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Research article

Convergence conditions for extreme solutions of an impulsive differential system

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Abstract: In this paper, we discuss the existence and convergence conditions of extreme solutions for a first-order differential boundary value system with an impulsive integral condition. By employing quasi-linear and monotone iterative methods, the existence, uniform convergence, and quadratic convergence for solution sequences are obtained.

Keywords: functional differential equation; impulsive integral condition; monotone iterative technique; quasilinearization method; quadratic convergence
Mathematics Subject Classification: 34B37, 34B05

1. Introduction

Impulsive differential systems are efficient mathematical models to describe a wide range of natural phenomena in real world, such as biology, physics, population dynamics, etc [1–3]. In fact, impulsive differential equations are basic tools to study evolutionary processes where the state undergoes sudden jumps at certain moments [4, 5]. In the past few years, an increasing number of scholars have become interested in the theory of impulsive differential equations and their potential applications [6–8].

Traditionally, we notice that in many impulsive differential equations, authors generally suppose that the impulsive condition is $\triangle z(\varepsilon_k) = A_k(z(\varepsilon_k))$, and the abrupt change in state at ε_k is closely dependent on the left limit of $z(\varepsilon_k)$. In recent years, the impulsive integral condition has received widespread attention [9–11]. For example, Tariboon proposed a new impulsive integral condition, and explored the existence conditions for extreme solutions in a functional differential system under these conditions [12]. Recently, Li et al. studied the stability of a fractional differential equation with a non-instantaneous integral impulse [13]. Compared with conventional impulsive conditions, impulsive integral conditions are more comprehensive and flexible in describing phenomena. They not only take the time or state at which impulses occur into account, but also focus on the cumulative effects of impulses throughout the entire action process [11–13]. Impulsive integral conditions can incorporate the cumulative effects of multiple instantaneous mutations into the system model through integration operations, thus more comprehensively reflecting the impact of impulses on the dynamic characteristics of the system, thereby covering multiple dimensions such as impulse intensity, duration, and action mode, and having broader applications in solving practical problems [11–13]. However, we noticed that the corresponding theory of the impulsive integral condition is far from complete; more specifically, its application in functional differential equations still has many works to consider.

The technique of combining a monotone iteration (MIT) with upper and lower solutions (ULS) is usually used to obtain solution sequences which uniformly converge to extreme solutions of impulsive differential equations [14,15]. For the utilization of the MIT method in functional differential equations with impulses, see [16,17]. Moreover, if we want to obtain the rapid convergence of solution sequences, such as quadratic convergence, we need to use the method of quasilinearization (QLM) [18–20]. The QLM method is a very powerful approximation technique, whose iteration sequences are not only uniform in convergence, but also meet rapid convergence [18–20]. For the application of the QLM method in impulsive functional differential equations, see [21]. We note that scholars generally use monotonic iterative methods to prove the existence of extreme solutions in impulsive integral condition problems [12, 22]; however, the QLM method is not readily used in this field.

Based on previous studies, in this paper, we develop a new impulsive integral condition, and use the quasilinearization method coupled with the MIT method to discuss the existence, uniform convergence, and quadratic convergence of solution sequences for a periodic boundary value differential system. The specific differential equations are described as follows:

$$\begin{cases} \upsilon'(\varepsilon) = f(\varepsilon, \upsilon(\varepsilon), \upsilon(\mathfrak{T}(\varepsilon))), & \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle\upsilon(\varepsilon_k) = A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \upsilon(s)ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} \upsilon(s)ds\right), & k = 1, 2, \cdots, m, \\ \upsilon(0) = \upsilon(P), \end{cases} \tag{1}$$

where $f \in C(\Theta \times R^2, R), 0 \leq \mathfrak{T}(\varepsilon) \leq \varepsilon$, $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < \varepsilon_{m+1} = P, A_k \in C(R, R)$, $0 < q_{k-1} \leq (\varepsilon_k - \varepsilon_{k-1})/2, \quad 0 \leq p_k \leq (\varepsilon_k - \varepsilon_{k-1})/2, \triangle\upsilon(\varepsilon_k) = \upsilon(\varepsilon_k^+) - \upsilon(\varepsilon_k^-), k = 1, 2, \cdots, m$. We denote $a = \max\limits_{k=1,2,\cdots,m} \{\varepsilon_k - \varepsilon_{k-1}\}$.

In Section 2, we obtain the solution expression for a class of linear systems, and prove two comparison results. In Section 3, we give the definition for a pair of ULS, and then derive main conclusions for a boundary value problem (BVP)(1).

2. Some lemmas

First, we recall the following spaces to define the solution for the BVP(1) [16, 17, 21]: letting $\Theta^- = \Theta\backslash\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$, $C(\Theta, R) = \{\upsilon : \Theta \to R; \upsilon(\varepsilon)$ maintains continuity at all points, with the exception of some ε_k, where $\upsilon(\varepsilon_k^+)$, and $\upsilon(\varepsilon_k^-)$ exist and $\upsilon(\varepsilon_k^-) = \upsilon(\varepsilon_k)$, $k = 1, \cdots, m\}$; $C'(\Theta, R) = \{\upsilon \in C(\Theta, R); \upsilon'$ remains continuous over Θ^-, where $\upsilon'(0^+), \upsilon'(P^-), \upsilon'(\varepsilon_k^+)$ and $\upsilon'(\varepsilon_k^-)$ exist, $k = 1, 2, \cdots, m\}$; $\mathbb{D}_0 = \{\upsilon \in C(\Theta, R)\}$, then $\mathbb{D}_0$ is a Banach space with a norm $\|\upsilon\|_{\mathbb{D}_0} = \sup\limits_{\varepsilon\in\Theta} |\upsilon(\varepsilon)|$; and $\mathbb{D} = C(\Theta, R) \cap C'(\Theta, R)$. If the function $\upsilon \in \mathbb{D}$ satisfies the BVP(1), then it is defined as a solution of then BVP(1).

