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Research article

Distance-based granular computing in networks modeled by intersection graphs

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Abstract: Networks are commonly represented as graphs, where vertices denote entities and edges capture relationships based on shared attributes. Granulation of a network is important for the structural analysis and understanding of its underlying patterns. In this paper, we introduce a distance-based granular computing framework for analyzing networks modeled by intersection graphs. We define these networks as information systems and investigate their granular structures using a distance-based representation. Based on the concepts of indiscernibility between two vertices using the distance from a set, we study indiscernibility partitions on the vertex set. Using the concept of discernibility between vertices, we define the distance-based discernibility matrix and explore its properties. We identify all minimal resolving sets using the discernibility matrix. Furthermore, using the proposed method, we study a transportation network for urban traffic planning.

Keywords: granular computing; network; intersection graphs; rough set; resolving set; reduct; discernibility matrix
Mathematics Subject Classification: 05A18, 05C12, 05C62

1. Introduction

In recent years, data generated from many real-world applications has been represented as networks of interconnected objects. This approach shifts away from traditional methods of analyzing independent data points, seeking to extract deeper insights by focusing on the relationships between entities. One of the most significant classes of such data networks is the social network. Social networks are formed by the relationships and interactions that connect individuals within communities. These connections can be based on a variety of factors, such as family ties, friendships, professional

collaborations, shared interests, or even online interactions. The analysis of such networks is a rapidly expanding interdisciplinary field that integrates mathematics, and machine learning to study the structure, dynamics, and interactions among individuals or groups within a network. Zhang et al. [41] analyzed the transformative impact of high-speed railway (HSR) construction on the spatial structure of urban agglomerations using social network analysis. Yang et al. [38] proposed an advanced approach to improve coordination in product development (PD) organizations through social network analysis, providing new insights into the influence of team similarity on organizational structure. Recent developments in graph theory, such as mutually orthogonal graph squares, have introduced new methods for constructing graph-orthogonal arrays [14] and graph-transversal designs with authentication codes [10]. These approaches enhance combinatorial design and security applications. Graph theory plays a fundamental role in the study and design of networks. Graphs provide a powerful tool for modeling these complex systems. Graph theory deals with network analysis and models the relationships between entities within a network.

Networks can have millions of nodes and edges, making them highly complex. Granular computing simplifies this complexity by organizing these nodes into clusters. These clusters represent communities or groups of nodes that are interconnected. For example, in a social network, granular computing can be used to group individuals based on interaction frequency or common interests. Instead of analyzing each individual separately, the system clusters users into communities, such as groups of friends or professional networks, allowing efficient information retrieval and recommendation systems. Zadeh introduced the concept of granular computing in 1979 within the framework of fuzzy set theory [39]. He suggested that granular computing serves as a unifying framework for complex systems, especially when information is vague, imprecise, or incomplete. This approach enables processing and reasoning at different granularities, which helps simplify computational tasks while preserving essential relationships. Rough set theory (RST) is a mathematical technique of granular computing, designed to analyze and process imprecise, uncertain, or incomplete information. The core idea of RST is an equivalence relation that divides the universal set into distinct, non-overlapping subsets known as equivalence classes. In the context of RST, these equivalence classes form the basic building blocks for organizing and processing information. RST has wide-ranging applications across various fields, including data mining, knowledge representation, machine learning, and decision studies.

The integration of granular computing techniques with graph theory is a powerful approach for dealing with complex, uncertain, and large-scale data. The idea of granulation for graphs was first introduced by Stell [27, 28]. Chiaselotti et al. [7, 8] examined the applications of granular computing to graph theory. Javaid et al. [18] applied the concept of orbits to analyze graphs within the framework of RST. Arshad et al. [4] investigated zero-divisor graphs associated with rings of integers modulo n through the technique of granular computing. Guan et al. [12] introduced a granular computing approach based on graph theory, converting decision tables into granular networks, and proposed a data reduction algorithm based on these networks. Liau [20] studied social networks using granular computing techniques. Yager [37] explored the integration of graph theory and granular computing, particularly fuzzy set theory, to develop intelligent social network analysis. Fatima et al. [11] used RST to study finite-dimensional vector spaces. Arshad and Javaid [3] introduced a metric-based granular computing approach for network analysis. Akram et al. [1] investigated granular computing based on fuzzy indiscernibility relations. Baldini et al. [6] proposed a class-specific metric learning approach

for graph embedding using information granulation.

Graph theory plays a crucial role in network analysis, where the concept of distance is fundamental in understanding structural properties. The metric dimension of a graph, which determines the smallest set of vertices that uniquely identifies all other vertices based on their distances, has been widely studied in various domains, including communication networks [32], transportation systems [15], and social interactions [30]. The metric dimension of a graph provides a fundamental measure of its structural complexity and has applications across diverse fields. This concept not only aids in network design and optimization but also helps solve real-world problems like traffic optimization [34], and sensor network deployment [35]. Distance plays a crucial role in network analysis as it helps quantify structural properties such as reachability, clustering, and centrality, which are fundamental for understanding navigation, optimization, and fault detection in real-world systems [36]. The concept of the metric dimension of a graph was initially proposed by Slater [26] and later studied independently by Harary and Melter [16]. Harary and Melter [16] used the concepts of location number and locating set, while Slater [26] used the terminology of resolving set and metric dimension. The problem of finding the metric dimension of a graph is computationally challenging [16]. Garey and Johnson [13] established that determining the metric dimension of a general graph is NP-hard, which inspired the development of approximation algorithms. Khuller et al. [19] made significant contributions in this area, focusing on applications in sensor placement and network optimization. Tomescu and Imran [31] studied the metric dimension of certain infinite regular graphs. Variants such as the strong metric dimension, introduced by Oellermann and Peters-Fransen [22], and the fault-tolerant metric dimension, studied by Hernado et al. [17], address specific challenges such as robustness and edge resolution. Singh et al. [29] studied the metric dimension and edge metric dimension of windmill graphs.

The study of resolving sets and metric dimension in graphs has been extensively explored using various traditional approaches, including brute-force methods, approximation algorithms [19], and heuristic-based techniques [16, 26]. Our proposed method leverages RST to systematically identify all minimal resolving sets, offering a structured and efficient alternative to existing approaches. In graph theory, research on metric dimension has primarily focused on finding a minimum resolving set, the smallest subset of vertices that uniquely distinguishes all others. However, some problems require identifying multiple resolving sets to preserve the graph's structure. This work introduces a novel approach to determine all possible resolving sets of a graph. Particularly, we study the intersection graphs associated with finite cyclic groups. Intersection graphs provide a powerful framework for modeling relationships in complex systems by representing entities as vertices and their interactions as edges based on set intersections. The study of graph representations of algebraic structures has been an active area of research, offering insights into the interplay between group theory and graph theory [5, 9]. Akbari [2] introduced the concept of an intersection graph of a group, where the vertices represent all non-trivial proper subgroups, and two distinct vertices are adjacent if their corresponding subgroups have a non-trivial intersection. Rajkumar and Devi [25] studied intersection graphs of cyclic subgroups of groups. Zelinka [40] studied the intersection graphs of finite abelian groups. Pal [23] classified intersection graphs based on their geometric representation. Moh'd et al. [21] introduced a simple-intersection graph of a ring where the vertices are the nonzero ideals of ring, and two vertices are adjacent if and only if their intersection is a nonzero simple ideal.

In [3], Arshad and Javaid developed a metric-based granular computing framework to analyze general networks by modeling them as information systems. Furthermore, we established the

equivalence between reducts from rough set theory and resolving sets in graph theory. Building on this equivalence, we proposed a novel methodology to compute all minimal resolving sets in general networks, providing both theoretical insights and practical algorithms for structural analysis and simplification of complex network systems.

This study explores groups and graphs, beginning with the introduction of the intersection graph of the cyclic group $\mathbb{Z}_n$ and applying the RST to the intersection graphs of $\mathbb{Z}_m$. An information table for the intersection graph is defined using the distances between vertices, along with an indiscernibility relation on the vertex set $\mathcal{V}$ with respect to a subset $\mathbb{A} \subseteq \mathcal{V}$. Our study introduces a distance-based granulation approach, where granules are formed based on vertex distance similarity. Unlike traditional rough set methods that rely on attribute-based indiscernibility relations, our framework leverages graph distance for granulation in networks. We aim to establish a relationship between the reduct of information systems associated with intersection graphs and their metric dimension. Graph-theoretic research has primarily focused on the metric dimension of a graph. However, in certain applications, we need to consider alternative resolving sets to distinguish the graph effectively. We introduce a novel method that uses rough set theory, particularly the concepts of the indiscernibility relation and discernibility matrix, to systematically identify all possible resolving sets of graphs. The indiscernibility relation helps in identifying subsets that can serve as resolving sets, while using the discernibility matrix, we identify all the resolving sets. This approach highlights the connection between attribute reduction in information systems and metric dimension, suggesting that metric dimension can be treated as an attribute reduction parameter in the context of graph theory.

2. Basic concepts

This section provides a brief overview of fundamental concepts that will be useful in the subsequent discussions.

2.1. Information system and indiscernibility relation

An information system is essentially a table where objects are represented as rows, and attributes are represented as columns. Each object is described by the values it assumes for each attribute, which allows us to organize and analyze data effectively. Formally, an information system is a quadruple (U, Att, f, Val), where U is a non-empty set of objects called the universe, Att is a non-empty set of attributes, and f is defined as $f : U \times Att \to Val$. Every attribute $a \in Att$ maps each object in U to a corresponding value, providing a framework for comparing and analyzing the objects.

The indiscernibility relation is a fundamental concept that formalizes the idea of indistinguishability among objects based on their attribute values. Given a subset of attributes $B \in Att$, the indiscernibility relation is defined as: $x \equiv_B y \Leftrightarrow f(x, e) = f(y, e)$ for all $e \in B$ and $x, y \in U$.

The indiscernibility relation B partitions the set U into disjoint subsets, called a partition of U. Every disjoint subset is called an equivalence class, also referred to as a granule. For an object $x \in U$, we denote its equivalence class (or granule) with respect to the indiscernibility relation $\equiv_B$ as $[x]_B$. The discernibility relation is the negation of the indiscernibility relation. For any two objects $x, y \in U$, the discernibility relation is defined as $x \not\equiv_B y \Leftrightarrow f(x, e) \neq f(y, e)$ for some $e \in B$.

