^2 \leq 1$ in the above example. The main difference between IFS and PFS regards the extent of MD and NMD associated with them. The assessment of IFS and PFS indicates that PFS demonstrates greater effectiveness than IFS in addressing unclear data within MADM challenges.

Research on PFS theory and methodology has grown in three main areas: the development of basic theories, the application of comparative analysis, and the incorporation of improved MADM methods. In [30], researchers explored the relationship between PFS and complex numbers. The extension of TOPSIS to PFS for MADM was investigated in [31]. Additionally, the introduction of Pythagorean membership grades for MADM was discussed in [32]. Asif et al. [33] studied Hamacher aggregation in PFSs, and Xiao et al. [34] introduced a q-rung orthopair fuzzy model for manufacturer selection. A design concept evaluation method based on fuzzy weighted zero inconsistency and combined compromise solutions was presented in [35]. New aggregation techniques using Einstein operations have been proposed in [36,37], and symmetric Pythagorean fuzzy weighted geometric/averaging operators have been applied to MADM problems in [38].

If a decision-maker expresses a preference for an item using a LIF number (u_4, u_3), where u_t denotes a linguistic term $t \in [0,6]$, the LIFS fails to tackle this preference for the linguistic variable as $4 + 3 > 6$. Garg [39] proposed the notion of an LPFS, which is defined by the MDs and NMDs that have a total of squares smaller than the cardinality of the set. Peng and Yang [40] introduced the MADM approach using Pythagorean fuzzy linguistic sets, while Lin et al. [41] developed interaction partitioned Bonferroni mean aggregation operators for enhanced MADM performance. Du et al. [42] further extended these concepts by proposing a novel interval-valued Pythagorean fuzzy linguistic decision-making method. The foundational concepts of fuzzy operators trace back to Dombi [43], who defined a generalized class of fuzzy operators and measures. These operators demonstrate substantial diversity in evaluating the impact of parameters. The utilization of Dombi AOs is a highly efficient approach to address MADM challenges. Several researchers have contributed significantly to the advancement of Dombi aggregation operators in decision-making frameworks. Liu et al. [44] introduced intuitionistic fuzzy Dombi Bonferroni mean operators for multi-attribute group decision-making. Akram et al. [45] and Jana et al. [46,47] extended the methodology to Pythagorean, Q-rung orthopair, and bipolar fuzzy environments. Jana et al. [48] further explored picture fuzzy Dombi operators, while Ashraf et al. [49] and Liu et al. [50] investigated spherical and interval-valued hesitant fuzzy models, respectively, for complex decision-making scenarios. Masmali et al. [51] utilized an all-encompassing mathematical framework to ascertain the optimal water purification method through the implementation of an optimization strategy in 2021. Seikh and Chatterjee [52] evaluated and selected E-learning websites using IF Dombi operators. In [53,54], Hussain et al. and Sarfraz explored the application of T-spherical fuzzy operators in decision-making under uncertainty.

1.3. Existing research gaps, motivating factors, and contributions of the present study

PFS provides substantial advantages over IFS, namely, the increased flexibility in MDs and NMDs. Although IFS limits the aggregate of these degrees to a maximum of 1, PFS permits the sum of their squares to be no more than 1. This provides a more nuanced and refined way of expressing uncertainty, making PFS more suitable for ambitious decision-making scenarios. However, PFS provides a framework for constraint management; it might not encompass linguistic or qualitative assessments of uncertainty in their entirety. This research challenge encourages the investigation of more generalized environments that can successfully handle decision-making situations. The incorporation of linguistic expressions into the PF environment accommodates the limitations of the exclusively numerical PF method, which could present a challenge for decision-makers who are more accustomed to qualitative or verbal forms of representation. This phenomenon not only increases the accessibility and interpretability of decision-making but also provides a more complete representation of uncertainty, which is consistent with human intuition and qualitative assessments. The above discussion motivates us to study the LPF environment in this article.

Aggregation operators are of critical importance in the process of condensing large datasets. They perform information consolidation by combining multiple data elements into a single, significant value. The Dombi AOs have immense importance in decision-making problems. These operators efficiently manage the inherent uncertainty in implicit information and allow for parameter adjustment to model various risk tolerances. Although the Dombi AOs are effective for aggregating information, they are limited in PF settings. Initially, they are dependent on a particular dimension to accurately represent the attitudinal nature of decision-making. Moreover, the Dombi parameter itself possesses mathematical conceptualization, which may prevent coherent communication and comprehension for individuals who are not familiar with mathematics. These decision-making challenges in the context of the Dombi aggregation environment can effectively be addressed through the techniques presented in this article.

Some of the main advantages of the newly suggested LPF Dombi AOs are as follows:

- The LPF Dombi weighted AOs employed in this article incorporate a versatile and adaptable parameter, enabling several forms of aggregation behaviors, such as prioritizing some inputs over others.
- The inclusion of a weighted vector in this setup enables the consideration of varying levels of priority for each criterion, leading to more precise and comprehensive aggregate results.
- These strategies allow experts to articulate their preferences using everyday language phrases by merging linguistic terms and the Dombi parameter. The primary characteristic of these approaches is their role as a bridge between the theoretical foundation of Dombi operators and the linguistic capabilities of human experts. Consequently, the aggregation of data in the LPF Dombi framework is perceived as more trustworthy and user-oriented due to this integration.

The following factors significantly contribute to the paper’s exposition:

1) An updated score and accuracy function for MADM challenges within the LPF environment are formulated. This advancement will enhance the ranking mechanism within the LPF system.
2) The LPFDWA and LPFDWG operators are introduced to handle intricate decision-making situations. These operators elucidate the interrelationships among various components in LPFNs and permit a more accurate estimation and evaluation of outcomes.
3) The basic structural characteristics of LPFDWA and LPFDWG operators, namely monotonicity, boundedness, and idempotency, have been formally demonstrated. This demonstrates the rationality of the proposed operators.
4) A comprehensive mathematical mechanism for MADM problems using the proposed strategies within the context of LPF information is designed. In addition, the validity of these newly defined approaches is established by applying them to the MADM problem of selecting the most efficient and reliable drone delivery firm.

5) A comprehensive comparative study is performed to evaluate the viability of the suggested strategy in contrast to existing techniques. The proposed methodology is coherent and dependable, as evidenced by the comparative results.

The structure of this paper is organized as follows: Section 2 offers a brief summary of the LPFS aggregation operators. Section 3 describes the score and accuracy functions for solving MADM issues in the LPF environment. The Dombi AO for LPFSs is presented, and its fundamental features are investigated in Section 4. In Section 5, we employ the newly formed operators to find the most efficient method for selecting the most reliable and efficient drone delivery firm. Section 6 presents a comparative analysis of current strategies to illustrate how these new methodologies compare in terms of effectiveness and feasibility. Finally, Section 7 presents a detailed explanation and examination of the conclusion, along with an outline of potential directions for further studies.

2. Preliminaries

Within this section, we delve into the foundational aspects of the subject. We offer an overview of the essential properties, operations, and techniques inherent to LPFSs defined on a non-empty universal set.

Definition 1. [18] An IFS I on a universe X is defined as:

$$I = \{(x, \mu_I(x), \vartheta_I(x)) |\; x \in X\},$$

where $\mu_I: X \to [0,1]$ and $\vartheta_I: X \to [0,1]$ represent the membership and non-membership functions, satisfying the condition $0 \le \mu_I(x) + \vartheta_I(x) \le 1$. The symbol $\pi_I(x) = 1 - \mu_I(x) - \vartheta_I(x), x$ describes the hesitancy degree of $x \in X$.

Definition 2. [45] A PFS P is defined over a universal set X as:

$$P = \{(x, \mu_P(x), \vartheta_P(x)) |\; x \in X\},$$

where $\mu_P: X \to [0,1]$ and $\vartheta_P: X \to [0,1]$ are the membership and non-membership functions, respectively, satisfying the condition $0 < (\mu_P(x))^2 + (\vartheta_P(x))^2 \le 1$. The hesitancy degree $x \in X$ is defined as $\pi_p(x) = \sqrt{1 - \mu_p^2(x) - \vartheta_p^2(x)}$.

Definition 3. [55] Consider a set $U = \{u_i | i = 0,1,2,\dots,t\}$ consisting of linguistic terms, where the cardinality of the set t is odd. The term u_i represents a possible qualitative value of the linguistic variable. For example, the linguistic variable "quality" may be described by a tripartite set of linguistic terms as $U = \{u_0 = \text{poor}, u_1 = \text{fair}, u_2 = \text{good}\}$. Let u_i and u_j represent two arbitrary linguistic terms of U. These terms must adhere to the following properties:

a) Set U to be an ordered set: $i < j \Leftrightarrow u_i < u_j$,
b) There are negation, maximum, and minimum operators as follows:
    1) $Neg(u_i) = u_j$ where $j = t - i$,
    2) $Max(u_i, u_j) = u_j \Leftrightarrow i < j$,
    3) $Min(u_i, u_j) = u_i \Leftrightarrow i < j$.