Next, we study the following linear system:

$$\begin{cases} \upsilon'(\varepsilon)+\imath\upsilon(\varepsilon)=\sigma(\varepsilon)-\jmath\upsilon(\mathfrak{I}(\varepsilon)), \qquad \varepsilon\neq\varepsilon_k, \varepsilon\in\Theta=[0,P], \\ \triangle\upsilon(\varepsilon_k)=-O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\upsilon(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\upsilon(s)ds\right) \\ +A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right)+O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right), \\ k=1,2,\cdots,m, \\ \upsilon(0)=\upsilon(P). \end{cases} \tag{2}$$

Let $\imath>0, \jmath\geq 0, 0\leq O_k<1$, and $\sigma(\varepsilon)\in\mathbb{D}_0, \psi(\varepsilon)\in\mathbb{D}$.

Lemma 2.1. *$\upsilon\in\mathbb{D}$ is a solution of system (2) when $\upsilon\in\mathbb{D}_0$ satisfies the following:*

$$\begin{aligned} \upsilon(\varepsilon) &= \int_0^P\Gamma(\varepsilon,s)[\sigma(s)-\jmath\upsilon(\mathfrak{I}(s))]ds \\ &+ \sum_{k=1}^m\Gamma(\varepsilon,\varepsilon_k)\left[-O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\upsilon(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\upsilon(s)ds\right)\right. \\ &+ A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right) \\ &+ \left.O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right)\right], \end{aligned}$$

where

$$\Gamma(\varepsilon,s)=\frac{1}{e^{\imath P}-1}\begin{cases} e^{\imath(P-\varepsilon+s)}, & 0\leq s\leq\varepsilon\leq P, \\ e^{\imath(s-\varepsilon)}, & 0\leq\varepsilon<s\leq P. \end{cases}$$

The method of derivation for the above Lemma 2.1 is analogous to the relevant results in [12]; the details are omitted here.

Lemma 2.2. *Suppose that there exist constants $\imath>0, \jmath\geq 0, 0\leq O_k<1, 0<q_{k-1}\leq(\varepsilon_k-\varepsilon_{k-1})/2, \quad 0\leq p_k\leq(\varepsilon_k-\varepsilon_{k-1})/2$, and $k=1,2,\cdots,m$, such that*

$$\frac{\jmath}{\imath}+\frac{e^{\imath P}}{e^{\imath P}-1}\sum_{k=1}^m O_k(p_k+q_{k-1})<1; \tag{3}$$

then, (2) exists as a unique solution.

Proof. For any $\upsilon\in\mathbb{D}_0$, we provide the following operator F:

$$\begin{aligned} (F\upsilon)(\varepsilon) &= \int_0^P\Gamma(\varepsilon,s)[\sigma(s)-\jmath\upsilon(\mathfrak{I}(s))]ds \\ &+ \sum_{k=1}^m\Gamma(\varepsilon,\varepsilon_k)\left[-O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\upsilon(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\upsilon(s)ds\right)\right. \\ &+ A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right) \\ &+ \left.O_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\psi(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\psi(s)ds\right)\right]. \end{aligned}$$

Clearly, $(F\upsilon) \in \mathbb{D}_0$. Since,

$$\int_0^P \Gamma(\varepsilon, s)ds = \frac{1}{\imath}, \qquad \max_{\varepsilon\in[0,P],s\in[0,P]} |\Gamma(\varepsilon, s)| = \frac{e^{\imath P}}{(e^{\imath P}-1)},$$

then for any $x, y \in \mathbb{D}_0$, we have the following:

$$\begin{aligned}
\|Fx - Fy\|_{\mathbb{D}_0} &= \sup_{\varepsilon\in\Theta} \left| \int_0^P \jmath\Gamma(\varepsilon, s)[-x(\mathfrak{T}(s)) + y(\mathfrak{T}(s))]ds \right. \\
&+ \sum_{k=1}^m \Gamma(\varepsilon, \varepsilon_k) \left[-O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} x(s)ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} x(s)ds \right) \right] \\
&- \left. \sum_{k=1}^m \Gamma(\varepsilon, \varepsilon_k) \left[-O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} y(s)ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} y(s)ds \right) \right] \right| \\
&\le \left(\frac{\jmath}{\imath} + \sup_{\varepsilon\in\Theta} |\Gamma(\varepsilon, \varepsilon_k)| \sum_{k=1}^m |O_k|(p_k + q_{k-1}) \right) \|x - y\|_{\mathbb{D}_0} \\
&= \left(\frac{\jmath}{\imath} + \frac{e^{\imath P}}{e^{\imath P} - 1} \sum_{k=1}^m O_k(p_k + q_{k-1}) \right) \|x - y\|_{\mathbb{D}_0}.
\end{aligned}$$

Therefore, by using condition (3) together with the Banach fixed point theorem, we know that F exists as a unique fixed point $\upsilon^* \in \mathbb{D}_0$. Then, as stated in Lemma 2.1, υ^* is a unique solution of system (2). □

Lemma 2.3. *(Comparison principle) Suppose that there exists $\imath > 0, \jmath \ge 0, 0 \le O_k < 1$, $0 < q_{k-1} \le (\varepsilon_k - \varepsilon_{k-1})/2, 0 \le p_k \le (\varepsilon_k - \varepsilon_{k-1})/2$, and $k = 1, 2, \cdots, m$, such that $\upsilon \in \mathbb{D}$ satisfies the following:*

$$\begin{cases} \upsilon'(\varepsilon) + \imath\upsilon(\varepsilon) + \jmath\upsilon(\mathfrak{T}(\varepsilon)) \le 0, & \varepsilon \ne \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle\upsilon(\varepsilon_k) \le -O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \upsilon(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \upsilon(s)ds \right), & k = 1, 2, \cdots, m, \\ \upsilon(0) \le \upsilon(P). \end{cases}$$

Moreover, the following inequality holds:

$$\jmath \int_0^P e^{\imath(\varepsilon - \mathfrak{T}(\varepsilon))}dt + \frac{1}{\imath} \sum_{k=1}^m O_k(e^{\imath a} - e^{\imath(a-q_{k-1})} + e^{\imath p_k} - 1) \le 1. \tag{4}$$

Then, $\upsilon(\varepsilon) \le 0$ for $\varepsilon \in \Theta$.