2.2. Rough set theory

Rough set theory provides a mathematical framework for dealing with uncertainty and ambiguity in data processing. It offers approaches for knowledge representation, data mining, and feature selection that focus on the fundamental links between objects based on their properties. In RST, each subset $X \subseteq U$ is connected with upper and lower approximations, defined as

$$\mathbb{L}_B(X) = \{x \in U : [x]_B \subseteq X\},$$
$$\mathbb{U}_B(X) = \{x \in U : [x]_B \cap X \neq \emptyset\}.$$

A set X is termed a rough set if its lower and upper approximations are distinct.

Example 2.1. Consider an information system where the universe of discourse U consists of five objects $U = \{x_1, x_2, x_3, x_4, x_5\}$ and the attribute set is $Att = \{a_1, a_2\}$. The attribute values for each object are given in Table 1.

Table 1. Information table.

Object	a_1	a_2
x_1	0	1
x_2	0	1
x_3	1	0
x_4	1	0
x_5	0	0

Suppose we define the target set as $X = \{x_1, x_2, x_3\}$. The equivalence classes induced by attributes a_1 and a_2 are $[x_1] = [x_2] = \{x_1, x_2\}$, $[x_3] = [x_4] = \{x_3, x_4\}$, $[x_5] = \{x_5\}$. The lower and upper approximations are $\mathbb{L}_B(X) = \{x_1, x_2\}$ and $\mathbb{U}_B(X) = \{x_1, x_2, x_3, x_4\}$. Since $\mathbb{L}_B(X) \neq \mathbb{U}_B(X)$, X is a rough set.

Rough set theory introduces the concepts of reducts and the core. A reduct of an information system is a minimal subset $B \subseteq Att$ such that the indiscernibility relation $\equiv_B$ holds the same equivalence classes as $\equiv_{Att}$. For an information system $\mathcal{I}$, a subset $R \subseteq V$ is called a reduct if $\Pi_R = \Pi_V$ and $\Pi_R \preceq \Pi_{R\setminus\{w\}}$, for all $w \in R$ [24]. The intersection of all different reducts is called the core. For two distinct objects $x, y \in U$, the discernibility matrix D_{ij} is the square matrix with the entry $D(i, j)$ given as follows:

$$D(i, j) = \{a \in Att : f(x, a) \neq f(y, a)\}.$$

2.3. Group theory

Group theory provides a unified framework for understanding and analyzing symmetry, structure, and operations across various mathematical contexts.

Definition 2.1. A group is a pair $(G, *)$, where G is a set and $*$ is a binary operation on G satisfying the following properties:
(i) $\forall x, y \in G,\ x * y \in G$.
(ii) $\forall x, y, z \in G,\ (x * y) * z = x * (y * z)$.
(iii) $\exists\, e \in G$ such that $\forall a \in G,\ e * x = x * e = x$.

(iv) $\forall\, x \in G,\ \exists\, y \in G$ such that $x * y = y * x = e$.
If the operation $*$ is also commutative, meaning $x * y = y * x$ for all $x, y \in G$, then the group is called an abelian group.

A group G is defined as an abelian group if it follows the commutative property, where $x * y = y * x$ for any elements x and y in G. Every group has a generating set. A subset S of G, denoted by $\langle S \rangle$, is called a generating set if every element in G can be derived from the elements of S. A group can have multiple generating sets. Based on its generating set, a group G is classified as either cyclic or non-cyclic. G is considered cyclic if it has a generating set S with $|S| = 1$, on the other hand, G is non-cyclic if its generating set S includes two or more elements, where $|S| \geq 2$. A subgroup is a subset of a group that satisfies the group axioms under the same operation as the original group.

Example 2.2. Let $G = \mathbb{Z}_{12}$ be the group of integers modulo 12 under addition. The non-trivial proper subgroups are $\langle 6 \rangle = \{0, 6\}$, $\langle 4 \rangle = \{0, 4, 8\}$, $\langle 3 \rangle = \{0, 3, 6, 9\}$, and $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$.

2.4. Graphs

We provide only the essential notations, assuming the reader has prior knowledge of the terminology. For more comprehensive explanations, please refer to [33].

Graph theory is the study of graphs, which are mathematical structures used to represent pairwise interactions between objects. A graph is defined as a pair $G = (V, E)$, where V is the set of vertices (or nodes) and E is the set of edges (or links) connecting pairs of vertices. Edges can be directed, resulting in a directed graph (or digraph), which indicates a one-way relationship between nodes. An undirected graph is one in which no edges have a direction. The degree of a vertex is defined as the number of edges connected to it. A path consists of a sequence of edges linking a series of vertices. A subgraph is derived by selecting a subset of vertices and edges from a graph while preserving their original relationships.

The shortest path length, or the number of edges between two vertices in a graph, is known as their distance and is denoted by $d(v_i, v_j)$. The diameter of a connected graph is defined as the greatest distance between any two vertices. For $v_i, v_j \in V$, if $d(v_i, u) \neq d(v_j, u)$, then $u \in V$ is said to resolve or distinguish v_i and v_j in V. A subset A of the vertex set V is called a resolving set of G if, for every pair $v_i, v_j \in V$, there exists at least one vertex $u \in A$ that resolves them. A resolving set with the smallest possible number of vertices is referred to as a metric basis for G, and its cardinality is called the metric dimension of G, denoted by $dim(G)$ [26]. Two vertices $v_i, v_j \in V$ are called twin vertices if $N[v_i] = N[v_j]$ or $N(v_i) = N(v_j)$. A vertex $v_i \in V$ is called a twin vertex if there exists a distinct vertex $v_j \neq v_i$ such that v_i and v_j are twins.

Let C be a finite cyclic group. The intersection graph of C, denoted as $\mathcal{G}_C = (\mathcal{V}, \mathcal{E})$, is a graph where the vertices represent all non-trivial proper subgroups of C, and two vertices are adjacent if their corresponding subgroups have a non-trivial intersection. The intersection graphs of cyclic group $\mathbb{Z}_m$ are denoted by $\mathcal{G}_{\mathbb{Z}_m}$ where m is a positive integer and $\mathcal{G}_{\mathbb{Z}_m}$ is a connected graph with $diam(\mathcal{G}_{\mathbb{Z}_m}) = 2$. For $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, $|\mathcal{V}| = \prod_{i=1}^{r}(s_i + 1) - 2$. Note that if two integers m and z can be expressed as $m = \prod_{i=1}^{r} p_i$ and $z = \prod_{j=1}^{r} q_j$, then the corresponding graphs $\mathcal{G}_{\mathbb{Z}_m}$ and $\mathcal{G}_{\mathbb{Z}_z}$ are isomorphic.

In this paper, we study two distinct families of intersection graphs associated with $\mathbb{Z}_m$, based on

the prime factorization of m. We use the notation $m = \prod_{j=1}^{r} p_j^{s_j}$, where p_j are distinct prime factors and $s_j \geq 1$ for graphs with twin vertices. However, when all prime powers are exactly 1, i.e., $m = \prod_{j=1}^{r} p_j$ the graph becomes twin-free. The intersection graphs associated with finite cyclic groups are complete, connected, star, and null graphs, depending on the prime factorization of the order of cyclic group. Graphs derived from groups provide a rich intersection between algebra and graph theory, offering a visual and structural way to explore the properties of groups.

3. Distance-based partition structure of intersection graphs

In this section, we represent the intersection graph of $\mathbb{Z}_m$ as an information table, where the distance between vertices serves as an information map to derive indiscernibility partitions. We establish an indiscernibility relation on the vertex set $\mathcal{V}$ with respect to a subset $S \subseteq \mathcal{V}$ and analyze the corresponding indiscernibility partitions induced on $\mathcal{V}$.

We start by presenting the concept of an information table for the intersection graph of $\mathbb{Z}_m$, which forms the foundation of this study.

Definition 3.1. An information table for the intersection graph $\mathcal{G}_C = (\mathcal{V}, \mathcal{E})$ in terms of the distance between vertices is a tabular representation where each entry corresponds to the distance-based relationships between pairs of vertices in $\mathcal{G}_C$. We define a distance-based information table $\mathcal{I}_C = (\mathcal{V}, S, \mathcal{F}, Val)$, where $S \subseteq \mathcal{V}$ and $\mathcal{V} = \{h_1, h_2, \cdots, h_n\}$, is the vertex set which is the set of all proper subgroups of $\mathcal{G}_C$, $Val = \{0, 1, 2, \cdots, diam(\mathcal{G}_C)\}$, and information map $\mathcal{F} : \mathcal{V} \times \mathcal{V} \to Val$ is defined as

$$\mathcal{F}(h_i, h_j) = \begin{cases} d(h_i, h_j), & \text{if } h_i \neq h_j, \\ 0, & \text{if } h_i = h_j. \end{cases}$$

In a traditional information system, objects are represented as rows, and attributes as columns, where each entry corresponds to the value of an attribute for a specific object. In a distance-based information table, rows and columns represent vertices (nodes) in a graph, where the entries correspond to the distance between pairs of vertices. From now onwards, we assume that $\mathcal{I}_C = \mathcal{I}_{\mathbb{Z}_m}$, where $\mathbb{Z}_m$ is a cyclic group.

Example 3.1. Consider the intersection graph of cyclic group $\mathbb{Z}_{24}$ in Figure 1 with the vertex set $\mathcal{V} = \{2, 3, 4, 6, 8, 12\}$ and the distance-based information system $\mathcal{I}_{\mathbb{Z}_{24}}$ in Table 2.

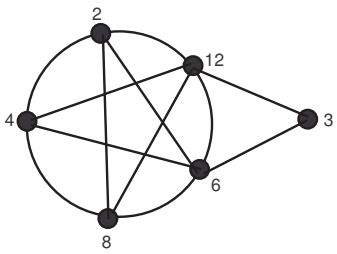

Figure 1. Intersection graph of $\mathbb{Z}_{24}$.

Table 2. Distance-based information system for $\mathcal{G}_{\mathbb{Z}_{24}}$.

