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Research article

Octonion-valued b-metric spaces and results on its application

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Abstract: This study introduces octonion-valued b-metric spaces as a natural extension of the octonion-valued metric spaces developed by establishing a partial ordering relation on octonions. Octonion-valued b-metric spaces are constructed by modifying the triangle inequality of a semi-metric space, where one side of the inequality is multiplied by a positive scalar $b \geq 1$. On the other hand, octonion-valued metric spaces generalize the concept of classical metric spaces by employing octonions, which provide a higher-dimensional and non-associative algebraic framework. Two key reasons make this novel generalization of metric spaces very interesting: First, octonions are not even a ring since they do not have the associative feature in multiplication; second, the spaces do not meet the standard triangle inequality. In addition to explanations on sequences, convergence, Cauchy characteristics, boundedness, theorems, and associated conclusions, examples are given to help visualize this recently formed metric space. Lastly, the building of a fixed point finds extensive applications in a variety of mathematical analytic subjects as well as applied mathematics domains like differential equations and dynamical systems. Because of this, octonion-valued b-metric spaces have been used to study the Banach fixed-point theorem and a few additional fixed-point theorems.

Keywords: Clifford analysis; octonion convergence; fixed point; generalized metric space
Mathematics Subject Classification: 40A05, 46A19, 47H10, 54E35, 54H25

1. Introduction

Shortly after Hamilton discovered quaternions, John T. Graves proposed octonions in 1843. Arthur Cayley then independently developed and expanded the idea. Under the direction of the Cayley-Dickson construction, hypercomplex number theory has expanded systematically, as seen by the progression from real numbers to complex numbers, quaternions, and octonions. From the one-dimensional reals to the two-dimensional complexes, the four-dimensional quaternions, and the eight-dimensional octonions, this iterative process doubles the dimensionality at each stage, revealing progressively more complicated algebraic structures. Octonions stand out due to their special mathematical characteristics. Octonions are neither commutative nor associative, in contrast to quaternions, which are non-commutative but associative, and real and complex numbers, which are both commutative. The fact that they are not associative suggests that the way words are grouped influences the result of multiplication, so that $(ab)\,c \neq a\,(bc)$. Because of this property, octonions are not included in traditional algebraic systems; instead, they fall within the more general category of alternative algebras, which meet the loose associative requirements set by the Moufang identities. Applications requiring multidimensional data interactions have benefited from the unique non-associative nature of octonions. In order to demonstrate their usefulness in physics, Kansu et al. [18] developed duality-invariant field equations for dyons that are comparable to Maxwell's equations. These equations, which make use of the eight-dimensional octonion framework, successfully capture the complex interaction between electric and magnetic elements in a single model. Octonions have become effective tools for analyzing high-dimensional data in machine learning. Deep octonion networks (DONs) were first presented by Wu et al. [35]. They used the multidimensional and compact characteristics of octonions to integrate a variety of features across neural network layers. In tasks like image classification, this method has demonstrated significant gains in performance and convergence efficiency. Octonions' usefulness to control systems, particularly in the dynamic control of robot manipulators, was further expanded by Takahashi et al. [33]. Precise multi-axis movement control is made possible by octonian-valued neural networks, which describe intricate spatial and temporal dynamics. Octonions' non-associative characteristics offer the adaptability needed for such complex modeling. To gain a thorough understanding of octonions, their subalgebraic structures, and their multidisciplinary applications encompassing octonion theory [8–10], quantum mechanics [4], and the field of physical algebra [28].

Conversely, fixed point theory has been thoroughly studied in a number of disciplines, including physics, engineering, and mathematics. Metric fixed point theory, which finds application in topology, analysis, and practical mathematics, is particularly significant. Furthermore, academics commonly employ two approaches to generalize the Banach contraction principle. While the second focuses on generalizing the underlying metric space, the first necessitates an expansion of the contraction condition applied. One of the key topics in fixed point theory, which is a generalization of the Banach contraction principle, is the presence of fixed points of contraction mappings in bipolar metric spaces. In recent years, these and related subjects have been studied extensively in the context of $\mathcal{F}$-metric spaces [2, 3, 36], and various other metric structures [24, 26, 34].

In this study, we introduce octonion-valued b-metric spaces as a natural and logical extension of octonion-valued metric spaces, constructed by first establishing a partial ordering relation on octonions. These generalized spaces are built upon the theoretical framework of octonion-valued metric spaces by relaxing the triangle inequality and taking into account the non-associative and non-commutative nature of octonions. In addition to comments on sequences, convergence, Cauchy features, boundedness, theorems, and associated conclusions, we then provide examples to aid in visualizing this newly created metric space. The Banach fixed point theorem and numerous other fixed point theorems for octonion-valued b-metric spaces are studied using a construction of a fixed point that we also provide. This construction has a broad range of applications in applied mathematics.

The structure of the paper is as follows. Concepts and qualities that will be helpful in the future are covered in Section 2. Summability theory and some concepts of convergence on b-metric spaces are covered in Section 3. The Banach fixed point theorem and its applications to octonion-valued b-metric spaces are the focus of Section 4, which also examines a number of other fixed point theorems.

2. Some definitions and notations

We now review the fundamental ideas along with a few definitions and symbols. Generalize the complex metric spaces defined by Azam et al. [7] by taking the codomain as the field of complex numbers.

Definition 1. *[7] Given a non-empty set S. If the transformation $\Omega_{\mathbb{C}} : S \times S \to \mathbb{C}$ on this set satisfies the following conditions,*

(1) $0_{\mathbb{C}} \preceq \Omega_{\mathbb{C}}(s,t)$, *for all* $s,t \in S$ *and* $\Omega_{\mathbb{C}}(s,t) = 0_{\mathbb{C}} \iff s = t$.
(2) $\Omega_{\mathbb{C}}(s,t) = \Omega_{\mathbb{C}}(t,s)$ *for all* $s,t \in S$.
(3) $\Omega_{\mathbb{C}}(s,t) \preceq \Omega_{\mathbb{C}}(s,v) + \Omega_{\mathbb{C}}(v,t)$ *for all* $s,t,v \in S$.

Then the pair $(S, \Omega_{\mathbb{C}})$ *is said to be a complex metric space.*
If a complex-valued metric space satisfies the condition

$$\Omega_{\mathbb{C}}(s,t) \preceq b \cdot (\Omega_{\mathbb{C}}(s,v) + \Omega_{\mathbb{C}}(v,t)), \tag{2.1}$$

for all $s,t,v \in S$*, which is a relaxed version of the triangle inequality for* $b \geq 1$ *derived using the partial ordering in the third property, such a space is called a complex-valued b-metric space. Detailed information about this space can be found in the literature, specifically in [27, 29].*

These are then generalized to quaternion-valued metric spaces, as defined by Ahmed et al. [11], taking the codomain as the skew field of quaternions, which serve as a non-commutative extension of these metric spaces to Clifford algebra analysis.

Definition 2. *[11] Given a nonempty set S. If the transformation $\Omega_{\mathbb{H}} : S \times S \to \mathbb{H}$ on this set satisfies the following conditions,*

(1) $0_{\mathbb{H}} \preceq \Omega_{\mathbb{H}}(s,t)$ *for all* $s,t \in S$ *and* $\Omega_{\mathbb{H}}(s,t) = 0_{\mathbb{H}} \iff s = t$,
(2) $\Omega_{\mathbb{H}}(s,t) = \Omega_{\mathbb{H}}(t,s)$ *for all* $s,t \in S$,

(3) $\Omega_{\mathbb{H}}(s,t) \preceq \Omega_{\mathbb{H}}(s,v) + \Omega_{\mathbb{H}}(v,t)$ *for all* $s,t,v \in S$.

Then $\Omega_{\mathbb{H}}$ *is said to be a quaternion-valued metric on* S*, and the pair* $(S,\Omega_{\mathbb{H}})$ *is said to be a quaternion-valued metric space.*

If a quaternion-valued metric space satisfies the condition

$$\Omega_{\mathbb{H}}(s,t) \preceq b \cdot (\Omega_{\mathbb{H}}(s,v) + \Omega_{\mathbb{H}}(v,t)), \tag{2.2}$$

for all $s,t,v \in S$*, which is a relaxed version of the triangle inequality for* $b \geq 1$ *derived using the partial ordering in the third property, such a space is called a quaternion-valued b-metric space. Detailed information about this space can be found in the literature, specifically in [20–23].*

We shall investigate $\mathbb{O}$, Octonions, a non-associative extension of the division algebra of quaternions, in the next section. We will now start by adding an extra basis element ℓ to the quaternion basis elements, which are represented as $\{1, i, j, k\}$. According to [15], this extension allows us to build the eight-dimensional octonion division algebra in detail, including its algebraic operations and diagrammatic representation.

