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Research article

Multiple solutions to a nonlocal sub-Laplacian system with critical growth and logarithmic perturbation

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Abstract: In this paper, we studied the existence of solutions for a nonlocal sub-Laplacian system with critical growth and logarithmic perturbation. That is to say, by using the symmetric mountain pass lemma, we proved that under some suitable conditions, the nonlocal sub-Laplacian system admits a sequence $\{z_k\}$ of nontrivial solutions satisfying $\lim_{k\to\infty} z_k = 0$. To the best of our knowledge, this result is new even in the Euclidean case.

Keywords: sub-Laplacian system; critical growth; logarithmic perturbation; symmetric mountain pass lemma; multiple solutions; Heisenberg group
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1. Introduction

The first Heisenberg group $\mathbb{H}^1$, whose points are denoted by $\xi = (z,t) = (x,y,t)$, is the Lie group $(\mathbb{R}^3, \circ)$ with the composition law defined by

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)).$$

Let $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. Then the sub-Laplace operator $\Delta_H = X^2 + Y^2$ and the horizontal gradient operator $\nabla_H = (X, Y)$. For any vector-valued function (ω, ν), the horizontal divergence $\mathrm{div}_H(\omega, \nu) = X\omega + Y\nu$. It is well-known that the sub-Laplace operator Δ_H is degenerate at any point of $\mathbb{H}^1$, and there are many different characteristics compared with the classical Laplacian operator Δ. Here, we refer the readers to [1] for more details on the Heisenberg group and the sub-Laplacian operator.

Now, let us begin to consider the following nonlocal sub-Laplacian system:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla_H u|^2 \mathrm{d}\xi\right)\Delta_H u = \lambda_1 u + \mu_1 u \log u^2 + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u, & \xi \in \Omega, \\ -\left(a + b\int_\Omega |\nabla_H v|^2 \mathrm{d}\xi\right)\Delta_H v = \lambda_2 v + \mu_2 v \log v^2 + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2} v, & \xi \in \Omega, \\ u = v = 0, & \xi \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{H}^1$, $\alpha,\beta > 1$, $\alpha + \beta = 4$, and λ_i, μ_i $(i = 1, 2)$ are some parameters. Due to the presence of integral term $\int_\Omega |\nabla_H \cdot|^2 \mathrm{d}\xi$ and the condition $\alpha+\beta = 4$, problem (1.1) is a nonlocal critical sub-Laplacian system and, of course, is also a typical Kirchhoff type system with sub-Laplacian operator Δ_H. This system arises in many different research fields such as Brownian motion, kinetic theory of gases, mathematical models in finance, and in human vision (see [1, Some Historical Overviews]). In addition, it is obvious that if $b = 0$, problem (1.1) reduces to the following sub-Laplacian system with critical exponent and logarithmic perturbation:

$$\begin{cases} -a\Delta_H u = \lambda_1 u + \mu_1 u \log u^2 + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u, & \xi \in \Omega, \\ -a\Delta_H v = \lambda_2 v + \mu_2 v \log v^2 + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2} v, & \xi \in \Omega, \\ u = v = 0, & \xi \in \partial\Omega. \end{cases} \tag{1.2}$$

Further, if $\lambda_i = \lambda$, $\mu_i = 0$, $\alpha = 4$, and $\beta = 0$ in (1.2), then problem (1.2) becomes the classical Brezis-Nirenberg problem with the sub-Laplacian operator, which was studied by Loiudice in [2].

On the other hand, we would love to mention that there are many studies on the logarithmic perturbation problems in the Euclidean case, e.g., Deng et al. [3] studied the following equation:

$$\begin{cases} -\Delta u = \lambda u + \mu u \log u^2 + |u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$. The authors proved that problem (1.3) has a positive least energy solution if $\lambda \in \mathbb{R}$, and $\mu > 0$. Later, Liu et al. [4] employed the subcritical approximation method to prove the existence of sign-changing solutions to problem (1.3) when $n \geq 6$, $\lambda \in \mathbb{R}$ and $\mu > 0$. Additionally, Hajaiej et al. [5, 6] investigated a coupled elliptic system and established the existence and nonexistence results under other conditions. Here, we refer the interested readers to [7–11] and the references therein for more details on the existence and multiplicity of solutions for partial differential equations with logarithmic-type nonlinearities in the Euclidean case. Nonetheless, as far as we know, there are few works dealing with the logarithmic perturbation problems shaped like problem (1.1) in the Heisenberg group. We have only found the reference [12], which obtained normalized solutions in the L^p-subcritical case for a critical Choquard equation with logarithmic nonlinearity.

2. Main results

In this section, we will present our main results. First, we define the Folland-Stein space $S_0^1(\Omega)$, which is a Hilbert space, by the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|^2 = \int_\Omega |\nabla_H u|^2 \mathrm{d}\xi$. Let H be the product spaces $S_0^1(\Omega) \times S_0^1(\Omega)$ with the norm $\|z\|^2 = \|u\|^2 + \|v\|^2$ for all $z = (u, v) \in H$. Then, H is also a Hilbert space, and if $1 \leq p < 4$, the embedding $H \hookrightarrow L^p(\Omega) \times L^p(\Omega)$ is continuous and compact, while if $p = 4$, the embedding is just continuous (see [13]). Let

$$\Phi_i(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda_i}{2}\int_\Omega u^2 \mathrm{d}\xi - \frac{\mu_i}{2}\int_\Omega u^2\left(\log u^2 - 1\right)\mathrm{d}\xi, \quad \forall u \in S_0^1(\Omega) \setminus \{0\}.$$

Then, the energy functional associated with problem (1.1) is defined by

$$J(z) = \Phi_1(u) + \Phi_2(v) - \frac{1}{4}\int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi, \quad \forall z = (u, v) \in H \setminus \{(0, 0)\}. \tag{2.1}$$

It follows from a standard argument that $J \in C^1(H \setminus \{(0, 0)\}, \mathbb{R})$. Moreover, for any $\phi = (\varphi, \psi) \in H$,

$$\langle J'(z), \phi\rangle = \langle \Phi_1'(u), \varphi\rangle + \langle \Phi_2'(v), \psi\rangle - \frac{\alpha}{4}\int_\Omega |u|^{\alpha-2}|v|^\beta u\varphi \mathrm{d}\xi - \frac{\beta}{4}\int_\Omega |u|^\alpha |v|^{\beta-2} v\psi \mathrm{d}\xi. \tag{2.2}$$

Next, it is necessary for us to clarify a basic concept. That is, we say that a function $z = (u, v) \in H$ is called a solution to problem (1.1) if and only if $\langle J'(z), \phi\rangle = 0$ for all $\phi = (\varphi, \psi) \in H$. Then, it follows from $\alpha + \beta = 4$ that every critical point of the energy functional $J(z)$ corresponds to a solution of problem (1.1).

In what follows, let S denote the best Sobolev embedding constant from $S_0^1(\Omega)$ to $L^4(\Omega)$, i.e.,

$$S = \inf_{u \in S_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_H u|^2 \mathrm{d}\xi}{\left(\int_\Omega |u|^4 \mathrm{d}\xi\right)^{\frac{1}{2}}}. \tag{2.3}$$

Therefore, by Young's inequality, (2.3), and $\alpha + \beta = 4$, it is easily seen that the following constant $S_{\alpha\beta}$ is well-defined:

$$S_{\alpha\beta} = \inf_{z \in H\setminus\{(0,0)\}} \frac{\int_\Omega \left(|\nabla_H u|^2 + |\nabla_H v|^2\right)\mathrm{d}\xi}{\left(\int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi\right)^{\frac{1}{2}}}. \tag{2.4}$$

Meanwhile, for convenience, let $|\Omega|$ denote the measure of Ω and

$$A_0 = \left\{(\lambda_1, \mu_1; \lambda_2, \mu_2) : \lambda_i, \mu_i \in \mathbb{R}, \frac{a^2 S_{\alpha\beta}^2}{2 - bS_{\alpha\beta}^2} + \frac{e|\Omega|}{2}\sum_{i=1}^{2} \mu_i e^{-\frac{\lambda_i}{\mu_i}} > 0\right\}.$$

Theorem 2.1. *Let Ω be a smooth bounded domain of $\mathbb{H}^1$, $a, b \geq 0$, $a + b > 0$, $\alpha, \beta > 1$, and $\alpha + \beta = 4$. If $\mu_1, \mu_2 < 0$ and one of the following (i) and (ii) holds:*

(i) $bS_{\alpha\beta}^2 - 2 < 0$ and $(\lambda_1, \mu_1; \lambda_2, \mu_2) \in A_0$;

(ii) $bS_{\alpha\beta}^2 - 2 \geq 0$ and $\lambda_i \in \mathbb{R}$, $i = 1, 2$.