	2	3	4	6	8	12
2	0	2	1	1	1	1
3	2	0	2	1	2	1
4	1	2	0	1	1	1
6	1	1	1	0	1	1
8	1	2	1	1	0	1
12	1	1	1	1	1	0

The vector of distances of a vertex h_i is the ordered k-tuple $(d(h_i, h_1), d(h_i, h_2), \cdots, d(h_i, h_n))$ and is denoted by $r(h_i|\mathcal{V})$. For any subset $S = \{s_1, s_2, \cdots, s_k\} \subseteq \mathcal{V}$, the distance vector of a vertex h_i is defined as $r(h_i|S) = d(h_i, s_1), d(h_i, s_2), \cdots, d(h_i, s_k))$. Clearly, the i^{th} vertex in S has 0 in its i^{th} coordinate, and all other coordinates are non-zero.

In Example 3.1, the distance vectors are $r(2|\mathcal{V}) = (0, 2, 1, 1, 1, 1)$, $r(3|\mathcal{V}) = (2, 0, 2, 1, 2, 1)$, $r(4|\mathcal{V}) = (1, 2, 0, 1, 1, 1)$, $r(6|\mathcal{V}) = (1, 1, 1, 0, 1, 1)$, $r(8|\mathcal{V}) = (1, 2, 1, 1, 0, 1)$, and $r(12|\mathcal{V}) = (1, 1, 1, 1, 1, 0)$. Consider a subset $S = \{2\} \subseteq \mathcal{V}$, we have $r(2|S) = (0)$, $r(3|S) = (2), r(4|S) = (1) = r(6|S) = r(8|S) = r(12|S)$. This representation of vertices based on distance corresponds to an equivalence relation among them. The indiscernibility relation fulfills all the properties required for an equivalence relation. The indiscernibility relation on $\mathcal{V}$ can be expressed as: two vertices h_i and h_j in $\mathcal{V}$ are considered indiscernible with respect to $S \subseteq \mathcal{V}$, denoted as $h_i \equiv_S h_j$ if and only if $r(h_i|S) = r(h_j|S)$. Equivalently, $h_i \equiv_S h_j \iff \mathcal{F}(h_i, s) = \mathcal{F}(h_j, s)$ for all $s \in S$. The collection of all vertices indiscernible to a vertex $h_i \in \mathcal{V}$ with respect to S is denoted by $[h_i]_S$ and defined as

$$[h_i]_S = \{h_j \in V : \mathcal{F}(h_j, s) = \mathcal{F}(h_i, s), \ \forall\, s \in S\}. \tag{3.1}$$

$[h_i]_S$ is said to be the equivalence class of h_i with respect to S. Each equivalence class is an information granule formed by indiscernible elements. These granules yield a partition on $\mathcal{V}$; that is, if $B \subseteq \mathcal{V}$ such that $B = [h_i]_S$, for some $h_i \in \mathcal{V}$, we say that B is an $S - granule$. Suppose $B_1, B_2, \cdots, B_k$ are the distinct granules in $\mathcal{V}$; we use the notation $\pi_S = B_1|B_2|\cdots|B_k$ to represent the indiscernibility partition of the vertex set $\mathcal{V}$. We say $(\mathcal{V}, S, \pi_S)$ is the S-granular referencing system.

The Algorithm 1 generates an information table $\mathcal{I}_{\mathbb{Z}_m}$ based on the distance between vertices in an intersection graph.

The following proposition establishes the relationship between vertices and distances in the graph $\mathcal{G}_{\mathbb{Z}_m}$.

Proposition 3.1. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, let $h_i, h_j \in \mathcal{V}$,
(i) $gcd(h_i, h_j) \neq 1$ if and only if $d(h_i, h_j) = 1$.
(ii) $gcd(h_i, h_j) = 1$ if and only if $d(h_i, h_j) = 2$.
(iii) $h_i = h_j$ if and only if $d(h_i, h_j) = 0$.

Two vertices, h_i and h_j are distance similar (twins) if $d(h_i, g) = d(h_j, g)$, $\forall\, g \in \mathcal{V} \setminus \{h_i, h_j\}$. The set of all twin vertices is called a twin class and denoted by $\mathcal{D}$. If a graph consists of k twin sets, we say π is a twin partition and $\pi = \mathcal{D}_1|\mathcal{D}_2|\cdots|\mathcal{D}_k$.

Algorithm 1. The information table $\mathcal{I}_{\mathbb{Z}_m}$

Input: $\mathcal{V} = \{h_1, h_2, \cdots h_n\}$ ▷ Vertices of $\mathcal{G}_{\mathbb{Z}_m}$
Output: $\mathcal{I}_{\mathbb{Z}_m}$ ▷ The information system $\mathcal{I}_{\mathbb{Z}_m}$ for $\mathcal{G}_{\mathbb{Z}_m}$.
for $1 \leq i \leq n$ **do**
  for $1 \leq j \leq k$ **do**
    if $(h_i \neq h_j \wedge d = d(h_i, h_j))$ **then**
      $\mathcal{F}(h_i, h_j) = d$
    else
      0
    end if
  end for
end for

For a given number $m = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$, let $P = \{p_1, p_2, \cdots, p_r\}$ be the set of all distinct primes used in the prime factorization of m. We let $\mathcal{D}(p_i) = \{p_i^{s_i} : 1 \leq r_i \leq s_r\}$ and generally, $\mathcal{D}(p_{i_1} p_{i_2} \cdots p_{i_k}) = \{p_{i_1}^{r_{i_1}} p_{i_2}^{r_{i_2}} \cdots p_{i_k}^{r_{i_k}} : 1 \leq r_{i_j} \leq s_{i_j} \text{ for } j = 1, 2, \ldots, k\}$. In the next result, we present the general representation of the partition of twin vertices.

Proposition 3.2. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, the twin partition is $\pi = \mathcal{D}(p_1)|\mathcal{D}(p_2)|\cdots|\mathcal{D}(p_r)|\ \mathcal{D}(p_1 p_2)|\cdots|\mathcal{D}(p_1 p_2 \cdots p_r)$ and $|\pi| = 2^r - 1$.

Proposition 3.2 is formulated specifically for intersection graphs derived from the cyclic group $\mathbb{Z}_m$, its applicability to non-cyclic groups must be examined to avoid biases inherent to cyclic structures. In non-cyclic groups such as dihedral groups D_n, symmetric groups S_n, and direct product groups, the concept of twin vertices remains valid as it is fundamentally based on neighborhood equivalence rather than cyclic properties. For instance, in D_n, twin vertices naturally emerge due to the reflection symmetries present in the group's structure. Similarly, in S_n, twin vertices appear when elements have identical orbit structures under conjugation. In direct product groups, twin vertices can be analyzed separately in each factor group, extending the partitioning approach used in Proposition 3.2. Since the theorem relies on equivalence classes formed by identical neighborhood relationships, it remains applicable beyond cyclic groups, though the specific partitions may vary in different groups.

The following corollary gives the number of vertices in a twin class.

Corollary 3.1. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$ and any vertex $p_{i_1}, p_{i_2}, \cdots p_{i_k} \in \mathcal{V}$, the cardinality of the class $\mathcal{D}(p_{i_1}, p_{i_2}, \cdots p_{i_k})$ is equal to $\phi(\frac{m}{p_{i_1}, p_{i_2}, \cdots p_{i_k}})$, where ϕ denotes Euler's totient function.

Example 3.2. Consider the intersection graph of cyclic group $\mathbb{Z}_{36}$ with the vertex set $\mathcal{V} = \{2, 3, 4, 6, 9, 12, 18\}$. The twin vertices are $\{2, 4\}, \{3, 9\}, \{6, 12, 18\}$ and $\pi = 2, 4|3, 9|6, 12, 18$.

Remark 3.1. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j$, note that no two distinct vertices are distance similar.

The following proposition establishes a connection between the indiscernibility relation and the pairwise distances between vertices in the graph.

Proposition 3.3. For $\mathcal{I}_{\mathbb{Z}_m}$, let $S \subseteq \mathcal{V}$ and $h_i, h_j \in \mathcal{V}$; then the following statements are equivalent:
(i) $h_i \equiv_S h_k$,
(ii) $d(h_i, s) = d(h_j, s), \forall s \in S$,
(iii) $r(h_i|S) = r(h_j|S)$.

The previous proposition also provides an interpretation for the indiscernibility relation $\equiv_S$. The equivalence relation $\equiv_S$ is then described as a type of distance-based similarity relation concerning the vertex subset S.

Proposition 3.4. For $\mathcal{I}_{\mathbb{Z}_m}$, the following statements hold:
(i) If $h_i \equiv_S h_j$, then $h_i \equiv_T h_j$ for all $T \subseteq S \subseteq \mathcal{V}$.
(ii) $h_i \equiv_P h_j$ iff $gcd(h_i, p_i) \neq 1 \neq gcd(h_j, p_i)$ or $gcd(h_i, p_i) = 1 = gcd(h_j, p_i)$ for $p_i \in P$.

Proof. (i) Suppose $h_i \equiv_S h_j$; then h_i and h_j are at the same distance from all elements of S. Since $T \subseteq S$, this implies that h_i and h_j are at same distance from all elements of T. Hence, $h_i \equiv_T h_j$.
(ii) The proof is structured by considering the following two cases:

Case 1. Suppose $gcd(h_i, p_i) \neq 1$ and $gcd(h_j, p_i) \neq 1$; then by Proposition 3.1, we have $d(h_i, p_i) = 1 = d(h_j, p_i)$ and by Proposition 3.3, $h_i \equiv_P h_j$.

Case 2. Suppose $gcd(h_i, p_i) = 1$ and $gcd(h_j, p_i) = 1$; then by Proposition 3.1, we have $d(h_i, p_i) = 2 = d(h_j, p_i)$ and by Proposition 3.3, $h_i \equiv_P h_j$.

Now, suppose conversely $h_i \equiv_P h_j$; then by Proposition 3.3, $d(h_i, p_i) = d(h_j, p_i)$, and by Proposition 3.1, $gcd(h_i, p_i) \neq 1 \neq gcd(h_j, p_i)$ or $gcd(h_i, p_i) = 1 = gcd(h_j, p_i)$ for all $p_i \in P$. This completes the proof.

Remark 3.2. For $\mathcal{I}_{\mathbb{Z}_m}$, let $S, T \subseteq \mathcal{V}$. Then, if $\pi_T = \pi_{\mathcal{V}}$, it follows that $\pi_S = \pi_{\mathcal{V}}$ for all $T \subseteq S$.

In the following result, we prove that the number of granules in π_S depends on the cardinality of S.