In Figure 1 is a diagram illustrating the multiplication of the generators of the octonions. According to this diagram, for example, $e_2e_3 = e_1$ and $e_3e_4 = e_7$ can be observed.

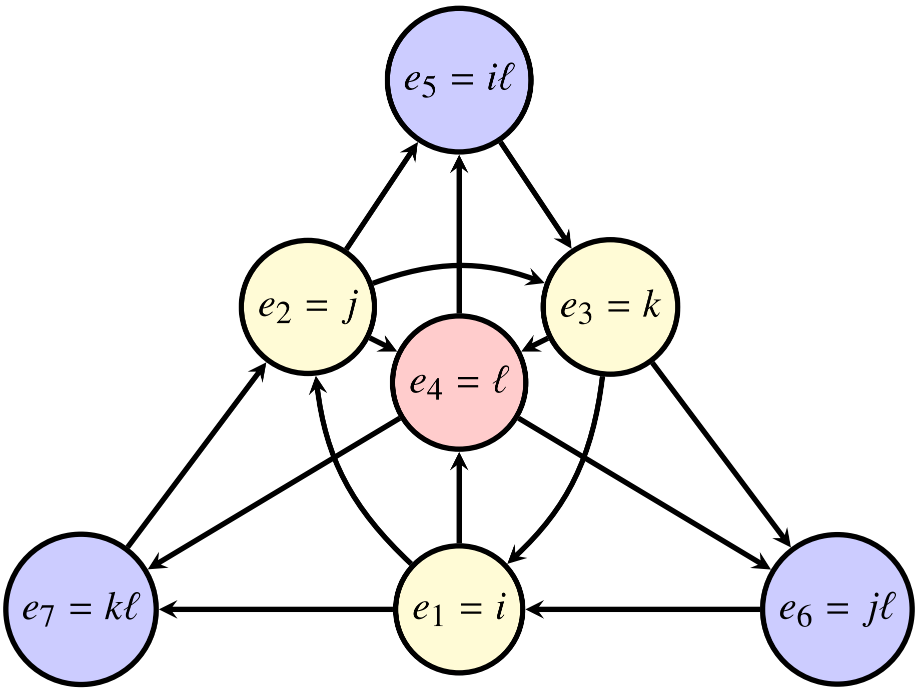

Figure 1. Octonion multiplication diagram.

Consequently, the following form may be used to express each element $\mathbb{o} \in \mathbb{O}$:

$$\mathbb{o} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7, \quad o_n \in \mathbb{R}, \quad \text{where } n = 0,1,2,3,4,5,6,7.$$

The basis elements of $\mathbb{O}$ are $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$.

Table 1 displays the specific multiplication of these foundational components.

Table 1. Cayley table for octonion multiplication.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The conjugate element $\bar{\mathbb{o}}$ is given by

$$\bar{\mathbb{o}} = o_0 - o_1e_1 - o_2e_2 - o_3e_3 - o_4e_4 - o_5e_5 - o_6e_6 - o_7e_7.$$

The norm of an arbitrary octonion is calculated as

$$||\mathbb{o}|| = \sqrt{\mathbb{o} \cdot \bar{\mathbb{o}}} = \sqrt{o_0^2 + o_1^2 + o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2}.$$

Additionally, the inverse of an arbitrary octonion $\mathbb{o}$ is given in the form

$$\mathbb{o}^{-1} = \frac{\bar{\mathbb{o}}}{||\mathbb{o}||^2}.$$

Similar to a movement vector, the imaginary component of each quaternion may be expressed as a vector in three-dimensional Euclidean space, whereas the real part of the quaternion indicates the time of the movement. An alternative viewpoint is also possible by redefining octonions as a pair made up of a scalar and a vector in a seven-dimensional Euclidean space. Octonions, being a more complicated structure, lose the associative quality from the group axioms in multiplication, but quaternions vary from real and complex numbers in their non-commutative multiplication. This adds to its interesting qualities by making division algebra over octonions non-associative.

The eight real numbers $(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ can be represented as an ordered set, with coordinate-wise addition and multiplication determined by a particular table. The real part in this case is the first component, o_0, while the imaginary part is the remaining seven-tuple $(o_1, o_2, o_3, o_4, o_5, o_6, o_7)$.

Accordingly, as previously mentioned, each quaternion may be expressed as $(o_0, \overrightarrow{u})$, where $\overrightarrow{u} = (o_1, o_2, o_3, o_4, o_5, o_6, o_7)$, and o_0 denotes the real part. The following characteristics are readily observable from this location:

$$\begin{aligned} \mathbb{o} :&= (o_0, \overrightarrow{u}), \quad \overrightarrow{u} \in \mathbb{R}^7; \quad o_0 \in \mathbb{R} \\ &= (o_0, (o_1, o_2, o_3, o_4, o_5, o_6, o_7)); \quad o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R} \\ &= o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7. \end{aligned}$$

Now, let us define a partial ordering relation $\preceq$ on the non-associative and non-commutative octonion algebra $\mathbb{O}$ as follows.

$\mathbb{O} \preceq \mathbb{O}'$ if and only if $\mathrm{Re}(\mathbb{O}) \leq \mathrm{Re}(\mathbb{O}')$, $\mathrm{Im}_e(\mathbb{O}) \leq \mathrm{Im}_e(\mathbb{O}')$, $\mathbb{O}, \mathbb{O}' \in \mathbb{H}$; $e = e_1, e_2, e_3, e_4, e_5, e_6, e_7$, where $\mathrm{Im}_{e_n} = o_n; n = 1, 2, 3, 4, 5, 6, 7$. To confirm that it is $\mathbb{O} \preceq \mathbb{O}'$, satisfying any one of the 256 conditions derived from the sum of all possible combinations of 8, from 0 to 8 respectively, will suffice.

Obtained from the 0 combinations of 8, meaning none of its components are equal, this 1 case constitutes

(1) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Obtained from the 1 combinations of 8, meaning only one component is equal; these 8 cases constitute

(2) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.
(3) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$.
(4) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{O}) = \mathrm{Im}_{e_2}(\mathbb{O}')$.
(5) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{O}) = \mathrm{Im}_{e_3}(\mathbb{O}')$.
(6) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{O}) = \mathrm{Im}_{e_4}(\mathbb{O}')$.
(7) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\mathbb{O}) = \mathrm{Im}_{e_5}(\mathbb{O}')$.
(8) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\mathbb{O}) = \mathrm{Im}_{e_6}(\mathbb{O}')$.
(9) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\mathbb{O}) = \mathrm{Im}_{e_7}(\mathbb{O}')$.

Obtained from the 2-combinations of 8, meaning only two components are equal; these 27 cases constitute

(10) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$.
(11) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{O}) = \mathrm{Im}_{e_2}(\mathbb{O}')$.
(12) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{O}) = \mathrm{Im}_{e_3}(\mathbb{O}')$.
(13) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{O}) = \mathrm{Im}_{e_4}(\mathbb{O}')$.
(14) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\mathbb{O}) = \mathrm{Im}_{e_5}(\mathbb{O}')$.
(15) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\mathbb{O}) = \mathrm{Im}_{e_6}(\mathbb{O}')$.
(16) $\mathrm{Re}(\mathbb{O}) = \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\mathbb{O}) = \mathrm{Im}_{e_7}(\mathbb{O}')$.
(17) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_2}(\mathbb{O}) = \mathrm{Im}_{e_2}(\mathbb{O}')$.
(18) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_3}(\mathbb{O}) = \mathrm{Im}_{e_3}(\mathbb{O}')$.
(19) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_4}(\mathbb{O}) = \mathrm{Im}_{e_4}(\mathbb{O}')$.
(20) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_5}(\mathbb{O}) = \mathrm{Im}_{e_5}(\mathbb{O}')$.
(21) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_6}(\mathbb{O}) = \mathrm{Im}_{e_6}(\mathbb{O}')$.
(22) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_1}(\mathbb{O}) = \mathrm{Im}_{e_1}(\mathbb{O}')$; $\mathrm{Im}_{e_7}(\mathbb{O}) = \mathrm{Im}_{e_7}(\mathbb{O}')$.
(23) $\mathrm{Re}(\mathbb{O}) < \mathrm{Re}(\mathbb{O}')$; $\mathrm{Im}_{e_n}(\mathbb{O}) < \mathrm{Im}_{e_n}(\mathbb{O}')$, $n = 1, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{O}) = \mathrm{Im}_{e_2}(\mathbb{O}')$; $\mathrm{Im}_{e_3}(\mathbb{O}) = \mathrm{Im}_{e_3}(\mathbb{O}')$.