Proof. Let $u(\varepsilon) = e^{\imath\varepsilon}\upsilon(\varepsilon)$; then,

$$\begin{cases} u'(\varepsilon) \le -\jmath e^{\imath(\varepsilon - \mathfrak{T}(\varepsilon))}u(\mathfrak{T}(\varepsilon)), & \varepsilon \ne \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle u(\varepsilon_k) \le -O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} e^{-\imath(s-\varepsilon_k)}u(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} e^{-\imath(s-\varepsilon_k)}u(s)ds \right), & k = 1, 2, \cdots, m, \\ u(0) \le e^{-\imath P}u(P). \end{cases} \tag{5}$$

On the contrary, we prove the following two cases:

Case (i): Assume the existence of $\varepsilon^* \in \Theta$ such that $u(\varepsilon^*) > 0$, and $u(\varepsilon) \ge 0$ hold for all other $\varepsilon \in \Theta$.

Clearly, $u(\varepsilon)$ is a non-increasing function; therefore, by the boundary value condition, we have $u(P) \leq u(0) \leq e^{-\imath P} u(P) < u(P)$, which is a contradiction.

Case (ii): Suppose that there exist two points $\varepsilon_1, \varepsilon_2 \in \Theta$ such that $u(\varepsilon_1) > 0$ and $u(\varepsilon_2) < 0$.

Set $\underline{\varepsilon} \in (\varepsilon_i, \varepsilon_{i+1}]$, $i \in \{1, 2, \cdots, m\}$ such that $u(\underline{\varepsilon}) = \inf(u(\varepsilon) : \varepsilon \in \Theta) < 0$. On the other hand, we set $\overline{\varepsilon} \in (\varepsilon_j, \varepsilon_{j+1})$, and assume that $u(\overline{\varepsilon}) > 0$ holds for some $j \in \{1, 2, \cdots, m\}$. Without a loss of generality, set $\underline{\varepsilon} < \overline{\varepsilon}$; then $i \leq j$. By integrating (5) from $\underline{\varepsilon}$ to $\overline{\varepsilon}$, we can obtain the following:

$$\begin{aligned}
u(\overline{\varepsilon}) - u(\underline{\varepsilon}) &\leq -\jmath \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} e^{\imath(\varepsilon - \mathfrak{T}(\varepsilon))} u(\mathfrak{T}(\varepsilon)) dt + \sum_{k=i+1}^{j} \Delta u(\varepsilon_k) \\
&\leq -u(\underline{\varepsilon}) \jmath \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} e^{\imath(\varepsilon - \mathfrak{T}(\varepsilon))} dt - u(\underline{\varepsilon}) \sum_{k=i+1}^{j} O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} e^{-\imath(s-\varepsilon_k)} ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} e^{-\imath(s-\varepsilon_k)} ds \right) \\
&\leq -u(\underline{\varepsilon}) \left[\jmath \int_0^P e^{\imath(\varepsilon - \mathfrak{T}(\varepsilon))} dt + \frac{1}{\imath} \sum_{k=1}^{m} O_k (e^{\imath a} - e^{\imath(a - q_{k-1})} + e^{\imath p_k} - 1) \right] \\
&\leq -u(\underline{\varepsilon}).
\end{aligned}$$

Therefore, a contradiction arises. □ □

Lemma 2.4. *Assuming there exists $\imath > 0, \jmath \geq 0, 0 \leq O_k < 1, 0 < q_{k-1} \leq (\varepsilon_k - \varepsilon_{k-1})/2, \quad 0 \leq p_k \leq (\varepsilon_k - \varepsilon_{k-1})/2$, and $k = 1, 2, \cdots, m$, such that (4) holds and $\upsilon \in \mathbb{D}$ satisfies the following:*

$$\begin{cases}
\upsilon'(\varepsilon) + \imath\upsilon(\varepsilon) + \jmath\upsilon(\mathfrak{T}(\varepsilon)) + \frac{\imath\varepsilon + \jmath\mathfrak{T}(\varepsilon) + 1}{P} [\upsilon(0) - \upsilon(P)] \leq 0, & \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\
\triangle\upsilon(\varepsilon_k) \leq -O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \upsilon(s) ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} \upsilon(s) ds \right) & \\
-\frac{1}{2P} [(\varepsilon_{k-1} + q_{k-1})^2 - \varepsilon_{k-1}^2 + \varepsilon_k^2 - (\varepsilon_k - p_k)^2][\upsilon(0) - \upsilon(P)], & k = 1, 2, \cdots, m, \\
\upsilon(0) > \upsilon(P);
\end{cases}$$

then, $\upsilon(\varepsilon) \leq 0$ holds on $\varepsilon \in \Theta$.

Proof. Set $u(\varepsilon) = \upsilon(\varepsilon) + \frac{\varepsilon}{P}[\upsilon(0) - \upsilon(P)]$; then, for all $\varepsilon \in \Theta$, $u(\varepsilon) \geq \upsilon(\varepsilon)$. By a direct computation, it is easy to obtain the following:

$$\begin{cases}
u'(\varepsilon) + \imath u(\varepsilon) + \jmath u(\mathfrak{T}(\varepsilon)) \leq 0, & \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\
\triangle u(\varepsilon_k) \leq -O_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} u(s) ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} u(s) ds \right), & k = 1, 2, \cdots, m, \\
u(0) \leq u(P);
\end{cases}$$

then, by Lemma 2.3, we have $u(\varepsilon) \leq 0$ (i.e., $\upsilon(\varepsilon) \leq 0$). □

3. Main results

First, we give a pair of new definitions for ULS. Second, we prove main results by combining the MIT and QLM methods.