Then, problem (1.1) *has a sequence $\{z_k\}$ of nontrivial solutions such that $J(z_k) \leq 0$, $z_k \neq 0$, for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} z_k = 0$.*

Obviously, when b is equal to zero in Theorem 2.1, the case (*ii*) is impossible. Therefore, from Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let Ω be a smooth bounded domain of $\mathbb{H}^1$, $a > 0$, $\alpha, \beta > 1$, and $\alpha + \beta = 4$. If $\mu_1, \mu_2 < 0$ and*

$$a^2 S_{\alpha\beta}^2 + e|\Omega| \sum_{i=1}^{2} \mu_i e^{-\frac{\lambda_i}{\mu_i}} > 0, \quad \lambda_i \in \mathbb{R},$$

then problem (1.2) *has a sequence $\{z_k\}$ of nontrivial solutions such that $J(z_k) \leq 0$, $z_k \neq 0$, for all $k \in \mathbb{N}$ and* $\lim_{k\to\infty} z_k = 0$.

Remark 2.3. *It is well-known that the logarithmic term $\lambda_i u \log u^2$ usually exerts a significantly greater influence than the term $\lambda_i u$ on the existence of solutions. Furthermore, when the Heisenberg group and the Euclidean space share the same topological dimension of $n = 3$, the critical exponent in the Heisenberg group case, denoted as $2_Q^* = 4$, is strictly smaller than that in the Euclidean case, which is $2^* = 6$ (see [14]). Therefore, it is more complicated and difficult to study the existence and multiplicity of nontrivial solutions for problem* (1.1)*, and the relevant results of the Euclidean case cannot be directly generalized to the Heisenberg group case. Moreover, to the best of our knowledge, Theorem 2.1 remains novel even in the Euclidean case and when b is equal to zero, e.g., Corollary 2.2 is not covered by the recent results presented in [5, Theorem 1.1]. In addition, Theorem 2.1 is established on the first Heisenberg group. In fact, it is also true on the n-th Heisenberg group when we appropriately adjust the range of certain parameters.*

The structure of the rest of this paper is as follows: In Section 3, we introduce some notations and basic facts. In Section 4, we will use Lemma 3.2 to prove the existence of a sequence $\{u_k\}$ of solutions to problem 1.1, more specifically, Section 4 is divided into two Subsections 4.1 and 4.2, where the proofs of the cases (*i*) and (*ii*) in Theorem 2.1 are presented in Subsections 4.1 and 4.2, respectively.

3. Preliminaries

In this section, we collect some definitions and basic facts. First, we give some basic inequalities, whose proofs are elementary and are omitted. Namely, for any $t > 0$ and $\delta > 0$, we have

$$-\frac{1}{e} \leq t \log t \leq \frac{1}{\delta e} t^{\delta+1}. \tag{3.1}$$

Meanwhile, for any $0 < t \leq 1$, we also have

$$|t \log t| \leq \frac{1}{e}. \tag{3.2}$$

For any $t_1, t_2 \in \mathbb{R}$, we also have

$$\frac{1}{2}(t_1^2 + t_2^2)^2 \leq t_1^4 + t_2^4 \leq (t_1^2 + t_2^2)^2. \tag{3.3}$$

In addition, if $\{u_n\}$ is a bounded sequence in $S_0^1(\Omega)$ such that $u_n \to u$ almost everywhere in Ω, then

$$\lim_{n\to\infty} \int_\Omega u_n^2 \log u_n^2 \mathrm{d}\xi = \int_\Omega u^2 \log u^2 \mathrm{d}\xi, \tag{3.4}$$

and

$$\lim_{n\to\infty}\int_{\Omega} u_n\phi\log u_n^2 \mathrm{d}\xi = \int_{\Omega} u\phi\log u^2 \mathrm{d}\xi, \quad \forall \phi\in S_0^1(\Omega). \tag{3.5}$$

In fact, it follows from $u_n \to u$ almost everywhere in Ω that $u_n^2\log u_n^2 \to u^2\log u^2$ almost everywhere in Ω as $n\to\infty$. Moreover, it follows from (3.1) and (3.2) that

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega}\left|u_n^2\log u_n^2\right|\mathrm{d}\xi &\le \lim_{n\to\infty}\frac{1}{e}\int_{\Omega}\left(1+\frac{1}{\delta}|u_n|^{2(1+\delta)}\right)\mathrm{d}\xi \\ &= \frac{1}{e}\int_{\Omega}\left(1+\frac{1}{\delta}|u|^{2(1+\delta)}\right)\mathrm{d}\xi,\end{aligned} \tag{3.6}$$

where δ is chosen to satisfy $0<\delta<1$. It follows from (3.6) and Lebesgue's dominated convergence theorem that (3.4) is true. Meanwhile, by an argument similar to the proof of (3.4), we can also deduce that (3.5) is true.

Definition 3.1. *[15, Section 7] Let E be a Banach space. We say a subset A of E is symmetric if and only if $z\in A$ and $-z\in A$. For a closed symmetric set A that does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to* $\mathbb{R}^k\backslash\{0\}$. *If there does not exist a finite such k, set $\gamma(A)=+\infty$. Moreover, set $\gamma(\emptyset)=0$.*

Let Γ_k denote the family of closed symmetric subsets A of E such that $0\notin A$ and $\gamma(A)\ge k$. Next, we introduce the the symmetric mountain pass lemma by Kajikiya [16, Theorem 1].

Lemma 3.2. *[16, Theorem 1] Let E be a Banach space and $J\in C^1(E,\mathbb{R})$. Assume that*

(A_1) $J(z)$ is even (i.e., $J(-z)=J(z)$ for all $z\in E$) and bounded from below, $J(0)=0$, and $J(z)$ satisfies the global $(P.S.)$ condition. That is, any sequence $\{z_k\}$ in E such that $\{J(z_k)\}$ is bounded and $\lim_{k\to\infty}J'(z_k)=0$ *has a convergent subsequence.*

(A_2) For each $k\in\mathbb{N}$, there exists an $A_k\in\Gamma_k$ such that $\sup_{z\in A_k}J(z)<0$.

Then there exists a sequence $\{z_k\}$ of critical points such that

$$J(z_k)\le 0,\ z_k\ne 0,\ \text{and}\ \lim_{k\to\infty}z_k=0.$$

Remark 3.3. *In Lemma 3.2, the functional $J(z)$ is required to satisfy the global $(P.S.)$ condition. However, by a careful examination of the proof of [16, Theorem 1], it is sufficient that the functional $J(z)$ satisfies the local $(P.S.)_c$ condition, e.g., for some $c^*>0$, any sequence $\{z_k\}$ in E satisfying* $\lim_{k\to\infty}J(z_k)=c<c^*$ *and* $\lim_{k\to\infty}J'(z_k)=0$ *has a convergent subsequence. That is to say, if we replace the $(P.S.)$ condition in (A_1) with the $(P.S.)_c$ condition, then Lemma 3.2 also holds.*

In addition, due to the fact that problem (1.1) is critical, the proof of the $(P.S.)_c$ condition usually needs the following concentration compactness principle for the system proved by Pucci and Temperini [17].