(24) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,3,5,6,7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(25) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,3,4,6,7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(26) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,3,4,5,7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(27) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,3,4,5,6$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(28) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,5,6,7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(29) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,4,6,7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(30) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,4,5,7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(31) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,4,5,6$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(32) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,3,6,7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(33) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,3,5,7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(34) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,3,5,6$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(35) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,3,4,7$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(36) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1,2,3,4,6$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.

The 56 cases where exactly 3 components are equal (taken from the 3-combinations of 8), the 70 cases with 4 equal components, the 56 cases with 5 equal components, and the 27 cases with 6 equal components may all be readily listed using a similar method. However, we won't go into great depth on the remaining 211 intermediate instances to save the post from becoming unduly boring. To keep things simple, let's simply concentrate on the 8 cases that have precisely 7 equal components, which corresponds to the 7-combinations of 8 where only one component is different.

(248) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,3,4,5,6,7$.
(249) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 2,3,4,5,6,7$; $\mathrm{Im}_{e_1}(\mathbb{o}) < \mathrm{Im}_{e_1}(\mathbb{o}')$.
(250) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,3,4,5,6,7$; $\mathrm{Im}_{e_2}(\mathbb{o}) < \mathrm{Im}_{e_2}(\mathbb{o}')$.
(251) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,4,5,6,7$; $\mathrm{Im}_{e_3}(\mathbb{o}) < \mathrm{Im}_{e_3}(\mathbb{o}')$.
(252) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,3,5,6,7$; $\mathrm{Im}_{e_4}(\mathbb{o}) < \mathrm{Im}_{e_4}(\mathbb{o}')$.
(253) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,3,4,6,7$; $\mathrm{Im}_{e_5}(\mathbb{o}) < \mathrm{Im}_{e_5}(\mathbb{o}')$.
(254) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,3,4,5,7$; $\mathrm{Im}_{e_6}(\mathbb{o}) < \mathrm{Im}_{e_6}(\mathbb{o}')$.
(255) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1,2,3,4,5,6$; $\mathrm{Im}_{e_7}(\mathbb{o}) < \mathrm{Im}_{e_7}(\mathbb{o}')$.

Lastly, let's look at the scenario that results from the 8-combinations of 8, in which the two octonions are similar since all associated components are equal.

(256) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Specifically, if $\|\mathbb{o}\| \neq \|\mathbb{o}'\|$ and any condition between (1) and (256) is satisfied,$\mathbb{o} \preceq \mathbb{o}'$ will be writen. If only condition (256) is satisfied, we will denote this by $\mathbb{o} \prec \mathbb{o}'$. We will briefly denote this situation as

$$\mathbb{o} \preceq \mathbb{o}' \implies \|\mathbb{o}\| \leq \|\mathbb{o}'\|. \tag{2.3}$$

We are able to introduce octonion-valued metric spaces by closely examining the 256 criteria mentioned previously.

Definition 3. *Given a nonempty set S. If the transformation $\Omega_{\mathbb{O}} : S \times S \to \mathbb{O}$ on this set satisfies following conditions,*

(1) $0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s,t)$ for all $s, t \in S$ and $\Omega_{\mathbb{O}}(s,t) = 0_{\mathbb{O}}$ if and only if $s = t$,
(2) $\Omega_{\mathbb{O}}(s,t) = \Omega_{\mathbb{O}}(t,s)$ for all $s, t \in S$,
(3) $\Omega_{\mathbb{O}}(s,t) \preceq \Omega_{\mathbb{O}}(s,v) + \Omega_{\mathbb{O}}(v,t)$ for all $s, t, v \in S$.

Then $\Omega_{\mathbb{O}}$ is called be an octonion-valued metric on S, and the pair $(S, \Omega_{\mathbb{O}})$ is called be an octonion-valued metric space.

Example 1. *Let $\Omega_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ be an octonion-valued function defined as $\Omega_{\mathbb{O}}(\mathbb{o}, \mathbb{o}') = |o_0 - o_0'| + |o_1 - o_1'|e_1 + |o_2 - o_2'|e_2 + |o_2 - o_2'|e_2 + |o_3 - o_3'|e_3 + |o_4 - o_4'|e_4 + |o_5 - o_5'|e_5 + |o_6 - o_6'|e_6 + |o_7 - o_7'|e_7$, where $\mathbb{o}, \mathbb{o}' \in \mathbb{O}$ with*

$$\mathbb{o} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7,$$

$$\mathbb{o}' = o_0' + o_1'e_1 + o_2'e_2 + o_3'e_3 + o_4'e_4 + o_5'e_5 + o_6'e_6 + o_7'e_7;$$

$$o_i, o_i' \in \mathbb{R};\ i = 0, 1, 2, 3, 4, 5, 6, 7.$$

Then $(\mathbb{O}, \Omega_{\mathbb{O}})$ defines an octonion-valued metric space.

Below, we provide an example of an octonion-valued metric that does not have a known numerical set as its domain.

Example 2. *Let $X = \{a, b, c\}$ be an arbitrary set with three elements. Define the distances between the elements of the set by*

$$\begin{aligned}
\Omega_{\mathbb{O}}(a,b) &= \Omega_{\mathbb{O}}(b,a) = 3 + 4e_1 - 6e_2 + 4e_3 + 3e_4 + 3e_5 - 2e_6 + e_7,\\
\Omega_{\mathbb{O}}(b,c) &= \Omega_{\mathbb{O}}(c,b) = 1 + 2e_1 + 3e_3 - 5e_4 - 3e_6 + 4e_7,\\
\Omega_{\mathbb{O}}(a,c) &= \Omega_{\mathbb{O}}(c,a) = 2 + 3e_1 + e_2 + e_3 - 2e_4 + 2e_5 - e_6 + 5e_7,\\
\Omega_{\mathbb{O}}(a,a) &= \Omega_{\mathbb{O}}(b,b) = \Omega_{\mathbb{O}}(c,c) = 0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7.
\end{aligned}$$

Since they are $\|\Omega_{\mathbb{O}}(a,b)\| = 10$, $\|\Omega_{\mathbb{O}}(a,c)\| = 7$, $\|\Omega_{\mathbb{O}}(c,b)\| = 8$,$\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{195}$, $\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(b,c)\| = \sqrt{200}$ and $\|\Omega_{\mathbb{O}}(c,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{169} = 13$, it can be seen through straightforward calculations that the conditions given in Definition 3 above are satisfied.

The concept of an octonion-valued metric space is a logical extension of the classical definition of a metric, as well as of complex and quaternion-valued metrics; this is clearly evident from the definitions and examples provided above. To demonstrate the relationships among them, we now present the following propositions.

Proposition 1. *Every quaternion-valued metric space can be embedded into an octonion-valued metric space.*

Proposition 2. *Every complex-valued metric space can be embedded into a quaternion-valued metric space and an octonion-valued metric space.*

Proposition 3. *Every metric space can be embedded into a complex-valued metric space, a quaternion-valued metric space, and an octonion-valued metric space.*

3. Main results

The triangle inequality and the partial order relation in the definition of octonion-valued metric spaces were discussed in the previous section. In this section, we present a new generalization of metric spaces defined by slightly relaxing it. This is an intriguing generalization using octonions that are neither commutative nor associative.

3.1. Octonion-valued b-metric spaces

Definition 4. *Given a nonempty set S. If the transformation $\Omega_{\mathbb{O}} : S \times S \to \mathbb{O}$ on this set satisfies the following conditions,*

(1) $0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s,t)$ *for all* $s,t \in S$ *and* $\Omega_{\mathbb{O}}(s,t) = 0_{\mathbb{O}}$ *if and only if* $s = t$*,*
(2) $\Omega_{\mathbb{O}}(s,t) = \Omega_{\mathbb{O}}(t,s)$ *for all* $s,t \in S$*,*
(3) $\Omega_{\mathbb{O}}(s,t) \preceq b \cdot (\Omega_{\mathbb{O}}(s,v) + \Omega_{\mathbb{O}}(v,t))$ *for all* $s,t,v \in S$, $1 \leq b \in \mathbb{R}$.

Then $\Omega_{\mathbb{O}}$ is called an octonion-valued b-metric on S, and the pair $(S, \Omega_{\mathbb{O}})$ is called be an octonion-valued b-metric space.

Example 3. *Examples 1 and 2 are instances of octonion-valued* 1*-metric spaces for the real scalar* $b = 1$.

Remark 1. *It should be explicitly noted that, as seen from Definitions 3 and 4, every octonion-valued metric space is an octonion-valued b-metric space in the special case where* $b = 1$.

The converse of the remark we provided above is not true, except for the special case of $b = 1$. The next example we will present is an octonion-valued b-metric space for $b = 2$, yet it is not an octonion-valued metric space.