Proposition 3.5. For $S \subseteq \mathcal{V}$, $1 \leq |\pi_S| \leq n$. Moreover, If $|S| \geq 1$, then $3 \leq |\pi_S| \leq n$.

Proof. We prove the result in the following three cases:

Case 1. Suppose $S = \emptyset$; all the vertices in $\mathcal{V}$ are indiscernible, and $\pi_S = h_1, h_2, \cdots, h_n$ implies that $|\pi_S| = 1$.

Case 2. For any two vertices $h_i, h_j \in \mathcal{V}$, suppose $d(h_i, s) \neq d(h_j, s)$ for all vertices $s \in S$, since the graph has n vertices, there are n different possible representations, and the maximum number of equivalence classes is n; therefore, $|\pi_S| = n$.

Case 3. Suppose $|S| = 1$, then for each $h_i \in \mathcal{V}$, there are three entries in representation $r(h_i|S)$ such that $\pi_S = D_0|D_1|D_2$, where $D_0 = S$, $D_1 = \{h \in \mathcal{V} \mid r(h_i|S) = (1)\}$, and $D_2 = \{h_i \in \mathcal{V} \mid r(h_i|S) = (2)\}$. Thus, $|\pi_S| = 3$.

In the subsequent result, we establish a general representation of the partition induced by twin vertices.

Proposition 3.6. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, let $S = \{h_1, h_2, \cdots h_s\} \subseteq \mathcal{V}$, then the following statements hold:
(i) If $|S \cap \mathcal{D}_i| \geq 1$ for all $i = \{1, 2, \cdots, k\}$, $|S| = s$, then $\pi_S = h_1|h_2|\cdots|h_s|\mathcal{D}_1 \setminus S|\mathcal{D}_2 \setminus S|\cdots|\mathcal{D}_k \setminus S$.
(ii) If $S \subseteq \mathcal{D}_i$, $|S| = s$ and $S \cap \mathcal{D}_j = \emptyset$ for all $j \in \{1, 2, \cdots, k \wedge i \neq j\}$, then $\pi_S = h_1|h_2|\cdots|h_s|\mathcal{A}|S \setminus \mathcal{A}$, where $\mathcal{A} = \{h_i : gcd(h_i, s) = 1 \forall s \in S\}$.

Proof. (i) Suppose $|S \cap \mathcal{D}_i| \geq 1$ for all $i = \{1, 2, \cdots, k\}$, and $h_i, h_j \in S$, then by definition of distance $d(h_i, s) \neq d(h_j, s)$ for all $s \in S$ implies $h_i \not\equiv_S h_j$. Now, suppose that if $h_i, h_j \in \mathcal{D}_i$ and $\mathcal{D}_i \cap S = \emptyset$, then $d(h_i, s) = d(h_j, s)$ for all $s \in S$ and we have $h_i \equiv_S h_j$. If $h_i \in S$ and $h_j \in \mathcal{V} \setminus S$, then by definition of distance $d(h_i, h_i) = 0$ implies $h_i \not\equiv_S h_j$. This completes the proof.
(ii) Suppose $S \subseteq \mathcal{D}_i$ and $S \cap \mathcal{D}_j = \emptyset$ for all $j \in \{1, 2, \cdots, k \wedge i \neq j\}$, and let $h_i, h_j \in S$, then by definition of distance $d(h_i, s) \neq d(h_j, s)$ for all $s \in S$ implies $h_i \not\equiv_S h_j$. For $h_i, h_j \in \mathcal{V} \setminus S$ and $gcd(h_i, s) = 1$ and $gcd(h_j, s) = 1$ for all $s \in S$, then $r(h_i|S) = (2, \cdots, 2) = r(h_j|S)$ implies that $h_i \equiv_S h_j$. If $gcd(h_i, s) \neq 1$ and $gcd(h_j, s) \neq 1$ for all $s \in S$, then $r(h_i|S) = (1, \cdots, 1) = r(h_j|S)$ implies that $h_i \equiv_S h_j$. Lastly, if $gcd(h_i, s) = 1$ and $gcd(h_j, s) \neq 1$ for all $s \in S$, then $r(h_i|S) \neq r(h_j|S)$ implies that $h_i \not\equiv_S h_j$. This completes the proof.

The following remark gives the partition structure of the vertex set for the case when $\mathcal{G}_{\mathbb{Z}_m}$ is a complete graph.

Remark 3.3. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = p_j^r$ with $r \geq 1$, suppose $S = \{s_1, s_2, \cdots, s_k\} \subseteq \mathcal{V}$, then $r(h_i|S) = (\cdots, 1, 1, \cdots) = r(h_i|S)$ for all $h_i, h_j \in \mathcal{V} \setminus S$ implies that $h_i \equiv_S h_j$. Now, if $s_i, s_j \in S$, then $r(s_i|S) = (\cdots, 0_{i^{th} component}, \cdots)$ and $r(s_j|S) = (\cdots, 0_{j^{th} component}, \cdots)$ implies that $s_i \not\equiv_S s_j$ and $\pi_S = h_1|\cdots|h_s|\mathcal{V}\setminus S$.

Let $G = (V, E)$ be a graph. An automorphism of G is a bijective function $\phi : V \to V$ such that for all $v_i, v_j \in V$, if $v_i v_j \in E$, then $\phi(v_i)\phi(v_j) \in E$. The set of all automorphisms of G forms a group under the composition of functions, denoted by $Aut(G)$. The following proposition states that the partition of the vertex set $\mathcal{V}$ induced by a subset S is preserved under the action of an automorphism of the graph $\mathcal{G}_{\mathbb{Z}_m}$.

Proposition 3.7. For $S \subseteq \mathcal{V}$, if $\phi \in Aut(\mathcal{G}_{\mathbb{Z}_m})$ and $\pi_S = X_1|X_2|\cdots|X_n$, then $\pi_{\phi(S)} = \phi(X_1)|\phi(X_2)|\cdots|\phi(X_n)$ where $\phi(X_i) = \{\phi(h) \mid h \in X_i\}$.

Proof. Since ϕ is a bijection from $\mathcal{V}$ to $\mathcal{V}$ that preserves adjacency. Since adjacency is preserved, automorphisms also preserve graph distances, and by isometry $d(h_1, h_2) = d(\phi(h_1), \phi(h_2))$ for all $h_1, h_2 \in \mathcal{V}$. Thus, for any $h_1, h_2 \in \mathcal{V}$ and $a \in S$, we have $d(h_1, a) = d(h_2, a) \implies d(\phi(h_1), \phi(a)) = d(\phi(h_2), \phi(a))$. This shows that $\phi(h_1)$ and $\phi(h_2)$ are in the same equivalence class under $\phi(S)$ if h_1 and h_2 are in the same equivalence class under S. Let $\pi_S = X_1|X_2|\cdots|X_k$, and for any block X_i, consider $\phi(X_i) = \{\phi(h_1) \mid h_1 \in X_i\}$. We claim that $\phi(X_i)$ is a block of $\pi_{\phi(S)}$. If $h_1, h_2 \in X_i$, then $h_1 \equiv_S h_2$, so $d(h_1, a) = d(h_2, a)$ for all $a \in S$. By the action of ϕ, $d(\phi(h_1), \phi(a)) = d(\phi(h_2), \phi(a))$ for all $a \in S$, meaning $\phi(h_1) \equiv_{\phi(S)} \phi(h_2)$. Therefore, $\phi(h_1), \phi(h_2) \in \phi(X_i)$, and $\phi(X_i)$ forms a block of $\pi_{\phi(S)}$. Since ϕ is bijective, every vertex in $\mathcal{V}$ belongs to exactly one block $\phi(X_i)$, and no two such blocks overlap. Thus, $\pi_{\phi(S)}$ consists of the blocks $\phi(X_1), \phi(X_2), \ldots, \phi(X_k)$. Finally, we conclude that $\pi_{\phi(S)} = \phi(X_1)|\phi(X_2)|\cdots|\phi(X_k)$, which proves the proposition.

Two different subsets can induce identical partitions on the vertex set $\mathcal{V}$. We say that two subsets are equivalent if they generate identical partitions. Our goal is to identify subsets of $\mathcal{V}$ that induce the same partition as the partition induced by $\mathcal{V}$ itself. The next results outlines the properties of two distinct subsets of $\mathcal{V}$ that lead to identical partitions on $\mathcal{V}$.

Proposition 3.8. For $\mathcal{I}_{\mathbb{Z}_m}$ where $m = \prod_{j=1}^{r} p_j$, then the following statements hold:
(i) $\pi_P = \pi_{\mathcal{V}}$.
(ii) $\pi_{\mathcal{V}\setminus P} = \pi_{\mathcal{V}}$.
(iii) For $r \geq 4$, $\pi_{P\setminus\{p_i\}} \cup \{p_i p_j : 1 \leq j \leq r, i \neq j\} = \pi_{\mathcal{V}}$.

Proof. (i) Suppose $h_i, h_j \in P$; then $h_i \not\equiv_P h_j$. Now, let $h_i \in \mathcal{V} \setminus P$ and $h_j \in P$; then $d(h_i, h_j) \neq d(h_j, h_j)$ implies that $h_i \not\equiv_P h_j$. For $h_i, h_j \in \mathcal{V} \setminus P$, then $r(h_i|P) \neq r(h_j|P)$ because if we express h_i in the form $p_{i_1} p_{i_2} \cdots p_{i_k}$, its representation will have ones at positions $i_1 i_2 \cdots i_k$ and h_j in the form $p_{j_1} p_{j_2} \cdots p_{j_k}$, then it will have representation with 1 corresponding to the positions $j_1 j_2 \cdots j_k$. This implies that $h_i \not\equiv_P h_j$.
(ii) Suppose $h_i, h_j \in \mathcal{V} \setminus P$, then $r(h_i|\mathcal{V} \setminus P) \neq r(h_j|\mathcal{V} \setminus P)$ implies that $h_i \not\equiv_{\mathcal{V}\setminus P} h_j$. Now, assume $h_i, h_j \in P$, then there exist an element x in $\mathcal{V} \setminus P$ such that $h_i \mid x$ and $h_j \nmid x$, then $d(h_i, x) = 1$ and $d(h_j, x) = 2$, yielding that $h_i \not\equiv_{\mathcal{V}\setminus P} h_j$. If $h_i \in P$ and $h_j \in \mathcal{V} \setminus P$, then $d(h_i, h_j) \neq d(h_j, h_j)$ which again implies $h_i \not\equiv_{\mathcal{V}\setminus P} h_j$. This establishes that $\pi_{\mathcal{V}\setminus P} = \pi_{\mathcal{V}}$.
(iii) Suppose $h_i, h_j \in \mathcal{V}$; then by (ii) we have $h_i \equiv_P h_j$. Let $A = P \setminus \{p_i\}$, then there exist $h_i, h_j \in \mathcal{V}$ such that $h_i = p_{j_1} p_{j_1} \cdots p_{j_k}$ and $h_j = p_{i} p_{j_1} \cdots p_{j_k}$, then $r(h_i|A) = r(h_j|A)$ implies that $\pi_A \neq \pi_{\mathcal{V}}$. Now, suppose $A = P \setminus \{p_i\} \cup \{p_i p_j : 1 \leq j \leq r, i \neq j\}$; then for all $h_i, h_j \in \mathcal{V}$, $r(h_i|A) \neq r(h_j|A)$ implies that $\pi_A = \pi_{\mathcal{V}}$, completing the proof.