Lemma 3.4. *[17, Theorem 1.2] Let $\{z_n\}$ be a sequence in H. Suppose that there exists $z\in H$ and two bounded nonnegative Radon measures μ and ν on $\mathbb{H}^1$ such that $z_n\rightharpoonup z$ weakly in H and*

$$\begin{cases}|\nabla_H u_n|^2+|\nabla_H v_n|^2\mathrm{d}\xi\rightharpoonup\mu \ \text{ in }\ \mathcal{M}(\mathbb{H}^1),\\ |u_n|^{\alpha}|v_n|^{\beta}\mathrm{d}\xi\rightharpoonup\nu\ \text{ in }\ \mathcal{M}(\mathbb{H}^1),\end{cases}$$

where $\mathcal{M}(\mathbb{H}^1)$ is the space of all bounded regular Borel measures on $\mathbb{H}^1$. Then, there exists an at most countable set J, a family of points $\{\xi_k\} \subset \mathbb{H}^1$, and two families of nonnegative numbers $\{\mu_k\}_{k\in J}$ and $\{\nu_k\}_{k\in J}$ such that

$$\mu \geq (|\nabla_H u|^2 + |\nabla_H v|^2)\mathrm{d}\xi + \sum_{k\in J}\mu_k\delta_{\xi_k},$$

$$\nu = |u|^\alpha|v|^\beta d\xi + \sum_{k\in J}\nu_k\delta_{\xi_k}, \quad \mu_k \geq S_{\alpha\beta}\nu_k^{\frac{1}{2}} \text{ for all } k \in J,$$

where $S_{\alpha\beta}$ is as in (2.4) *and δ_{ξ_k} is the Dirac function at the point ξ_k of $\mathbb{H}^1$.*

Remark 3.5. *Lemma 3.4 is a special case of [17, Theorem 2.1]. Meanwhile, the concentration compactness principle is an important tool for addressing nonlinear problems that lack compactness. Therefore, in addition to [17, Theorem 2.1], we would also love to mention the concentration compactness principle of [18, Theorem 1.1], which can be applied to the study of certain elliptic systems with critical exponents and Hardy terms in the Heisenberg group.*

4. Proof of main results

In this section, we will use Lemma 3.2 to obtain the existence of nontrivial solutions for problem 1.1. More specially, this section will be divided into two Subsections 4.1 and 4.2 later. The proofs of the cases (*i*) and (*ii*) of Theorem 2.1 are treated in Subsections 4.1 and 4.2, respectively.

To begin with, it is necessary for us to clarify that $\langle \Phi_1'(u), \varphi\rangle$ and $\langle \Phi_2'(v), \psi\rangle$ in (2.2) for any $\phi = (\varphi, \psi) \in H$. That is,

$$\langle \Phi_1'(u), \varphi\rangle = (a + b\|u\|^2)\int_\Omega \nabla_H u \nabla_H \varphi \mathrm{d}\xi - \lambda_1 \int_\Omega u\varphi \mathrm{d}\xi - \mu_1 \int_\Omega u\varphi \log u^2 \mathrm{d}\xi, \tag{4.1}$$

$$\langle \Phi_2'(v), \psi\rangle = (a + b\|v\|^2)\int_\Omega \nabla_H v \nabla_H \psi \mathrm{d}\xi - \lambda_2 \int_\Omega v\psi \mathrm{d}\xi - \mu_2 \int_\Omega v\psi \log v^2 \mathrm{d}\xi. \tag{4.2}$$

In addition, for any $u \in S_0^1(\Omega) \setminus \{0\}$, by (3.1), one has

$$\begin{aligned}\Phi_i(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda_i}{2}\int_\Omega u^2 \mathrm{d}\xi - \frac{\mu_i}{2}\int_\Omega u^2\left(\log u^2 - 1\right)\mathrm{d}\xi \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu_i}{2}e^{1-\frac{\lambda_i}{\mu_i}}\int_\Omega (e^{\frac{\lambda_i}{\mu_i}-1}u^2)\log(e^{\frac{\lambda_i}{\mu_i}-1}u^2)\mathrm{d}\xi \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\mu_i}{2}e^{-\frac{\lambda_i}{\mu_i}}|\Omega|.\end{aligned} \tag{4.3}$$

In the following, we divided into two subsections to complete the proof of Theorem 2.1.

4.1. The case $bS_{\alpha\beta}^2 < 2$

In this subsection, we always assume $bS_{\alpha\beta}^2 < 2$ and $(\lambda_1, \mu_1; \lambda_2, \mu_2) \in A_0$. We will complete the proof of the case (*i*) of Theorem 2.1. To begin with, it follows from (2.1), (2.4), (3.3), and (4.3) that

$$J(z) = \Phi_1(u) + \Phi_2(v) - \frac{1}{4}\int_\Omega |u|^\alpha|v|^\beta \mathrm{d}\xi$$

$$
\begin{aligned}
&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\mu_1}{2}e^{-\frac{\lambda_1}{\mu_1}}|\Omega| - \frac{1}{4}\int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi \\
&\quad + \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 + \frac{\mu_2}{2}e^{-\frac{\lambda_2}{\mu_2}}|\Omega| \\
&\geq \frac{a}{2}\|z\|^2 - \frac{2-bS_{\alpha\beta}^2}{8S_{\alpha\beta}^2}\|z\|^4 + \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \qquad (4.4)
\end{aligned}
$$

Now, we define a function $f : [0, +\infty) \to \mathbb{R}$, $\forall t \in [0, +\infty)$,

$$
f(t) = \frac{a}{2}t^2 - \frac{2-bS_{\alpha\beta}^2}{8S_{\alpha\beta}^2}t^4 + \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \qquad (4.5)
$$

It is easily seen that there exists a $t_M = \sqrt{\frac{2aS_{\alpha\beta}^2}{2-bS_{\alpha\beta}^2}} > 0$, such that

$$
\begin{cases}
f'(t) > 0, & t \in (0, t_M), \\
f'(t) = 0, & t = t_M, \\
f'(t) < 0, & t \in (t_M, +\infty).
\end{cases} \qquad (4.6)
$$

In other words, $f(t)$ attains its maximum at t_M and the maximum

$$
f_M = f(t_M) = \frac{a^2S_{\alpha\beta}^2}{4-2bS_{\alpha\beta}^2} + \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}.
$$

Since $(\lambda_1, \mu_1; \lambda_2, \mu_2) \in A_0$, that is,

$$
\frac{a^2S_{\alpha\beta}^2}{2-bS_{\alpha\beta}^2} + \frac{e|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}} > 0. \qquad (4.7)
$$

Note that $\mu_i (i = 1, 2) < 0$ and $\frac{e}{2} > 1$. Therefore, it follows from (4.7) that

$$
\begin{aligned}
\frac{a^2S_{\alpha\beta}^2}{4-2bS_{\alpha\beta}^2} + \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}} &= \frac{1}{2}\left(\frac{a^2S_{\alpha\beta}^2}{2-bS_{\alpha\beta}^2} + |\Omega|\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}\right) \\
&\geq \frac{1}{2}\left(\frac{a^2S_{\alpha\beta}^2}{2-bS_{\alpha\beta}^2} + \frac{e|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}\right) \\
&> 0. \qquad (4.8)
\end{aligned}
$$

That is to say, the maximum f_M of $f(t)$ is greater than zero. Let

$$
f_0 = \frac{1}{2}f_M = \frac{a^2S_{\alpha\beta}^2}{8-4bS_{\alpha\beta}^2} + \frac{|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}.
$$