Example 4. *Let* $\Omega_{\mathbb{O}}^b : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ *be an octonion-valued function defined as* $\Omega_{\mathbb{O}}^b(\mathbb{o}, \mathbb{o}') = |o_0 - o_0'|^2 + |o_1 - o_1'|^2 e_1 + |o_2 - o_2'|^2 e_2 + |o_3 - o_3'|^2 e_3 + |o_4 - o_4'|^2 e_4 + |o_5 - o_5'|^2 e_5 + |o_6 - o_6'|^2 e_6 + |o_7 - o_7'|^2 e_7$*, where* $\mathbb{o}, \mathbb{o}' \in \mathbb{O}$ *with*

$$\mathbb{o} = o_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7,$$

$$\mathbb{o}' = o'_0 + o'_1 e_1 + o'_2 e_2 + o'_3 e_3 + o'_4 e_4 + o'_5 e_5 + o'_6 e_6 + o'_7 e_7;$$

$$o_i, o'_i \in \mathbb{R}; \ i = 0, 1, 2, 3, 4, 5, 6, 7.$$

Then $(\mathbb{O}, \Omega_{\mathbb{O}})$ *defines an octonion-valued b-metric space.*

Indeed, note that if we take

$$\begin{aligned} \mathbb{o} &= 3 + 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + 3e_6 + 3e_7, \\ \mathbb{o}' &= 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7, \\ \mathbb{o}'' &= 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7, \end{aligned}$$

although they are comparable under the partial ordering relation defined on octonions,

$$\begin{aligned} \Omega^b_{\mathbb{O}}(\mathbb{o}, \mathbb{o}'') &= 4 + 4e_1 + 4e_2 + 4e_3 + 4e_4 + 4e_5 + 4e_6 + 4e_7, \\ \Omega^b_{\mathbb{O}}(\mathbb{o}, \mathbb{o}') &= 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7, \\ \Omega^b_{\mathbb{O}}(\mathbb{o}', \mathbb{o}'') &= 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7, \\ \Omega^b_{\mathbb{O}}(\mathbb{o}, \mathbb{o}') + \Omega^b_{\mathbb{O}}(\mathbb{o}', \mathbb{o}'') &= 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7, \end{aligned}$$

which would violate the third property of the axioms for being an octonion-valued metric space as stated in Definition 3, making it not an octonion-valued metric space. However, if we take $b = 2$*, in this case, the partial ordering* $\preceq$ *satisfies the axioms in Definition 4.*

As can be seen from the definitions and example above, the definition we provided is a natural generalization of the classical *b*-metric definition, as well as complex and quaternion-valued *b*-metrics. To express the connections between them, let us present the following propositions.

Proposition 4. *Every quaternion-valued b-metric space can be embedded into an octonion-valued b-metric space.*

Proof. If we take $o_4 = o_5 = o_6 = o_7 = 0$ in the definition given in Definition 4, the desired result can be directly observed from Figure 1, and Definitions 2 and 3. □

Proposition 5. *Every complex-valued b-metric space can be embedded into a quaternion-valued b-metric space and an octonion-valued b-metric space.*

Proof. If we take $o_2 = o_3 = o_4 = o_5 = o_6 = o_7 = 0$ in the definition given in Definition 4, the desired result can be directly observed from Figure 1 and Definitions 1 and 2. □

Proposition 6. *Every b-metric space can be embedded into a complex-valued b-metric space, a quaternion-valued b-metric space, and an octonion-valued b-metric space.*

Proof. If we take $o_1 = o_2 = o_3 = o_4 = o_5 = o_6 = o_7 = 0$ in the definition given in Definition 4, the desired result can be directly observed from Figure 1, Definitions 1 and 2 and the definition of classical metric space. □

Categorically speaking, the diagrammatic representation of the above propositions and the transitions between these different metric space categories are as Figure 2:

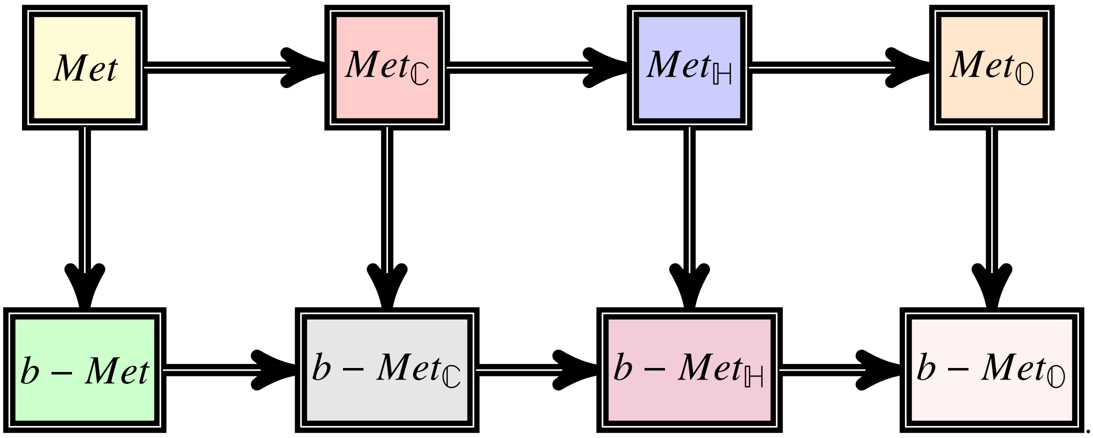

Figure 2. Interconnections between various generalizations of metric spaces.

From usual metric spaces, the transition to complex-valued metric spaces is achieved through the generalization of scalar fields. Further generalization to quaternion-valued metric spaces extends the integral domain, and non-associative, higher-dimensional extensions lead to octonion-valued metric spaces. Relaxing the triangle inequality for $b \geq 1$ introduces the categories of classical, complex-valued, quaternion-valued, and octonion-valued b-metric spaces. These transitions are facilitated by inclusion functors, while reverse transitions occur through forgetful functors. Here, we focus on the calculus aspects of octonion-valued b-metric spaces rather than their algebraic and categorical properties.

Thus, we can now proceed to define some fundamental concepts related to the definition above.

Definition 5. *Any point $s \in S$ is called an interior point of set $A \subset S$ whenever there exists $0_{\mathbb{O}} \prec r \in \mathbb{O}$ such that*

$$B(s, r) = \{t \in S : \Omega_{\mathbb{O}}(s, t) \prec r\} \subset A.$$

Definition 6. *Any point $s \in S$ is called a limit point of $A \subset S$ whenever for every $0_{\mathbb{O}} \prec r \in \mathbb{O}$*

$$B(s, r) \cap (A - \{s\}) \neq \emptyset.$$

Definition 7. *Set O is said to be an open set whenever each element of O is an interior point of O. Subset $C \subset S$ is called a closed set whenever each limit point of C belongs to C. The family*

$$F = \{B(s, r) : s \in S, 0_{\mathbb{O}} \prec r\},$$

is a subbase for Hausdorff topology τ on S.

3.2. The concept of convergence in the octonion-valued b-metric spaces

The octonion-valued b-metric spaces that we previously constructed will be analyzed in this part, along with the idea of convergence in these special mathematical structures. Using octonions, which provide a higher-dimensional, non-associative algebraic framework, octonion-valued b-metric spaces extend ordinary metric and b-metric spaces. Convergence and associated features will be defined, and the impact of octonions' non-associative nature on metric structure and convergence behaviors will be further investigated.

Definition 8. *Let $s \in S$, and s_k be a sequence in the set S. If for each $\mathbb{o} \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \mathbb{o}$ there is $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_k, s) \prec \mathbb{o}$, then (s_k) is called a convergence sequence. Then, in this case, the (s_k) sequence converges to the limit point s; as notation, $s_k \to s$ as $k \to \infty$ or $\lim_{k\to\infty} s_k = s$.*

Theorem 1. *Given an octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$, let (s_k) be a sequence in S. Then (s_k) converges to s if and only if $\|\Omega_{\mathbb{O}}(s_k, s)\| \to 0$ for $k \to \infty$.*

Proof. Let the sequence (s_k) converge to point s. Given a real number $\varepsilon > 0$, suppose that

$$\mathbb{o} = \frac{\varepsilon}{2\sqrt{2}} + e_1\frac{\varepsilon}{2\sqrt{2}} + e_2\frac{\varepsilon}{2\sqrt{2}} + e_3\frac{\varepsilon}{2\sqrt{2}} + e_4\frac{\varepsilon}{2\sqrt{2}} + e_5\frac{\varepsilon}{2\sqrt{2}} + e_6\frac{\varepsilon}{2\sqrt{2}} + e_7\frac{\varepsilon}{2\sqrt{2}}.$$

In this case, $0_{\mathbb{O}} \prec \mathbb{o} \in \mathbb{O}$, and there exists a natural number K such that $\Omega_{\mathbb{O}}(s_k, s) \prec \mathbb{o}$ for every $k > K$. Thus, $\|\Omega_{\mathbb{O}}(s_k, s)\| < \|\mathbb{o}\| = \varepsilon$ for all $k > K$. Hence, $\|\Omega_{\mathbb{O}}(s_k, s)\| \to 0$ as $k \to \infty$.