Definition 4. [55] Consider $U = \{u_i | i = 0,1,2,\dots,t\}$ to be a discrete linguistic terms set. A continuous linguistic term set (CLTS) is defined as $\overline{U} = \{u_j | u_0 \le u_j \le u_t, j \in [0,t]\}$; within this context, if $u_i \in U$, then it is denoted as an original linguistic term, and if $u_i \notin U$, then it is termed as virtual linguistic term.

Definition 5. [39] Let X be a universal set and $\overline{U} = \{u_j | u_0 \le u_j \le u_t, j \in [0,t]\}$ be a CLTS, then a LPFS A is defined as:

$$A = \{(x, u_\mu(x), u_\vartheta(x) | x \in X\},$$

where $u_\mu(x), u_\vartheta(x) \in \overline{U}$ such that $u_\mu(x)$ represents the linguistic MD and $u_\vartheta(x)$ represents the linguistic NMD for $x \in X$. We denote the pair $\left(u_\mu(x), u_\vartheta(x)\right)$ as $\alpha = (u_\mu, u_\vartheta)$ known as a LPFN, satisfying $0 \leq \mu \leq t, 0 \leq \vartheta \leq t$ and $0 \leq \mu^2 + \vartheta^2 \leq t^2$ for any $x \in X$. The degree of indeterminacy π_A is defined as $\pi_A(x) = u_{\sqrt{t^2-\mu^2-\vartheta^2}}$.

Definition 6. [39] Let $\alpha = (u_\mu, u_\vartheta)$, $\alpha_1 = (u_{\mu_1}, u_{\vartheta_1})$, and $\alpha_2 = (u_{\mu_2}, u_{\vartheta_2})$ be any three LPFNs where $u_\mu, u_\vartheta, u_{\mu_1}, u_{\vartheta_1}, u_{\mu_2}, u_{\vartheta_2} \in \overline{U} = \{u_j | u_0 \leq u_j \leq u_t, j \in [0, t]\}$. These LPFNs satisfy the following fundamental laws:

1) $\alpha_1 = \alpha_2$ iff $u_{\mu_1} = u_{\mu_2}$ and $u_{\vartheta_1} = u_{\vartheta_2}$.
2) $\alpha_1 < \alpha_2$iff $u_{\mu_1} < u_{\mu_2}$and $u_{\vartheta_1} > u_{\vartheta_2}$.
3) $\alpha^c = (u_\vartheta, u_\mu)$, where $\alpha^c$is the complement of α.
4) $\alpha_1 \vee \alpha_2 = (max(u_{\mu_1}, u_{\mu_2}), min(u_{\vartheta_1}, u_{\vartheta_2}))$.
5) $\alpha_1 \wedge \alpha_2 = (min(u_{\mu_1}, u_{\mu_2}), max(u_{\vartheta_1}, u_{\vartheta_2}))$.

Now, we present operational laws for LPFNs with respect to t-norm and s-norm.

Definition 7. [39] Let $\alpha = (u_\mu, u_\vartheta)$, $\alpha_1 = (u_{\mu_1}, u_{\vartheta_1})$ and $\alpha_2 = (u_{\mu_2}, u_{\vartheta_2})$ be any three LPFNs where $u_\mu, u_\vartheta, u_{\mu_1}, u_{\vartheta_1}, u_{\mu_2}, u_{\vartheta_2} \in \overline{U} = \{u_j | u_0 \leq u_j \leq u_t, j \in [0, t]\}$ and $\lambda > 0$ is a real number, then

1) $\alpha_1 \oplus \alpha_2 = (u_{t\sqrt{\mu_1^2/t^2+\mu_2^2/t^2-\mu_1^2\mu_2^2/t^4}}, u_{t(\vartheta_1\vartheta_2/t^2)})$;
2) $\alpha_1 \otimes \alpha_2 = (u_{t(\mu_1\mu_2/t^2)}, u_{t\sqrt{\vartheta_1^2/t^2+\vartheta_2^2/t^2-\vartheta_1^2\vartheta_2^2/t^4}})$;
3) $\lambda\alpha = (u_{t\sqrt{1-(1-\mu^2/t^2)^\lambda}}, u_{t(\vartheta/t)^\lambda})$;
4) $\alpha^\lambda = (u_{t(\mu/t)^\lambda}, u_{t\sqrt{1-(1-\vartheta^2/t^2)^\lambda}})$.

Definition 8. [45] The Dombi t-norm and s-norm are described as

$$T(a, b) = \frac{1}{1+\left[\left(\frac{1-a}{a}\right)^\Psi + \left(\frac{1-b}{b}\right)^\Psi\right]^{\frac{1}{\Psi}}}, \tag{2.1}$$

$$S(a, b) = 1 - \frac{1}{1+\left[\left(\frac{a}{1-a}\right)^\Psi + \left(\frac{b}{1-b}\right)^\Psi\right]^{\frac{1}{\Psi}}}. \tag{2.2}$$

Here, $\Psi \geq 1$ is a real number and $(a, b) \in [0,1] \times [0,1]$, and (2.1) is called Dombi product and (2.2) is called Dombi sum.

3. Development of a novel ranking mechanism for LPFNs

In this section, we formulate a pair of new score and accuracy functions for LPFNs to address MADM challenges. The purpose of developing this ranking mechanism is to facilitate the comparison of two LPFNs, as these numbers are represented as ordered pairs and cannot be directly compared. To enhance the decision-making process, it is crucial to establish a refined score function that improves the evaluation, accuracy, and credibility assessment of LPFNs.

Definition 9. Let $\alpha = (u_\mu, u_\vartheta)$ with $u_\mu, u_\vartheta \in \overline{U} = \{u_j | u_0 \leq u_j \leq u_t, j \in [0, t]\}$ be a LPFN, then the score function S for LPFN α is defined as follows:

$$S(\alpha) = u_{\sqrt{\frac{t^2-\vartheta^2}{2t^2-\mu^2-\vartheta^2}}}, \text{ where } \sqrt{\frac{t^2-\vartheta^2}{2t^2-\mu^2-\vartheta^2}} \in [0,1]. \tag{3.1}$$

The accuracy function A for LPFN α is defined as follows:

$$A(\alpha) = u_{\sqrt{\frac{t^2+\mu^2}{2t^2-\mu^2+\vartheta^2}}}, \text{ where } \sqrt{\frac{t^2+\mu^2}{2t^2-\mu^2+\vartheta^2}} \in \left[\frac{1}{\sqrt{3}}, \sqrt{2}\right]. \tag{3.2}$$

The comparison rules for any two LPFNs α and β by means of the above definition are described as follows:

1. If $S(\alpha)>S(\beta)$, then $\alpha \succ \beta$ where $\succ$ means "preferred to";
2. If $S(\alpha)=S(\beta)$, and
   - $A(\alpha) = A(\beta)$ then $\alpha = \beta$;
   - $A(\alpha) > A(\beta)$ then $\alpha \succ \beta$.

The subsequent example shows the validity of the above ranking mechanism.

Example 1. Let $\alpha = (u_5, u_1)$ and $\beta = (u_7, u_5)$ represent two LPFNs defined on $\overline{U} = \{u_j | u_0 \leq u_j \leq u_8, j \in [0,8]\}$. By substituting the value of LPFNs α and β in Eq (3.1), we get

$$S(\alpha) = u_{\sqrt{\frac{8^2-1^2}{2(8)^2-5^2-1^2}}} = u_{0.7859};\ S(\beta) = u_{\sqrt{\frac{8^2-5^2}{2(8)^2-7^2-5^2}}} = u_{0.8498}.$$

In view of the obtained score values of α and β and using comparison rule 1 of Definition 9, we conclude that $\beta \succ \alpha$. This discernment signifies that β is preferred to α.

4. Dombi aggregation operators for LPFNs and their properties

This section explains the key characteristics of the Dombi operators within an LPF framework. It further presents two Dombi-based weighted aggregation operators, the LPFDWA operator and the LPFDWG operator, both formulated based on the Dombi operational laws for LPFNs. Moreover, the structural attributes of these operators are examined in detail.