Then f_0 is also greater than zero. Therefore, it follows from (4.6) that there exists a unique $t_0 \in (0, t_M)$ such that $f(t_0) = f_0$. In the following, we need an appropriate truncation on the critical term. That is to say, we may define a smooth truncation function $g : [0, +\infty) \to \mathbb{R}$ such that

$$g(t) = \begin{cases} 1, & t \in [0, t_0^2], \\ 0 \le g(t) \le 1, & t \in [t_0^2, t_M^2], \\ \frac{bS_{\alpha\beta}^2}{2} + \frac{2aS_{\alpha\beta}^2}{t} + \frac{4S_{\alpha\beta}^2}{t^2}\left(\frac{|\Omega|}{2}\sum_{i=1}^2 \mu_i e^{-\frac{\lambda_i}{\mu_i}} - f_M\right), & t \in [t_M^2, +\infty). \end{cases}$$

Now, let us define the following functional in H. That is, for any $z = (u, v) \in H$,

$$J_g(z) = \Phi_1(u) + \Phi_2(v) - \frac{1}{4} g(\|z\|^2) \int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi. \tag{4.9}$$

Then, it follows from [19, Theorem C.1] and the definition of g that $J_g \in C^1(H, \mathbb{R})$. Furthermore, for any $\phi = (\varphi, \psi) \in H$, we have

$$\begin{aligned} \langle J_g'(z), \phi \rangle = & \langle \Phi_1'(u), \varphi \rangle + \langle \Phi_2'(v), \psi \rangle - \frac{1}{2} g'(\|z\|^2) \int_\Omega \nabla_H u \nabla_H \varphi \mathrm{d}\xi \int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi \\ & - \frac{\alpha}{4} g(\|z\|^2) \int_\Omega |u|^{\alpha-2} |v|^\beta u \varphi \mathrm{d}\xi - \frac{\beta}{4} g(\|z\|^2) \int_\Omega |u|^\alpha |v|^{\beta-2} v \psi \mathrm{d}\xi \\ & - \frac{1}{2} g'(\|z\|^2) \int_\Omega \nabla_H v \nabla_H \psi \mathrm{d}\xi \int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi. \end{aligned} \tag{4.10}$$

Lemma 4.1. *Let $a, b \ge 0$, $a + b > 0$ and $\mu_1, \mu_2 < 0$. Then $J_g(z)$ is bounded from below in H.*

Proof. It follows from (2.4), (3.3), and (4.3) that, for any $z = (u, v) \in H \backslash \{(0, 0)\}$,

$$\begin{aligned} J_g(z) &= \Phi_1(u) + \Phi_2(v) - \frac{1}{4} g(\|z\|^2) \int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi \\ &\ge \frac{a}{2}\|z\|^2 + \frac{b}{8}\|z\|^4 - \frac{1}{4S_{\alpha\beta}^2} g(\|z\|^2)\|z\|^4 + \frac{|\Omega|}{2}\sum_{i=1}^2 \mu_i e^{-\frac{\lambda_i}{\mu_i}} \\ &= \frac{a}{2}\|z\|^2 - \left(\frac{1}{4S_{\alpha\beta}^2} g(\|z\|^2) - \frac{b}{8}\right)\|z\|^4 + \frac{|\Omega|}{2}\sum_{i=1}^2 \mu_i e^{-\frac{\lambda_i}{\mu_i}}. \end{aligned} \tag{4.11}$$

Define $\widetilde{f} : [0, +\infty) \to \mathbb{R}$, $\forall t \in [0, +\infty)$,

$$\widetilde{f}(t) = \frac{a}{2}t^2 - \left(\frac{1}{4S_{\alpha\beta}^2} g(t^2) - \frac{b}{8}\right)t^4 + \frac{|\Omega|}{2}\sum_{i=1}^2 \mu_i e^{-\frac{\lambda_i}{\mu_i}}. \tag{4.12}$$

By the definition of $g(t)$ and a straightforward calculation, we have

$$\begin{cases} \widetilde{f}(t) = f(t), & t \in [0, t_0], \\ \widetilde{f}(t) \ge f(t), & t \in [t_0, t_M], \\ \widetilde{f}(t) = f_M, & t \in [t_M, +\infty). \end{cases} \tag{4.13}$$

Hence, it follows from (4.11) and (4.13) that

$$J_g(z) \geq \widetilde{f}(\|z\|^2) \geq \widetilde{f}(0) = \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}},$$

which proves that $J_g(z)$ is bounded from below in H.

Lemma 4.2. *Under the conditions of this subsection, assume that*

$$c < c^* = \frac{a^2 S_{\alpha\beta}^2}{8 - 4bS_{\alpha\beta}^2} + \frac{e|\Omega|}{8}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \tag{4.14}$$

Then $J_g(z)$ satisfies the $(P.S.)_c$ condition.

Proof. Let $z_n = (u_n, v_n)$ be a $(P.S.)_c$ sequence of $J_g(z)$. That is,

$$J_g(z_n) \to c \ \text{ and } \ J_g'(z_n) \to 0 \ \text{ as } \ n \to \infty. \tag{4.15}$$

It follows from (4.14) and the first expression of (4.15) that

$$J_g(z_n) < \frac{a^2 S_{\alpha\beta}^2}{8 - 4bS_{\alpha\beta}^2} + \frac{e|\Omega|}{8}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}, \tag{4.16}$$

for n large enough. In addition, by the definition of $f(t)$, f_0, and (4.13), for any $t \in [t_0, +\infty)$, one has

$$\widetilde{f}(t) \geq f_0 = \frac{1}{2}f_M = \frac{a^2 S_{\alpha\beta}^2}{8 - 4bS_{\alpha\beta}^2} + \frac{|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \tag{4.17}$$

Now we claim that $\|z_n\| < t_0$ for n large enough.

In fact, if not, it follows from (4.17), $\mu_i < 0$, and the process of (4.11) that

$$\begin{aligned} J_g(z_n) \geq \widetilde{f}(\|z_n\|^2) &\geq \frac{a^2 S_{\alpha\beta}^2}{8 - 4bS_{\alpha\beta}^2} + \frac{|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}} \\ &\geq \frac{a^2 S_{\alpha\beta}^2}{8 - 4bS_{\alpha\beta}^2} + \frac{e|\Omega|}{8}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \end{aligned} \tag{4.18}$$

Obviously, there is a contradicts between (4.16) and (4.18), that is to say, $\|z_n\| < t_0$ for n large enough. Therefore, it follows from the definition of $g(t)$ and the claim that

$$g(\|z_n\|^2) = 1 \ \text{ and } \ g'(\|z_n\|^2) = 0, \tag{4.19}$$

for n large enough, that by (3.1), (4.1), (4.2), (4.19), $\alpha + \beta = 4$, and the definition of $J_g(z_n)$, for n large enough, we have

$$J_g(z_n) - \frac{1}{4}\langle J_g'(z_n), z_n\rangle = \Phi_1(u_n) - \frac{1}{4}\langle \Phi_1'(u_n), u_n\rangle + \Phi_2(v_n) - \frac{1}{4}\langle \Phi_1'(v_n), v_n\rangle$$

$$
\begin{aligned}
&= \frac{a}{4}\|u_n\|^2 - \frac{\lambda_1}{4}\|u_n\|_2^2 + \frac{\mu_1}{2}\|u_n\|_2^2 - \frac{\mu_1}{4}\int_\Omega u_n^2 \log u_n^2 \mathrm{d}\xi \\
&\quad + \frac{a}{4}\|v_n\|^2 - \frac{\lambda_2}{4}\|v_n\|_2^2 + \frac{\mu_2}{2}\|v_n\|_2^2 - \frac{\mu_2}{4}\int_\Omega v_n^2 \log v_n^2 \mathrm{d}\xi \\
&= \frac{a}{4}\|z_n\|^2 - \frac{\mu_1}{4}e^{2-\frac{\lambda_1}{\mu_1}}\int_\Omega e^{\frac{\lambda_1}{\mu_1}-2}u_n^2 \log\left(e^{\frac{\lambda_1}{\mu_1}-2}u_n^2\right)\mathrm{d}\xi \\
&\quad - \frac{\mu_2}{4}e^{2-\frac{\lambda_2}{\mu_2}}\int_\Omega e^{\frac{\lambda_2}{\mu_2}-2}v_n^2 \log\left(e^{\frac{\lambda_2}{\mu_2}-2}v_n^2\right)\mathrm{d}\xi \\
&\geq \frac{a}{4}\|z_n\|^2 + \frac{e|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}.
\end{aligned} \tag{4.20}
$$