On the other hand, suppose that $\|\Omega_{\mathbb{O}}(s_k, s)\| \to 0$ as $k \to \infty$. In this case, given $\mathbb{o} \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \mathbb{o}$, there is a real number $\delta > 0$ such that, as $\mathbb{o}' \in \mathbb{O}$,

$$\|\mathbb{o}'\| < \delta \Longrightarrow \mathbb{o}' \prec \mathbb{o}.$$

Corresponding to this δ, there exists a natural number K such that $\|\Omega_{\mathbb{O}}(s_k, s)\| < \delta$ for every $k > K$. This implies that $\Omega_{\mathbb{O}}(s_k, s) \prec \mathbb{o}$ for every $k > K$, hence the sequence (s_k) converges to point s. □

Theorem 2. *Let (s_k) be a sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$, and let the limit of this sequence be $s_0 \in S$. In this case, the limit s_0 is unique.*

Proof. Suppose that both $\lim_{k\to\infty} s_k = s_0$ and $\lim_{k\to\infty} s_k = t_0$. By the third axiom in the definition of the octonion-valued b-metric space, we have

$$0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s_0, t_0) \preceq b \cdot (\Omega_{\mathbb{O}}(s_0, s_n) + \Omega_{\mathbb{O}}(s_n, t_0)),$$

and by the partial ordering, we obtain

$$0 \leq \|\Omega_{\mathbb{O}}(s_0, t_0)\| \leq \|b \cdot (\Omega_{\mathbb{O}}(s_0, s_n) + \Omega_{\mathbb{O}}(s_n, t_0))\| \leq b \cdot (\|\Omega_{\mathbb{O}}(s_0, s_n)\| + \|\Omega_{\mathbb{O}}(s_n, t_0)\|) = b \cdot (0 + 0) = 0.$$

From this, it follows that $\|\Omega_{\mathbb{O}}(s_0, t_0)\| = 0$. Thus, we have $\Omega_{\mathbb{O}}(s_0, t_0) = 0_{\mathbb{O}}$. Finally, by the first axiom of the octonion-valued b-metric space, we obtain $s_0 = t_0$. This completes the proof. □

Corollary 1. *In both cases, the quaternion-valued b-metric space $(S, \Omega_{\mathbb{H}})$ and the complex-valued b-metric space $(S, \Omega_{\mathbb{C}})$, the limit is unique.*

Proof. This can be directly seen from Propositions 4 and 5, respectively. □

Definition 9. *Let $(S, \Omega_{\mathbb{O}})$ be an octonion-valued b-metric space and A be a non-empty subset of S. In this b-metric space, the diameter of the set A is denoted by*

$$Diam(A) = sup\Big\{||\Omega_{\mathbb{O}}(s,t)|| : s,t \in A\Big\}.$$

If $Diam(A) < \infty$, then the set A is said to be bounded in S. A sequence (s_k) in S is called a bounded sequence in S if the set of all terms of the sequence (s_k) is bounded in S.

Theorem 3. *Let (s_k) be a sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$, and let the limit of this sequence be $s_0 \in S$. In this case, the sequence (s_k) is bounded.*

Proof. For every $k \in \mathbb{N}$ and arbitrary $a \in S$, by the third axiom of the octonion-valued $b-$metric space definition, since it is

$$\Omega_{\mathbb{O}}(s_k, a) \preceq b \cdot (\Omega_{\mathbb{O}}(s_0, s_k) + \Omega_{\mathbb{O}}(a, s_0)),$$

the inequality

$$||\Omega_{\mathbb{O}}(s_k, a)|| \leq b \cdot (||\Omega_{\mathbb{O}}(s_0, s_n) + \Omega_{\mathbb{O}}(a, s_0)||)$$

holds. Since $\lim_{k\to\infty} s_k = s_0$, for $0_{\mathbb{O}} \preceq \varepsilon$, there exists $k_1 \in \mathbb{N}$ such that for all $k > k_1$, $\Omega_{\mathbb{O}}(s_0, s_k) \preceq 1_{\mathbb{O}}$ holds. In this case, since it is $||\Omega_{\mathbb{O}}(s_k, s_0)|| \leq 1$, if we take

$$M = max\Big\{||\Omega_{\mathbb{O}}(s_1, s_0)||, ||\Omega_{\mathbb{O}}(s_2, s_0)||, \cdots, ||\Omega_{\mathbb{O}}(s_{k_1}, s_0)||, 1 + ||\Omega_{\mathbb{O}}(a, s_0)||\Big\},$$

then for $\forall k \in \mathbb{N}$, it will be $||\Omega_{\mathbb{O}}(s_k, a)|| \leq M$. This means that (s_k) is a bounded sequence in S. This completes the proof. □

Corollary 2. *A convergent sequence is bounded in both the quaternion-valued b-metric space $(S, \Omega_{\mathbb{H}})$ and the complex-valued b-metric space $(S, \Omega_{\mathbb{C}})$.*

Proof. This can be directly seen from Definition 9, and Propositions 4 and 5, respectively. □

Theorem 4. *Let (s_k) be a sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. If the sequence (s_k) converges to the point s_0, then any arbitrary subsequence (s_{k_n}) also converges, and this subsequence converges to the point s_0.*

Proof. Let (s_{k_n}) be an arbitrary subsequence of the sequence (s_k). Given that $\lim_{k\to\infty} s_k = s_0$, for every $0_{\mathbb{O}} \prec \varepsilon$, there exists some $k_\varepsilon \in \mathbb{N}$ such that for all $k > k_\varepsilon$, it is $\Omega_{\mathbb{O}}(s_k, s_0) \prec \varepsilon$. Therefore, as $\lim_{n\to\infty} k_n = \infty$ approaches, there exists some $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$, it is $\Omega_{\mathbb{O}}(s_{k_n}, s_0) \prec \varepsilon$. From this, we obtain $\lim_{k_n\to\infty} s_{k_n} = s_0$. This completes the proof. □

Corollary 3. *In both the quaternion-valued b-metric space $(S, \Omega_{\mathbb{H}})$ and the complex-valued b-metric space $(S, \Omega_{\mathbb{C}})$, every subsequence of a convergent sequence converges to the same point.*

Proof. This can be directly seen from Theorem 4, Propositions 4 and 5, respectively. □

Definition 10. *If there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_{k+m}, s_k) \prec \mathbb{o}$, then (s_k) is said to be a Cauchy sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. If every Cauchy sequence is convergent in $(S, \Omega_{\mathbb{O}})$, then $(S, \Omega_{\mathbb{O}})$ is said to be a complete octonion-valued b-metric space.*

Note that not every octonion-valued b-metric space must be complete. The following example of an octonion-valued b-metric space supports this.

Example 5. *Let $\Omega_{\mathbb{O}} : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{O}$ be an octonion-valued function defined as*

$$\Omega_{\mathbb{O}}(n,m) = |\frac{1}{n} - \frac{1}{m}|^2 + |\frac{2}{n} - \frac{2}{m}|^2 e_1 + |\frac{3}{n} - \frac{3}{m}|^2 e_2 + |\frac{4}{n} - \frac{4}{m}|^2 e_3 + |\frac{5}{n} - \frac{5}{m}|^2 e_4 + |\frac{6}{n} - \frac{6}{m}|^2 e_5 + |\frac{7}{n} - \frac{7}{m}|^2 e_6 + |\frac{8}{n} - \frac{8}{m}|^2 e_7,$$

where $n, m \in \mathbb{N}^+$. Then $(\mathbb{N}^+, \Omega_{\mathbb{O}})$ defines an octonion-valued b-metric space.

However, since it is $0 \notin \mathbb{N}^+$, this octonion-valued b-metric space is not complete.