The following result provides a generalization for intersections graphs containing twin vertices, where the partitions induced by vertex subset and $\mathcal{V}$ itself coincide.

Proposition 3.9. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$ if $|\mathcal{D}_i \setminus S| \leq 1$ for all $i = \{1, 2, \cdots k\}$, then $\pi_S = h_1|h_2|\cdots|h_n = \pi_{\mathcal{V}}$.

Proof. For $S \subseteq \mathcal{V}$, suppose that $|\mathcal{D}_i \setminus S| \leq 2$, that is, there exist twin vertices $h_i, h_j \in \mathcal{D}_i \setminus S$. By the definition of twin vertices, we have $d(h_i, s) = d(h_j, s)$, $\forall s \in S$. This implies that $h_i \equiv_S h_j$, which is a contradiction to the fact that $\pi_S = h_1|h_2|\cdots|h_n$.

The above result shows that if we take $k - 1$ vertices from each twin set of k vertices, then it gives the same partition as $\mathcal{V}$.

In the previous result, we explored subsets that induce the same partition as the entire vertex set $\mathcal{V}$. Now, we are interested in identifying the minimal subsets of $\mathcal{V}$ that generate the same partition as $\mathcal{V}$. This problem is similar to the metric dimension problem, where the objective is to determine the smallest subset of vertices, known as resolving sets, that uniquely distinguish the positions of all vertices in a graph, such that $\pi_S = h_1|h_2|\cdots|h_n$. This concept is closely related to the idea of a reduct, with both resolving sets and reducts aiming to identify the smallest sets that preserve the complete information.

In light of the above correspondence between resolving sets and reducts, we now present key results that gives the reducts. In the subsequent theorem, we present the reducts for intersection graphs with twin vertices.

Theorem 3.1. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, any reduct set R satisfies $|R \cap \mathcal{D}_i| = |\mathcal{D}_i| - 1$ for all $i \in \{1, 2, \cdots, k\}$.

Proof. Suppose, to the contrary, that $|R \cap \mathcal{D}_i| = |\mathcal{D}_i| - 2$ for some $h_i \in \mathcal{V}$. Let $h_j, h_j' \in \mathcal{D}_i \setminus R$. Since all the vertices in $\mathcal{G}_{\mathbb{Z}_m}$ are twin, this implies $h_j \equiv_A h_j'$ for all $A \subseteq \mathcal{V} \setminus \{h_j, h_j'\}$. This is a contradiction. Hence, R is a reduct of $\mathcal{I}_{\mathbb{Z}_m}$, where $|R \cap \mathcal{D}_i| = |\mathcal{D}_i| - 1$ for all $i \in \{1, 2, \ldots, k\}$.

The following result is a direct implication of the preceding theorem.

Corollary 3.2. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, the size of the reduct R is $|R| = \sum_{i=1}^{k} (|\mathcal{D}_i| - 1)$.

Remark 3.4. For $\mathcal{G} = \mathbb{Z}_m$, where $m = p_j^s$ with $s \geq 1$, then $RED = \mathcal{V} \setminus \{h_i\}$ for any $h_i \in \mathcal{V}$.

In the following results, we present the reducts for intersection graphs with twin-free vertices.

Theorem 3.2. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j$ with $r \geq 4$, the set of prime numbers P is a reduct.

Proof. Suppose $p_i, p_j \in \mathcal{V}$, then $r(p_i|P) = (0_{ith}, 2, \cdots, 2)$ and $r(p_j|P) = (2, 0_{jth}, \cdots, 2)$ and we have $r(p_i|P) \neq r(p_j|P)$ implies that $p_i \not\equiv_P p_j$. For $p_ip_j, p_ip_k \in \mathcal{V}$, then $r(p_ip_j|P) = (1_{ith}, 1_{jth}, 2, \cdots, 2) \neq r(p_ip_k|P) = (1_{ith}, 2, 1_{kth}, \cdots, 2)$ implies that $p_ip_j \not\equiv_P p_ip_k$. Similarly, for $p_{i_1}p_{i_2}\cdots p_{i_s}, p_{j_1}p_{j_2}\cdots p_{j_s} \in \mathcal{V}$, then $r(p_{i_1}p_{i_2}\cdots p_{i_s}|P) \neq r(p_{j_1}p_{j_2}\cdots p_{j_s}|P)$ implies that $p_{i_1}p_{i_2}\cdots p_{i_s} \not\equiv_P p_{j_1}p_{j_2}\cdots p_{j_s}$. For $p_i, p_{i_1}p_{i_2}\cdots p_{i_s} \in \mathcal{V}$, then $r(p_i|P) = (0_{ith}, 2, \cdots, 2)$ and $r(p_{i_1}p_{i_2}\cdots p_{i_s}|P) = (1, \cdots 1, 2 \cdots 2)$ implies $r(p_i|P) \neq r(p_{i_1}p_{i_2}\cdots p_{i_s}|P)$. From the above arguments, we conclude that for all $h_i, h_j \in \mathcal{V}$ we have $h_i \not\equiv_P h_j$.

The following corollary directly follows from the above result.

Corollary 3.3. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j$ with $r \geq 4$, the cardinality of the minimal resolving set is r.

Remark 3.5. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j$ with $r = 3$, reducts are of the form
(i) $R = \{p_i, p_j, i \neq j\}$.
(ii) $|R \cap P| = 1$ and $|R \cap \mathcal{V} \setminus P| = 2$.
(iii) $R = \mathcal{V} \setminus P$.

From the previous results, we observe that multiple distinct subsets of the vertex set $\mathcal{V}$ can generate identical partitions on $\mathcal{V}$. Additionally, we notice the cases where two different subsets produce partitions with the same number of blocks, each having identical cardinalities. These types of partitions are termed size-isomorphic partitions. To formalize this, let $S, T \subseteq \mathcal{V}$ be subsets that induce partitions π_S and π_T on $\mathcal{V}$, respectively. If the number of blocks in π_S equals the number of blocks in π_T and the cardinality of each corresponding block is the same, then π_S and π_T are considered size isomorphic. In the following result, we establish that any two partitions induced by subsets from the same twin class are size isomorphic, thereby demonstrating a structural symmetry within the network. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, suppose $S, T \subseteq \mathcal{D}_i$ and $|S| = |T| = k$; then there are k blocks of single elements and all other elements are distributed likewise with respect to S and T, yielding partitions of the same size, and hence π_S and π_T are size isomorphic.

In this section, we explored the distance-based partition structure of intersection graphs, focusing on the partitions formed by subsets S such that $\pi_S = h_1 \mid h_2 \mid \cdots \mid h_n = \pi_{\mathcal{V}}$. We identified minimal subsets that give the same partition as $\mathcal{V}$, which coincides with the concept of a reduct (resolving set).

4. Discernibility matrix and essential sets

In this section, we introduce a novel approach to solve the metric dimension problem by identifying all the reducts using the discernibility matrix. We start by introducing the discernibility matrix, and we show that all the resolving sets can be derived by using the discernibility function. To formalize our approach, we introduce the discernibility matrix, which captures the pairwise distances between vertices and helps the identification of resolving sets.

The discernibility relation emerges from the negation of the indiscernibility relation, serving as the basis for constructing the discernibility matrix. We define the discernibility matrix $\Delta_{\mathcal{V}}$ as follows:

Definition 4.1. For $\mathcal{I}_{\mathbb{Z}_m}$, the discernibility matrix $\Delta_{\mathcal{V}}$ is the $|\mathcal{V}| \times |\mathcal{V}|$ matrix, and the entries are $\Delta_{\mathcal{V}}(h_i, h_j)$, where

$$\Delta_{\mathcal{V}}(h_i, h_j) = \{g \in \mathcal{V} : d(h_i, g) \neq d(h_j, g)\}.$$

Note that $\Delta_{\mathcal{V}}$ is symmetric. For the remainder of this section, we will consider a lower triangular matrix. Vertices with the same entries in the discernibility matrix may be structurally equivalent, meaning they play similar roles in the network. The collection $DM(\mathcal{G})$ consists of all distinct entries of $\Delta_{\mathcal{V}}$. The numeric discernibility matrix associated with $\mathcal{I}_{\mathbb{Z}_m}$ is denoted by $\mathcal{N}_{\mathcal{V}}$, and the entries in $\mathcal{N}_{\mathcal{V}}$ are defined as $\mathcal{N}(h_i, h_j) = |\Delta_{\mathcal{V}}(h_i, h_j)|$.

The following result outlines the relationship between the entries in the discernibility matrix and the indiscernibility of vertices.

Proposition 4.1. For $\mathcal{I}_{\mathbb{Z}_m}$, let $S \subseteq \mathcal{V}$ and $h_i, h_j \in \mathcal{V}$, we have:
(i) If $S = \Delta_{\mathcal{V}}(h_i, h_j)$, then $u \equiv_{\mathcal{V}\setminus S} v$.
(ii) If $h_i \equiv_{\mathcal{V}\setminus S} h_j$, then $\Delta_{\mathcal{V}}(h_i, h_j) \subseteq S$.
(iii) $\Delta_{\mathcal{V}}(h_i, h_j) \cap S = \emptyset$ if and only if $h_i \equiv_S h_j$.