Definition 10. For any real number $\Psi \geq 1$, $\lambda > 0$ and for any three LPFNs $\alpha = (u_\mu, u_\vartheta)$, $\alpha_1 = (u_{\mu_1}, u_{\vartheta_1})$, and $\alpha_2 = (u_{\mu_2}, u_{\vartheta_2})$ defined on $\overline{U} = \{u_j | u_0 \leq u_j \leq u_t, j \in [0, t]\}$. In this context, the Dombi operational laws for LPFNs are derived using Dombi t-norm and s-norm developed in [45] as follows:

1. $$\alpha_1 \oplus \alpha_2 = \left(u_{t\sqrt{1-\frac{1}{1+\left\{\left(\frac{\frac{\mu_1^2}{t^2}}{1-\frac{\mu_1^2}{t^2}}\right)^\Psi + \left(\frac{\frac{\mu_2^2}{t^2}}{1-\frac{\mu_2^2}{t^2}}\right)^\Psi\right\}^{\frac{1}{\Psi}}}}},\ u_{t\sqrt{\frac{1}{1+\left\{\left(\frac{1-\frac{\vartheta_1^2}{t^2}}{\frac{\vartheta_1^2}{t^2}}\right)^\Psi + \left(\frac{1-\frac{\vartheta_2^2}{t^2}}{\frac{\vartheta_2^2}{t^2}}\right)^\Psi\right\}^{\frac{1}{\Psi}}}}} \right).$$

It follows that

$$\alpha_1 \oplus \alpha_2 = \left(u_{t\sqrt{1-\frac{1}{1+\left\{\left(\frac{\mu_1^2}{t^2-\mu_1^2}\right)^{\Psi}+\left(\frac{\mu_2^2}{t^2-\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\left(\frac{t^2-\vartheta_1^2}{\vartheta_1^2}\right)^{\Psi}+\left(\frac{t^2-\vartheta_2^2}{\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.1}$$

Moreover,

2. $$\alpha_1 \otimes \alpha_2 = \left(u_{t\sqrt{\frac{1}{1+\left\{\left(\frac{1-\frac{\mu_1^2}{t^2}}{\frac{\mu_1^2}{t^2}}\right)^{\Psi}+\left(\frac{1-\frac{\mu_2^2}{t^2}}{\frac{\mu_2^2}{t^2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\left(\frac{\frac{\vartheta_1^2}{t^2}}{1-\frac{\vartheta_1^2}{t^2}}\right)^{\Psi}+\left(\frac{\frac{\vartheta_2^2}{t^2}}{1-\frac{\vartheta_2^2}{t^2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

It follows that

$$\alpha_1 \otimes \alpha_2 = \left(u_{t\sqrt{\frac{1}{1+\left\{\left(\frac{t^2-\mu_1^2}{\mu_1^2}\right)^{\Psi}+\left(\frac{t^2-\mu_2^2}{\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\left(\frac{\vartheta_1^2}{t^2-\vartheta_1^2}\right)^{\Psi}+\left(\frac{\vartheta_2^2}{t^2-\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.2}$$

$$\lambda\alpha = \left(u_{t\sqrt{1-\frac{1}{1+\left\{\lambda\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\lambda\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.3}$$

Proof. We confirm the theorem by employing mathematical induction on λ.
For the initial case when $\lambda = 2$, then in view of Eq (4.1), we have

$$2\alpha = \alpha \oplus \alpha$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}+\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}+\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{2\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{2\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

This verifies the result in the case of $\lambda = 2$.

Moving ahead with the induction process, we assume that statement is valid for $\lambda = m$, that is

$$m\alpha = \left(u_{t\sqrt{1-\frac{1}{1+\left\{m\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{m\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Let $\lambda = m + 1$, then the application of Eq (4.1) gives:

$$(m+1)\,\alpha = m\alpha \oplus \alpha$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{m\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}+\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{m\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}+\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{(m+1)\left(\frac{\mu^2}{t^2-\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{(m+1)\left(\frac{t^2-\vartheta^2}{\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

As the result is established for $\lambda = m + 1$, it is consequently true for every positive integer λ.

$$\alpha^{\lambda} = \left(u_{t\sqrt{\frac{1}{1+\left\{\lambda\left(\frac{t^2-\mu^2}{\mu^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\lambda\left(\frac{\vartheta^2}{t^2-\vartheta^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.4}$$

The validity of the relation 4 is established by adopting the above mathematical procedure and using Eq (4.2).

4.1. Key features of Dombi weighted averaging operator in the context of LPFNs

We define the LPFDWA operator and examine its basic features in the ensuing section.

Definition 11. Consider a set $\Phi = \{\alpha_i = (u_{\mu_i}, u_{\vartheta_i}), i = 1,2,\dots,n\}$ having n LPFNs and $w = (w_1, w_2, \dots, w_n)^T$ is the corresponding weight vector satisfying $\sum_{i=1}^{n} w_i = 1$ with $0 \le w_i \le 1$ and operational parameter $\Psi \ge 1$. Then the LPFDWA operator is defined by a mapping $LPFDWA$: $\Phi^n \to \Phi$ such that:

$$LPFDWA(\alpha_1, \alpha_2, \dots, \alpha_n) = w_1\alpha_1 \oplus w_2\alpha_2 \oplus \dots \oplus w_n\alpha_n. \tag{4.5}$$

Theorem 1. Assume that $\alpha_i = (u_{\mu_i}, u_{\vartheta_i})$, where $i = 1,2,\dots,n$, are LPFNs and $w = (w_1, w_2, \dots, w_n)^T$ is the corresponding weight vector satisfying $\sum_{i=1}^{n} w_i = 1$ with $0 \le w_i \le 1$ and $\Psi \ge 1$. The aggregated value achieved through the LPFDWA operator is an LPFN and can be expressed as:

$$LPFDWA(\alpha_1, \alpha_2, \dots, \alpha_n) = \oplus_{i=1}^{n} w_i\alpha_i$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.6}$$

Proof. We confirm the theorem by employing mathematical induction.

Suppose that $n = 2$, we have $\alpha_1 = (u_{\mu_1}, u_{\vartheta_1})$ and $\alpha_2 = (u_{\mu_2}, u_{\vartheta_2})$. Utilizing Definition 11, we get

$$w_1\alpha_1 = \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_1\left(\frac{\mu_1^2}{t^2-\mu_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_1\left(\frac{t^2-\vartheta_1^2}{\vartheta_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right),$$

$$w_2\alpha_2 = \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_2\left(\frac{\mu_2^2}{t^2-\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_2\left(\frac{t^2-\vartheta_2^2}{\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

In accordance with Definition 11, the aggregated value of α_1 and α_2 is calculated in the following manner:

$$LPFDWA(\alpha_1, \alpha_2) = w_1\alpha_1 \oplus w_2\alpha_2$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_1\left(\frac{\mu_1^2}{t^2-\mu_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_1\left(\frac{t^2-\vartheta_1^2}{\vartheta_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right) \oplus \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_2\left(\frac{\mu_2^2}{t^2-\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_2\left(\frac{t^2-\vartheta_2^2}{\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

It can be inferred that:

$$w_1\alpha_1 \oplus w_2\alpha_2 = \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_1\left(\frac{\mu_1^2}{t^2-\mu_1^2}\right)^{\Psi}+w_2\left(\frac{\mu_2^2}{t^2-\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_1\left(\frac{t^2-\vartheta_1^2}{\vartheta_1^2}\right)^{\Psi}+w_2\left(\frac{t^2-\vartheta_2^2}{\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Consequently,

$$LPFDWA(\alpha_1, \alpha_2) = \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{2} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{2} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

The result holds true when n equals 2.

Moving ahead with the induction process, we assume that statement is valid for $n = r$, that is

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_r) = \oplus_{i=1}^{r} w_i\alpha_i$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Let $n = r + 1$, then,

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}) = \oplus_{i=1}^{r} w_i\alpha_i \oplus w_{r+1}\alpha_{r+1}$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right) \oplus \left(u_{t\sqrt{1-\frac{1}{1+\left\{w_{r+1}\left(\frac{\mu_{r+1}^2}{t^2-\mu_{r+1}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{w_{r+1}\left(\frac{t^2-\vartheta_{r+1}^2}{\vartheta_{r+1}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Consequently,

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_{r+1}) = \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r+1} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r+1} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Hence, the validity of the theorem has been established for $n = r + 1$, affirming that Theorem 1 holds for all integer values of n.

Example 2. Consider three customers who want to rank the food quality of a restaurant. The opinion of three customers is summarized in the form of LPFNs, $\alpha_1 = (u_1, u_3)$, $\alpha_2 = (u_3, u_5)$, and $\alpha_3 = (u_2, u_4)$ defined on CLTS $\overline{U} = \{u_i | u_0 < u_i < u_6, i \in [0,6]\}$ with the corresponding weight vectors of the three customers $w = (0.2,0.3,0.5)^T$ and $\Psi = 3$. Then the LPFDWA operator can be effectively utilized to aggregate the three LPFNs and hence we have,

$$LPFDWA(\alpha_1, \alpha_2, \alpha_3) = \oplus_{i=1}^{3} w_i \alpha_i$$

$$= \left(u_{6\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{\mu_i^2}{6^2-\mu_i^2}\right)^{3}\right\}^{\frac{1}{3}}}}}, u_{6\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{6^2-\vartheta_i^2}{\vartheta_i^2}\right)^{3}\right\}^{\frac{1}{3}}}}} \right) = \left(u_{6\sqrt{1-\frac{1}{1+\left\{0.2\left(\frac{1^2}{6^2-1^2}\right)^3+0.3\left(\frac{3^2}{6^2-3^2}\right)^3+0.5\left(\frac{2^2}{6^2-2^2}\right)^3\right\}^{\frac{1}{3}}}}}, u_{6\sqrt{\frac{1}{1+\left\{0.2\left(\frac{6^2-3^2}{3^2}\right)^3+0.3\left(\frac{6^2-5^2}{5^2}\right)^3+0.5\left(\frac{6^2-4^2}{4^2}\right)^3\right\}^{\frac{1}{3}}}}} \right).$$

Thus,

$$LPFDWA(\alpha_1, \alpha_2, \alpha_3) = (u_{2.5924}, u_{3.5499}).$$

Hence, we conclude that the preceding discussion demonstrates the validity of the fact indicated in

Theorem 1.