Theorem 5. *Every convergent sequence in an octonion-valued b-metric space is a Cauchy sequence.*

Proof. Let (s_k) be a sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. Suppose that $\lim_{k\to\infty} s_k = s_0$. In this case, for every $0_{\mathbb{O}} \prec \varepsilon$, there exists some $k_\varepsilon \in \mathbb{N}$ such that for all $k, l > k_\varepsilon$, by the definition of the partial ordering relation given above, and since $0_{\mathbb{O}} \prec \varepsilon \in \mathbb{O}$, the octonion $\frac{\varepsilon}{2 \cdot b}$ holds for $0_{\mathbb{O}} \prec \frac{\varepsilon}{2 \cdot b}$, moreover, they are $\Omega_{\mathbb{O}}(s_k, s_0) \prec \frac{\varepsilon}{2 \cdot b}$ and $\Omega_{\mathbb{O}}(s_l, s_0) \prec \frac{\varepsilon}{2 \cdot b}$. Therefore, as $k, l > k_\varepsilon$, by the third axiom of the octonion-valued metric space definition, since it is

$$\Omega_{\mathbb{O}}(s_k, s_l) \preceq b \cdot (\Omega_{\mathbb{O}}(s_0, s_k) + \Omega_{\mathbb{O}}(a, s_0)) = b \cdot (\frac{\varepsilon}{2 \cdot b} + \frac{\varepsilon}{2 \cdot b}) = \varepsilon.$$

Since $\Omega_{\mathbb{O}}(s_k, s_l) \preceq \varepsilon$ holds for every $0_{\mathbb{O}} \prec \varepsilon$, (s_k) is a Cauchy sequence. This completes the proof. □

Proposition 7. *Every convergent sequence is also a Cauchy sequence in both quaternion-valued b-metric spaces and complex-valued b-metric spaces.*

Proof. This can be directly seen from Theorem 5, Propositions 4 and 5, respectively. □

Theorem 6. *Given an octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$, let (s_k) be a sequence in S. Then (s_k) is a Cauchy sequence if and only if $\|\Omega_{\mathbb{O}}(s_k, s_{k+m})\| \to 0$ as $k \to \infty$.*

Proof. We assume that (s_k) is a Cauchy sequence in S. As a given real number $\varepsilon > 0$, suppose that

$$\mathbb{o} = \frac{\varepsilon}{2\sqrt{2}} + e_1 \frac{\varepsilon}{2\sqrt{2}} + e_2 \frac{\varepsilon}{2\sqrt{2}} + e_3 \frac{\varepsilon}{2\sqrt{2}} + e_4 \frac{\varepsilon}{2\sqrt{2}} + e_5 \frac{\varepsilon}{2\sqrt{2}} + e_6 \frac{\varepsilon}{2\sqrt{2}} + e_7 \frac{\varepsilon}{2\sqrt{2}}.$$

In this case, $0_{\mathbb{O}} \prec \mathbb{o} \in \mathbb{O}$, and there exists a natural number K such that $\Omega_{\mathbb{O}}(s_k, s_{k+m}) \prec \mathbb{o}$ for every $k > K$. Then, $\|\Omega_{\mathbb{O}}(s_k, s_{k+m})\| < \|\mathbb{o}\| = \varepsilon$ for every $k > K$. Thereby, $\|\Omega_{\mathbb{O}}(s_k, s_{k+m})\| < \|\mathbb{o}\| \to 0$ as $k \to \infty$.

On the other hand, we assume that $||\Omega_{\mathbb{O}}(s_k, s_{k+m})|| < ||\mathbb{o}|| \to 0$ as $k \to \infty$. So, given $\mathbb{o} \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \mathbb{o}$, there is a real number $\delta > 0$ such that as $\mathbb{o}' \in \mathbb{O}$,

$$||\mathbb{o}'|| < \delta \Longrightarrow \mathbb{o}' \prec \mathbb{o}.$$

Corresponding to this δ, there exists a natural number K such that $||\Omega_{\mathbb{O}}(s_k, s_{k+m})|| < \delta$ for every $k > K$, implying that $\Omega_{\mathbb{O}}(s_k, s_{k+m}) \prec \mathbb{o}$ for every $k > K$. So, (s_k) is a Cauchy sequence. Thus, the proof is complete. □

Theorem 7. *Let (s_k) be a Cauchy sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. In this case, the sequence (s_k) is bounded.*

Proof. Assume that (s_k) is a Cauchy sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. In this case, $0_{\mathbb{O}} \preceq \varepsilon$, there exists $k_1 \in \mathbb{N}$ such that for all $k, l > k_1$, $\Omega_{\mathbb{O}}(s_l, s_k) \prec \varepsilon \preceq 1_{\mathbb{O}}$ holds. Therefore, since it is $||\Omega_{\mathbb{O}}(s_k, s_l)|| \leq 1$, if we take

$$M = max\Big\{||\Omega_{\mathbb{O}}(s_1, s_{k_1})||, ||\Omega_{\mathbb{O}}(s_2, s_{k_1})||, \cdots, ||\Omega_{\mathbb{O}}(s_{k_1}, s_{k_1})||\Big\},$$

for every $k \in \mathbb{N}$ we have $||\Omega_{\mathbb{O}}(s_n, s_{k_1})|| < M + 1$. So, for $\forall k, l \in \mathbb{N}$ by the third axiom of the octonion-valued b-metric space definition, since it is

$$\Omega_{\mathbb{O}}(s_k, s_l) \preceq b \cdot (\Omega_{\mathbb{O}}(s_k, s_{k_1}) + \Omega_{\mathbb{O}}(s_{k_1}, s_l)),$$

the inequality

$$||\Omega_{\mathbb{O}}(s_k, s_l)|| \leq b \cdot (||\Omega_{\mathbb{O}}(s_k, s_{k_1}) + \Omega_{\mathbb{O}}(s_{k_1}, s_l)||) \leq b \cdot (||\Omega_{\mathbb{O}}(s_k, s_{k_1})|| + ||\Omega_{\mathbb{O}}(s_{k_1}, s_l)||) \leq b \cdot (M + 1)$$

holds. This means that (s_k) is a bounded Cauchy sequence in S. This completes the proof. □

Corollary 4. *A Cauchy sequence is bounded in both the quaternion-valued b-metric space $(S, \Omega_{\mathbb{H}})$ and the complex-valued b-metric space $(S, \Omega_{\mathbb{C}})$.*

Proof. This can be directly seen from Theorem 7, Propositions 4 and 5, respectively. □

Theorem 8. *If (s_k) is a Cauchy sequence in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$, has the subsequence (s_{k_n}) converges to the point s_0, then Cauchy sequence (s_k) also converges, and this Cauchy sequence converges to the point s_0.*

Proof. Let (s_{k_n}) be an arbitrary convergence subsequence of the Cauchy sequence (s_k) in the octonion-valued b-metric space $(S, \Omega_{\mathbb{O}})$. Given that $\lim\limits_{k_n \to \infty} s_{k_n} = s_0$, for every $0_{\mathbb{O}} \prec \varepsilon$, there exists some $k_1 = k_1(\varepsilon) \in \mathbb{N}$ such that for all $k > k_1$, it is $\Omega_{\mathbb{O}}(s_{k_n}, s_0) \prec \frac{\varepsilon}{2 \cdot b}$. Furthermore, since (s_k) is a Cauchy sequence, there exists some $k_\varepsilon \in \mathbb{N}$ such that for all $k, l > k_\varepsilon$, it is $\Omega_{\mathbb{O}}(s_k, s_l) \prec \frac{\varepsilon}{2 \cdot b}$. Moreover, as $\lim\limits_{n \to \infty} k_n = \infty$ approaches, there exists some $n_2 = n_2(\varepsilon) \in \mathbb{N}$ such that for $\forall n > k_2$, $k_n > k_\varepsilon$. From these three results,

$$\Omega_{\mathbb{O}}(s_n, s_0) \preceq b \cdot (\Omega_{\mathbb{O}}(s_n, s_{k_n}) + \Omega_{\mathbb{O}}(s_{k_n}, s_0)) \prec b \cdot (\frac{\varepsilon}{2 \cdot b} + \frac{\varepsilon}{2 \cdot b}) = \varepsilon$$

is obtained for $n > n_\varepsilon = max\{k_\varepsilon, n_1, n_2\}$.

From this, we obtain $\lim\limits_{k \to \infty} s_k = s_0$. This completes the proof. □

Corollary 5. *In both the quaternion-valued b-metric space* $(S, \Omega_{\mathbb{H}})$ *and the complex-valued b-metric space* $(S, \Omega_{\mathbb{C}})$*, if an arbitrary Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges to the same point.*

Proof. This can be directly seen from Theorem 8, Propositions 4 and 5, respectively. □

4. Applications of the concept of convergence

Numerous domains of mathematics, including fixed point theory, differential equations, numerical analysis, general topology, functional analysis, probability theory, machine learning and optimization, and algorithm analysis, have found use for the idea of convergence. We will concentrate on the applications of convergence in fixed point theory and provide some associated theorems and conclusions, because it is not practical for us to cover all of these applications here.