The following result provides a representation of the entries in the discernibility matrix of $\mathcal{I}$ associated with the intersection graphs of $\mathbb{Z}_m$.

Theorem 4.1. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, the entries of $\Delta_{\mathcal{V}}$ are $\Delta_{\mathcal{V}}(h_i, h_j) = \{g : (gcd(g, h_i) = 1 \wedge gcd(g, h_j) \neq 1) \vee (gcd(g, h_i) \neq 1 \wedge gcd(g, h_j) = 1)\} \cup \{h_i, h_j\}$.

Proof. Suppose $h_i, h_j \in \mathcal{V}$ such that $gcd(g, h_i) = 1$ or $gcd(g, h_j) \neq 1$; then by Proposition 3.1, $d(h_i, g) \neq d(h_j, g)$ as a result $h_i, h_j \in \Delta_{\mathcal{V}}(h_i, h_j)$. Similarly, if $(gcd(g, h_i) \neq 1$ or $gcd(g, h_j) = 1$, then again by Proposition 3.1, $d(h_i, g) \neq d(h_j, g)$ implies $h_i, h_j \in \Delta_{\mathcal{V}}(h_i, h_j)$. It is clear from the information system that $\{h_i, h_j\} \subseteq \Delta_{\mathcal{V}}(h_i, h_j)$. Hence, the entries of $\Delta_{\mathcal{V}}$ are given by $\Delta_{\mathcal{V}}(h_i, h_j) = \{g : (gcd(g, h_i) = 1 \wedge gcd(g, h_j) \neq 1) \vee (gcd(g, h_i) \neq 1 \wedge gcd(g, h_j) = 1)\} \cup \{h_i, h_j\}$.

The next result is a direct consequence of the preceding theorem.

Corollary 4.1. For $\mathcal{G} = \mathbb{Z}_m$, let h_i and h_j be the twin vertices; then the entry of the discernibility matrix associated with h_i and h_j consists of only the two vertices themselves such that $\Delta_{\mathcal{V}}(h_i, h_j) = \{h_i, h_j\}$ for all $h_i, h_j \in \mathcal{V}$.

The discernibility function is a Boolean function derived from the discernibility matrix used to compute all the minimal resolving sets of the intersection graphs. The discernibility function $\zeta_{\mathcal{V}}$ for the information table $\mathcal{I}$ is given as follows:

$$\zeta_{\mathcal{V}} = \wedge\{\vee\Delta_{\mathcal{V}}(h_i, h_j) : \Delta_{\mathcal{V}}(h_i, h_j) \neq \emptyset\},$$

where $\vee\Delta_{\mathcal{V}}(h_i, h_j)$ represents the disjunction (OR) over attributes that distinguish h_i and h_j, and $\wedge\{\vee\Delta_{\mathcal{V}}(h_i, h_j)\}$ represents the conjunction (AND) over all $\vee\Delta_V(v_i, v_j)$.

Our approach leverages granular computing to handle large-scale networks by grouping vertices into granules based on distance similarity. Instead of analyzing every vertex pair independently, we identify distance-similar vertices and process them collectively, reducing the number of pairwise distance

calculations required for distinguishability analysis. To further enhance computational efficiency, we incorporate a greedy algorithm which iteratively selects resolving set elements, avoiding the need to compute the full discernibility matrix. We use the discernibility function to identify resolving sets based on the minimal entries of the discernibility matrix. This targeted approach eliminates the necessity of computing the entire matrix, significantly reducing computational overhead.

The Algorithm 2 computes all the resolving sets of $\mathcal{G}$ using the discernibility matrix.

Algorithm 2. Resolving sets of a graph using discernibility matrix

Input: $DM(\mathcal{G}) = \{S_i : 1 \leq i \leq s\}$ ▷ Collection of all the distinct entries of the discernibility matrix of $\mathcal{G}$.
Output: RED ▷ Resolving set (reduct) of $\mathcal{G}$.
Initialize: $i = 1$
for $i \leq s$ **do**
  Initialize: $j = 2$
  for $j \leq s$ **do**
    if $S_j \subset S_i$ **then**
      $DM(\mathcal{G}) = DM(\mathcal{G}) \setminus S_i$
      $j = j + 1$
    else
      $DM(\mathcal{G}) = DM(\mathcal{G})$
      $j = j + 1$
    end if
  end for
  $i = i + 1$
end for
Return $DM(\mathcal{G})$
$DM(\mathcal{G}) = \{S_l : 1 \leq l \leq h\}$ ▷ updated $DM(\mathcal{G})$
Initialize: $l = 1$
Initialize: $RED = \emptyset$
Initialize: $R = \emptyset$
for $l \leq h$ **do**
  if $R \cap S_l = \emptyset$ **then**
    $R = R \cup \{v\}$, for some $v \in S_l$
    $l = l + 1$
  else
    $RED = RED \cup R$
    $l = l + 1$
  end if
end for
Return R

Algorithm 2 iterates over all distinct entries in the discernibility matrix and performs set-based operations to construct the resolving set. The redundancy removal step involves nested iterations

over the entries, leading to a worst-case time complexity of $O(S^2)$. The second phase, which selects elements for the resolving set, runs in $O(s)$ time. Therefore, the overall time complexity of Algorithm 2 is $O(S^2)$. The pseudocode is given in Algorithm 3.

Algorithm 3. Pseudocode for finding resolving sets of a graph using the discernibility matrix

Input: $DM(\mathcal{G}) = \{S_i : 1 \leq i \leq s\}$ ▷ Distinct entries of the discernibility matrix
Output: RED ▷ Resolving set (reduct) of $\mathcal{G}$

```
Initialize i ← 1
while i ≤ s do
    Initialize j ← i + 1
    while j ≤ s do
        if S_j ⊂ S_i then
            Remove S_i from DM(G)
        end if
        j ← j + 1
    end while
    i ← i + 1
end while
Update DM(G) = {S_l : 1 ≤ l ≤ h}
Initialize l ← 1
Initialize RED ← ∅
Initialize R ← ∅
while l ≤ h do
    if R ∩ S_l = ∅ then
        Select an arbitrary v ∈ S_l
        R ← R ∪ {v}
    else
        RED ← RED ∪ R
    end if
    l ← l + 1
end while
return RED
```

In [7], Chiaselotti et al. introduced the concept of essential sets, also referred to as extended cores, for cases where the core is empty. Analogously, we can define essential sets in our context as follows:

Definition 4.2. For a graph $\mathcal{G}$, a set $E \subseteq \mathcal{V}$ is an essential set if for all $F \subseteq E$, $\pi_{\mathcal{V}\setminus E} = \pi_{\mathcal{V}}$ and $\pi_{\mathcal{V}\setminus F} = \pi_{\mathcal{V}}$.

Let $ESS(\mathcal{G})$ denote the set of all essential sets of the graph $\mathcal{G}$, and let $|ESS(\mathcal{G})|$ represent its cardinality in $\mathcal{G}$. Chiaselotti et al. [7] explored the relationship between essential sets in an information system and the entries of the associated discernibility matrix, demonstrating that the minimal entries of this matrix correspond to the essential sets of $\mathcal{G}$.

In the next results, we describe essential sets associated with the intersection graphs with twin vertices and with twin-free vertices.

Proposition 4.2. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j^{s_j}$ with $s_j \geq 1$, we have
(i) $ESS(\mathcal{I}) = \{h_i, h_i' : h_i, h_i \in \mathcal{D}_j\}$;
(ii) $E_{dim}(\mathcal{I}_(\mathbb{Z}_m) = 2$;
(iii) $|ESS(\mathcal{I})| = \sum_{l=1}^{k} \binom{|\mathcal{D}_j|}{2}$, where $|\mathcal{D}_j| \geq 2$.

Proof. (i) To prove that $\{h_i, h_j : h_i, h_i' \in \mathcal{D}_j\}$ is an essential set, we first show that $\pi_{\mathcal{V}} \neq \pi_{\mathcal{V}\setminus\{h_i,h_i'\}}$. For $h_i, h_i' \in \mathcal{V}$ then from Proposition 3.1 $d(h_i, h_i) = 0$ and $d(h_i', h_i) = 2$ implies that $\mathcal{F}(h_i, h_i) \neq \mathcal{F}(h_i', h_i)$ gives that $h_i \not\equiv_{\mathcal{V}} h_i'$. But $\mathcal{F}(h_i, u) = \mathcal{F}(h_i', u)$, for all $u \in \mathcal{V} \setminus \{h_i, h_i'\}$, gives $h_i \equiv_{\mathcal{V}\setminus\{h_i,h_i'\}} h_i'$. So $\pi_{\mathcal{V}} \neq \pi_{\mathcal{V}\setminus\{h_i,h_i'\}}$. By Proposition 3.9, $\pi_{\mathcal{V}\setminus\{h_i\}} = \pi_{\mathcal{V}}$; hence, $\{h_i, h_i'\}$ is an essential set.
(ii) The proof directly follows from (i).
(iii) It is evident from (i) that each subset of cardinality two of each distance-similar class is an essential set. So from each class $\mathcal{D}_i$ with $|\mathcal{D}_i| \geq 2$, there exist $\binom{|\mathcal{D}_j|}{2}$ essential sets. Consequently, the total number of essential sets is given by the sum $\sum_{i=1}^{k} \binom{|\mathcal{D}_j|}{2}$. □

The above result shows that each pair of twin vertices forms an essential set.

Remark 4.1. For $\mathcal{G} = \mathbb{Z}_m$, where $m = \prod_{j=1}^{r} p_j$ with $s_j \geq 1$, all the distinct entries of the discernibility matrix are essential sets other than V.

The following example demonstrates how to identify resolving sets within a graph using the discernibility function. This method offers a systematic approach for determining all minimal resolving sets by examining pairwise relationships between vertices and their distinguishing attributes.