It follows from (4.15) and (4.20) that $\{z_n\}$ is bounded in H. Hence, we may assume that

$$
(u_n, v_n) \rightharpoonup (u, v) \ \text{ weakly in } \ H. \tag{4.21}
$$

Passing to the subsequence, we may also assume that

$$
\begin{cases}
u_n \rightharpoonup u, \ \ v_n \rightharpoonup v \ \text{ weakly in } \ L^4(\Omega), \\
u_n \to u, \ \ v_n \to v \ \text{ almost everywhere in } \ \Omega, \\
u_n \to u, \ \ v_n \to v \ \text{ strongly in } \ L^p(\Omega) \ \text{ for } \ 1 \leq p < 4.
\end{cases} \tag{4.22}
$$

Further, since $J_g(z_n) = J_g(|z_n|)$, we may also assume that $u_n, v_n \geq 0$ and $u, v \geq 0$. Hence, it follows from the concentration compactness principle for system (Lemma 3.4) that

$$
\begin{cases}
(|\nabla_H u_n|^2 + |\nabla_H v_n|^2)d\xi \rightharpoonup \mu \geq (|\nabla_H u|^2 + |\nabla_H v|^2)\mathrm{d}\xi + \sum_{k\in J}\mu_k\delta_{\xi_k}, \\
|u_n|^\alpha |v_n|^\beta \mathrm{d}\xi \rightharpoonup \nu = |u|^\alpha |v|^\beta d\xi + \sum_{k\in J}\nu_k\delta_{\xi_k},
\end{cases} \tag{4.23}
$$

and

$$
\mu_k, \ \nu_k \geq 0, \qquad \mu_k \geq S_{\alpha\beta}\nu_k^{\frac{1}{2}}, \tag{4.24}
$$

where J is an at most countable index set, $\xi_k \in \Omega$, and δ_{ξ_k} is the Dirac function at the point ξ_k of $\mathbb{H}^1$.

We claim that $J = \emptyset$.

In fact, if we assume on the contrary that $J \neq \emptyset$ and fix $k \in J$, then, for $\rho > 0$ small enough, it follows from [20, Lemma 3.2] that there exists a cut-off function $\phi : C_0^\infty(\Omega) \to [0, 1]$ such that

$$
\begin{cases}
\phi(\xi) = 1 \ \text{ for any } \ \xi \in B_H(\xi_k, \rho), \\
\phi(\xi) = 0 \ \text{ for any } \ \xi \in \Omega \backslash B_H(\xi_k, 2\rho), \\
|\nabla_H \phi(\xi)| \leq \frac{2}{\rho} \ \text{ for any } \ \xi \in \Omega.
\end{cases}
$$

For convenience, let

$$
L_\phi^i(u) = \left(a + b\|u\|^2\right)\int_\Omega u\nabla_H u \nabla_H \phi \mathrm{d}\xi - \lambda_i \int_\Omega u^2\phi \mathrm{d}\xi - \mu_i \int_\Omega u^2 \phi \log u^2 \mathrm{d}\xi, \ \ i = 1, 2.
$$

Then, for n large enough, it follows from (3.3), (4.1), (4.2), and (4.19) that

$$
\langle J_g'(z_n), \phi z_n\rangle = \langle \Phi_1'(u_n), \phi u_n\rangle + \langle \Phi_2'(v_n), \phi v_n\rangle - \int_\Omega |u_n|^\alpha |v_n|^\beta \phi \mathrm{d}\xi
$$

$$
\begin{aligned}
&= \left(a + b\|u_n\|^2\right)\int_\Omega |\nabla_H u_n|^2\phi \mathrm{d}\xi + \left(a + b\|v_n\|^2\right)\int_\Omega |\nabla_H v_n|^2\phi \mathrm{d}\xi \\
&\quad - \int_\Omega |u_n|^\alpha |v_n|^\beta \phi \mathrm{d}\xi + L^1_\phi(u_n) + L^2_\phi(v_n) \\
&\geq a\int_\Omega (|\nabla_H u_n|^2 + |\nabla_H v_n|^2)\phi \mathrm{d}\xi + b\left(\int_\Omega |\nabla_H u_n|^2\phi \mathrm{d}\xi\right)^2 + b\left(\int_\Omega |\nabla_H v_n|^2\phi \mathrm{d}\xi\right)^2 \\
&\quad - \int_\Omega |u_n|^\alpha |v_n|^\beta \phi \mathrm{d}\xi + L^1_\phi(u_n) + L^2_\phi(v_n) \\
&\geq a\int_\Omega (|\nabla_H u_n|^2 + |\nabla_H v_n|^2)\phi \mathrm{d}\xi + \frac{b}{2}\left(\int_\Omega (|\nabla_H u_n|^2 + |\nabla_H v_n|^2)\phi \mathrm{d}\xi\right)^2 \\
&\quad - \int_\Omega |u_n|^\alpha |v_n|^\beta \phi \mathrm{d}\xi + L^1_\phi(u_n) + L^2_\phi(v_n). \qquad (4.25)
\end{aligned}
$$

Now, let us estimate the last two terms $L^1_\phi(u_n)$ and $L^2_\phi(v_n)$ in (4.25). In fact, it follows from the boundedness of $\{z_n\}$, Hölder's inequality, and the definition of ϕ that

$$
\begin{aligned}
|L^1_\phi(u_n)| &\leq C_1\left\{\int_{\Omega'} |u_n \nabla_H u_n \nabla_H \phi| \mathrm{d}\xi + \int_{\Omega'} u_n^2\phi \mathrm{d}\xi + \int_{\Omega'} u_n^2 \phi \log u_n^2 \mathrm{d}\xi\right\} \\
&\leq C_2\left\{\left(\int_{\Omega'} |u_n \nabla_H \phi|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} + \int_{\Omega'} u^2\phi \mathrm{d}\xi + \int_{\Omega'} u^2\phi \log u^2 \mathrm{d}\xi\right\} + o_n(1) \\
&\leq C_3\left\{\left(\int_{\Omega'} |u|^4 \mathrm{d}\xi\right)^{\frac{1}{4}} + \int_{\Omega'} u^2 \mathrm{d}\xi + \int_{\Omega'} u^{2(\delta+1)} \mathrm{d}\xi\right\} + o_n(1), \qquad (4.26)
\end{aligned}
$$

where $C_i (i = 1, 2, 3)$ are some positive constants, $\Omega' = \Omega \cap B(\xi_k, 2\rho)$, and $\lim_{n\to\infty} o_n(1) = 0$. Hence, if follows from (4.26) that

$$L^1_\phi(u_n) \to 0 \text{ as } n \to \infty \text{ and } \rho \to 0. \qquad (4.27)$$

In the same way, we also have

$$L^2_\phi(v_n) \to 0 \text{ as } n \to \infty \text{ and } \rho \to 0. \qquad (4.28)$$

By (4.15), (4.23), (4.27), (4.28), and letting $n \to \infty$ and $\rho \to 0$ in (4.25), we have

$$0 \geq a\mu_k + \frac{b}{2}\mu_k^2 - \nu_k. \qquad (4.29)$$

Substituting (4.24) into (4.29) yields a range of values for ν_k, and then substituting the resulting ν_k back into (4.24), we have

$$\mu_k \geq \frac{2aS^2_{\alpha\beta}}{2 - bS^2_{\alpha\beta}}. \qquad (4.30)$$

Besides, it follows from (4.15), (4.20), and the definition of ϕ that

$$c = \lim_{n\to\infty}\left\{J_g(z_n) - \frac{1}{4}\langle J'_g(z_n), z_n\rangle\right\}$$

$$\geq \lim_{n\to\infty}\left\{\frac{a}{4}\|z_n\|^2 + \frac{e|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}\right\}$$

$$\geq \lim_{n\to\infty}\left\{\frac{a}{4}\int_\Omega(|\nabla_H u_n|^2 + |\nabla_H u_n|^2)\phi + \frac{e|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}\right\}. \tag{4.31}$$