4.1. Basic fixed point theorems in octonion-valued b-metric spaces

Convergence is a fundamental aspect in the study of fixed point theorems, where iterative methods are used to show the existence and uniqueness of fixed points in various spaces. This is particularly useful in optimization problems, economics, and game theory. In this subsection, we will present several theorems using the newly provided definitions and theorems above. These theorems in the octonion-valued b-metric space we defined will be generally useful in the formulation and proof of basic fixed point theorems, which we will discuss subsequently. In metric spaces fixed point theory, along with its applications and numerous generalizations, have been studied [16, 19, 30]. In addition to studies conducted in metric spaces, some results have also been established in Banach spaces [12, 13], and various other types of spaces [1, 14, 17], for different classes of functions, for instance, Greguš type theorems [5, 31],

In this section, we present some fundamental fixed point theorems in the framework of complete octonion-valued b-metric spaces. After constructing the notion of a b-metric over octonions, we provide a concrete example of a contraction mapping within this newly defined structure to illustrate the applicability of the developed theory. The results presented here primarily focus on extensions of the classical Banach fixed point theorem to the octonion-valued b-metric setting. We have established several basic fixed point theorems that serve as initial developments in this framework.

It is important to note that we have limited our analysis to Banach-type contractions and some elementary generalizations. Further generalizations, such as Kannan-type or Chatterjea-type contractions, and the exploration of broader classes of fixed point theorems within octonion-valued b-metric spaces, remain open for future research. Therefore, the study of various types of contractions in this setting constitutes a rich and promising direction for further investigations.

Theorem 9. *Let* $(S, \Omega_{\mathbb{O}})$ *be a complete b-metric space over octonions, where* $b \geq 1$*. Suppose* $F : S \longrightarrow S$ *is a mapping such that for all* $s, t \in S$*, the following condition holds:*

$$\Omega_{\mathbb{O}}(F(s), F(t)) \preceq \alpha \Omega_{\mathbb{O}}(s, t), \tag{4.1}$$

where $\alpha \in [0, \frac{1}{b})$. Under this condition, F possesses a unique fixed point in S.

Proof. Let F satisfy condition (4.1), and let $s_0 \in S$ be an arbitrary point. Define the sequence (s_k) by $s_k = F^k(s_0)$. Using (4.1), it follows that

$$\Omega_{\mathbb{O}}(s_k, s_{k+1}) \preceq \alpha \Omega_{\mathbb{O}}(s_{k-1}, s_k). \tag{4.2}$$

Applying (4.1) iteratively, we get

$$\Omega_{\mathbb{O}}(s_{k-1}, s_k) \preceq \alpha \Omega_{\mathbb{O}}(s_{k-2}, s_{k-1}),$$

and by substituting this into (4.2),

$$\Omega_{\mathbb{O}}(s_k, s_{k+1}) \preceq \alpha^2 \Omega_{\mathbb{O}}(s_{k-2}, s_{k-1}).$$

Continuing this process yields

$$\Omega_{\mathbb{O}}(s_k, s_{k+1}) \preceq \alpha^n \Omega_{\mathbb{O}}(s_0, s_1). \tag{4.3}$$

Now, using the third property of the b-metric space on octonions and condition (4.3) for all $k, l \in \mathbb{N}$ with $k < l$, we deduce

$$\Omega_{\mathbb{O}}(s_k, s_l) \preceq b \sum_{i=k}^{l-1} \Omega_{\mathbb{O}}(s_i, s_{i+1}) \preceq b\alpha^k \Omega_{\mathbb{O}}(s_0, s_1)[1 + b\alpha + (b\alpha)^2 + \cdots + (b\alpha)^{l-k-1}].$$

By summing the geometric series and simplifying, we find

$$\Omega_{\mathbb{O}}(s_k, s_l) \preceq \frac{b\alpha^k}{1 - b\alpha} \Omega_{\mathbb{O}}(s_0, s_1).$$

Since $\alpha \in [0, \frac{1}{b})$ and $b > 1$, we have

$$\lim_{k\to\infty} \frac{b\alpha^k}{1 - b\alpha} \Omega_{\mathbb{O}}(s_0, s_1) = 0_{\mathbb{O}}.$$

This implies $\Omega_{\mathbb{O}}(s_k, s_l) \to 0$, meaning (s_k) is an octonion-valued Cauchy sequence. By the completeness of $(S, \Omega_{\mathbb{O}})$, there exists a unique $u \in S$ such that (s_k) converges to u.

To prove that u is a fixed point of F, consider $\Omega_{\mathbb{O}}(u, Fu)$. For any $k \in \mathbb{N}$, we have

$$\Omega_{\mathbb{O}}(u, Fu) \preceq b[\Omega_{\mathbb{O}}(u, s_k) + \Omega_{\mathbb{O}}(s_k, Fu)].$$

Substituting from (4.2) and taking $k \to \infty$, since $s_k \to u$, we obtain $\Omega_{\mathbb{O}}(u, Fu) = 0$, which implies $Fu = u$.

Finally, we show the uniqueness. Suppose $w \neq u$ is another fixed point of F. Then, using (4.1),

$$\Omega_{\mathbb{O}}(w, u) = \Omega_{\mathbb{O}}(Fw, Fu) \preceq \alpha \Omega_{\mathbb{O}}(w, u).$$

Since $\alpha \in [0, \frac{1}{b})$, we have $|\Omega_{\mathbb{O}}(w, u)| \leq \alpha |\Omega_{\mathbb{O}}(w, u)|$, implying $|\Omega_{\mathbb{O}}(w, u)| = 0$, or $w = u$. Thus, u is the unique fixed point of F. □

Example 6. *Let $\Omega^b_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ be an octonion-valued b-metric defined as in Example 4. Let $\mathbb{o} = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$ be a fixed octonion, and define the function $F : \mathbb{O} \to \mathbb{O}$ by*

$$F(x) = \frac{x + \mathbb{o}}{4}.$$

$(\mathbb{O}, \Omega^b_{\mathbb{O}})$ is an octonion-valued complete b-metric space satisfying the condition

$$\Omega_{\mathbb{O}}(F(s), F(t)) \preceq \alpha \Omega_{\mathbb{O}}(s, t),$$

where since $b = 2$, it follows that $\alpha = \frac{1}{16} \in [0, \frac{1}{2})$. Under this condition, F possesses a unique fixed point in $\mathbb{O}$. It can be seen through simple calculations that this fixed point is $\frac{\mathbb{o}}{3} = \frac{1}{3} + \frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3 + \frac{1}{3}e_4 + \frac{1}{3}e_5 + \frac{1}{3}e_6 + \frac{1}{3}e_7$.

Theorem 10. *Let $(S, \Omega_{\mathbb{O}})$ be a complete octonion-valued b-metric space, where $b > 1$ is an integer. Assume $F : S \longrightarrow S$ is a continuous mapping satisfying the condition:*

$$\Omega_{\mathbb{O}}(s, F(s)) \preceq \frac{1}{b^l}(\phi(s) - \phi(F(s))), \tag{4.4}$$

for all $s \in S$ and integers $l \geq 0$, where $\phi : S \to \mathbb{O}$ is a function. Under this condition, the sequence $\{F^k(s)\}$ converges to a fixed point of F for all $s \in S$.

Proof. For any fixed $s \in S$, let $s_k = F^k(s)$ with $k \in \mathbb{N}$. From (4.4), we derive

$$0_{\mathbb{O}} \preceq \frac{1}{b^l}(\phi(s) - \phi(F(s))) \iff \phi(s) \preceq \phi(F(s)),$$

for all $s \in S$. Consequently, the sequence $(\phi(F^k(s)))$ satisfies:

$$\phi(s_{k+1}) = \phi(F^{k+1}(s)) = \phi(F(F^k(s))) = \phi(F(s_k)) \preceq \phi(s_k).$$

This shows that $(\phi(F^k(s)))$ is monotonically decreasing and bounded below. Therefore, it converges to some $0_{\mathbb{O}} \preceq \mathbb{o} \in \mathbb{O}$, i.e.,

$$\lim_{k\to\infty} \phi(F^k(s)) = \mathbb{o}.$$

Now, for $l, k \in \mathbb{N}$ with $l > k$, using the third axiom of the b-metric space on octonions and condition (4.4), we get

$$\begin{aligned}
\Omega_{\mathbb{O}}(s_k, s_l) &\preceq b \cdot (\Omega_{\mathbb{O}}(s_k, s_{k+1}) + \Omega_{\mathbb{O}}(s_{k+1}, s_l)) = b \cdot \Omega_{\mathbb{O}}(s_k, s_{k+1}) + b \cdot \Omega_{\mathbb{O}}(s_{k+1}, s_l) \\
&\preceq b \cdot \Omega_{\mathbb{O}}(s_k, s_{k+1}) + b \cdot \Big(b \cdot (\Omega_{\mathbb{O}}(s_{k+1}, s_{k+2}) + \Omega_{\mathbb{O}}(s_{k+2}, s_l))\Big) \\
&\cdots \\
&\cdots \\
&\cdots \\
&\preceq b \cdot \Omega_{\mathbb{O}}(s_k, s_{k+1}) + b^2 \cdot \Omega_{\mathbb{O}}(s_{k+1}, s_{k+2}) + \cdots + b^{l-k} \cdot \Omega_{\mathbb{O}}(s_{l-1}, s_l)
\end{aligned}$$

and substituting $\Omega_{\mathbb{O}}(s_i, s_{i+1}) \preceq \frac{1}{b^l}(\phi(s_i) - \phi(s_{i+1}))$, we have:

$$\begin{aligned}\Omega_{\mathbb{O}}(s_k, s_l) &\preceq \frac{b}{b^l}[\phi(s_k) - \phi(s_{k+1})] + \frac{b^2}{b^l}[\phi(s_{k+1}) - \phi(s_{k+2})] + \cdots + \frac{b^{l-k}}{b^l}[\phi(s_{l-1}) - \phi(s_l)] \\ &= \frac{b}{b^l} \cdot \phi(s_k) + \frac{b^2 - b}{b^l} \cdot \phi(s_{k+1}) + \frac{b^3 - b^2}{b^l} \cdot \phi(s_{k+2}) + \cdots + \frac{b^{l-k} - b^{l-k-1}}{b^l}\phi(s_{l-1}) - \frac{b^{l-k}}{b^l}\phi(s_l).\end{aligned}$$