Proposition 1. Assume that $\alpha_i = \left(u_{\mu_i}, u_{\vartheta_i}\right)$, where $i = 1,2,\ldots,n$, are LPFNs and $w = (w_1, w_2, \ldots, w_n)^T$ is the corresponding weight vector of α_i, where $0 \le w_i \le 1$ such that $\sum_{i=1}^{n} w_i = 1$ and $\Psi \ge 1$.

P1 (*Idempotency*). If $\alpha_i = \left(u_{\mu_i}, u_{\vartheta_i}\right) = \left(u_{\mu}, u_{\vartheta}\right) = \alpha$, for all i, then

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha. \tag{4.7}$$

P2 (*Monotonicity*). Assume that $\beta_i = \left(u_{\mu_{i'}}, u_{\vartheta_{i'}}\right)$ is the LPFN. If $u_{\mu_i} \le u_{\mu_{i'}}$ and $u_{\vartheta_i} \ge u_{\vartheta_{i'}}$. Then,

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \le LPFDWA(\beta_1, \beta_2, \ldots, \beta_n). \tag{4.8}$$

P3 (*Boundedness*). If $\alpha^- = \underset{i}{min}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\alpha^+ = \underset{i}{max}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are two LPFNs. Then,

$$\alpha^- \le LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+. \tag{4.9}$$

Equation (4.7) in fact describes that applying the weighted aggregation operator to the same input LPFS multiple times produces the same result as applying it once.

Equation (4.8) shows that the output of the aggregation operators behaves consistently with the changes in the input LPFS's MD and NMD values.

Equation (4.9) ensures that the output of the weighted aggregation operator remains within certain limits and is bound.

Proof. Since $\alpha_i = \left(u_{\mu_i}, u_{\vartheta_i}\right)$, where $i = 1,2,\ldots,n$, are LPFNs, which implies that $u_{\mu_i}, u_{\vartheta_i} \in \overline{U} = \left\{u_j | u_0 \le u_j \le u_t, j \in [0, t]\right\}$ and $\mu_i^2 + \vartheta_i^2 \le t^2$. Then,

P1. By applying the given conditions, we have $\alpha_i = \alpha$

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \oplus_{i=1}^{n} w_i \alpha_i$$

$$= \left(u_{t\sqrt{1 - \frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{\mu_i^2}{t^2 - \mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1 + \left\{\sum_{i=1}^{n} w_i \left(\frac{t^2 - \vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$= \left(u_{t\sqrt{1 - \frac{1}{1 + \left(\frac{\mu^2}{t^2 - \mu^2}\right)\left\{\sum_{i=1}^{n} w_i\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1 + \left(\frac{t^2 - \vartheta^2}{\vartheta^2}\right)\left\{\sum_{i=1}^{n} w_i\right\}^{\frac{1}{\Psi}}}}} \right).$$

Consequently,

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(u_{t\sqrt{1 - \frac{1}{1 + \left(\frac{\mu^2}{t^2 - \mu^2}\right)}}}, u_{t\sqrt{\frac{1}{1 + \left(\frac{t^2 - \vartheta^2}{\vartheta^2}\right)}}} \right) = \alpha.$$

P2. Considering the provided condition, we may deduce that $u_{\mu_i} \le u_{\mu_{i'}}$ for all i.

$$\Rightarrow u_{\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}} \leq u_{\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}}$$

$$\Rightarrow u_{\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}} \leq u_{\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}$$

$$\Rightarrow u_{\left(1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}\right)} \leq u_{\left(1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}\right)}$$

$$\Rightarrow u_{\left(\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)} \geq u_{\left(\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)}$$

$$\Rightarrow u_{\left(1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)} \leq u_{\left(1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)}$$

$$\Rightarrow u_{\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \leq u_{\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}$$

$$\Rightarrow u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \leq u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{\mu_i^{2\prime}}{t^2-\mu_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}. \tag{4.10}$$

Moreover, in view of the given condition, we have $u_{\vartheta_i} \geq u_{\vartheta_i'}$ for all i, and by adapting the above mathematical procedure, we get

$$u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{t^2-\vartheta_i^2}{\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \geq u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{n} w_i\left(\frac{t^2-\vartheta_i^{2\prime}}{\vartheta_i^{2\prime}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}. \tag{4.11}$$

Upon comparing Eqs (4.10) and (4.11) and applying Definition 6, the following is obtained:

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq LPFDWA(\beta_1, \beta_2, \ldots, \beta_n).$$

P3. Let us apply the LPFDWA operator to the collection of LPFNs as follows:

$$LPFDWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = (u_\mu, u_\vartheta).$$

Assume that $\alpha^- = (u_{\mu^-}, u_{\vartheta^-})$ and $\alpha^+ = (u_{\mu^+}, u_{\vartheta^+})$, where $u_{\mu^-} = \underset{i}{min}(u_{\mu_i})$, $u_{\vartheta^-} = \underset{i}{max}(u_{\vartheta_i})$ and $u_{\mu^+} = \underset{i}{max}(u_{\mu_i})$, $u_{\vartheta^+} = \underset{i}{min}(u_{\vartheta_i})$.

Since for each LPFN, $\underset{i}{min}(u_{\mu_i}) \leq u_{\mu_i} \leq \underset{i}{max}(u_{\mu_i})$

$$\Rightarrow \min_i \left(u_{\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}} \right) \le u_{\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}} \le \max_i \left(u_{\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}} \right)$$

$$\Rightarrow \min_i \left(u_{\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}} \right) \le u_{\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}} \le \max_i \left(u_{\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}} \right)$$

$$\Rightarrow \max_i \left(u_{\left(\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)} \right) \le u_{\left(\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)} \le \min_i \left(u_{\left(\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}\right)} \right)$$

$$\Rightarrow \min_i \left(u_{\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right) \le u_{\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}$$

$$\le \max_i \left(u_{\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$\Rightarrow u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu^{-2}}{t^2-\mu^{-2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \le u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu_i^2}{t^2-\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \le u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{\mu^{+2}}{t^2-\mu^{+2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}$$

$$\Rightarrow u_{\mu^-} \le u_{\mu_i} \le u_{\mu^+}. \tag{4.12}$$

In addition, by utilizing the aforementioned mathematical process for the relationship $max(u_{\vartheta_i}) \le u_{\vartheta_i} \le min(u_{\vartheta_i})$, this leads to the following result:

$$u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{t^2-\vartheta_i^{-2}}{\vartheta_i^{-2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \le u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{t^2-\vartheta_i^{2}}{\vartheta_i^{2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \le u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^n w_i\left(\frac{t^2-\vartheta_i^{+2}}{\vartheta_i^{+2}}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}$$

$$\Rightarrow u_{\vartheta^-} \le u_{\vartheta_i} \le u_{\vartheta^+}. \tag{4.13}$$

By comparing Eqs (4.12) and (4.13), we get

$$\alpha^{-} \leq LPDFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^{+}.$$

4.2. Key features of Dombi weighted geometric operator in the context of LPFNs

In this section, we define the LPFDWG operator and examine its basic features.

Definition 12. Consider a set $\Phi = \{\alpha_i = (u_{\mu_i}, u_{\vartheta_i}), i = 1,2,\ldots,n\}$ having n LPFNs and $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector of α_i, where $0 \leq w_i \leq 1$ such that $\sum_{i=1}^{n} w_i = 1$ and operational parameter $\Psi \geq 1$. Then the LPFDWG operator is defined by a mapping $LPFDWG$: $\Phi^n \to \Phi$ such that:

$$LPFDWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = {\alpha_1}^{w_1} \otimes {\alpha_2}^{w_2} \otimes \ldots \otimes {\alpha_n}^{w_n}. \tag{4.14}$$

Theorem 2. Assume that $\alpha_i = (u_{\mu_i}, u_{\vartheta_i})$, where $i = 1,2,\ldots,n$, are LPFNs and $w = (w_1, w_2, \ldots, w_n)^T$ is the corresponding weight vector of α_i, where $0 \leq w_i \leq 1$ such that $\sum_{i=1}^{n} w_i = 1$ and $\Psi \geq 1$. The aggregated value achieved through the LPFDWG operator is an LPFN and can be expressed as:

$$LPFDWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \otimes_{i=1}^{n} {\alpha_i}^{w_i}$$

$$= \left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{n} w_i \left(\frac{t^2-\mu_i^2}{\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{n} w_i \left(\frac{\vartheta_i^2}{t^2-\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right). \tag{4.15}$$

Proof. We confirm the theorem by employing mathematical induction.