Example 4.1. Consider $\mathcal{G} = \mathbb{Z}_{30}$ with the vertex set $\mathcal{V} = \{2, 3, 5, 6, 10, 15\}$. The distinct non-zero entries of the discernibility matrix are

$$DM(\mathcal{I}_{\mathbb{Z}_{30}}) = \{\{2, 3, 10, 15\}, \{2, 5, 6, 15\}, \{2, 5, 10, 15\}, \{2, 3, 5, 15\}, \{3, 5, 6, 10\}, \{2, 6, 10, 15\}, \{2, 3, 5, 10\}, \{3, 5, 6, 15\}, \{2, 5, 6, 10\}, \{2, 3, 5, 6\}\}.$$

From Theorem 3.5, $RED = \{\{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 6, 10\}, \{2, 6, 15\}, \{2, 10, 15\}, \{3, 6, 10\}, \{3, 6, 15\}, \{3, 10, 15\}, \{5, 6, 10\}, \{5, 6, 15\}, \{5, 10, 15\}, \{6, 10, 15\}\}$ is a collection of all resolving sets. Similarly, we can find these resolving sets by using Algorithm 2. We start from applying the discernibility function on $DM(\mathcal{I}_{\mathbb{Z}_{30}})$ as

$$\zeta_{dis} = \{\{2 \vee 3 \vee 10 \vee 15\} \wedge \{2 \vee 5 \vee 6 \vee 15\} \wedge \{2 \vee 5 \vee 10 \vee 15\} \wedge \{2 \vee 3 \vee 5 \vee 15\} \wedge \{3 \vee 5 \vee 6 \vee 10\} \wedge \{2 \vee 6 \vee 10 \vee 15\} \wedge \{2 \vee 3 \vee 5 \vee 10\} \wedge \{3 \vee 5 \vee 6 \vee 15\} \wedge \{2 \vee 5 \vee 6 \vee 10\} \wedge \{2 \vee 3 \vee 5 \vee 6\}\}.$$

By applying conjunction and disjunction, the obtained resolving sets are

$$Resolving\ Sets = \{\{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 6, 10\}, \{2, 6, 15\}, \{2, 10, 15\}, \{3, 6, 10\}, \{3, 6, 15\}, \{3, 10, 15\}, \{5, 6, 10\}, \{5, 6, 15\}, \{5, 10, 15\}, \{6, 10, 15\}\}.$$

In the previous section, we studied the equivalence relation, which is reflexive, symmetric, and transitive. We now turn our attention to partial order relation, which is reflexive, antisymmetric, and transitive.

5. Granular structures, dependency measures and approximations

In this section, we study the granular structures formed by the indiscernibility relation and study how different vertex subsets yields different partitions. We discuss measures of dependency of two subsets of a vertex set and examine the relationship between indiscernibility partitions.

The connection between two subsets $S, T \subseteq \mathcal{V}$ defines a partial order relation, denoted by $\preceq$, among the corresponding indiscernibility partitions as follows: $S \subseteq T \iff \pi_T \preceq \pi_S$. A partial order relation $\preceq$ is a type of binary relation that adheres to the principles of reflexivity, antisymmetry, and transitivity. For every $h_i \in \mathcal{V}$, the relation $\preceq$ is defined as $\pi_S \preceq \pi_T \iff [h_i]_S \subseteq [h_i]_T$. When $\pi_S \preceq \pi_T$, we describe π_S as being finer than π_T. If $\pi_S \preceq \pi_T$ and $\pi_S \neq \pi_T$, then π_S is strictly finer than π_T, written as $\pi_S \prec \pi_T$. It is important to note that when $S = \emptyset$, the partition induced by S is expressed as $\pi_S = h_1, h_2, \cdots, h_n$.

Remark 5.1. For $\mathcal{I}$, the partition derived from $\mathcal{V}$ corresponds to the finest partition, whereas the partition induced by the empty set is regarded as the coarsest partition.

Granular structures, commonly referred to as lattices, provide a framework to visualize the relationships among different partitions or ways of organizing elements within a network. A lattice demonstrates how one partition can be more detailed or refined compared to another. Let $\Pi_{\mathcal{V}} = \{\pi_S : S \subseteq \mathcal{V}\}$ denote the set of all distinct partitions of $\mathcal{V}$. The pair $\Pi_{ind(\mathcal{V})} := (\Pi_{\mathcal{V}}, \preceq)$ is called the indiscernibility partition lattice. Within this lattice, the ordering reflects how partitions refine or coarsen each other. The meet operation identifies the greatest partition, $\pi_S \wedge \pi_T$, that refines both π_S and π_T, while the join operation finds the smallest partition, $\pi_S \vee \pi_T$, that coarsens both π_S and π_T. A partially ordered set (poset) is classified as a complete lattice if it supports both meet and join operations. Moreover, two lattices are isomorphic if a bijective mapping between their elements preserves the order structure.

For $m = \prod_{j=1}^{r} p_j$, let B_i be the set of all divisors of m that contain exactly i prime factors, where $i \in \{1, 2, \ldots, r\}$. The next result establishes the isomorphism between the indiscernibility partition lattices corresponding to any two subsets ofB_i.

Proposition 5.1. For $\mathcal{G}_{\mathbb{Z}_m}$, where $m = \prod_{j=1}^{r} p_j$, let $S, T \subseteq B_i$ such that $|S| = |T|$, then $\Pi_{ind(S)}= (\Pi_S, \preceq)$ is isomorphic to $\Pi_{ind(T)} = (\Pi_T, \preceq)$.

Proof. Let $\eta : \Pi_{ind(S)} \to \Pi_{ind(T)}$ be a mapping such that $|S| = |T|$. By the definition of B_i, it follows that $|\pi_S| = |\pi_T|$. For each $S_i \in p(S)$ (power set of S), there exists a corresponding $T_i \in p(T)$ satisfying $\eta(\pi_{S_i}) = \pi_{T_i}$, which establishes η is bijective. Assume $\pi_{S_i} \preceq \pi_{S_j}$ for some $\pi_{S_i}, \pi_{S_j} \in \Pi_{ind(S)}$. This implies that every block of π_{S_i} is a subset of a block in π_{S_j}. Since η maps each block of π_{S_i} to a block of π_{T_i}, where each block of π_{T_i} is similarly contained in a block of π_{T_j}, it follows that $\eta(\pi_{S_i}) \preceq \eta(\pi_{S_j})$. Applying the same reasoning in the reverse direction confirms that η preserves the partial order relation.

From the preceding result, the following remark follows.

Remark 5.2. Consider $\mathcal{G}_{\mathbb{Z}_m}$ with $m = \prod_{j=1}^{r} p_j^{s_j}$, $s_j \geq 1$, and $\mathcal{G}_{\mathbb{Z}_z}$ with $z = \prod_{i=1}^{k} q_i^{s_i}$, $s_i \geq 1$, such that $m \neq z$ with associated sets of primes P and Q, respectively, let $S \subseteq P$ and $T \subseteq Q$ such that $|S| = |T|$, then $\Pi_{ind(S)}= (\Pi_S, \preceq)$ is isomorphic to $\Pi_{ind(T)} = (\Pi_T, \preceq)$.

The next proposition shows that the indiscernibility partition lattices generated by two distinct subsets within a distance-similar class are isomorphic. This means that the structure of the partition ordering is identical between the two lattices.

Proposition 5.2. For $\mathcal{I}$, let $S, T \subseteq \mathcal{D}_i$ such that $|S| = |T|$, then $\Pi_{ind(S)} = (\Pi_S, \preceq)$ is isomorphic to $\Pi_{ind(T)} = (\Pi_T, \preceq)$.

Proof. Assume that $\eta : \Pi_{ind(S)} \to \Pi_{ind(T)}$ with $|S| = |T|$. By the definition of $\mathcal{D}_i$, it follows that $|\pi_S| = |\pi_T|$. For each $S_i \in p(S)$, there exists a corresponding $T_i \in p(T)$ such that $\eta(\pi_{S_i}) = \pi_{T_i}$, which implies that η is a bijection. To prove that η preserves the relation $\preceq$, suppose $\pi_{S_i} \preceq \pi_{S_j}$ for some $\pi_{S_i}, \pi_{S_j} \in \Pi_{ind(S)}$. This implies that every granule of π_{S_i} is contained within a granule of π_{S_j}. Since η maps granules of π_{S_i} to granules of π_{T_i}, and each granule of π_{T_i} is contained within a granule of π_{T_j}, it follows that $\eta(\pi_{S_i}) \preceq \eta(\pi_{S_j})$. A similar argument shows that η maintains the partial order relation.

Two subsets of vertices are considered equivalent if they yield identical partitions. Let $S, T \subseteq \mathcal{V}$; we define the equivalence as: $S \approx T \iff \pi_S = \pi_T$. The equivalence class of S is denoted by $[S]^{\approx}$ and is defined as: $[S]^{\approx} = \{T \subseteq \mathcal{V} : S \approx T\}$. The maximum partitioner of S, denoted by $Max(S)$, is defined as the union of all elements in $[S]^{\approx}$, where $Max(S)$ is the largest set within this equivalence class. Specifically, $Max(S) = \{u \in \mathcal{V} : (h_i, h_k \in \mathcal{V} \wedge h_i \equiv_S h_k) \Rightarrow \mathcal{F}(h_i, u) = \mathcal{F}(h_k, u)\}$. In a similar manner, a set $B \in [S]^{\approx}$ is called the minimum partitioner of S, represented as $Min(S)$, if $\pi_B = \pi_S$ and for all $B' \subset B$, we have $\pi_{B'} \neq \pi_B$.

The following result shows the relationship between subsets and their maximum partitioners.

Proposition 5.3. For $\mathcal{I}$, let $S, T \subseteq \mathcal{V}$; then:
(i) $S \approx T \Leftrightarrow Max(S) = Max(T)$.
(ii) $\pi_S \preceq \pi_T \Leftrightarrow Max(T) \subseteq Max(S)$.
(iii) $\pi_{S \cup T} = \pi_{Max(S) \cup Max(T)}$.
(iv) $S \cap T \subseteq Max(S \cap T)$.

The following result establishes the connection between the maximum partitioner and the resolving sets (reducts) of $\mathcal{I}$ corresponding to $\mathcal{G}_{\mathbb{Z}_m}$.

Proposition 5.4. For $\mathcal{I}$, let $S \subseteq \mathcal{V}$, we have:
(i) If S is a resolving set (i.e., $S \in RED$, then $Max(S) = \mathcal{V}$).
(ii) If S is not a resolving set (i.e., $S \notin RED$, then $Max(S) = S$).
(iii) $Min(S) \subseteq S \subseteq Max(S)$.