Definition 3.1. *The function $\nabla_0 \in \mathbb{D} \cap \mathbb{D}_0$ is defined as a lower solution of the BVP (1) if*

$$\begin{cases}
\nabla_0{}'(\varepsilon) \leq f(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon))) - H^{\nabla_0}(\varepsilon), & \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\
\triangle\nabla_0(\varepsilon_k) \leq A_k \left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s) ds + \int_{\varepsilon_k - p_k}^{\varepsilon_k} \nabla_0(s) ds \right) - O_k H_k^{\nabla_0}, & k = 1, 2, \cdots, m,
\end{cases}$$

for any $v \in C(\Theta, R)$,

$$H^{v}(\varepsilon)=\begin{cases}0, & v(0)\leq v(P),\\ \frac{\iota\varepsilon+\jmath\mathfrak{T}(\varepsilon)+1}{P}[v(0)-v(P)], & v(0)>v(P),\end{cases}$$

$$H_k^{v}=\begin{cases}0, & v(0)\leq v(P),\\ \frac{1}{2P}[(\varepsilon_{k-1}+q_{k-1})^2-\varepsilon_{k-1}^2+\varepsilon_k^2-(\varepsilon_k-p_k)^2][v(0)-v(P)], & v(0)>v(P).\end{cases}$$

Definition 3.2. *The function $\triangle_0 \in \mathbb{D}\cap\mathbb{D}_0$ is defined as an upper solution of the BVP (1) if*

$$\begin{cases}\triangle_0{}'(\varepsilon)\geq f(\varepsilon,\triangle_0(\varepsilon),\triangle_0(\mathfrak{T}(\varepsilon)))-H^{\triangle_0}(\varepsilon), & \varepsilon\neq\varepsilon_k, \varepsilon\in\Theta=[0,P],\\ \triangle\triangle_0(\varepsilon_k)\geq A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\triangle_0(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\triangle_0(s)ds\right)-O_kH_k^{\triangle_0}, & k=1,2,\cdots,m.\end{cases}$$

Theorem 3.1. *Suppose that assumptions (A_1)–(A_3) hold:*

(A_1): $\nabla_0(\varepsilon), \triangle_0(\varepsilon)$ are the lower and upper solutions of the BVP (1), and satisfy $\nabla_0(\varepsilon)\leq\triangle_0(\varepsilon)$ on Θ;

(A_2): The function f meets $f_v(\varepsilon,\upsilon(\varepsilon),\upsilon(\mathfrak{T}(\varepsilon)))<0, f_{\upsilon\mathfrak{T}}(\varepsilon,\upsilon(\varepsilon),\upsilon(\mathfrak{T}(\varepsilon)))\leq 0$, and the quadratic form is defined as

$$K(f(\varepsilon,\upsilon,z))=(\upsilon-v)^2f_{\upsilon\upsilon}(\varepsilon,\upsilon_1,\upsilon_2)+2(\upsilon-v)(z-u)f_{\upsilon\upsilon\mathfrak{T}}(\varepsilon,\upsilon_1,\upsilon_2)+(z-u)^2f_{\upsilon\mathfrak{T}\upsilon\mathfrak{T}}(\varepsilon,\upsilon_1,\upsilon_2)\leq 0,$$

where $\nabla_0\leq v\leq\upsilon_1\leq\upsilon\leq\triangle_0, \nabla_0\leq u\leq\upsilon_2\leq z\leq\triangle_0$, and $\varepsilon\neq\varepsilon_k, \varepsilon\in\Theta$.

(A_3): For $k=1,2,\cdots,m$, the functions A_k meet $-1<A_k^{'}(.)\leq 0$ and $A_k^{''}(.)\geq 0$.

Then, there exist two monotone sequences $\{\nabla_n(\varepsilon)\}$ and $\{\triangle_n(\varepsilon)\}$ of the lower and upper solutions, respectively, which converge uniformly and quadratically to the extreme solutions of the BVP (1) in $[\nabla_0,\triangle_0]$.

Proof. Using Taylor′s theorem and (A_2), we obtain the following:

$$f(\varepsilon,\upsilon(\varepsilon),\upsilon(\mathfrak{T}(\varepsilon)))\leq Q(\varepsilon,\upsilon(\varepsilon),\mathfrak{T}(\varepsilon),v(\varepsilon)),$$

where

$$\begin{aligned}Q(\varepsilon,\upsilon(\varepsilon),\mathfrak{T}(\varepsilon),v(\varepsilon)) &= f(\varepsilon,v(\varepsilon),v(\mathfrak{T}(\varepsilon)))+f_v(\varepsilon,v(\varepsilon),v(\mathfrak{T}(\varepsilon)))(\upsilon(\varepsilon)-v(\varepsilon))\\ &+ f_{\upsilon\mathfrak{T}}(\varepsilon,v(\varepsilon),v(\mathfrak{T}(\varepsilon)))(\upsilon(\mathfrak{T}(\varepsilon))-v(\mathfrak{T}(\varepsilon))).\end{aligned}$$

Similarly, using Taylor′s theorem together with (A_3), we obtain the following:

$$\begin{aligned}&A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}x(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}x(s)ds\right)-A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\upsilon(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\upsilon(s)ds\right)\\ \geq\ & A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}\upsilon(s)ds+\int_{\varepsilon_k-p_k}^{\varepsilon_k}\upsilon(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}(x(s)-\upsilon(s))ds\right.\\ +\ & \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k}(x(s)-\upsilon(s))ds\right),\end{aligned}$$

where $\nabla_0(\varepsilon_k)\leq\upsilon(\varepsilon_k)\leq x(\varepsilon_k)\leq\triangle_0(\varepsilon_k)$.

Now, we provide two sequences, $\nabla_i(\varepsilon)$ and $\triangle_i(\varepsilon)$, which satisfy the following:

$$\begin{cases} \nabla_i{}'(\varepsilon) - f_\upsilon(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\nabla_i(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\nabla_i(\mathfrak{T}(\varepsilon)) \\ = f(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon))) - f_\upsilon(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\nabla_{i-1}(\varepsilon) \\ -f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\nabla_{i-1}(\mathfrak{T}(\varepsilon)), \qquad \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle\nabla_i(\varepsilon_k) = A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{i-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{i-1}(s)ds\right) + \\ A_k'\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{i-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{i-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}(\nabla_i(s) - \nabla_{i-1}(s))ds \right. \\ \left. + \int_{\varepsilon_k-p_k}^{\varepsilon_k}(\nabla_i(s) - \nabla_{i-1}(s))ds\right), \qquad k = 1, 2, \cdots, m, \\ \nabla_i(0) = \nabla_i(P), \end{cases} \tag{6}$$

$$\begin{cases} \triangle_i{}'(\varepsilon) - f_\upsilon(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\triangle_i(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\triangle_i(\mathfrak{T}(\varepsilon)) \\ = f(\varepsilon, \triangle_{i-1}(\varepsilon), \triangle_{i-1}(\mathfrak{T}(\varepsilon))) - f_\upsilon(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\triangle_{i-1}(\varepsilon) \\ -f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{i-1}(\varepsilon), \nabla_{i-1}(\mathfrak{T}(\varepsilon)))\triangle_{i-1}(\mathfrak{T}(\varepsilon)), \qquad \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle\triangle_i(\varepsilon_k) = A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \triangle_{i-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \triangle_{i-1}(s)ds\right) + \\ A_k'\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{i-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{i-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}(\triangle_i(s) - \triangle_{i-1}(s))ds \right. \\ \left. + \int_{\varepsilon_k-p_k}^{\varepsilon_k}(\triangle_i(s) - \triangle_{i-1}(s))ds\right), \qquad k = 1, 2, \cdots, m, \\ \triangle_i(0) = \triangle_i(P). \end{cases} \tag{7}$$

Obviously, from Lemmas 2.1 and 2.2, we can see that both (6) and (7) have a unique solution.