For $n = 2$, we have $\alpha_1 = (u_{\mu_1}, u_{\vartheta_1})$ and $\alpha_2 = (u_{\mu_2}, u_{\vartheta_2})$. Utilizing Definition 12, we get

$$\alpha_1^{w_1} = \left(u_{t\sqrt{\frac{1}{1+\left\{w_1 \left(\frac{t^2-\mu_1^2}{\mu_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{w_1 \left(\frac{\vartheta_1^2}{t^2-\vartheta_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right);$$

$$\alpha_2^{w_2} = \left(u_{t\sqrt{\frac{1}{1+\left\{w_2 \left(\frac{t^2-\mu_2^2}{\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{w_2 \left(\frac{\vartheta_2^2}{t^2-\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

In accordance with Definition 12, the aggregated value of α_1 and α_2 is calculated in the following manner:

$$LPFDWG(\alpha_1, \alpha_2) = \alpha_1^{w_1} \otimes \alpha_2^{w_2}$$

$$=\left(u_{t\sqrt{\frac{1}{1+\left\{w_1\left(\frac{t^2-\mu_1^2}{\mu_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{w_1\left(\frac{\vartheta_1^2}{t^2-\vartheta_1^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right)\otimes\left(u_{t\sqrt{\frac{1}{1+\left\{w_2\left(\frac{t^2-\mu_2^2}{\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{w_2\left(\frac{\vartheta_2^2}{t^2-\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

It can be inferred that:

$$\alpha_1^{w_1}\otimes\alpha_2^{w_2}=\left(u_{t\sqrt{\frac{1}{1+\left\{w_1\left(\frac{t^2-\mu_1^2}{\mu_1^2}\right)^{\Psi}+w_2\left(\frac{t^2-\mu_2^2}{\mu_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{w_1\left(\frac{\vartheta_1^2}{t^2-\vartheta_1^2}\right)^{\Psi}+w_2\left(\frac{\vartheta_2^2}{t^2-\vartheta_2^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

Consequently,

$$LPFDWG(\alpha_1,\alpha_2)=\left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{2}w_i\left(\frac{t^2-\mu_i^2}{\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{2}w_i\left(\frac{\vartheta_i^2}{t^2-\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

Hence, it holds for $n=2$.

Moving ahead with the induction process, we assume that the statement of theorem holds for $n=r$, that is,

$$LPFDWG(\alpha_1,\alpha_2,\ldots,\alpha_r)=\otimes_{i=1}^{r}{\alpha_i}^{w_i}$$

$$=\left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r}w_i\left(\frac{t^2-\mu_i^2}{\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r}w_i\left(\frac{\vartheta_i^2}{t^2-\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

Let $n=r+1$, then

$$LPFDWG(\alpha_1,\alpha_2,\ldots,\alpha_r,\alpha_{r+1})=\otimes_{i=1}^{r}{\alpha_i}^{w_i}\otimes{\alpha_{r+1}}^{w_{r+1}}$$

$$=\left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r}w_i\left(\frac{t^2-\mu_i^2}{\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r}w_i\left(\frac{\vartheta_i^2}{t^2-\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right)\otimes$$

$$\left(u_{t\sqrt{\frac{1}{1+\left\{w_{r+1}\left(\frac{t^2-\mu_{r+1}^2}{\mu_{r+1}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{w_{r+1}\left(\frac{\vartheta_{r+1}^2}{t^2-\vartheta_{r+1}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Consequently,

$$LPFDWG(\alpha_1, \alpha_2, \dots, \alpha_{r+1}) = \left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{r+1} w_i\left(\frac{t^2-\mu_i^2}{\mu_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{r+1} w_i\left(\frac{\vartheta_i^2}{t^2-\vartheta_i^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right).$$

Hence, the validity of the theorem has been established for $n = r + 1$, affirming that Theorem 2 holds for all integer values of n.

Example 3. Applying the LPFDWG operator to the dataset presented in Example 2, we obtain

$$LPFDWG(\alpha_1, \alpha_2, \alpha_3) = \otimes_{i=1}^{3} \alpha_i^{w_i}$$

$$= \left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{6^2-\mu_i^2}{\mu_i^2}\right)^{3}\right\}^{\frac{1}{3}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{i=1}^{3} w_i\left(\frac{\vartheta_i^2}{6^2-\vartheta_i^2}\right)^{3}\right\}^{\frac{1}{3}}}}} \right)$$

$$= \left(u_{6\sqrt{\frac{1}{1+\left\{0.2\left(\frac{6^2-1^2}{1^2}\right)^3+0.3\left(\frac{6^2-3^2}{3^2}\right)^3+0.5\left(\frac{6^2-2^2}{2^2}\right)^3\right\}^{\frac{1}{3}}}}}, u_{6\sqrt{1-\frac{1}{1+\left\{0.2\left(\frac{3^2}{6^2-3^2}\right)^3+0.3\left(\frac{5^2}{6^2-5^2}\right)^3+0.5\left(\frac{4^2}{6^2-4^2}\right)^3\right\}^{\frac{1}{3}}}}} \right)$$

$$= (u_{1.2887}, u_{4.6823}).$$

Proposition 2. Assume that $\alpha_i = \left(u_{\mu_i}, u_{\vartheta_i}\right)$, where $i = 1,2,\dots,n$, are LPFNs and $w = (w_1, w_2, \dots, w_n)^T$ is the corresponding weight vector of α_i, where $0 \le w_i \le 1$ such that $\sum_{i=1}^{n} w_i = 1$ and $\Psi \ge 1$.

P1 (*Idempotency*). If $\alpha_i = \left(u_{\mu_i}, u_{\vartheta_i}\right) = \left(u_\mu, u_\vartheta\right) = \alpha$, for all i, then

$$LPFDWG(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha. \tag{4.16}$$

P2 (*Monotonicity*). Assume that $\beta_i = \left(u_{\mu_i'}, u_{\vartheta_i'}\right)$ is the LPFN. If $u_{\mu_i} \le u_{\mu_i'}$ and $u_{\vartheta_i} \ge u_{\vartheta_i'}$. Then,

$$LPFDWG(\alpha_1, \alpha_2, \dots, \alpha_n) \le LPFDWG(\beta_1, \beta_2, \dots, \beta_n). \tag{4.17}$$

P3 (*Boundedness*). If $\alpha^{-}=\underset{i}{min}(\alpha_1,\alpha_2,..,\alpha_n)$ and $\alpha^{+}=\underset{i}{max}(\alpha_1,\alpha_2,...,\alpha_n)$ are two LPFNs. Then,

$$\alpha^{-}\leq LPFDWG(\alpha_1,\alpha_2,...,\alpha_n)\leq\alpha^{+}. \tag{4.18}$$

Proof. The proof of this proposition can be obtained by applying the same reasoning used in Proposition 1.

5. Utilization of the LPF Dombi aggregation operators in MADM contexts

In the subsequent sections, we develop a decision-making methodology tailored to a scenario where we apply the Dombi AO to the information presented in the form of LPFNs with the weight vector of attributes. We denote the set of alternatives $\chi=\{\chi_1,\chi_2,...,\chi_m\}$ and a set of attributes $T=\{T_1,T_2,...,T_n\}$, each associated with the weight vector $w=\{w_1,w_2,...,w_n\}^T$, where $w_j>0$, for all $j=(1,2,...,n)$ such that $\sum_{j=1}^{n}w_j=1$. Consider an LPF decision matrix $F=\left[(\alpha_{ij})\right]_{m\times n}$ $=\left[\left(u_{\mu_{ij}},u_{\vartheta_{ij}}\right)\right]_{m\times n}$ where $u_{\mu_{ij}},u_{\vartheta_{ij}}\in\overline{U}=\{u_q|u_0\leq u_q\leq u_t,q\in[0,t]\}$ are the MD and NMD of $x\in X$ to the LPF design matrix F awarded by a specialist based on how an alternative χ_i satisfies the criteria T_j.

To efficiently address the MADM challenges using suggested LPF aggregation operators, the algorithm is formulated as follows:

Step 1: Formulate the LPF decision matrix $F=\left[(\alpha_{ij})\right]_{m\times n}$ containing entries as LPFNs associated with the given alternatives on all attributes.

Step 2: To calculate the aggregated value φ_i for all alternatives, use the LPFDWA operator as follows:

$$\varphi_i=LPFDWA(\alpha_{i1},\alpha_{i2},...,\alpha_{in})$$

$$=\left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{\mu_{ij}^2}{t^2-\mu_{ij}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{t^2-\vartheta_{ij}^2}{\vartheta_{ij}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

In a similar way, to calculate the aggregated value φ_i for all alternatives, use the LPFDWG operator as follows:

$$\varphi_i=LPFDWG(\alpha_{i1},\alpha_{i2},...,\alpha_{in})$$

$$=\left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{t^2-\mu_{ij}^2}{\mu_{ij}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}},u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{\vartheta_{ij}^2}{t^2-\vartheta_{ij}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}\right).$$

Step 3: Utilize the formula specified in Definition 9 to determine the score value for each α_i.