By (4.30), $(\lambda_1, \mu_1; \lambda_2, \mu_2) \in A_0$, and letting $\rho \to 0$ in (4.31), one has

$$\begin{aligned} c &\geq \frac{a}{4}\mu_k + \frac{e|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}} \geq \frac{a^2 S_{\alpha\beta}^2}{4-2bS_{\alpha\beta}^2} + \frac{e|\Omega|}{4}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}} \\ &\geq \frac{a^2 S_{\alpha\beta}^2}{8-4bS_{\alpha\beta}^2} + \frac{e|\Omega|}{8}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \end{aligned} \tag{4.32}$$

It follows from (4.7) that (4.32) contradicts with (4.14). Therefore, $J = \emptyset$ yields

$$\int_\Omega |u_n|^\alpha |v_n|^\beta \mathrm{d}\xi \to \int_\Omega |u|^\alpha |u|^\beta \mathrm{d}\xi \ \text{ as } \ n \to \infty. \tag{4.33}$$

Hence, for n large enough, it follows from (3.4), (3.5), (4.10), (4.19), (4.22), and (4.33) that

$$\begin{aligned} \langle J_g'(z_n), (u_n - u, 0)\rangle &= \left(a + b\|u_n\|^2\right)\int_\Omega \nabla_H u_n \nabla_H(u_n - u)\mathrm{d}\xi - \lambda_1\int_\Omega u_n(u_n - u)\mathrm{d}\xi \\ &\quad - \mu_1\int_\Omega u_n(u_n - u)\log u_n^2 \mathrm{d}\xi - \frac{\alpha}{4}\int_\Omega |u_n|^{\alpha-2}|v_n|^\beta u_n(u_n - u)\mathrm{d}\xi \\ &= \left(a + b\|u_n\|^2\right)\int_\Omega \nabla_H u_n \nabla_H(u_n - u)\mathrm{d}\xi + o_n(1) \\ &= \left(a + b\|u_n\|^2\right)\left(\|u_n\|^2 - \|u\|^2\right) + o_n(1). \end{aligned} \tag{4.34}$$

In the second equals sign of (4.34), we use the fact that

$$|u_n|^{\alpha-2}|v_n|^\beta u_n \to |u|^{\alpha-2}|v|^\beta u \ \text{ in } \ L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega), \tag{4.35}$$

which is proved in [17, 4.12]. Similarly, for n large enough, we also have

$$\langle J_g'(z_n), (0, v_n - v)\rangle = \left(a + b\|v_n\|^2\right)\left(\|v_n\|^2 - \|v\|^2\right) + o_n(1). \tag{4.36}$$

It follows from (4.15), (4.34), (4.36), and the boundedness of $\{z_n\}$ that

$$\lim_{n\to\infty}\|u_n\|^2 = \|u\|^2, \ \ \lim_{n\to\infty}\|v_n\|^2 = \|v\|^2. \tag{4.37}$$

Hence, by (4.22) and (4.37), one has

$$\lim_{n\to\infty}\|z_n - z\|^2 = \lim_{n\to\infty}(\|u_n - u\|^2 + \|v_n - v\|^2) = 0, \tag{4.38}$$

which means that $z_n \to z$ in H as $n \to \infty$. That is, $J_g(z)$ satisfies the $(P.S.)_c$ condition.

To apply Lemma 3.2, it remans to prove that the condition (A_2) in Lemma 3.2 holds.

Lemma 4.3. *Assume that $a, b \geq 0$, $a + b > 0$, and $\mu_i < 0$. Then for any $k \in \mathbb{N}$, there is an $A_k \in \Gamma_k$ such that* $\sup_{z\in A_k} J_g(z) < 0$.

Proof. Let E_k be a k-dimensional subspace of H and $z_k = (u_k, v_k) \in E_k\backslash\{(0,0)\}$, $\|z_k\| < \min\{1, t_0\}$ for all $k \in \mathbb{N}$. Then, we have $g(\|z_k\|^2) = 1$. Set

$$\phi_k = (\varphi_k, \psi_k) = \left(\frac{u_k}{\|u_k\|}, \frac{v_k}{\|v_k\|}\right),$$

and then $\|\varphi_k\| = \|\psi_k\| = 1$. Therefore, by (2.3), (3.6), and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega \varphi_k^2 \log \varphi_k^2 \mathrm{d}\xi &\leq \frac{1}{e}\int_\Omega \left(1 + \frac{1}{\delta}|\varphi_k|^{2(1+\delta)}\right)\mathrm{d}\xi = \frac{\sqrt{|\Omega|}}{e} + \frac{1}{e\delta}\int_\Omega |\varphi_k|^{2(1+\delta)}\mathrm{d}\xi \\ &\leq \frac{\sqrt{|\Omega|}}{e} + \frac{1}{e\delta}|\Omega|^{\frac{1-\delta}{2}} + \frac{1}{e\delta}\left(\int_\Omega |\varphi_k|^4 \mathrm{d}\xi\right)^{\frac{1+\delta}{2}} \\ &\leq \frac{\sqrt{|\Omega|}}{e} + \frac{1}{e\delta}|\Omega|^{\frac{1-\delta}{2}} + \frac{1}{e\delta S^{1+\delta}} \triangleq C_4, \end{aligned} \tag{4.39}$$

where δ is chosen to satisfy $0 < \delta < 1$. In addition, we pay special attention to the fact that E_k is a finite dimensional subspace of H ($\dim E_k = k$), and hence all the norms are equivalent. That is to say, there are two positive constants γ_1 and γ_2 such that

$$\gamma_1 = \gamma_1\|\varphi_k\|^2 \leq \|\varphi_k\|_2^2 \leq \gamma_2\|\varphi_k\|^2 = \gamma_2. \tag{4.40}$$

Recall that $\mu_1 < 0$, $\|\varphi_k\| = 1$, and $\log\|u_k\|^2 \leq \log\|z_k\|^2 < 0$. Then, by (4.39) and (4.40), one has

$$\begin{aligned}\Phi_1(u_k) &= \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{\lambda_1}{2}\|u_k\|_2^2 + \frac{\mu_1}{2}\|u_k\|_2^2 - \frac{\mu_1}{2}\int_\Omega u_k^2 \log u_k^2 \mathrm{d}\xi \\ &\leq \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 + \frac{|\lambda_1|}{2}\|u_k\|_2^2 - \frac{\mu_1}{2}\int_\Omega u_k^2 \log u_k^2 \mathrm{d}\xi \\ &\leq \frac{a\Lambda_1 + |\lambda_1|}{2\Lambda_1}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{\mu_1}{2}\|u_k\|^2 \log\|u_k\|^2 \int_\Omega \varphi_k^2 \mathrm{d}\xi \\ &\quad - \frac{\mu_1}{2}\|u_k\|^2 \int_\Omega |\varphi_k^2 \log \varphi_k^2| \mathrm{d}\xi \\ &\leq \frac{a\Lambda_1 + |\lambda_1|}{2\Lambda_1}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{\mu_1}{2}\|u_k\|^2 \log\|u_k\|^2 \int_\Omega \varphi_k^2 \mathrm{d}\xi - \frac{\mu_1 C_4}{2}\|u_k\|^2 \\ &\leq \frac{a\Lambda_1 + |\lambda_1| - \mu_1\Lambda_1 C_4}{2\Lambda_1}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{\mu_1\gamma_1}{2}\|u_k\|^2 \log\|z_k\|^2, \end{aligned} \tag{4.41}$$

where Λ_1 is the first eigenvalue of $-\Delta_H$ with the Dirichlet boundary condition. Meanwhile, if we replace φ_k with ψ_k in (4.39) and (4.40), then they are also true. Therefore, by an argument similar to the proof of (4.41), we also have

$$\Phi_2(v_k) \leq \frac{a\Lambda_1 + |\lambda_2| - \mu_2\Lambda_1 C_4}{2\Lambda_1}\|v_k\|^2 + \frac{b}{4}\|v_k\|^4 - \frac{\mu_2\gamma_1}{2}\|v_k\|^2 \log\|z_k\|^2. \tag{4.42}$$