Next, we will accomplish the proof through five steps:

Step 1. We will prove that $\nabla_i \leq \nabla_{i+1}$ $(i = 0, 1, 2, \cdots)$ and $\triangle_i \leq \triangle_{i-1}$ $(i = 1, 2, \cdots)$.

Let $i = 1$ in (6); then, ∇_1 meets the following:

$$\begin{cases} \nabla_1{}'(\varepsilon) - f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\mathfrak{T}(\varepsilon)) = f(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon))) \\ -f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\mathfrak{T}(\varepsilon)), \qquad \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P], \\ \triangle\nabla_1(\varepsilon_k) = A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right) + \\ A_k'\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}}(\nabla_1(s) - \nabla_0(s))ds \right. \\ \left. + \int_{\varepsilon_k-p_k}^{\varepsilon_k}(\nabla_1(s) - \nabla_0(s))ds\right), \qquad k = 1, 2, \cdots, m, \\ \nabla_1(0) = \nabla_1(P). \end{cases}$$

We set $p(\varepsilon) = \nabla_0(\varepsilon) - \nabla_1(\varepsilon)$. Below, we will prove this in two scenarios:

Case (i): $\nabla_0(0) \leq \nabla_0(P)$.

$$\begin{aligned} p'(\varepsilon) &- f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\mathfrak{T}(\varepsilon)) = \nabla_0'(\varepsilon) - \nabla_1'(\varepsilon) \\ &- f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\varepsilon) + f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\varepsilon) \\ &- f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\mathfrak{T}(\varepsilon)) + f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\mathfrak{T}(\varepsilon)) \\ &\leq f(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon))) - f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\mathfrak{T}(\varepsilon)) \\ &- f(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon))) + f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\varepsilon) + f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\mathfrak{T}(\varepsilon)) \\ &- f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\varepsilon) + f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\varepsilon) \\ &- f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_0(\mathfrak{T}(\varepsilon)) + f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\mathfrak{T}(\varepsilon)) = 0, \\ \Delta p(\varepsilon_k) &= \Delta\nabla_0(\varepsilon_k) - \Delta\nabla_1(\varepsilon_k) \\ &\leq A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right) - A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right) \end{aligned}$$

$$
\begin{aligned}
&- A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (\nabla_1(s)-\nabla_0(s))ds\right.\\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (\nabla_1(s)-\nabla_0(s))ds\right)\\
&= A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p(s)ds\right),\\
p(0) &\leq p(P).
\end{aligned}
$$

Therefore, from Lemma 2.3, we know that $p(\varepsilon) \leq 0$ (i.e., $\nabla_0 \leq \nabla_1$).

Case (ii): $\nabla_0(0) > \nabla_0(P)$.

By a direct computation, we obtain the following:

$$
\begin{cases}
p'(\varepsilon) - f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\mathfrak{T}(\varepsilon)) \\
+\frac{-f_\upsilon(\varepsilon,\nabla_0(\varepsilon),\nabla_0(\mathfrak{T}(\varepsilon)))\varepsilon - f_{\upsilon\mathfrak{T}}(\varepsilon,\nabla_0(\varepsilon),\nabla_0(\mathfrak{T}(\varepsilon)))\mathfrak{T}(\varepsilon)+1}{P}[p(0)-p(P)] \leq 0, \qquad \varepsilon \neq \varepsilon_k, \varepsilon \in \Theta = [0, P],\\
\triangle p(\varepsilon_k) \leq A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p(s)ds\right)\\
-\frac{1}{2P}[(\varepsilon_{k-1}+q_{k-1})^2 - \varepsilon_{k-1}^2 + \varepsilon_k^2 - (\varepsilon_k - p_k)^2][p(0)-p(P)], \qquad k = 1, 2, \cdots, m,\\
p(0) > p(P).
\end{cases}
$$

Then, by the Lemma 2.4, we have $p(\varepsilon) \leq 0$ (i.e., $\nabla_0 \leq \nabla_1$).

By the same way, we can show that $\triangle_1 \leq \triangle_0$. Then, by mathematic induction, we obtain $\nabla_i \leq \nabla_{i+1}$ ($i = 0, 1, 2, \cdots$) and $\triangle_i \leq \triangle_{i-1}$ ($i = 1, 2, \cdots$).

Step 2. We show that $\nabla_1 \leq \triangle_1$ for all $\varepsilon \in \Theta$.

Let $p(\varepsilon) = \nabla_1(\varepsilon) - \triangle_1(\varepsilon)$; then, by $(A_1) - (A_3)$, we have the following:

$$
\begin{aligned}
p^{'}(\varepsilon) &- f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))p(\mathfrak{T}(\varepsilon)) = \nabla_1^{'}(\varepsilon) - \triangle_1^{'}(\varepsilon)\\
&- f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\varepsilon) + f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\triangle_1(\varepsilon)\\
&- f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\nabla_1(\mathfrak{T}(\varepsilon)) + f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))\triangle_1(\mathfrak{T}(\varepsilon))\\
&= f(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon))) - f(\varepsilon, \triangle_0(\varepsilon), \triangle_0(\mathfrak{T}(\varepsilon))) + f_\upsilon(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))(\triangle_0(\varepsilon) - \nabla_0(\varepsilon))\\
&+ f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_0(\varepsilon), \nabla_0(\mathfrak{T}(\varepsilon)))(\triangle_0(\mathfrak{T}(\varepsilon)) - \nabla_0(\mathfrak{T}(\varepsilon))) \leq 0,\\
\Delta p(\varepsilon_k) &= \Delta\nabla_1(\varepsilon_k) - \Delta\triangle_1(\varepsilon_k)\\
&= A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right) - A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \triangle_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \triangle_0(s)ds\right)\\
&+ A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (\nabla_1(s)-\nabla_0(s))ds\right.\\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (\nabla_1(s)-\nabla_0(s))ds\right)\\
&- A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (\triangle_1(s)-\triangle_0(s))ds\right.\\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (\triangle_1(s)-\triangle_0(s))ds\right)\\
&\leq A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_0(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_0(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p(s)ds\right),
\end{aligned}
$$

$$p(0) \leq p(P).$$

Therefore, as stated in Lemma 2.3, we have $p(\varepsilon) \leq 0$ (i.e., $\nabla_1 \leq \triangle_1$).

Step 3. It is clear that from the two above steps, we can obtain two monotone sequences, $\nabla_i(\varepsilon)$ and $\triangle_i(\varepsilon)$, which satisfy the following:

$$\nabla_0(\varepsilon) \leq \nabla_1(\varepsilon) \leq \cdots \leq \nabla_n(\varepsilon) \leq \cdots \leq \triangle_n(\varepsilon) \leq \cdots \leq \triangle_1(\varepsilon) \leq \triangle_0(\varepsilon), \quad \varepsilon \in \Theta,$$

where $\nabla_i(\varepsilon), \triangle_i(\varepsilon) \in \mathbb{D} \cap \mathbb{D}_0$, and satisfy (6) and (7), respectively.

It can be readily demonstrated that $\nabla_n(\varepsilon)$, $\triangle_n(\varepsilon)$ are equi-continuous and uniformly bounded. Therefore, by the Ascoli-Arzela criterion [23], we know that functions $r(\varepsilon)$ and $\rho(\varepsilon)$ exist, and the following expression uniformly holds on all $\varepsilon \in \Theta$:

$$\lim_{n\to\infty} \nabla_n(\varepsilon) = r(\varepsilon), \quad \lim_{n\to\infty} \triangle_n(\varepsilon) = \rho(\varepsilon).$$

Obviously, by letting $i \to \infty$ in (6) and (7), $r(\varepsilon)$ and $\rho(\varepsilon)$ are two solutions of the BVP(1).

Step 4. We prove that $r(\varepsilon), \rho(\varepsilon)$ are the minimal and maximal solutions of the BVP(1), respectively.

Set $x(\varepsilon)$ as an any solution of the BVP(1) and $\nabla_0(\varepsilon) \leq x(\varepsilon) \leq \triangle_0(\varepsilon)$. We assume that $\nabla_n(\varepsilon) \leq x(\varepsilon) \leq \triangle_n(\varepsilon)$ holds for an integer n; then, we prove that $\nabla_{n+1}(\varepsilon) \leq x(\varepsilon) \leq \triangle_{n+1}(\varepsilon)$.

Let $p(\varepsilon) = \nabla_{n+1}(\varepsilon) - x(\varepsilon)$; then,

$$\begin{aligned}
p'(\varepsilon) &- f_\upsilon(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))p(\varepsilon) - f_{\upsilon\mathfrak{I}}(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))p(\mathfrak{I}(\varepsilon)) = \nabla'_{n+1}(\varepsilon) - x'(\varepsilon)\\
&- f_\upsilon(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))\nabla_{n+1}(\varepsilon) + f_\upsilon(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))x(\varepsilon)\\
&- f_{\upsilon\mathfrak{I}}(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))\nabla_{n+1}(\mathfrak{I}(\varepsilon)) + f_{\upsilon\mathfrak{I}}(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))x(\mathfrak{I}(\varepsilon))\\
&= f(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon))) - f(\varepsilon, x(\varepsilon), x(\mathfrak{I}(\varepsilon))) - f_\upsilon(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))(\nabla_n(\varepsilon) - x(\varepsilon))\\
&- f_{\upsilon\mathfrak{I}}(\varepsilon, \nabla_n(\varepsilon), \nabla_n(\mathfrak{I}(\varepsilon)))(\nabla_n(\mathfrak{I}(\varepsilon)) - x(\mathfrak{I}(\varepsilon))) \leq 0,\\
\Delta p(\varepsilon_k) &= \Delta\nabla_{n+1}(\varepsilon_k) - \Delta x(\varepsilon_k)\\
&= A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right) - A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} x(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} x(s)ds\right)\\
&+ A'_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (\nabla_{n+1}(s) - \nabla_n(s))ds\right.\\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (\nabla_{n+1}(s) - \nabla_n(s))ds\right)\\
&= A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right) - A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} x(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} x(s)ds\right)\\
&+ A'_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (x(s) - \nabla_n(s))ds\right.\\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (x(s) - \nabla_n(s))ds\right)\\
&+ A'_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p(s)ds\right)\\
&\leq A'_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_n(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p(s)ds\right),
\end{aligned}$$

$$p(0) \quad \leq \quad p(P).$$

Therefore, as stated in Lemma 2.3, we get $p(\varepsilon) \leq 0$ (i.e., $\nabla_{n+1} \leq x$). By the same token, we know that $x \leq \triangle_{n+1}$. Therefore, $\nabla_{n+1}(\varepsilon) \leq x(\varepsilon) \leq \triangle_{n+1}(\varepsilon)$ holds. Finally, by letting $n \to \infty$, we can get that $r(\varepsilon) \leq x(\varepsilon) \leq \rho(\varepsilon)$.

Step 5. We show that the two above monotone sequences satisfy a quadratic convergence.

Without a loss of generality, we only prove that ∇_n satisfies a quadratic convergence.