Step 4: Evaluate each alternative based on its corresponding score value to identify the most optimal choice.

The schematic representation of the proposed decision-making framework is presented in Figure 1.

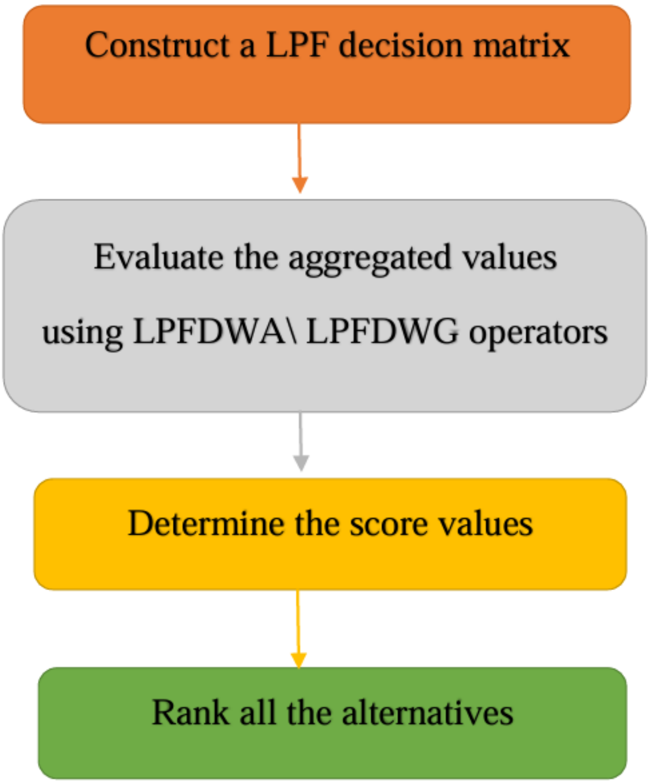

Figure 1. Flowchart of the proposed decision-making algorithm using LPF Dombi weighted aggregation operators for solving MADM problems.

5.1. Illustrative example

Over the last several years, e-commerce firms have experienced an explosion in the daily quantity of packages for transportation [56,57], along with a rise in the number of highly demanding consumer requirements. Regarding this matter, the transportation method got excessively costly, especially for the last kilometer. In order to maintain competitiveness and address growing needs, firms started exploring novel autonomous delivery solutions for the final means of transportation. One such potential alternative for the logistics sector is the use of autonomous unmanned aerial vehicles [58−60] or drones. Their purpose is to autonomously carry products from one point to another. These devices utilize advanced technology like GPS, sensors, and artificial intelligence to efficiently navigate and deliver products. Recent advancements, such as anomaly detection methods based on wavelet decomposition and stacked denoising autoencoders [61], as well as adaptive control mechanisms based on deep reinforcement learning [62], have further enhanced the operational efficiency of drones. Drone delivery has the potential to be used in a wide range of businesses, offering the advantages of quicker and more economical deliveries for the final stage of the journey. With the proven effectiveness of drones in surveillance and remote sensing, drone delivery systems are now being developed as an innovative alternative to decrease both delivery costs and delivery time.

There are numerous uses for drone delivery in various fields, including retail, healthcare, and emergency services. For example, drones are used to quickly deliver items purchased online in order to maintain total client loyalty. A drone may carry essential medicine to distant or challenging locations, thus closing the healthcare gap. In addition, drones play a significant role in responding to critical circumstances, as drones carrying supplies fly quicker than transit allows. In the future, autonomous drone sharing systems will become an inevitable logistical solution, particularly due to the new laws and recommendations implemented by the Flight World Organization regarding the organization of operations for these unique unmanned airline systems. With the increasing need for fast and effective delivery solutions, it is crucial to optimize drone delivery strategies. This case study explores the complexities of improving efficiency and dependability, recognizing the potential advantages these developments might offer to different businesses. By implementing strategic optimization techniques, drone delivery possesses the capacity not only to fulfill but exceed existing expectations, therefore influencing the future of logistics and transportation.

The principal aims of this case study are:

1) To optimize operational efficiency by streamlining the delivery procedure, consequently decreasing the duration of item transportation and operating expenditures.
2) To enhance the reliability of drone delivery operations through the reduction of disruptions and the improvement of overall dependability.
3) To optimize technological components and refine algorithms for obstacle detection, navigation systems, battery management, and course planning.

The following key features are taken into consideration to accomplish the goals of drone delivery:

1) Incorporating technology:
- Work along with prominent drone manufacturers to integrate advanced navigation technologies, guaranteeing accurate and effective routes.
- Utilize innovative obstacle detection technology to improve safety and prevent collisions in the delivery process.

2) Optimizing battery management:
- Perform extensive testing to evaluate and enhance battery efficacy, with the goal of achieving longer flight durations and minimizing the need for frequent recharging.
- Establish collaborations with battery technology firms to include state-of-the-art techniques for storing and utilizing energy.

3) Route planning algorithm:
- Capitalize on the expertise and knowledge of data scientists and AI specialists in order to enhance the performance of route planning algorithms. This involves integrating up-to-date environmental data, traffic patterns, and delivery density as determinants of influence.
- Utilize machine learning techniques to iteratively enhance and adjust route planning tactics using past data.

The application of the refined drone delivery techniques produces substantial outcomes:

- The refined path-planning algorithms result in a noteworthy 20% decrease in typical delivery times.
- The implementation of advanced obstacle detection technologies results in a 30% reduction in occurrences and enhanced overall safety.
- By implementing battery management improvements, flight durations are extended by 25%, thereby lowering the frequency of recharging.

Drone delivery effectively enhances the efficiency and dependability of its delivery techniques, showcasing a dedication to innovation in the logistics sector. Drone delivery has established itself as a leader in efficient and dependable last-mile deliveries by adopting sophisticated technology and consistently improving its operations. This allows it to effectively satisfy the ever-changing demands of contemporary logistics. The effectiveness of this optimization technique highlights the revolutionary capacity for incorporating technology into conventional delivery systems.

5.2. Numerical implementation

In this section, we solve the MADM problem of optimizing the efficiency and reliability of drone delivery technology using the proposed AOs within the context of the LPF environment.

A certain transportation firm faces difficulties in resolving the problem of transporting crucial and delicate material from one place to another. Significant quantities of goods are transported annually using conventional manual techniques and specialized logistics. However, these approaches become insufficient in situations that need careful handling and urgent delivery. In order to address these issues, the firm hires an expert with the objective of identifying the most suitable firm from among various drone delivery firms that could effectively fulfill their requirements.

Let $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ be the set of four drone delivery firms (alternatives) selected by the expert where

1) χ_1: Flytrex
2) χ_2: Amazon Prime Air
3) χ_3: Zipline
4) χ_4: Google wing.

The effectiveness and dependability of these alternatives are assessed based on several attributes $\{T_1, T_2, T_3, T_4\}$, where

1) $T_1 =$ Weather conditions: They impact drone delivery by monitoring environmental factors like wind, rain, and temperature to ensure safe and efficient operations.
2) $\mathrm{T}_2 =$ Obstacle detection: It enables drones to identify and avoid physical barriers using advanced sensors, ensuring collision-free navigation during delivery.
3) $\mathrm{T}_3 =$ Battery management: It tracks and optimizes power usage to extend flight range, ensuring drones complete deliveries and return safely before running out of charge.
4) $\mathrm{T}_4 =$ Path planning: It determines the safest and most efficient delivery routes by considering real-time data on obstacles, weather, and airspace regulations.

The linguistic terms set $U = \{u_0 = extremely\, unreliable, u_1 = very\, unreliable, u_2 = unreliable, u_3 = slightly\, unreliable, u_4 = neutral, u_5 = slightly\, reliable, u_6 = reliable, u_7 = very\, reliable, u_8 = extremely\, reliable\}$.

Step 1: Summarize the expert assessments provided by the decision-maker for each alternative in the form of an LPF decision matrix (see Table 3) with respect to every attribute, having entries as LPFNs.

Table 3. Decision matrix representing expert ratings of drone delivery firms using LPFNs.

Alternatives	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	$\boldsymbol{T_3}$	$\boldsymbol{T_4}$
$\boldsymbol{\chi_1}$	(u_2, u_3)	(u_3, u_5)	(u_3, u_2)	(u_4, u_5)
$\boldsymbol{\chi_2}$	(u_6, u_1)	(u_7, u_3)	(u_6, u_3)	(u_5, u_5)
$\boldsymbol{\chi_3}$	(u_4, u_4)	(u_7, u_1)	(u_3, u_7)	(u_3, u_5)
$\boldsymbol{\chi_4}$	(u_7, u_1)	(u_4, u_5)	(u_2, u_4)	(u_3, u_3)

The associated weight vector of four attributes is $w = (0.1,0.3,0.4,0.2)^T$, where $\sum_{j=1}^{4} w_j = 1$.