Hence, it follows from the definition of $g(t)$, (4.41), and (4.42) that

$$\begin{aligned}J_g(z_k) &= \Phi_1(u_k) + \Phi_2(v_k) - \frac{1}{4}g(\|z_k\|^2)\int_\Omega |u_k|^\alpha|v_k|^\beta \mathrm{d}\xi \le \Phi_1(u_k) + \Phi_2(v_k)\\ &\le \frac{a_0}{2}\|z_k\|^2 + \frac{b}{4}\|z_k\|^4 - \frac{b_0}{2}\|z_k\|^2 \log\|z_k\|^2\\ &= \frac{\|z_k\|^2}{2}\left(a_0 + \frac{b}{2}\|z_k\|^2 - b_0 \log\|z_k\|^2\right),\end{aligned} \tag{4.43}$$

where $b_0 = \max\{\mu_1\gamma_1, \mu_2\gamma_1\}$ and

$$a_0 = \max\left\{\frac{a\Lambda_1 + |\lambda_1| - \mu_1\Lambda_1 C_4}{2\Lambda_1}, \frac{a\Lambda_1 + |\lambda_2| - \mu_2\Lambda_1 C_4}{2\Lambda_1}\right\}.$$

Noting that $b_0 < 0$ and $\lim_{t\to 0^+}\log t = -\infty$, then by (4.43), we may choose $0 < \rho_k < t_0$ and $M_k > 0$ such that if $z_k \in E_k$ and $\|z_k\| = \rho_k$, $J_g(z_k) \le -M_k < 0$ for each $k \in \mathbb{N}$. Let

$$A_k = \{z_k \in E_k : \|z_k\| = \rho_k\}.$$

It follows from the Borsuk-Ulam theorem (see [19, Proposition 5.4]) that $\gamma(A_k) = k$. Therefore, we have $A_k \in \Gamma_k$ and $\sup_{z\in A_k} J_g(z) < 0$. This completes the proof of Lemma 4.3.
Proof of Theorem 2.1 with the case (i). Obviously, by the conditions of Theorem 2.1, one has

$$J_g \in C^1(H, \mathbb{R}),\ J_g(0) = 0, \text{ and } J_g(z) \text{ is even in } H.$$

Moreover, if $bS^2_{\alpha\beta} - 2 < 0$ and $(\lambda_1, \mu_1; \lambda_2, \mu_2) \in A_0$, Lemmas 4.1–4.3 hold, where the $(P.S.)$ condition in (A_1) is replaced by the $(P.S.)_c$ condition. Therefore, it follows from Lemma 3.2 and Remark 3.5 that there exists a sequence of critical points $\{z_k\}$ of $J_g(z)$ such that

$$J_g(z_k) \le 0,\ z_k \ne 0,\ \text{and}\ \lim_{k\to\infty} z_k = 0.$$

Hence, passing to the subsequence, we may choose that the norm of the sequence of critical points $\{z_k\}$ is less than t_0. Note that

$$J_g(z) = J(z) \text{ for any } z \in H \setminus \{(0,0)\} \text{ and } \|z\| < t_0.$$

Therefore, we conclude that $\{z_k\}$ are also the critical points of $J(z)$. This completes the proof of Theorem 2.1 with the case (i).

4.2. The case $bS^2_{\alpha\beta} \ge 2$

In this subsection, we always assume $bS^2_{\alpha\beta} \ge 2$, $a > 0$, $\lambda_i \in \mathbb{R}$, and $\mu_i < 0$. We prove the case (ii) of Theorem 1.1. First, we give the global compactness result for the functional $J(z)$.

Lemma 4.4. *Let $bS^2_{\alpha\beta} - 2 \ge 0$, $\lambda_i \in \mathbb{R}$, and $\mu_i < 0$ $(i = 1, 2)$. Then, the functional $J(z)$ satisfies the global $(P.S.)$ condition.*

Proof. Let $\{z_n\}$ be a $(P.S.)$ sequence of $J(z)$. That is,

$$|J(z_n)| < M \in \mathbb{R}^+ \text{ and } J'(z_n) \to 0 \text{ as } n \to \infty. \tag{4.44}$$

Similar to the proof of (4.4), we also have

$$J(z_n) \geq \frac{a}{2}\|z_n\|^2 + \frac{bS_{\alpha\beta}^2 - 2}{8S_{\alpha\beta}^2}\|z_n\|^4 + \frac{|\Omega|}{2}\sum_{i=1}^{2}\mu_i e^{-\frac{\lambda_i}{\mu_i}}. \tag{4.45}$$

Noting that $bS^2 - 2 \geq 0$ and $a > 0$, then it follows from (4.44) and (4.45) that $\{z_n\}$ is bounded in H, which means that we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } H. \tag{4.46}$$

Passing to the subsequence, we may also assume that

$$\begin{cases} u_n \rightharpoonup u, \ \ v_n \rightharpoonup v \text{ weakly in } L^4(\Omega), \\ u_n \to u, \ \ v_n \to v \text{ almost everywhere in } \Omega, \\ u_n \to u, \ \ v_n \to v \text{ strongly in } L^p(\Omega) \text{ for } 1 \leq p < 4. \end{cases} \tag{4.47}$$

Let $w_n = u_n - u$ and $w_n' = v_n - v$. Passing to the subsequence, from (4.47), we may assume

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1), \quad \|v_n\|^2 = \|w_n'\|^2 + \|v\|^2 + o_n(1), \tag{4.48}$$

where $\lim_{n\to\infty} o_n(1) = 0$, and the same applies below. In addition, we claim that

$$\int_\Omega |u_n|^\alpha |v_n|^\beta \mathrm{d}\xi = \int_\Omega |w_n|^\alpha |w_n'|^\beta \mathrm{d}\xi + \int_\Omega |u|^\alpha |v|^\beta \mathrm{d}\xi + o_n(1). \tag{4.49}$$

The proof of the claim is similar to the proof of [17, Lemma 3.1]. However, for the reader's convenience, we give the details of the proof. In fact, let $I = [0, 1]$ and define $f_n, g_n : \Omega \times I \to \mathbb{R}$,

$$f_n(\xi, s) = |u_n - su|^{\alpha-2}|v_n|^\beta w_n, \quad g_n(\xi, s) = |w_n|^\alpha |v_n - sv|^{\beta-2}(v_n - sv), \quad \forall \in \Omega \times I.$$

It follows from Fubini's theorem that $f_n u \in L^1(\Omega \times I)$ and $g_n v \in L^1(\Omega \times I)$. Therefore, by Tonelli's theorem, we have

$$\begin{aligned} \alpha \iint\limits_{\Omega\times I} f_n u \mathrm{d}\xi \mathrm{d}s + \beta \iint\limits_{\Omega\times I} g_n v \mathrm{d}\xi \mathrm{d}s &= \alpha \iint\limits_{\Omega\times I} |u_n - su|^{\alpha-2}|v_n|^\beta w_n u \mathrm{d}\xi \mathrm{d}s \\ &\quad + \beta \iint\limits_{\Omega\times I} |w_n|^\alpha |v_n - sv|^{\beta-2}(v_n - sv) v \mathrm{d}\xi \mathrm{d}s \\ &= \int\limits_\Omega |v_n|^\beta \mathrm{d}\xi \int_0^1 \left(-\frac{d}{ds}|u_n - su|^\alpha\right) \mathrm{d}s \\ &\quad + \int\limits_\Omega |w_n|^\alpha \mathrm{d}\xi \int_0^1 \left(-\frac{d}{ds}|v_n - sv|^\beta\right) \mathrm{d}s \\ &= \int_\Omega |u_n|^\alpha |v_n|^\beta \mathrm{d}\xi - \int_\Omega |w_n|^\alpha |w_n'|^\beta \mathrm{d}\xi. \end{aligned} \tag{4.50}$$