Let $p_n(\varepsilon) = r(\varepsilon) - \nabla_n(\varepsilon) \geq 0$; then, we obtain the following system:

$$\begin{aligned}
p_n{}'(\varepsilon) \quad &- \quad f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))p_n(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))p_n(\mathfrak{T}(\varepsilon)) = r^{'}(\varepsilon) - \nabla^{'}_n(\varepsilon) \\
&- \quad f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))r(\varepsilon) - f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))r(\mathfrak{T}(\varepsilon)) \\
&+ \quad f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\varepsilon) + f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\mathfrak{T}(\varepsilon)) \\
&= \quad f(\varepsilon, r(\varepsilon), r(\mathfrak{T}(\varepsilon))) - f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\varepsilon) \\
&- \quad f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\mathfrak{T}(\varepsilon)) - f(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon))) \\
&+ \quad f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_{n-1}(\varepsilon) \\
&+ \quad f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_{n-1}(\mathfrak{T}(\varepsilon)) - f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))r(\varepsilon) \\
&- \quad f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))r(\mathfrak{T}(\varepsilon)) + f_\upsilon(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\varepsilon) \\
&+ \quad f_{\upsilon\mathfrak{T}}(\varepsilon, \nabla_{n-1}(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)))\nabla_n(\mathfrak{T}(\varepsilon)) \\
&= \quad \frac{1}{2}[p^2_{n-1}(\varepsilon)f_{\upsilon\upsilon}(\varepsilon, \upsilon_1, \upsilon_2) + 2p_{n-1}(\varepsilon)p_{n-1}(\mathfrak{T}(\varepsilon))f_{\upsilon\upsilon\mathfrak{T}}(\varepsilon, \upsilon_1, \upsilon_2) + p^2_{n-1}(\mathfrak{T}(\varepsilon))f_{\upsilon\mathfrak{T}\upsilon\mathfrak{T}}(\varepsilon, \upsilon_1, \upsilon_2)],
\end{aligned}$$

where $\nabla_{n-1}(\varepsilon) \leq \upsilon_1 \leq r(\varepsilon), \nabla_{n-1}(\mathfrak{T}(\varepsilon)) \leq \upsilon_2 \leq r(\mathfrak{T}(\varepsilon))$.

$$\begin{aligned}
\Delta p_n(\varepsilon_k) &= \Delta r(\varepsilon_k) - \Delta\nabla_n(\varepsilon_k) \\
&= A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} r(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} r(s)ds\right) \\
&- A_k\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds\right) \\
&- A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (r(s) - \nabla_{n-1}(s))ds\right. \\
&+ \left.\int_{\varepsilon_k-p_k}^{\varepsilon_k} (r(s) - \nabla_{n-1}(s))ds\right) \\
&+ A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p_n(s)ds\right) \\
&= A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p_n(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p_n(s)ds\right) \\
&+ \frac{1}{2}A_k^{''}(\xi)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p_{n-1}(s)ds\right)^2, \\
p_n(0) &= p_n(P),
\end{aligned}$$

where

$$\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds < \xi < \int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} r(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} r(s)ds.$$

We denote

$$\sigma(p_{n-1}(\varepsilon)\ \ ,\ \ p_{n-1}(\Im(\varepsilon))) = \frac{1}{2}[p_{n-1}^2(\varepsilon)f_{\upsilon\upsilon}(\varepsilon,\upsilon_1,\upsilon_2) + 2p_{n-1}(\varepsilon)p_{n-1}(\Im(\varepsilon))f_{\upsilon\upsilon\Im}(\varepsilon,\upsilon_1,\upsilon_2)$$
$$+\ p_{n-1}^2(\Im(\varepsilon))f_{\upsilon\Im\upsilon\Im}(\varepsilon,\upsilon_1,\upsilon_2)].$$

Then, by Lemma 2.1, the solution of the above system is as follows:

$$\begin{aligned} p_n(\varepsilon) &= \int_0^P \Gamma(\varepsilon,s)[\sigma(p_{n-1}(s),p_{n-1}(\Im(s))) + f_{\upsilon\Im}(s,\nabla_{n-1}(s),\nabla_{n-1}(\Im(s)))p_n(\Im(s))]ds \\ &+ \sum_{k=1}^m \Gamma(\varepsilon,\varepsilon_k)\left[A_k^{'}\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} \nabla_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} \nabla_{n-1}(s)ds\right)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} (p_n(s))ds\right.\right. \\ &+ \left.\left.\int_{\varepsilon_k-p_k}^{\varepsilon_k}(p_n(s))ds\right) + \frac{1}{2}A_k^{''}(\xi)\left(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p_{n-1}(s)ds\right)^2\right]. \end{aligned}$$

Let $|f_{\upsilon\upsilon}| \le \delta_1, |f_{\upsilon\upsilon\Im}| \le \delta_2, |f_{\upsilon\Im\upsilon\Im}| \le \delta_3$; then,

$$\begin{aligned} \sigma(p_{n-1}(\varepsilon),p_{n-1}(\Im(\varepsilon))) &\le \frac{1}{2}\delta_1 p_{n-1}^2(\varepsilon) + \delta_2 p_{n-1}(\varepsilon)p_{n-1}(\Im(\varepsilon)) + \frac{1}{2}\delta_3 p_{n-1}^2(\Im(\varepsilon)) \\ &\le \frac{1}{2}(\delta_1+\delta_2)p_{n-1}^2(\varepsilon) + \frac{1}{2}(\delta_2+\delta_3)p_{n-1}^2(\Im(\varepsilon)). \end{aligned}$$

We take the norm of p_{n-1} on Θ by $\|p_{n-1}\|_{\mathbb{D}_0} = \max_\Theta\{p_{n-1}(\varepsilon), p_{n-1}(\Im(\varepsilon))\}$.

Due to

$$(\int_{\varepsilon_{k-1}}^{\varepsilon_{k-1}+q_{k-1}} p_{n-1}(s)ds + \int_{\varepsilon_k-p_k}^{\varepsilon_k} p_{n-1}(s)ds)^2 \le (p_k+q_{k-1})^2\|p_{n-1}\|_{\mathbb{D}_0}^2,$$

by the expression of $p_n(\varepsilon)$, it is clear that there exist a constant μ such that

$$\|p_n\|_{\mathbb{D}_0} \le \mu\|p_{n-1}\|_{\mathbb{D}_0}^2.$$

Therefore, p_i is a quadratic convergence. This completes the proof. □

4. Conclusions

In 2010, Tariboon proposed a new impulsive integral condition, and discussed the existence of extreme solutions for a delay pulse differential system [12]. Subsequently, this impulsive integral condition has also been applied to other differential equation systems. For example, Liu et al. introduced this impulsive integral condition into an integral-differential equation with an integral boundary value condition, and obtained the sufficient condition for the existence of extreme solutions [22]. Unlike previous studies, in this paper, we have modified the impulsive integration condition. Interestingly, we found that the extremum solution still exists in the pulse functional system under the modified impulse integral condition. More importantly, we derived that the monotonic sequence not only uniformly converges to the extremum solution, but also has a quadratic convergence. We note that a quadratic convergence has not been discussed in previous studies [12, 22]. Due to the faster speed of a quadratic convergence compared to a uniform convergence, and the fact that a quadratic convergence is rarely considered in impulse integral conditions, we hope that the methods and conclusions proposed in this paper can be better applied in practical problems.