Step 2: Obtain the aggregated value φ_i for each alternative χ_i by implementing the LPFDWA operator on the data presented in Table 3, with the parameter set to $\Psi = 3$.

Let $i = 1$, then

$$\varphi_1 = LPFDWA(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})$$

$$= \left(u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(\frac{\mu_{1j}^2}{t^2-\mu_{1j}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(\frac{t^2-\vartheta_{1j}^2}{\vartheta_{1j}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$= \left(u_{8\sqrt{1-\frac{1}{1+\left\{0.1\left(\frac{2^2}{8^2-2^2}\right)^3+0.3\left(\frac{3^2}{8^2-3^2}\right)^3+0.4\left(\frac{3^2}{8^2-3^2}\right)^3+0.2\left(\frac{4^2}{8^2-4^2}\right)^3\right\}^{\frac{1}{3}}}}}, \right.$$

$$\left. u_{8\sqrt{\frac{1}{1+\left\{0.1\left(\frac{8^2-3^2}{3^2}\right)^3+0.3\left(\frac{8^2-5^2}{5^2}\right)^3+0.4\left(\frac{8^2-2^2}{2^2}\right)^3+0.2\left(\frac{8^2-5^2}{5^2}\right)^3\right\}^{\frac{1}{3}}}}} \right);$$

$$\varphi_1 = (u_{3.3909}, u_{2.2980}).$$

The results generated by this process are shown in Table 4.

Table 4. Aggregated values of alternatives using the LPFDWA operator.

Alternatives	$\boldsymbol{\varphi_i}$
$\boldsymbol{\chi_1}$	$(u_{3.3915}, u_{2.2980})$
$\boldsymbol{\chi_2}$	$(u_{6.6618}, u_{1.4532})$
$\boldsymbol{\chi_3}$	$(u_{6.6272}, u_{1.2175})$
$\boldsymbol{\chi_4}$	$(u_{6.2115}, u_{1.4542})$

Step 3: Determine the score for each φ_i following the guidelines of Definition 9.

For instance, $i = 1$. We have

$$S(\varphi_1) = u_{\sqrt{\frac{t^2-\vartheta^2}{2t^2-\mu^2-\vartheta^2}}},$$

$$= u_{\sqrt{\frac{8^2-2.2980^2}{2(8)^2-3.3915^2-2.2980^2}}},$$

$$= u_{0.7266}.$$

Similarly, the score values of the remaining alternatives are calculated by adopting the above mathematical procedure.

$$S(\varphi_2) = u_{0.8713},\ S(\varphi_3) = u_{0.8699} \text{ and } S(\varphi_4) = u_{0.8419}.$$

Step 4: As the score values for φ_i were determined, the ranking sequence was established as $S(\varphi_2) > S(\varphi_3) > S(\varphi_4) > S(\varphi_1)$. Consequently, all feasible alternatives have been ranked in the following order:

$$\chi_2 > \chi_3 > \chi_4 > \chi_1.$$

Hence, Amazon Prime Air emerges as a preferred alternative.

Likewise, the MADM problem within the framework of the LPFDWG operator is addressed through the following steps:

Step 2: Obtain the aggregated value φ_i, for each alternative χ_i by implementing the LPFDWG operator on the data presented in Table 3, with the parameter set to $\Psi = 3$.

Let $i = 1$, then

$$\varphi_1 = LPFDWG(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})$$

$$= \left(u_{t\sqrt{\frac{1}{1+\left\{\sum_{j=1}^{4} w_j \left(\frac{t^2-\mu_{1j}^2}{\mu_{1j}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}}, u_{t\sqrt{1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{\vartheta_{1j}^2}{t^2-\vartheta_{1j}^2}\right)^{\Psi}\right\}^{\frac{1}{\Psi}}}}} \right)$$

$$= \left(u_{8\sqrt{\frac{1}{1+\left\{0.1\left(\frac{8^2-2^2}{2^2}\right)^3+0.3\left(\frac{8^2-3^2}{3^2}\right)^3+0.4\left(\frac{8^2-3^2}{3^2}\right)^3+0.2\left(\frac{8^2-4^2}{4^2}\right)^3\right\}^{\frac{1}{3}}}}}, \right.$$

$$\left. u_{8\sqrt{1-\frac{1}{1+\left\{0.1\left(\frac{3^2}{8^2-3^2}\right)^3+0.3\left(\frac{5^2}{8^2-5^2}\right)^3+0.4\left(\frac{2^2}{8^2-2^2}\right)^3+0.2\left(\frac{5^2}{8^2-5^2}\right)^3\right\}^{\frac{1}{3}}}}} \right).$$

$$\varphi_1 = (u_{2.6739}, u_{4.6477}).$$

The results generated by this process are shown in Table 5.

Table 5. Aggregated values of alternatives using the LPFDWG operator.

Alternatives	$\boldsymbol{\varphi_i}$
$\boldsymbol{\chi_1}$	$(u_{2.6739}, u_{4.6477})$
$\boldsymbol{\chi_2}$	$(u_{5.6554}, u_{4.2060})$
$\boldsymbol{\chi_3}$	$(u_{3.2160}, u_{6.7255})$
$\boldsymbol{\chi_4}$	$(u_{2.2907}, u_{4.4763})$

Step 3: Determine the score for each φ_i following the guidelines of Definition 9.

For instance, $i = 1$. We have

$$S(\varphi_1) = u_{\sqrt{\frac{t^2-\vartheta^2}{2t^2-\mu^2-\vartheta^2}}},$$

$$= u_{\sqrt{\frac{8^2-4.6477^2}{2(8)^2-2.6739^2-4.6477^2}}},$$

$$= u_{0.6536}.$$

Similarly, the score values of the remaining alternatives are calculated by adopting the above mathematical procedure.

$$S(\varphi_2) = u_{0.7689},\ S(\varphi_3) = u_{0.5090} \text{and}\ S(\varphi_4) = u_{0.6542}.$$

Step 4: As the score values for φ_i were determined, the ranking sequence was established as $S(\varphi_2) > S(\varphi_4) > S(\varphi_1) > S(\varphi_3)$. Consequently, all feasible alternatives have been ranked in the following order:

$$\chi_2 > \chi_4 > \chi_1 > \chi_3.$$

Hence, Amazon Prime Air emerges as a preferred alternative.

The above discussion shows that Amazon Prime Air is the most efficient and reliable drone delivery firm.

6. Comparative analysis

In this section, a comparison study is undertaken to evaluate the performance of the presented methodologies. This analysis is performed by comparing these strategies with the established approaches, namely the LIF weighted averaging (LIFWA) and LIF weighted geometric (LIFWG) operators delineated in [24], as well as the LPF weighted averaging (LPFWA) and LPF weighted geometric (LPFWG) operators elaborated in [39]. The outcomes of this comparison are displayed in Tables 6 and 7.

Table 6. Comparative analysis of aggregated values across established and proposed methods.

Operators	χ_1	χ_2	χ_3	χ_4
LIFWA [24]	$(u_{3.3102}, u_{3.2931})$	$(u_{6.2383}, u_{2.9770})$	$(u_{4.9829}, u_{3.4517})$	$(u_{3.7177}, u_{3.5151})$
LIFWG [24]	$(u_{3.0514}, u_{3.8340})$	$(u_{6.0589}, u_{3.3311})$	$(u_{3.9811}, u_{5.4345})$	$(u_{3.0266}, u_{3.9424})$
LPFWA [39]	$(u_{3.1638}, u_{3.2931})$	$(u_{6.2604}, u_{2.9770})$	$(u_{5.2208}, u_{3.4517})$	$(u_{3.9992}, u_{3.5151})$
LPFWG [39]	$(u_{3.0514}, u_{4.0035})$	$(u_{6.0590}, u_{3.4572})$	$(u_{3.9811}, u_{5.7117})$	$(u_{3.0266}, u_{4.0467})$
LPFDWA	$(u_{3.3915}, u_{2.2980})$	$(u_{6.6618}, u_{1.4532})$	$(u_{6.6272}, u_{1.2175})$	$(u_{6.2115}, u_{1.4542})$
LPFDWG	$(u_{2.6739}, u_{4.6477})$	$(u_{5.6554}, u_{4.2060})$	$(u_{3.2160}, u_{6.7255})$	$(u_{2.2907}, u_{4.4763})$

Table 7. Scoring and ranking analysis of alternatives utilizing existing and novel approaches.