Moreover, by (4.47), as $n \to \infty$, we have

$$f_n \to (1-s)^{\alpha-1}|u|^{\alpha-2}|v|^{\beta}u \ \text{ and } \ g_n \to 0 \ \text{ almost everywhere in } \ \Omega \times I. \tag{4.51}$$

It follows from (4.51) and Hölder's inequality that

$$\iint\limits_{\Omega\times I} |f_n|^{\frac{\alpha+\beta}{\alpha+\beta-1}} \mathrm{d}\xi \mathrm{d}s \le \left(\iint\limits_{\Omega\times I} |u_n - su|^{\alpha+\beta} \mathrm{d}\xi \mathrm{d}s \right)^{\frac{\alpha-1}{\alpha+\beta-1}} \left(\iint\limits_{\Omega\times I} |v_n|^{\alpha+\beta} \mathrm{d}\xi \mathrm{d}s \right)^{\frac{\beta}{\alpha+\beta-1}} \le C, \tag{4.52}$$

since $\alpha+\beta = 4$, where C is a positive constant. Similarly, we also have

$$\iint\limits_{\Omega\times I} |g_n|^{\frac{\alpha+\beta}{\alpha+\beta-1}} \mathrm{d}\xi \mathrm{d}s \le C. \tag{4.53}$$

Therefore, from (4.51)–(4.53), one has

$$f_n \rightharpoonup (1-s)^{\alpha-1}|u|^{\alpha-2}|v|^{\beta}u \ \text{ and } \ g_n \rightharpoonup 0 \ \text{ weakly in } \ L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\Omega \times I). \tag{4.54}$$

Hence, from this, we get

$$\alpha \iint\limits_{\Omega\times I} f_n u \mathrm{d}\xi \mathrm{d}s = \alpha \iint\limits_{\Omega\times I} (1-s)^{\alpha-1}|u|^{\alpha}|v|^{\beta} \mathrm{d}\xi \mathrm{d}s + o_n(1) = \int_{\Omega} |u|^{\alpha}|v|^{\beta} \mathrm{d}\xi + o_n(1), \tag{4.55}$$

and

$$\beta \iint\limits_{\Omega\times I} g_n v \mathrm{d}\xi \mathrm{d}s = o_n(1). \tag{4.56}$$

It follows from (4.50), (4.55), and (4.56) that the claim holds. That is, (4.49) holds. Therefore, it follows from (4.35) and (4.47)–(4.49) that

$$\begin{aligned}
\langle J'(z_n), (w_n, 0)\rangle &= \left(a + b\|u_n\|^2\right) \int_{\Omega} \nabla_H u_n \nabla_H w_n \mathrm{d}\xi - \lambda_1 \int_{\Omega} u_n w_n \mathrm{d}\xi \\
&\quad - \mu_1 \int_{\Omega} u_n w_n \log u_n^2 \mathrm{d}\xi - \frac{\alpha}{4} \int_{\Omega} |u_n|^{\alpha-2}|v_n|^{\beta} u_n w_n \mathrm{d}\xi \\
&= \left(a + b\|u_n\|^2\right) \|w_n\|^2 - \frac{\alpha}{4} \int_{\Omega} |w_n|^{\alpha}|w_n'|^{\beta} \mathrm{d}\xi + o_n(1) \\
&\ge \left(a + b\|u_n\|^2\right) \|w_n\|^2 - \frac{\alpha}{4S_{\alpha\beta}^2} \|(w_n, w_n')\|^4 + o_n(1).
\end{aligned} \tag{4.57}$$

In the same way, we also have

$$\langle J'(z_n), (0, w_n')\rangle \ge \left(a + b\|v_n\|^2\right) \|w_n'\|^2 - \frac{\beta}{4S_{\alpha\beta}^2} \|(w_n, w_n')\|^4 + o_n(1). \tag{4.58}$$

From (4.57), (4.58), and $\alpha+\beta = 4$, one has

$$\langle J'(z_n), (w_n, w_n')\rangle \ge \left(a + b\|u_n\|^2\right) \|w_n\|^2 + \left(a + b\|v_n\|^2\right) \|w_n'\|^2$$

$$
\begin{aligned}
& -\frac{\alpha+\beta}{4S_{\alpha\beta}^{2}}\|(w_n, w_n')\|^4 + o_n(1) \\
\geq & \left(a + b\|u_n\|^2\right)\|w_n\|^2 + \left(a + b\|v_n\|^2\right)\|w_n'\|^2 \\
& -\frac{2}{S_{\alpha\beta}^{2}}(\|w_n\|^4 + \|w_n'\|^4) + o_n(1) \\
= & a\|w_n\|^2 + \frac{bS_{\alpha\beta}^{2} - 2}{S_{\alpha\beta}^{2}}\|w_n\|^4 + b\|w_n\|^2\|u\|^2 + o_n(1) \\
& + a\|w_n'\|^2 + \frac{bS_{\alpha\beta}^{2} - 2}{S_{\alpha\beta}^{2}}\|w_n'\|^4 + b\|w_n'\|^2\|v\|^2.
\end{aligned} \tag{4.59}
$$

Now, let

$$\lim_{n\to\infty}\|w_n\|^2 = l_1 \text{ and } \lim_{n\to\infty}\|w_n'\|^2 = l_2.$$

Therefore, from (4.44), the boundedness of $\{(w_n, w_n')\}$, and letting $n \to \infty$ in (4.59), we get

$$0 \geq al_1 + \frac{bS_{\alpha\beta}^{2} - 2}{S_{\alpha\beta}^{2}}l_1^2 + bl_1\|u\|^2 + al_2 + \frac{bS_{\alpha\beta}^{2} - 2}{S_{\alpha\beta}^{2}}l_2^2 + bl_2\|v\|^2. \tag{4.60}$$

Note that $bS_{\alpha\beta}^{2} - 2 \geq 0$ and $l_i (i = 1, 2)$ are all nonnegative. Hence, it follows from (4.60) that $l_1 = l_2 = 0$. That is to say,

$$u_n \to u \text{ and } v_n \to v \text{ in } S_0^1(\Omega) \text{ as } n \to \infty,$$

which means that $z_n \to z$ strongly in H. This completes the proof of Lemma 4.4.
Proof of Theorem 2.1 with the case (ii). Obviously, by the conditions of Theorem 2.1, one has

$$J \in C^1(H, \mathbb{R}),\ J(0) = 0, \text{ and } J(z) \text{ is even in } H.$$

Moreover, if $bS_{\alpha\beta}^{2} \geq 2$, $\lambda_i \in \mathbb{R}$, and $\mu_i < 0$ $(i = 1, 2)$, Lemma 4.3 is valid. In addition, it follows from the argument similar to the proof of Lemma 4.3 that for all $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{z\in A_k} J(z) < 0$. Therefore, it follows from Lemma 3.2 that $J(z)$ has a sequence of critical points $\{z_k\}$ converging to zero with $J(z_k) \leq 0$ and $z_k \neq 0$ for all $k \in \mathbb{N}$. This completes the proof of Theorem 2.1 with the case (ii).

5. Conclusions

In this study, we have investigated a nonlocal sub-Laplacian system with critical growth and logarithmic perturbation. By employing the symmetric mountain pass lemma, an appropriate truncation of the critical term, and a careful analysis of the structure of the energy functional, we obtained the sufficient conditions for the existence of a sequence $\{z_k\}$ of nontrivial solutions satisfying $\lim_{k\to\infty} z_k = 0$ for this system. The results of this paper are new even for the Euclidean case.

Author contributions

Yu-Cheng An: Writing-original draft, investigation, formal analysis; Bi-Jun An: writing-review and editing. All authors have real and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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