Author contributions

Bing Hu: Writing-original draft, Methodology, Conceptualization, Validation, Investigation, Writing-review & editing. Yikai Liao: Writing-review & editing, Supervision, Validation, Methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations*, Singapore: World scientific, 1995.
2. H. Xu, Q. Zhu, W. X. Zheng, Exponential stability of stochastic nonlinear delay systems subject to multiple periodic impulses, *IEEE Trans. Autom. Control*, **69** (2024), 2621–2628. https://doi.org/10.1109/TAC.2023.3335005
3. I. Stamova, G. T. Stamov, *Applied impulsive mathematical models, Cham*, Switzerland: Springer, 2016.
4. Z. Li, Global attractiveness and quasi-invariant sets of impulsive neutral stochastic functional differential equations driven by fBm, *Neurocomputing*, **177** (2016), 620–627. https://doi.org/10.1016/j.neucom.2015.11.070
5. W. Hu, Q. Zhu, H. R. Karimi, Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems, *IEEE Trans. Autom. Control*, **64** (2019), 5207–5213. https://doi.org/10.1109/TAC.2019.2911182
6. M. Zaczynska, H. Abramovich, C. Bisagni, Parametric studies on the dynamic buckling phenomenon of a composite cylindrical shell under impulsive axial compression, *J. Sound. Vib.*, **482** (2020), 115462. https://doi.org/10.1016/j.jsv.2020.115462
7. Z. Wang, J. Sun, J. Chen, Input-to-state stability of impulsive switched nonlinear time-delay systems with two asynchronous switching phenomena, *Int. J. Robust. Nonlin.*, **30** (2020), 4463–4484. https://doi.org/10.1002/rnc.4995

8. Q. Zhu, Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Le'vy processes, *IEEE Trans. Autom. Control*, **70** (2025), 1176–1183. https://doi.org/10.1109/TAC.2024.3448128

9. A. Kumar, H. V. S. Chauhan, C. Ravichandran, K. S. Nisar, D. Baleanu, Existence of solutions of non-autonomous fractional differential equations with integral impulse condition, *Adv. Differ. Equ.*, **2020** (2020), 434. https://doi.org/10.1186/s13662-020-02888-3

10. A. Zada, S. Ali, Y. Li, Ulam-type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition, *Adv. Differ. Equ.*, **2017** (2017), 317. https://doi.org/10.1186/s13662-017-1376-y

11. X. Li, H. Akca, X. Fu, Uniform stability of impulsive infinite delay differential equations with applications to systems with integral impulsive conditions, *Appl. Math. Comput.*, **219** (2013), 7329–7337. https://doi.org/10.1016/j.amc.2012.12.033

12. J. Tariboon, Boundary value problems for first order functional differential equations with impulsive integral conditions, *J. Comput. Appl. Math.*, **234** (2010), 2411–2419. https://doi.org/10.1016/j.cam.2010.03.007

13. G. Li, Y. Zhang, Y. Guan, W. Li, Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse, *Math. Biosci. Eng.*, **20** (2023), 7020–7041. https://doi.org/10.3934/mbe.2023303

14. C. Thaiprayoon, S. K. Ntouyas, J. Tariboon, Monotone iterative technique for nonlinear impulsive conformable fractional differential equations with delay, *Commun. Math. Appl.*, **12** (2021), 11. https://doi.org/10.26713/CMA.V12I1.587

15. B. Li, H. Gou, Monotone iterative method for the periodic boundary value problems of impulsive evolution equations in Banach spaces, *Chaos Solitons Fract.*, **110** (2018), 209–215. https://doi.org/10.1016/j.chaos.2018.03.027

16. Z. He, X. He, Monotone iterative technique for impulsive integro-differential equations with periodic boundary conditions, *Comput. Math. Appl.*, **48** (2004), 73–84. https://doi.org/10.1016/j.camwa.2004.01.005

17. Z. Luo, J. J. Nieto, New results for the periodic boundary value problem for impulsive integro-differential equations, *Nonlinear Anal.-Theory*, **70** (2009), 2248–2260. https://doi.org/10.1016/j.na.2008.03.004

18. B. Hu, Z. Wang, M. Xu, D. Wang, Quasilinearization method for an impulsive integro-differential system with delay, *Math. Biosci. Eng.*, **19** (2022), 612–623. https://doi.org/10.3934/mbe.2022027

19. B. Ahmad, A quasilinearization method for a class of integro-differential equations with mixed nonlinearities, *Nonlinear Anal. Real World Appl.*, **7** (2006), 997–1004. https://doi.org/10.1016/j.nonrwa.2005.09.003

20. R. K.Pandey, S. Tomar, An effective scheme for solving a class of nonlinear doubly singular boundary value problems through quasilinearization approach, *J. Comput. Appl. Math.*, **392** (2021), 113411. https://doi.org/10.1016/j.cam.2021.113411

21. B. Ahmad, J. J. Nieto, Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions, *Nonlinear Anal.-Theory*, **69** (2008), 3291–3298. https://doi.org/10.1016/j.na.2007.09.018

22. Z. Liu, J. Han, L. Fang, Integral boundary value problems for first order integro-differential equations with impulsive integral conditions, *Comput. Math. Appl.*, **61** (2011), 3035–3043. https://doi.org/10.1016/j.camwa.2011.03.094

23. V. Lakshmikantham, P. S. Simeonov, *Theory of impulsive differential equations*, Singapore: World scientific, 1989.