Since $\lim_{k\to\infty} \phi(s_k) = \mathbb{o}$, it follows that

$$\lim_{k,l\to\infty} \Omega_{\mathbb{O}}(s_k, s_l) = 0_{\mathbb{O}}.$$

This shows that (s_k) is a Cauchy sequence in S. By Theorem 6 and the completeness of $(S, \Omega_{\mathbb{O}})$, there exists a point $u \in S$ such that:

$$\lim_{k\to\infty} F^n(s) = u.$$

Finally, by the continuity of F, we conclude that:

$$u = F(u).$$

Thus, u is the fixed point of F. □

Theorem 11. *Let $(S, \Omega_{\mathbb{O}})$ be an octonion-valued complete b-metric space, and let $\psi : \mathbb{O} \longrightarrow \mathbb{O}$ be a monotone nondecreasing function such that $\lim_{k\to\infty} \psi^k(\mathbb{o}) = 0_{\mathbb{O}}$ for all $0_{\mathbb{O}} \prec \mathbb{o} \in \mathbb{O}$. If $F : S \longrightarrow S$ satisfies the contraction condition*

$$\Omega_{\mathbb{O}}(F(s), F(t)) \preceq \psi(\Omega_{\mathbb{O}}(s, t)), \tag{4.5}$$

for all $s, t \in S$, then F has a unique fixed point $u \in S$, and $\lim_{k\to\infty} \Omega_{\mathbb{O}}(F^k(s), u) = 0$ for all $s \in S$.

Proof. To demonstrate the existence of a fixed point, let $s \in S$ and define the sequence $s_k = F^k(s)$ for $k \in \mathbb{N}$. If $s_1 = F(s) = s$, then s is a fixed point of F. Assume $s_1 = F(s) \neq s$. By the contraction condition:

$$\Omega_{\mathbb{O}}(s_k, s_{k+1}) \preceq \psi(\Omega_{\mathbb{O}}(s_{k-1}, s_k)) \preceq \psi^2(\Omega_{\mathbb{O}}(s_{k-2}, s_{k-1})) \preceq \cdots \preceq \psi^k(\Omega_{\mathbb{O}}(s, F(s))).$$

Since $\lim_{k\to\infty} \psi^k(\mathbb{o}) = 0_{\mathbb{O}}$ for all $0_{\mathbb{O}} \prec \mathbb{o} \in \mathbb{O}$, we have:

$$0_{\mathbb{O}} \preceq \lim_{k\to\infty} \Omega_{\mathbb{O}}(s_k, s_{k+1}) = \lim_{k\to\infty} \psi^k(\Omega_{\mathbb{O}}(s, F(s))) = 0_{\mathbb{O}}.$$

Thus:

$$\lim_{k\to\infty} \Omega_{\mathbb{O}}(s_k, s_{k+1}) = 0_{\mathbb{O}}. \tag{4.6}$$

We now show that (s_k) is a Cauchy sequence. Let $0_{\mathbb{O}} \prec \varepsilon \in \mathbb{O}$ and $b > 0$. Given that $\psi(\varepsilon) \prec \frac{\varepsilon}{2b}$, we have:

$$\Omega_{\mathbb{O}}(s_k, s_{k+1}) \preceq \frac{\varepsilon}{2b}.$$

For the set $B_\varepsilon[s_k] = \{s \in S : \Omega_{\mathbb{O}}(s, s_{k+1}) \preceq \varepsilon\}$, if $w \in B_\varepsilon[s_k]$, then $\Omega_{\mathbb{O}}(w, s_{k-1}) \preceq \varepsilon$ and

$$\begin{aligned}\Omega_{\mathbb{O}}(F(w), s_{k+1}) &\preceq b \cdot \Big[\Omega_{\mathbb{O}}(F(w), s_k) + \Omega_{\mathbb{O}}(s_k, s_{k+1})\Big]\\ &\preceq b \cdot \Big[\Omega_{\mathbb{O}}(F(w), F(s_{k-1})) + \Omega_{\mathbb{O}}((F(s_{k-1}), s_{k+1}))\Big]\\ &\preceq b \cdot \Big[\psi(\Omega_{\mathbb{O}}(w, s_k)) + \Omega_{\mathbb{O}}(s_k, s_{k+1})\Big]\\ &\preceq b \cdot \Big[\psi(\varepsilon) + \frac{\varepsilon}{2 \cdot b}\Big]\\ &\preceq b \cdot \Big[\frac{\varepsilon}{2 \cdot b} + \frac{\varepsilon}{2 \cdot b}\Big] = \varepsilon,\end{aligned}$$

the repeated application of the contraction condition ensures that,

$$\lim_{k\to\infty} \Omega_{\mathbb{O}}(s_k, s_{k+1}) = 0_{\mathbb{O}}, \quad \text{and} \quad \lim_{k,l\to\infty} \Omega_{\mathbb{O}}(s_k, s_l) = 0_{\mathbb{O}}.$$

This implies that (s_k) is a Cauchy sequence. Since $(S, \Omega_{\mathbb{O}})$ is complete, there exists $u \in S$ such that

$$\lim_{k\to\infty} F^k(s) = u. \tag{4.7}$$

To show that u is a fixed point of F, we use the continuity of F. By definition:

$$\Omega_{\mathbb{O}}(u, F(u)) = \lim_{k\to\infty} \Omega_{\mathbb{O}}(F^k(s), F^{k+1}(s)).$$

From (1), $\lim_{k\to\infty} \Omega_{\mathbb{O}}(F^k(s), F^{k+1}(s)) = 0_{\mathbb{O}}$, which implies:

$$\Omega_{\mathbb{O}}(u, F(u)) = 0_{\mathbb{O}} \implies F(u) = u.$$

Thus, u is a fixed point of F.

To demonstrate the uniqueness of a fixed point, assume $u, v \in S$ are two distinct fixed points of F. Using the contraction condition (4.5), we have:

$$\Omega_{\mathbb{O}}(u, v) = \Omega_{\mathbb{O}}(F(u), F(v)) \preceq \psi(\Omega_{\mathbb{O}}(u, v)) \preceq \cdots \preceq \psi^k(\Omega_{\mathbb{O}}(u, v)).$$

Taking the limit as $k \to \infty$, and using the fact that $\lim_{k\to\infty} \psi^k(t) = 0_{\mathbb{O}}$ for all $0_{\mathbb{O}} \prec ⅼ \in \mathbb{O}$, we get,

$$\Omega_{\mathbb{O}}(u, v) = 0_{\mathbb{O}} \implies u = v.$$

As a result, the mapping F has a unique fixed point $u \in S$, and the sequence $\{F^k(s)\}$ converges to u for all $s \in S$. □

Remark 2. *It is well known that every field forms a vector space over itself and every ring forms a module over itself. However, let us explicitly note that octonions, lacking multiplicative associativity, do not even qualify as a ring, and therefore cannot form a module over themselves. This makes the metric spaces we have defined and the related results particularly interesting.*

5. Conclusions

In this work, we developed the theory of octonion-valued b-metric spaces, introduced fundamental topological concepts, and extended the Banach fixed point theorem into this new setting. By constructing a partial ordering on octonions and proving the existence of fixed points under contraction conditions, we have demonstrated that fixed point theory can be effectively applied even in non-associative algebras. Our findings open the way for future research, including the study of Kannan-type or Chatterjea-type contractions, and potential applications in areas such as non-associative geometry and theoretical physics, where octonionic structures naturally arise.

Author Contributions

Xiu-Liang Qiu: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing, funding acquisition; Selim Çetin: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing; Ömer Kişi: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing; Mehmet Gürdal: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing; Qing-Bo Cai: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing, funding acquisition. All authors contributed equally to this work and have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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