The partial order relation is tied to the ideas of the positive region and dependency measure. Given subsets $S, T \subseteq \mathcal{V}$, the positive region of S with respect to T is defined as $POS_T(S) = \{h_j \in \mathcal{V} : [h_j]_T \subseteq [h_j]_S\}$. The dependency degree of S on T is quantified by $\kappa_T(S) = \frac{|POS_T(S)|}{|\mathcal{V}|}$, a value that ranges from 0 to 1. Specifically, if $S, T \subseteq \mathcal{V}$ and $\pi_S \preceq \pi_T$, then $POS_S(T) = \mathcal{V}$. This happens because when the partition induced by S is finer than the one induced by T, each element x is completely identifiable (or positively classified) by the information in T, as $[h_i]_S \subseteq [h_j]_T$. As a result, the dependency measure $\gamma_S(T)$ equals 1, with the positive region encompassing the entire set $\mathcal{V}$. In the subsequent results, we examine the positive region and the degree of dependency for various vertex subsets within the information system $\mathcal{I}$.

Proposition 5.5. For an information system $\mathcal{I}$, if S and T are resolving sets, then $POS_S(T) = \mathcal{V}$.

Proof. Let S and T be resolving sets. By definition, the partitions $\pi_S = h_1|h_2|\cdots|h_k = \pi_T$ hold for all $h_i \in \mathcal{V}$, and the condition $[h_i]_S \subseteq [h_j]_T$ implies that $POS_S(T) = \mathcal{V}$. □

Next, we consider the positive region for two distinct subsets of a distance-similar class.

Proposition 5.6. For non-empty subsets S, T of $\mathcal{V}$, the following hold:

(i) For $S, T \subseteq \mathcal{D}_i$, if $S \not\subseteq T$ and $S \not\supseteq T$, then $POS_S(T) = \emptyset$ and $POS_T(S) = \emptyset$.

(ii) For $S = \mathcal{D}_i$ and $S \supset T$, then $POS_S(T) = \mathcal{V}$.

Proof. (i) Suppose S, T are non-empty subsets of $\mathcal{D}_i$ such that $S \not\subseteq T$ and $S \not\supseteq T$. There exist elements $h_i, h_j \in S$ and $h_i, h_j \notin T$ such that $h_i \not\equiv_S h_j$ and $h_i \equiv_T h_j$. Therefore, we have $POS_S(T) = \emptyset$ and $POS_T(S) = \emptyset$.
(ii) Suppose $S = \mathcal{D}_i$ and $S \supset T$. By the definition of distance-similarity, $\pi_S \preceq \pi_T$, which implies that $POS_S(T) = \mathcal{V}$. □

We now illustrate the concept of resolving set using examples from real-life networks.

6. Applications in networking

6.1. Urban traffic planning using traffic pattern networks modeled by intersection graphs

A traffic pattern network is a graph-based representation of vehicle flow within an urban traffic system. It highlights critical aspects such as frequent routes, congestion hotspots, and interactions between intersections or road segments. This network plays a pivotal role in understanding and managing urban traffic efficiently by focusing on key elements like connectivity, intersection management, route optimization, and congestion prediction. The network is constructed using data from various sources, with nodes representing intersections and edges representing roads. It is utilized for traffic planning and for optimizing the placement of traffic monitoring devices to ensure smooth traffic operations. The primary objective is to achieve maximum traffic coverage at minimal cost.

Intersection graphs can be used to model traffic pattern networks effectively. In this framework, vertices represent intersections, while edges denote roads connecting these intersections. Resolving sets are employed to determine the optimal locations for placing traffic monitoring devices. Devices positioned at the vertices of a resolving set can uniquely track traffic across all intersections within the network. Identifying resolving sets with minimal cardinality–referred to as reducts in rough set theory–is particularly advantageous. These reducts aid in selecting key intersections for maximum coverage, developing emergency response systems, predicting congestion, and monitoring traffic using data collected from the resolving set intersections.

We model the traffic network of the city as an intersection graph, where each intersection in the city is represented as a vertex, and each road connecting two intersections is represented as an edge. The objective is to determine the minimal set of intersections (vertices) where traffic monitoring devices should be placed to ensure that all other intersections are uniquely identifiable based on their distances from the selected intersections. Consider a network of 6 intersections (denoted by $I_1, I_2, I_3, I_4, I_5, I_6$) connected by roads. The road connections can be represented as: $(I_1, I_4), (I_1, I_5), (I_2, I_4), (I_2, I_6), (I_3, I_5), (I_3, I_6)$. The network is shown in Figure 2.

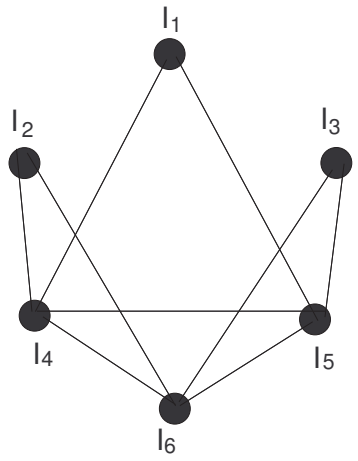

Figure 2. Traffic pattern network.

We now construct an information table based on the definition given in Section 3 that captures the pairwise distances between all intersections in the network. Using this information table, we identify the smallest subset of intersections where sensors can be placed so that every other intersection can be uniquely identified based on its distances to the selected intersections.

From the distance matrix in Table 3, we can see that placing sensors at intersections I_1 and I_2 will uniquely identify all other intersections in the graph. This is because the distances I_1 and I_2 to all other intersections are distinct. Thus, the metric dimension of this traffic network is 2, and one of the resolving sets is $\{I_1, I_2\}$. By placing sensors at intersections I_1 and I_2, we can monitor the entire traffic network with only 2 sensors. This results in a significant reduction in the number of sensors required compared to traditional methods, where sensors might be placed at every intersection.

Table 3. Distance-based information system for traffic pattern network.

	I_1	I_2	I_3	I_4	I_5	I_6
I_1	0	2	2	1	1	2
I_2	2	0	2	1	2	1
I_3	2	2	0	2	1	1
I_4	1	1	2	0	1	1
I_5	1	2	1	1	0	1
I_6	2	1	1	1	1	0

In urban transport planning, optimizing traffic flow and minimizing congestion are critical challenges. Traditional methods rely on shortest path algorithms or centrality measures, but they often fail to capture the structural dependencies between different road segments. The proposed method, based on distance-based granular computing and discernibility matrices, provides a systematic way to analyze the transport network as an intersection graph, where vertices represent critical road intersections or transportation hubs, and edges capture shared traffic flows or connectivity. By applying the discernibility matrix approach, we can efficiently determine resolving sets, which identify key intersections that uniquely distinguish all other locations in the network. This helps in strategic

placement of traffic sensors for real-time monitoring, designing optimal public transport routes by identifying critical transit nodes, and improving emergency response planning by ensuring efficient access to high-traffic areas.

6.2. Protein-protein interaction (PPI) networks modeled by intersection graphs

Protein-protein interaction (PPI) networks represent the functional relationships between proteins, where proteins are modeled as vertices and interactions between them form the edges of the graph. Understanding the structure of PPI networks is essential for functional annotation, drug discovery, and identifying critical proteins involved in diseases. In such networks, resolving sets play a crucial role in distinguishing proteins based on their interaction patterns.

Let $\mathcal{G}_{PPI} = (\mathcal{V}, \mathcal{E})$ represent a PPI network, where $\mathcal{V}$ is the set of proteins, and $\mathcal{E}$ represents interaction relationships between proteins. We construct an intersection graph by defining a biological similarity measure, for instance, two proteins are connected if they share a common biological function, pathway, or domain. This ensures that the intersection graph captures meaningful structural and functional relationships among proteins. Using the proposed a distance-based granular computing approach, we can compute resolving sets by forming discernibility matrices. Given a resolving set $S \subseteq \mathcal{V}$, each protein in $\mathcal{V}$ can be uniquely identified by its distance to the proteins in S. This ensures that the network's structure is well-represented while minimizing redundancy in functional classification.

6.3. Social influence networks modeled by intersection graphs

Social networks, such as those found on platforms like Twitter and Facebook, can be modeled as graphs where users are represented as vertices and edges indicate relationships (e.g., friendships, follower-following connections). Understanding influence propagation within such networks is crucial for marketing strategies, opinion dynamics, and information diffusion studies.

Let $\mathcal{G}_{social} = (\mathcal{V}, \mathcal{E})$ be a social influence network, where $\mathcal{V}$ represents individuals and $\mathcal{E}$ represents social connections between them. We construct an intersection graph based on shared interest groups, discussion topics, or engagement in similar activities. Two users are connected in this intersection graph if they belong to at least one common subgroup (e.g., same community, mutual interactions in discussions). Using distance-based granular computing approach, we determine the resolving set that identifies key influencers within the network. A resolving set $S \subseteq \mathcal{V}$ ensures that each individual in $\mathcal{V}$ has a unique distance signature relative to the influencers in S. This approach allows for the identification of opinion leaders, who play a crucial role in spreading information.

7. Conclusions

Network analysis is essential for understanding relationships between entities in networks, and graph theory provides a powerful framework to study these relationships. In this paper, we explored a distance-based granular computing approach for analyzing networks modeled by intersection graphs. By representing networks as information systems, we investigated granulation in networks through the concepts of indiscernibility and discernibility. We introduced a discernibility matrix and examined its properties, providing a novel framework for analyzing the metric dimension problem in networks. Two methods were proposed to identify minimal resolving sets, one based on reducts and the other

using the discernibility matrix. These methods not only enable the determination of the metric dimension but also allow for the identification of all resolving sets of minimal cardinality, offering a comprehensive solution to the metric dimension problem. Additionally, the practical applicability of the proposed methods was demonstrated through their application to a transportation network, showcasing their potential in urban traffic planning. The results underscore the effectiveness of the approach in uncovering underlying network patterns and facilitating the design of efficient computational frameworks for granular network analysis.

While the framework is applicable to large-scale networks, its computational efficiency can be further improved. The construction of the discernibility matrix involves distance calculations between vertex pairs, which may pose challenges for extremely large and dense graphs. Although our primary focus has been on intersection graphs of cyclic groups, an important direction for future work is extending this framework to non-cyclic groups and other network structures to further validate its applicability. Additionally, extending the proposed framework to dynamic networks would allow for the study of temporal variations in network structures and how granular partitions change over time.

Author contributions

Rehab Alharbi, Hibba Arshad, Imran Javaid, Ali. N. A. Koam and Azeem Haider: Writing–review and editing, Writing–original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. All authors have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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