Operators	χ_1	χ_2	χ_3	χ_4	Ranking
LIFWA [24]	$u_{0.7036}$	$u_{0.8290}$	$u_{0.7554}$	$u_{0.7122}$	$\chi_2 > \chi_3 > \chi_4 > \chi_1$
LIFWG [24]	$u_{0.6885}$	$u_{0.8122}$	$u_{0.6459}$	$u_{0.6849}$	$\chi_2 > \chi_1 > \chi_4 > \chi_3$
LPFWA [39]	$u_{0.7043}$	$u_{0.8304}$	$u_{0.7657}$	$u_{0.7199}$	$\chi_2 > \chi_3 > \chi_4 > \chi_1$
LPFWG [39]	$u_{0.6835}$	$u_{0.8099}$	$u_{0.6281}$	$u_{0.6817}$	$\chi_2 > \chi_1 > \chi_4 > \chi_3$
LPFDWA	$u_{0.7266}$	$u_{0.8713}$	$u_{0.8699}$	$u_{0.8419}$	$\chi_2 > \chi_3 > \chi_4 > \chi_1$
LPFDWG	$u_{0.6536}$	$u_{0.7689}$	$u_{0.5090}$	$u_{0.6542}$	$\chi_2 > \chi_4 > \chi_1 > \chi_3$

Comparison 1: The methodology described in [24] has several limitations in effectively representing the relationship between MD and NMD in a cohesive framework. On the other hand, the LPF Dombi operators overcome this drawback by offering a flexible approach that incorporates both MDs and NMDs. This is achieved if the aggregate of their squares remains less than the number of elements of the set. So, the LPF Dombi aggregation operators are useful when dealing with uncertainty in a more complex way, going beyond what the normal LIF environment can do. The dynamic nature of this tool makes it highly helpful in decision-making procedures that include complex and imprecise information.

Comparison 2: The methodology described in [39] demonstrates inherent limitations in addressing intricate decision-making paradigms because it fails to deal with subtle complexities. On the other hand, the proposed models are flexible as they provide a more adaptable method. By inserting a parameter, it becomes possible to modify the effects of different factors, thereby offering a more customized approach to dealing with uncertainty in the aggregation process. In situations where a more precise management of ambiguity is required, this flexibility is particularly important, as it surpasses the capacity of a standard aggregation operator.

Comparison 3: Spearman's rank correlation coefficient is frequently employed to assess the strength and direction of relationships between ranked variables. However, Spearman's technique is confined to ordinal data and inadequately addresses uncertainty or imprecision, rendering it less appropriate for the analysis of LPF information. Our proposed solutions, including LPFD AOs, offer a more robust and adaptable approach by accounting for both membership and non-membership grades. In contrast to Spearman's correlation, our methodologies maintain expert viewpoints presented in linguistic terms and adeptly describe uncertainty and ambiguity in decision-making. Moreover, our methodologies consider the interdependencies across factors, facilitating a more thorough assessment. Our strategies produce more precise and dependable findings by including the entirety of ambiguity and expert subjectivity. Consequently, in comparison to Spearman's rank correlation coefficient, our methodologies provide a more advanced framework for addressing intricate decision-making challenges that involve LPF data.

The preceding explanation clearly emphasizes the greater applicability of the strategies provided in this article in comparison to the current procedures. This is demonstrated by the ability of the recently introduced LPF Dombi aggregation operator to adeptly accommodate shifts in preferences, thus mitigating the inherent loss of information associated with traditional LPF operators. The integration of parametric parameters highlights the versatility of the newly developed operators. The LPF Dombi aggregation operators offer an improved approach compared to the LPFS and LIFS. It successfully handles linguistic terms and Pythagorean uncertainty simultaneously, making it applicable in many decision-making scenarios.

The graphical interpretation of the information given in Tables 6 and 7 is depicted in Figure 2.

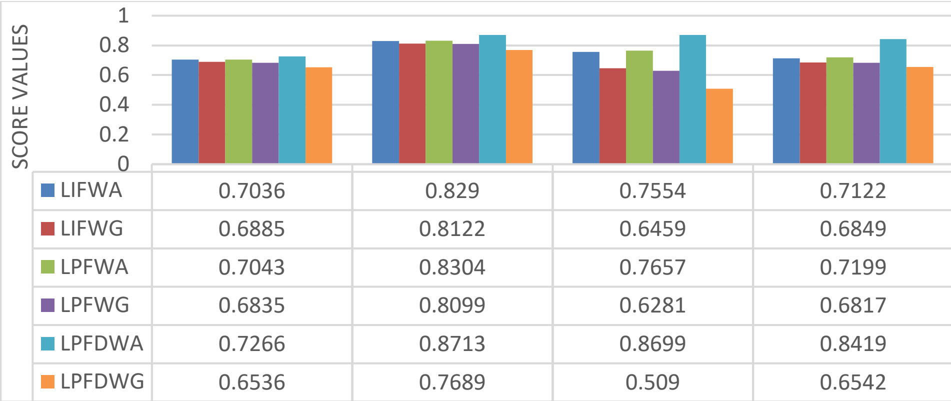

Figure 2. Pictorial depiction of the ranking of the alternatives using different operators.

6.1. Advantages of the current study

The current study provides significant advantages compared to existing methods by addressing essential gaps in decision-making amid ambiguity. The suggested LPF Dombi operators offer a flexible and accurate approach for managing both quantitative and qualitative data. Incorporating LPFNs allows experts to articulate their preferences in linguistic terms, hence enhancing the decision-making process to be more intuitive and representative of real-world situations. The novelty of this study is to facilitate significant flexibility through adaptable parameters by integrating the Dombi t-norm and t-conorm, enabling enhanced adjustment among competing attributes. This contrasts with traditional methods, which frequently depend on inflexible algebraic frameworks and lack mechanisms to adequately represent hesitancy or expert uncertainty. The suggested operators utilize weighted measures to reflect the differing significance of expert opinions, hence improving their relevance to decision-making issues. In comparison to traditional methods like the SAW technique, which use a linear model and have challenges with unknown data, the proposed model provides a more adaptable, competitive, and comprehensible framework.

Table 8 describes the advantages of the proposed strategies compared to the existing techniques.

Table 8. Advantages of the proposed methods compared to the existing approaches.

Criteria	Traditional operators (LIFWA, SAW, LPFWA)	Proposed LPF Dombi operators (LPFDWA, LPFDWG)
Handling of uncertainty	Limited modeling of hesitation and ambiguity	Effectively captures hesitation and expert uncertainty via LPFNs
Data type support	Primarily quantitative or limited linguistic	Integrates both qualitative (linguistic) and quantitative information
Aggregation Flexibility	Rigid algebraic operations	Adaptive aggregation via Dombi t-norm/t-conorm with adjustable parameters
Linguistic interpretability	Less intuitive, lacks real-world alignment	Enhanced realism through direct use of linguistic scales
Weight incorporation	Uniform or fixed weights	Weighted mechanism reflecting varied expert importance
Decision robustness	Prone to bias under conflicting data	Balanced outcomes through flexible compensation strategies
Ranking stability	May result in less stable rankings	Improved ranking discrimination and consistency

6.2. Managerial implications of the current study

The LPF Dombi aggregation operators discussed in this work provide significant management insights for optimizing drone delivery systems through improved decision-making in uncertain and dynamic contexts. These models enable managers to evaluate qualitative and quantitative data, enhancing route planning, reducing delivery time, and increasing operational accuracy. They facilitate risk assessment by recognizing potential threats, such as drone issues, enabling proactive measures to assure dependable delivery. The adaptability of the Dombi operators facilitates scaling, rendering them appropriate for both small and large-scale activities. This extensive decision-support system assists managers in resource allocation, mitigates risks, and improves customer satisfaction via accurate and efficient delivery solutions.

7. Conclusions

In this study, the notions of the LPFDWA and LPFDWG operators have been introduced. An enhanced score function has been formulated to select the optimally appropriate choice in a decision-making process. Additionally, various structural characteristics of newly defined operators have been analyzed. A detailed mathematical protocol has been devised for MADM challenges using recently proposed techniques under LPF data. Furthermore, the efficacy of these freshly articulated techniques has been demonstrated by providing a solution to the MADM issue of selecting the most efficient drone delivery service. A comparison study has also been conducted to demonstrate the usefulness of the suggested methods in relation to the current body of information.

7.1. Study limitations

The techniques presented in this study have notable advantages but are not without certain shortcomings:

- These techniques fail when the squared sum of linguistic MD and NMD is greater than the square cardinality of the linguistic terms set.
- This study lacks a dynamic adjustment mechanism, limiting its suitability for varying-interval data collection in MADM.

7.2. Potential goals of future studies

To address the limitations of this work, future research will extend the scope of recently introduced methodologies to broader models, including linguistic Fermatean fuzzy sets and linguistic dynamic Fermatean fuzzy sets. This article's suggested methods will find effective application across diverse domains such as AI and healthcare diagnostics, environmental modeling, and human–machine interaction. Moreover, the scope of recently proposed techniques will be explored in [63,64].

Author contributions

Asima Razzaque: Conceptualization, formal analysis, methodology, writing − original draft, review & editing; Umme Kalsoom: Conceptualization, investigation, methodology, writing − original draft, review & editing; Dilshad Alghazzawi: Formal analysis, validation, review & editing; Abdul

Razaq: Supervision, project administration, review & editing; Ghaliah Alhamzi: Resources, visualization, review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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